Research Article

Computing Edge Weights of Symmetric Classes of Networks

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Accessibility, robustness, and connectivity are the salient structural properties of networks. The labelling of networks with numeric numbers using the parameters of edge or vertex weights plays an eminent role in the study of the aforesaid properties. The systems interlinked in a network are transformed into a graphical network, and specific numeric labels assigned to the converted network under certain rules assist us in the regulation of data traffic, bandwidth, and coding/decoding of signals. Two major classes of such network labellings are magic and antimagic. The notion of super $(a, 0)$ edge-antimagic labelling on networks was identified in the late nineties. The present article addresses super $(a, 0)$ edge-antimagicness of union of the networks' star S_n , the path P_n , and copies of paths and the rooted product of cycle C_n with $K_{2,m}$. We also provide super $(a, 0)$ edge-antimagic labelling of the rooted product of cycle C_n and planar pancyclic networks. Further, we design a super $(a, 0)$ edge-antimagic labelling on a pancyclic network containing chains of C_6 and three different symmetrically designed lattices. Moreover, our findings have also been recapitulated in the shape of 3-D plots and tables.

1. Introduction

In this section, we shall define our problem and explain the objective of this study in Section 1.1, followed by Section 1.2, consisting of the definitions and results which we will use in our findings. Some previously performed work in this area will also be discussed in this section. Moreover, Section 1.3 concerns with applications of antimagic and magic labelling in various branches of networking, engineering, and computer science.

1.1. Problem Definition and Objective of the Study. In the fields of networking and computer science, the magic and antimagic labelling on networks are designed due to their extensive applications. Numerous results have been obtained on numeric labelling of several operations on networks such as Cartesian, lexicographical, corona, and modular products of various kinds of connected networks

(see [1,2] for instance). The present article addresses super $(a, 0)$ edge-antimagic labelling of the rooted product of $K_{2,m}$ and C_n taking its disjoint union with the star, path, and copies of paths. We shall also design super $(a, 0)$ edge-antimagic labelling on rooted product of specifically designed planar pancyclic networks with cycle C_n . Moreover, we shall design super $(a, 0)$ edge-antimagic labelling on planar pancyclic networks containing chains of C_6 and three different symmetric lattice networks (notated as $\mathbb{L}_n^1$, $\mathbb{L}_n^2$, and $\mathbb{L}_n^3$). Except lattice networks, interestingly, all networks discussed in this note are planar. The overlapping probability of various networking elements minimizes in the course of planar networks. In organizations, this issue of entities' overlapping is one of the major reasons of inefficiency. The test ready antimagic labellings obtained in this note on particular networks can be utilized in various projects of computer science and engineering admitting suitable and equivalently designed schemes of networking.

1.2. Definitions and Preliminaries. Some useful definitions and preliminary results in the context of this note shall be discussed in this section. We will also mention some relevant studies previously done in this field.

An ordered 2-tuple comprising two sets, i.e., set of nodes termed as vertex set $V(\mathbb{G})$ and connection between these vertices termed as edge set $E(\mathbb{G})$, is called a network $\mathbb{G}$, where $E(\mathbb{G})$ is contained in $V(\mathbb{G}) \times V(\mathbb{G})$. A network $\mathbb{G}$ can either be connected or consists of connected components. We will consider nonempty and simple networks throughout, having $V(\mathbb{G})$ as its vertex set and $E(\mathbb{G})$ as its edge set with order $|V(\mathbb{G})| = p$ and size $|E(\mathbb{G})| = q$. The network $\mathbb{G}$ in this case is termed as a (p, q) -network. Reference [3] is referred to for further discernment into the network terminologies.

A labelling is a function that maps +ive integers (non-zero) onto the component(s) of a network under specified constraints. If the components include vertices and edges both, then this labelling is termed as total. The labellings are referred to as vertex or edge labellings if they cover, respectively, $V(\mathbb{G})$ or $E(\mathbb{G})$ alone in the domain. Magic and antimagic labelling are two main classes of labelling. Precisely, equal or unequal vertex/edge weights refer to magic or antimagic labellings, respectively.

Definition 1. For a (p, q) -network $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$, the bijective function δ from $V(\mathbb{G}) \cup E(\mathbb{G})$ onto $\{1, 2, \dots, p + q\}$ is termed as (a, d) edge-antimagic total labelling with the constraint that the edge weights $\delta(x) + \delta(xy) + \delta(y)$, for each $xy \in E(\mathbb{G})$, constitute a sequence of consecutive positive integers, where a is the minimum edge weight and common difference is d . $\mathbb{G}$ is referred to be an (a, d) edge-antimagic total network, if such a labelling exists.

Definition 2. A super (a, d) edge-antimagic total labelling is in fact an (a, d) edge-antimagic total labelling in which minimum labels $1, 2, \dots, p$ are assigned to vertices of the (p, q) -network $\mathbb{G}$. $\mathbb{G}$ is termed to be a super (a, d) edge-antimagic total network in this case.

In Definitions 1 and 2, the minimum edge weight a becomes constant c at $d = 0$, for all edges $xy \in E(\mathbb{G})$, which is referred to as magic sum or magic constant for the network $\mathbb{G}$.

Definition 3. A network $\mathbb{G}$ is termed as a pancyclic network if it contains cycle of every order from 3 to $|V(\mathbb{G})|$.

Definition 4. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two simple networks. The network obtained by taking $|V(\mathbb{G}_1)|$ copies of $\mathbb{G}_2$ and then for each point (vertex) v_j in V (called the root vertex) ($\mathbb{G}_1$), v_j is being replaced with the j_{th} copy of $\mathbb{G}_2$, termed as the rooted product of the networks $\mathbb{G}_1$ and $\mathbb{G}_2$. It is notated as $\mathbb{G}_1 \circ \mathbb{G}_2$.

Further in the article, the abbreviations are being used as given in Table 1.

We further provide some specific definitions within the corresponding section of our main results' section.

The idea of magic labelling on networks was identified by Sadláček in 1963 [4]. The notion of antimagic labelling, for vertex sums of networks, was presented by Ringel and

TABLE 1: Notations and their abbreviations used in the article.

Terminology	Abbreviation
(a, d) edge-antimagic	(a, d) -EAM
Super (a, d) edge-antimagic	S- (a, d) -EAM

Hartsfield [5] later. Kotzig and Rosa brought into the light the concept of magic valuations of networks in [6] which was in fact the $(a, 0)$ -EAM total labelling on networks (studied by Ringel and Llado [7] in 1996). The notion of S- $(a, 0)$ -EAM total labelling of networks was defined by Enomoto et al. [8] with the terminology super edge-magic labelling. Simanjantuk et al. highlighted (a, d) -EAM total labelling of networks in [9] in year 2000.

The literature of $(a, 0)$ -EAM total labelling of networks includes the following interesting and useful conjectures.

Conjecture 1. All trees admit $(a, 0)$ -EAM total labelling [6].

Conjecture 2. All trees admit S- $(a, 0)$ -EAM total labelling [8].

Graph theorists, in the support of Conjecture 2, have been rectifying several particular classes of trees. Using an encryption of a computer programme, this conjecture has been verified for the trees having at most 17 vertices by Lee and Shah [10]. Specifically, the derivations can be seen for stars, subdivided stars [11, 12, 13, 14, 15], w -trees [16, 17, 18], banana trees [19], caterpillars [20], subdivided caterpillars [21], and the union of books and stars [22]. Further relevant works can be found in [23–25]. However, this conjecture is still open for working. Enomoto et al. proved that if a simple (p, q) -network $\mathbb{G}$ is S- $(a, 0)$ -EAM total, then $2p - 3$ is at least q [8]. They further derived that the network $K_{m,n}$ is S- $(a, 0)$ -EAM total $\Leftrightarrow m$ or n is 1. Figueroa-Centeno et al. derived that the union of networks $K_{1,m} \cup K_{1,n}$ is S- $(a, 0)$ -EAM total if either $m = \eta_1(n + 1)$ or $n = \eta_2(n + 1)$ [26]. The network C_n is also proved to be S- $(a, 0)$ -EAM total only if $n \equiv 1 \pmod{2}$ in [8]. In [27], $C_3 \cup C_n$ has been proven to be S- $(a, 0)$ -EAM total only when $6 \leq n \equiv 0 \pmod{2}$. The generalized prism $D_{m,n}$ is proven to be S- $(a, 0)$ -EAM total for all odd values of m in [28]. Baig et al. classified a class of planar pancyclic networks in [29] and exhibited its S- $(a, 0)$ -EAM total labelling for all possible values of the parameters involved. An immensely advantageous lemma on S- $(a, 0)$ -EAM total networks is as follows. Liu et al. studied the bounds of the minimum and maximum edge weights for super (a, d) -EAM total labelling on a generalized class of subdivided caterpillars in [30] for various values of d . In [31], Ahmad et al. studied the super $(a, 0)$ -EAM total labelling of certain Toeplitz graphs combined with isolated vertices nK_1 , for various values of n (also known as super edge-magic deficiency of networks). The properties and existence of super (a, d) vertex-antimagic labelling of regular graphs have been discussed in [32]. In [33], Ahmad et al. constructed the α -labelling, a special case of graceful labelling (labelling in which distinct edge weights are considered with respect to the difference of vertices' labels) on trees, and transformed this labelling to edge-antimagic vertex labelling

of trees. In [34], $S-(a, 0)$ -EAM total labelling on the graphs $G \cup nK_1$ has been studied, where G represents the unicyclic graph, whereas $S-(a, 0)$ -EAM total labelling of networks like zig-zag triangle and disjoint union of combs and stars has been studied in [35].

Lemma 1 (see [28]). A $(\wp, \mathfrak{T})$ -network $\mathbb{G}$ is $S-(a, 0)$ -EAM total if and only if there is a bijection $\delta: V(\mathbb{G}) \rightarrow \{1, 2, \dots, \wp\}$ such that the set $S = \{\delta(x) + \delta(y) | xy \in E(\mathbb{G})\}$ consists of $\mathfrak{T}$ consecutive integers. In such a case, δ extends to an $S-(a, 0)$ -EAM total labelling of $\mathbb{G}$ with magic constant $\mathfrak{R} = \wp + \mathfrak{T} + s$, where $s = \min(S)$ and $S = \{\mathfrak{R} - (\wp + 1), \mathfrak{R} - (\wp + 2), \dots, \mathfrak{R} - (\wp + \mathfrak{T})\}$.

In Lemma 1, the sum $\delta(x) + \delta(y)$ is called as edge sum for each edge $xy \in E(\mathbb{G})$. This lemma will be used frequently in our derivations, as it keeps this sufficient to label the vertices of a network only to make the network $S-(a, 0)$ -EAM total, if the edge sums are positive consecutive integers. The following result is also very pertinent as far as $S-(a, 0)$ -EAM total networks are concerned.

Theorem 1. A simple network G admits an $S-(a, 0)$ -EAM total labelling $\Leftrightarrow G$ admits an $S-(a-|E(G)|+1, 2)$ -EAM total labelling [36].

1.3. Applications in Networking, Computer Science, and Engineering. In software engineering, network labelling keeps on attaining an improved role in the security codes' encryption in order to encounter the attacks of trojans onto the precious data designed by hackers and also in designing of algorithm that helps the transmission of data to various networks and similar devices. The configurations of software in the encryption of their updated version is being improved by the use of reference labels and test ready labels nowadays. For connected components of networks in binary graphics, the mechanism which is predominantly nurturing the creation of clearer graphics involves labelling [37]. The study of magic labelling has been appearing to be more useful gradually in the data mining. The task of collection of data for the derivation of latest information gets more uncomplicated by designating equal weightage data as a single element. Resultantly, in organizations, the data mining task is becoming facile and more simplistic with far less consumption of time and effort due to the usage of magic labelling.

1.3.1. Networking. The primary hallmarks in networking are the functioning and optimization of the networks that demand management, construction, and concrete planning of networks at its base. Wireless and wired networking are two fundamental types of networking. The importance and large-scale usage of wired networking cannot be denied in the present era as well. The application of robust tools like network labelling is getting attention due to an escalation in the usage of wireless networking, in order to attain more precision in this field (see [38]). The modern era is of network communication whose part and parcel is radio transmission. The

interference, making the job of channel assignment more complicated, is one of the major concerns in radio transmission. The transmission of concurrent networks that are constraint-free, admitting same instance surfacing, is the central reason of this unwanted interruption [39,40]. The magic labelling assists in the allotment of constant weights to the networks that are concurrent. Such interferences are eliminated by using this procedure. The radio labelling on networks is playing a tremendous part in the reduction of interference issue in wireless networking from the last decade or so. For the automatic routing in networks, the $(a, 0)$ -antimagic labelling is particularly very useful. In this regard, a suitable constant edge-weight function is designed on a particular network, which helps routing for automatic detection of the succeeding node in the network (see [41]).

1.3.2. Telecommunication. In modern era, telecommunication involves most successful application of network labelling commercially [42]. In network telecommunication, a utility coverage region is split into a polygonal area described as a cell. Such a cell serves as a separate station. Using its radio transceiver, the base cell is designed to be a hub with the capacity to interface with other mobile stations. The defiance task here for the base cell is to facilitate with the ability to re-use utmost channels, avoiding any violation of the constraints. This challenge is being tackled by assigning a label to each user, while the communication loop of this user acquires a distinct label. Resultantly, any pair of communication terminals identifies the link label of connection path automatically by simple use of graceful antimagic or magic labelling. The label of the path specifies uniquely the two users which it interlinks conversely (see [43]).

1.3.3. Urban Planning. Consider the wheel W_6 , the helm H_6 , and prism D_5 in Figure 1 as a specific example. The edges of the networks W_6 and H_6 are labelled with consecutive labels ranging from 1 up to the size of the network such that the label appearing on all the vertices is distinct, i.e., we are provided here with the vertex antimagic labelling of the networks, whereas with edge weight 29 (constant), edge-magic labelling on D_5 is given [44,45]. As an example, the chambers are identified by point (vertices) and admissible pathways to approach these chambers are identified by edges, in a surveillance design of highly secured building. A total disturbance in the labelling will occur if a person attempts to breach a single legal pathway. The magic constant, in the scenario of design like D_5 , gets disordered promptly in case of violation in the pathway. This disorder, through programming software, will abruptly alert to the security concerned that the legitimate pathway has been breached. Once such magic or antimagic labellings are designed on a network, they can be used for surveillance of all the networks having the same hubs and connections. Antimagic and magic labelling both are equally valuable in this regard. In urban planning, this is one of the large-scale

FIGURE 1: Vertex-antimagic total labelling of the wheel W_6 and helm H_6 and edge-magic labelling total of the prism D_5 [44,45].

usages of the concept of labelling. That is, as a model for surveillance of the extensively secure areas, these labellings perform their distinct role [46].

1.3.4. Robotics. The routing and functioning of inducted robots at places like restaurants and factories in the form of production lines and machine units derive assistance by making use of any such suitable labelling function. In order to keep robotic components kinetic or make them stationary, these labelling functions assist to opt which operation to be skipped at which instant and vice versa. The antimagic labelling and distance-based dimensions alike tools help to minimize the time and maximize the accuracy of robots in their routing [47]. In the industry, these tools are causing a massive reduction in the cost.

2. Main Results

This section contains our main findings. It is divided into four subsections further. In Section 2.1, the $S-(a, 0)$ -EAM total labelling on the union of $K_{2,m} \circ C_n$ with copies of paths, the star, and the path shall be designed, whereas in Section 2.2, we derive an $S-(a, 0)$ -EAM total labelling on the rooted product of network C_n and pancyclic networks H_1 and H_2 (planar also). Further, $S-(a, 0)$ -EAM total

labelling of planar pancyclic network Γ_n and symmetric lattice networks $\mathbb{L}_n^1$, $\mathbb{L}_n^2$, and $\mathbb{L}_n^3$ shall be exhibited in Sections 2.3 and 2.4, respectively.

2.1. $S-(a, 0)$ -EAM Total Labelling of the Disjoint Union of $C_n \circ K_{2,m}$ and P_n , S_n , and mP_2 . Our main motivation to explore the findings in this section is the following open problem of Ngurah et al. [48].

Open Problem. For $n \geq 2$ and $m \geq 3$, is there any $S-(a, 0)$ -EAM total labelling of $nK_{2,m}$?

In fact, the rooted product $C_n \circ K_{2,m}$ contains n copies of the complete bipartite network $K_{2,m}$. A cycle C_n is a 2-regular network of order n , whereas the complete bipartite network $K_{2,m}$ is class-wise regular in which one partitioned class of vertices is 2-regular and other is m -regular, where $|V(K_{2,m})| = m + 2$.

Theorem 2. For $2 \leq m \equiv 0 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1$ admits an $S-(a, 0)$ -EAM total labelling having $a = 3mn + 8n + 2$.

Proof. Consider the network $G_1 \cong (C_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1$ with connection scheme as below:

$$\begin{aligned}
 V(G_1) &= \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i, z_i: 1 \leq i \leq n\} \\
 &\quad \cup \{p_i: 1 \leq i \leq n\} \cup \{c_i: 1 \leq i \leq (n - 1/2)\} \cup \{c\}, \\
 E(G_1) &= \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\
 &\quad \cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{c p_i: 1 \leq i \leq n\} \\
 &\quad \cup \{y_i y_{i+1}: 1 \leq i \leq n - 1\} \cup \{y_1 y_n\}.
 \end{aligned} \tag{1}$$

Here, $|V(G_1)| = (7n + 2mn + 1/2)$ and $|E(G_1)| = 2n(m + 1)$. Define a labelling $f_1: V(G_1) \rightarrow \{1, 2, \dots, (7n + 2mn + 1/2)\}$ as

(i) For $m = 2$:

$$f_1(x_j^i) = \begin{cases} \frac{5n+i}{2}, & j = 1, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{4n+i}{2}, & j = 1, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j = 2, 1 \leq i \leq n. \end{cases} \quad (2)$$

(ii) For $m > 2$:

$$f_1(x_j^i) = \begin{cases} \frac{5n+i}{2}, & j = \frac{m}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{4n+i}{2}, & j = \frac{m}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j = \frac{m}{2} + 1, 1 \leq i \leq n, \\ n(j+3) - (i-1), & 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2} - 1, \\ nj - (i-1) + n, & 1 \leq i \leq n, \frac{m}{2} + 2 \leq j \leq m. \end{cases} \quad (3)$$

(iii) The remaining labels for $m \geq 2$ are as follows:

$$f_1(y_i) = \begin{cases} \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_1(z_i) = \begin{cases} n(m+1) + \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1. \end{cases} \quad (4)$$

$$f_1(p_i) = i, \quad 1 \leq i \leq n,$$

$$f_1(c) = \frac{7n+2mn+1}{2},$$

$$f_1(c_i) = mn + 3n + i, \quad 1 \leq i \leq \frac{n-1}{2}.$$

All edge sums generated by the labelling scheme f_1 constitute a sequence of consecutive integer $(5n + 3/2), (5n + 5/2), (5n + 7/2), \dots, (4mn + 9n + 1/2)$. So, by Lemma 1, f_1 extends to an $S-(a, 0)$ -EAM total labelling of the network G_1 having magic constant $a = 3mn + 8n + 2$.

Theorem 3. For $3 \leq m \equiv 1 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1$ admits an $S-(a, 0)$ -EAM total labelling with $a = 3mn + 8n + 2$.

Proof. Consider $G_2 \cong (C_n \circ K_{2,m}) \cup K_{1,n} \cup (n - 1/2)K_1$, a network with odd m as follows:

$$\begin{aligned} V(G_2) &= \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i, z_i: 1 \leq i \leq n\} \\ &\cup \{p_i: 1 \leq i \leq n\} \cup \{c_i: 1 \leq i \leq (n - 1/2)\} \cup \{c\}, \\ E(G_2) &= \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{c p_i: 1 \leq i \leq n\} \\ &\cup \{y_i y_{i+1}: 1 \leq i \leq n - 1\} \cup \{y_1 y_n\}. \end{aligned} \quad (5)$$

Here, $|V(G_2)| = (7n + 2mn + 1/2)$ and $|E(G_2)| = 2n(m + 1)$. We define a labelling $f_2: V(G_2) \longrightarrow \{1, 2, \dots, (7n + 2mn + 1/2)\}$ as follows:

(i) For $m = 3$:

$$f_2(x_j^i) = \begin{cases} \frac{5n+i}{2}, & j = 2, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{4n+i}{2}, & j = 2, i \equiv 0 \pmod{2}, 2 \leq i \leq n - 1, \\ n(m+2) - (i-1), & j = 3, 1 \leq i \leq n, \\ 4n - (i-1), & j = 1, 1 \leq i \leq n. \end{cases} \quad (6)$$

(ii) For $m > 3$:

$$f_2(x_j^i) = \begin{cases} \frac{5n+i}{2}, & j = \frac{m+1}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{4n+i}{2}, & j = \frac{m+1}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n - 1, \\ n(m+2) - (i-1), & j = \frac{m+3}{2}, 1 \leq i \leq n, \\ n(j+3) - (i-1), & 1 \leq i \leq n, 1 \leq j \leq \frac{m-1}{2}, \\ nj - (i-1) + n, & 1 \leq i \leq n, \frac{m+5}{2} \leq j \leq m. \end{cases} \quad (7)$$

(iii) The remaining labels for $m \geq 3$ are as follows:

$$\begin{aligned}
 f_2(y_i) &= \begin{cases} \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases} \\
 f_2(z_i) &= \begin{cases} n(m+1) + \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases} \\
 f_2(p_i) &= i, \quad 1 \leq i \leq n, \\
 f_2(c) &= \frac{7n+2mn+1}{2}, \\
 f_2(c_i) &= mn+3n+i, \quad 1 \leq i \leq \frac{n-1}{2}.
 \end{aligned} \tag{8}$$

All edge sums generated by the above labelling scheme f_2 constitute a sequence of consecutive integer $(5n+3/2), (5n+5/2), (5n+7/2), \dots, (4mn+9n+1/2)$. By Lemma 1, f_2 extends to an S -($a, 0$)-EAM total labelling of the network G_2 having $a = 3mn + 8n + 2$.

Theorem 4. For $2 \leq m \equiv 0 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup nP_2$ admits an S -($a, 0$)-EAM total labelling with $a = (6mn + 17n + 3/2)$.

Proof. Let $G_3 \cong (C_n \circ K_{2,m}) \cup nP_2$ be a network for $mn \equiv 1 \pmod{2}$ with vertices:

$$\begin{aligned}
 V(G_3) &= \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i, z_i: 1 \leq i \leq n\} \\
 &\quad \cup \{p_i, q_i: 1 \leq i \leq n\}, \\
 E(G_3) &= \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\
 &\quad \cup \{p_i q_i: 1 \leq i \leq n\} \\
 &\quad \cup \{y_i y_{i+1}: 1 \leq i \leq n-1\} \cup \{y_1 y_n\}.
 \end{aligned} \tag{9}$$

Here, $|V(G_3)| = n(m+4)$ and $|E(G_3)| = 2n(m+1)$. A labelling function $f_3: V(G_3) \rightarrow \{1, 2, \dots, n(m+4)\}$ is defined as follows:

(i) For $m = 2$:

$$f_3(x_j^i) = \begin{cases} \frac{1}{2}(5n+i), & j=1, 1 \leq i \leq n, i \equiv 1 \pmod{2}, \\ \frac{1}{2}(4n+i), & j=1, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j=2, 1 \leq i \leq n. \end{cases} \tag{10}$$

(ii) For $m > 2$:

$$f_3(x_j^i) = \begin{cases} \frac{1}{2}(5n+i), & j = \frac{m}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(4n+i), & j = \frac{m}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j = \frac{m}{2} + 1, 1 \leq i \leq n, \\ n(j+3) - (i-1), & 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2} - 1, \\ nj - (i-1) + n, & 1 \leq i \leq n, \frac{m}{2} + 2 \leq j \leq m. \end{cases} \tag{11}$$

(iii) The remaining labels for $m \geq 2$ are as follows:

$$f_3(y_i) = \begin{cases} \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_3(z_i) = \begin{cases} n(m+1) + \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1. \end{cases}$$

$$f_3(q_i) = \begin{cases} n(m+1) + \frac{5n-i+2}{2}: & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{6n-i+2}{2}: & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_3(p_i) = i, \quad 1 \leq i \leq n. \quad (12)$$

All edge sums generated by the above labelling scheme f_3 constitute a sequence of consecutive integer $(5n+3/2)$, $(5n+5/2)$, $(5n+7/2)$, $\dots$, $(4mn+9n+1/2)$. So, by Lemma 1, f_3

extends to an S -($a, 0$)-EAM total labelling of the network G_3 admitting magic constant $a = (6mn + 17n + 3/2)$.

Theorem 5. For $3 \leq m \equiv 1 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup nP_2$ admits an S -($a, 0$)-EAM total labelling with $a = (6mn + 17n + 3/2)$.

Proof. Consider the network $G_4 \cong (C_n \circ K_{2,m}) \cup nP_2$, for odd m with the construction:

$$V(G_4) = \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i, z_i: 1 \leq i \leq n\} \\ \cup \{p_i, q_i: 1 \leq i \leq n\},$$

$$E(G_4) = \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\ \cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{p_i q_i: 1 \leq i \leq n\} \\ \cup \{y_i y_{i+1}: 1 \leq i \leq n-1\} \cup \{y_1 y_n\}. \quad (13)$$

Here, $|V(G_4)| = n(m+4)$ and $|E(G_4)| = 2n(m+1)$. Define a function $f_4: V(G_4) \rightarrow \{1, 2, \dots, n(m+4)\}$ as follows:

(i) For $m = 3$:

$$f_4(x_j^i) = \begin{cases} \frac{1}{2}(5n+i), & j = 2, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(4n+i), & j = 2, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j = 3, 1 \leq i \leq n, \\ 4n - (i-1), & j = 1, 1 \leq i \leq n. \end{cases} \quad (14)$$

(ii) For $m > 3$:

$$f_4(x_j^i) = \begin{cases} \frac{1}{2}(5n+i), & j = \frac{m+1}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(4n+i), & j = \frac{m+1}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ n(m+2) - (i-1), & j = \frac{m+3}{2}, 1 \leq i \leq n, \\ n(j+3) - (i-1), & 1 \leq i \leq n, 1 \leq j \leq \frac{m-1}{2}. \\ nj - (i-1) + n: & 1 \leq i \leq n, \frac{m+5}{2} \leq j \leq m. \end{cases} \quad (15)$$

(iii) The remaining labels for $m \geq 3$ are as follows:

$$f_4(y_i) = \begin{cases} \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_4(z_i) = \begin{cases} n(m+1) + \frac{i+2n+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{i+3n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_4(q_i) = \begin{cases} n(m+1) + \frac{5n-i+2}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{6n-i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_4(p_i) = i, \quad 1 \leq i \leq n. \quad (16)$$

All edge sums generated by the above labelling scheme f_4 constitute a sequence of consecutive integer $(5n+3/2), (5n+5/2), (5n+7/2), \dots, (4mn+9n+1/2)$. So,

by Lemma 1, f_4 extends to an S- $(a, 0)$ -EAM total labelling of the network G_4 admitting $a = (6mn+17n+3/2)$.

Theorem 6. For $2 \leq m \equiv 0 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup P_{n+1}$ admits an S- $(a, 0)$ -EAM total labelling with $a = (6mn+17n+3/2)$.

Proof. Consider the network $G_5 \cong (C_n \circ K_{2,m}) \cup P_{n+1}$, for $n \geq 3$ odd, with connections:

$$\begin{aligned} V(G_5) &= \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\cup \{y_i, z_i: 1 \leq i \leq n\} \cup \{p_i: 1 \leq i \leq n+1\}, \\ E(G_5) &= \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_1 y_n\} \\ &\cup \{p_i p_{i+1}: 1 \leq i \leq n\} \cup \{y_i y_{i+1}: 1 \leq i \leq n-1\}. \end{aligned} \quad (17)$$

Here, $|V(G_5)| = 3n+mn+1$ and $|E(G_5)| = 2n(m+1)$. A labelling $f_5: V(G_5) \rightarrow \{1, 2, \dots, mn+3n+1\}$ is defined as follows:

(i) For $m = 2$:

$$f_5(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1), & j=1, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(3n+i+1), & j=1, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ \frac{1}{2}(2mn+3n-2i+3), & j=2, 1 \leq i \leq n. \end{cases} \quad (18)$$

(ii) For $m > 2$:

$$f_5(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1), & j = \frac{m}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(3n+i+1), & j = \frac{m}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ \frac{1}{2}(2mn+3n-2i+3), & j = \frac{m}{2}+1, 1 \leq i \leq n, \\ \frac{1}{2}(2nj+5n-2i+3), & 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2}-1, \\ \frac{1}{2}(2nj+n-2i+3): & 1 \leq i \leq n, \frac{m}{2}+2 \leq j \leq m. \end{cases} \quad (19)$$

(iii) The remaining labels for $m \geq 2$ are as follows:

$$f_5(y_i) = \begin{cases} \frac{n+i+2}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{2n+i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_5(z_i) = \begin{cases} n(m+1) + \frac{n+i+2}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{2n+i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases}$$

$$f_5(p_i) = \begin{cases} \frac{i+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(2mn+5n+i+1), & i \equiv 0 \pmod{2}, 2 \leq i \leq n+1. \end{cases} \quad (20)$$

All edge sums generated by the above labelling scheme f_5 constitute a sequence of consecutive integer $(3n+5/2)$, $(3n+7/2)$, $(3n+9/2)$, $\dots$, $(4mn+7n+3/2)$. Therefore, by Lemma 1, f_5 extends to an S -($a, 0$)-EAM total labelling of

the network G_5 admitting magic constant $a = (6mn + 13n + 7/2)$.

Theorem 7. For $3 \leq m \equiv 1 \pmod{2}$ and $3 \leq n \equiv 1 \pmod{2}$, the network $(C_n \circ K_{2,m}) \cup P_{n+1}$ admits an S -($a, 0$)-EAM total labelling with $a = (6mn + 13n + 7/2)$.

Proof. Consider the network $G_6 \cong (C_n \circ K_{2,m}) \cup P_{n+1}$, for both m and $n \geq 3$, as follows:

$$\begin{aligned} V(G_6) &= \{x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{y_i, z_i: 1 \leq i \leq n\} \cup \{p_i: 1 \leq i \leq n+1\}, \\ E(G_6) &= \{y_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{z_i x_j^i: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_1 y_n\} \\ &\quad \cup \{p_i p_{i+1}: 1 \leq i \leq n\} \cup \{y_i y_{i+1}: 1 \leq i \leq n-1\}. \end{aligned} \quad (21)$$

Here, $|V(G_6)| = 3n + mn + 1$ and $|E(G_6)| = 2n(m+1)$. A labelling $f_6: V(G_6) \longrightarrow \{1, 2, \dots, 3n + mn + 1\}$ is designed as follows:

(i) For $m = 3$:

$$f_6(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1), & j = 2, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(3n+i+1), & j = 2, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ \frac{1}{2}(2mn+3n-2i+3), & j = 3, 1 \leq i \leq n, \\ \frac{1}{2}(7n-2i+3), & j = 1, 1 \leq i \leq n. \end{cases} \quad (22)$$

(ii) For $m > 3$:

$$f_6(x_j^i) = \begin{cases} \frac{1}{2}(4n+i+1), & j = \frac{m+1}{2}, i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(3n+i+1), & j = \frac{m+1}{2}, i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \\ \frac{1}{2}(2mn+3n-2i+3), & j = \frac{m+3}{2}, 1 \leq i \leq n, \\ \frac{1}{2}(2nj+5n-2i+3), & 1 \leq i \leq n, 1 \leq j \leq \frac{m-1}{2}, \\ \frac{1}{2}(2nj+n-2i+3), & 1 \leq i \leq n, \frac{m+5}{2} \leq j \leq m. \end{cases} \quad (23)$$

(iii) The remaining labels for $m \geq 3$ are as follows:

$$\begin{aligned} f_6(y_i) &= \begin{cases} \frac{n+i+2}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{2n+i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases} \\ f_6(z_i) &= \begin{cases} n(m+1) + \frac{n+i+2}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ n(m+1) + \frac{2n+i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases} \\ f_6(p_i) &= \begin{cases} \frac{i+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{1}{2}(2mn+5n+i+1), & i \equiv 0 \pmod{2}, 2 \leq i \leq n+1. \end{cases} \end{aligned} \quad (24)$$

The edge sums generated by the above labelling scheme f_6 constitute a sequence of consecutive integer $(3n+5/2), (3n+7/2), (3n+9/2), \dots, (4mn+7n+3/2)$. By Lemma 1, f_6 extends to an $S-(a,0)$ -EAM total labelling of the network G_6 having magic constant $a = (6mn+13n+7/2)$.

Observations. The network C_n is $S-(a,0)$ -EAM total for $n \equiv 1 \pmod{2}$ only [8], pointing out that C_4 is not $S-(a,0)$ -EAM total. The networks in Theorems 2, 4, and 6 contain interesting substructures. Keeping $m = 2$ fixed for these results, $S-(a,0)$ -EAM total families of networks involving n copies of C_4 can be obtained. For instance, $S-(a,0)$ -EAM total labelling (again by using Lemma 1) of $(C_5 \circ C_4) \cup K_{1,5} \cup 2K_1$, $(C_5 \circ C_4) \cup 5P_2$, and $(C_5 \circ C_4) \cup P_6$ is presented in Figure 2.

The following results from Theorems 2–7 are direct consequences of Theorem 1.

Theorem 8. $\forall m \geq 2$ and $3 \leq n \equiv 1 \pmod{2}$, $(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$ admits an $S-(mn+6n+3,2)$ -EAM total labelling.

Theorem 9. $\forall m \geq 2$ and $3 \leq n \equiv 1 \pmod{2}$, $(C_n \circ K_{2,m}) \cup nP_2$ admits an $S-((2mn+13n+5/2),2)$ -EAM total labelling.

Theorem 10. $\forall m \geq 2$ and $3 \leq n \equiv 1 \pmod{2}$, $(C_n \circ K_{2,m}) \cup P_{n+1}$ admits an $S-((2mn+9n+9/2),2)$ -EAM total labelling.

2.2. $S-(a,0)$ -EAM Total Labelling of Rooted Product of Pancyclic Networks with C_n . The present section deals with

$S-(a,0)$ -EAM total labelling of the rooted product of two specific planar non-isomorphic pancyclic networks and the cycle C_n .

A specific pancyclic network H_1 is defined as follows.

Definition 5. H_1 is a pancyclic network having the following construction.

$$\begin{aligned} V(H_1) &= \{y, x_1, x_2, x_3, x_4, x_5, x_6, z\}, \\ E(H_1) &= \{x_1x_3, x_3x_5, x_2x_4, x_4x_6, x_1x_2, \\ &\quad \cdot x_3x_4, x_5x_6, x_2x_3, x_4x_5\} \cup \{yx_1, yx_2, zx_5, zx_6\}. \end{aligned} \quad (25)$$

Theorem 11. For $n \equiv 1 \pmod{2}$, the rooted product $C_n \circ H_1$ admits an $S-(a,0)$ -EAM total labelling having magic constant $a = (45n+3/2)$.

Proof

(i) For $n = 1$, $C_1 \circ H_1 \cong H_1$. The vertex labelling $\{y, x_1, x_2, x_3, x_4, x_5, x_6, y, z: 2, 1, 4, 3, 6, 5, 8, 7\}$ extends to an $S-(24,0)$ -EAM total labelling of $C_n \circ H_1$, by Lemma 1.

(ii) For $n \geq 3$.

Consider the network $C_n \circ H_1$ with $|V(C_n \circ H_1)| = 8n$ and $|E(C_n \circ H_1)| = 14n$ connected as per the following scheme:

$$\begin{aligned} V(C_n \circ H_1) &= \{x_i, v_i: 1 \leq i \leq n\} \cup \{y_i, z_i, w_i: 1 \leq i \leq 2n\}, \\ E(C_n \circ H_1) &= \{y_i y_{i+1}, w_i w_{i+1}, z_i z_{i+1}: 1 \leq i \leq 2n-1, i \equiv 1 \pmod{2}\} \\ &\quad \cup \{y_i z_i, z_i w_i: 1 \leq i \leq 2n\} \cup \{x_i x_{i+1}: 1 \leq i \leq n-1\} \\ &\quad \cup \{x_1 x_n\} \cup \{z_i y_{i+1}, w_i z_{i+1}: 1 \leq i \leq 2n-1, i \equiv 1 \pmod{2}\} \\ &\quad \cup \{x_i y_{2i-1}, x_i y_{2i}, v_i w_{2i-1}, v_i w_{2i}: 1 \leq i \leq n\}. \end{aligned} \quad (26)$$

FIGURE 2: An S -($a, 0$)-EAM total labelling (using Lemma 1) of the cyclic networks (a) $(C_5 \circ C_4) \cup K_{1,5} \cup 2K_1$, (b) $(C_5 \circ C_4) \cup 5P_2$, and (c) $(C_5 \circ C_4) \cup P_6$.

Consider a labelling $\psi_1: V(C_n \circ H_1) \longrightarrow \{1, 2, \dots, |V(C_n \circ H_1)| = 8n\}$ defined as

$$\begin{aligned}
 \psi_1(x_i) &= \begin{cases} \frac{i+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{i+n+1}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1, \end{cases} \\
 \psi_1(y_i) &= \begin{cases} \frac{6n-i+2}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq 2n, \\ \frac{6n+i+1}{4}, & i \equiv 1 \pmod{4}, 1 \leq i \leq 2n-1, \\ \frac{4n+i+1}{4}, & i \equiv 3 \pmod{4}, 3 \leq i \leq 2n-3, \end{cases} \\
 \psi_1(z_i) &= \begin{cases} \frac{16n+i}{4}, & i \equiv 0 \pmod{4}, 4 \leq i \leq 2n-2, \\ \frac{12n+i+3}{4}, & i \equiv 1 \pmod{4}, 1 \leq i \leq 2n-1, \\ \frac{18n+i}{4}, & i \equiv 2 \pmod{4}, 2 \leq i \leq 2n, \\ \frac{14n+i+3}{4}, & i \equiv 3 \pmod{4}, 3 \leq i \leq 2n-3, \end{cases} \\
 \psi_1(w_i) &= \begin{cases} \frac{12n-i+1}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq 2n-1, \\ \frac{26n+i+2}{4}, & i \equiv 0 \pmod{4}, 4 \leq i \leq 2n-2, \\ \frac{24n+i+2}{4}, & i \equiv 2 \pmod{4}, 2 \leq i \leq 2n, \end{cases} \\
 \psi_1(v_i) &= \begin{cases} \frac{15n+i}{2}, & i \equiv 1 \pmod{2}, 1 \leq i \leq n, \\ \frac{14n+i}{2}, & i \equiv 0 \pmod{2}, 2 \leq i \leq n-1. \end{cases}
 \end{aligned} \tag{27}$$

All edge sums generated by the above labelling scheme ψ_1 form a sequence of consecutive integer $(n + 3/2), (n + 5/2), (n + 7/2), \dots, (29n + 1/2)$. Therefore, by Lemma 1, ψ_1 extends to an S- $(a, 0)$ -EAM total labelling of the network $C_n \circ H_1$ having magic constant $(45n + 3/2)$.

Definition 6. We define a pancyclic network $H_2 \# H_1$ having vertex set:

$$\begin{aligned} V(H_2) &= \{y, x_1, x_2, x_3, x_4, x_5, x_6, z\}, \\ E(H_2) &= \{x_1x_3, x_3x_5, x_2x_4, x_4x_6, x_1x_2, x_5x_6, x_2x_3, x_4x_5\} \cup \{yx_1, yx_2, zx_5, zx_6, yz\}. \end{aligned} \quad (28)$$

Theorem 12. For $n \equiv 1 \pmod{2}$, the rooted product $C_n \circ H_2$ admits an S- $(a, 0)$ -EAM total labelling with magic constant $a = (45n + 3/2)$.

Proof

- (i) For $n = 1$, $C_1 \circ H_2 \cong H_2$.
- (ii) For $n \geq 3$.

Consider $C_n \circ H_2$ with $|V(C_n \circ H_2)| = 8n$ and $|E(C_n \circ H_2)| = 14n$ with the following connection:

$$\begin{aligned} V(C_n \circ H_2) &= \{x_i, v_i: 1 \leq i \leq n\} \cup \{w_i, y_i, z_i: 1 \leq i \leq 2n\}, \\ E(P_n \circ H_2) &= \{y_i y_{i+1}, w_i w_{i+1}: 1 \leq i \leq 2n-1, i \equiv 1 \pmod{2}\} \\ &\cup \{y_i z_i, z_i w_i: 1 \leq i \leq 2n\} \cup \{x_i x_{i+1}: 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \\ &\cup \{z_i y_{i+1}, w_i z_{i+1}: 1 \leq i \leq 2n-1, i \equiv 1 \pmod{2}\} \\ &\cup \{x_i y_{2i-1}, x_i y_{2i}, v_i w_{2i-1}, v_i w_{2i}, x_i v_i: 1 \leq i \leq n\}. \end{aligned} \quad (29)$$

The labelling scheme for $n = 1$ and $n \geq 3$ is the same as ψ_1 designed in Theorem 11.

A direct derivation from Theorem 1 is given as follows.

Theorem 13. For $n \equiv 1 \pmod{2}$, $C_n \circ H_1$ and $C_n \circ H_2$ are S- $((17n + 5/2), 2)$ -EAM total.

2.3. S- $(a, 0)$ -EAM Total Labelling of a Pancyclic Class of Networks: Extension of a Result Appearing in [29]. In [29], Baig et al. provided a result regarding S- $(a, 0)$ -EAM total labelling of a pancyclic class of networks involving chains of cycle C_4 . Here, we shall introduce a pancyclic family of networks involving chains of cycle C_6 , while our point of convergence is the S- $(a, 0)$ -EAM total labelling of this class. Thus, we further extend the results of Baig et al. [29].

Definition 7. The pancyclic network Γ_n is a network with order $|V(\Gamma_n)| = 6n$ and $|E(\Gamma_n)| = 12n - 3$, with structure as follows:

$$\begin{aligned} V(\Gamma_n) &= \{z_i, x_i, y_i: 1 \leq i \leq 2n\}, \\ E(\Gamma_n) &= \{y_i y_{i+1}, x_i x_{i+1}, z_i z_{i+1}: 1 \leq i \leq 2n-1\} \cup \{x_i y_i, x_i z_i, y_i z_i: 1 \leq i \leq 2n\}. \end{aligned} \quad (30)$$

Figure 3 reveals general formation of Γ_n .

In Figure 4, we have shown the network Γ_2 and its contained cycles of orders 3, 4, $\dots$, 12.

In the upcoming result, we show that the pancyclic network Γ_n is S- $(a, 0)$ -EAM total.

Theorem 14. For all positive integers n , the pancyclic network Γ_n is S- $(a, 0)$ -EAM total having magic constant 18n.

Proof

- (i) For $n=1$, the labelling $\{z_1, z_2, x_1, x_2, y_1, y_2: 3, 5, 2, 4, 1, 6\}$ extends to an S- $(18, 0)$ -EAM total labelling of Γ_1 by Lemma 1.
- (ii) For $n \geq 3$.

Define here a labelling $g: V(\Gamma_n) \longrightarrow \{1, 2, \dots, 6n\}$ as

$$\begin{aligned} g(x_i) &= \begin{cases} 3i: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i-1: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\ g(y_i) &= \begin{cases} 3i-2: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\ g(z_i) &= \begin{cases} 3i-1: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i-2: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n. \end{cases} \end{aligned} \quad (31)$$

All edge sums generated by the above labelling scheme constitute a sequence of consecutive integers 3, 4, $\dots$, $12n-1$. Therefore, by Lemma 1, g extends to an S- $(a, 0)$ -EAM total labelling of Γ_n having magic constant $a = 18n$.

Again from Theorem 1, we have a direct consequence as follows.

Theorem 15. For all positive integers n , the pancyclic network Γ_n admits an S- $(6n + 4, 2)$ -EAM total labelling.

FIGURE 3: The general formation of the pancyclic network Γ_n . FIGURE 4: The network Γ_2 having order 12 and its contained cycles $C_3, C_4, \dots, C_{12}$.

2.4. S-(a,0)-EAM Total Labelling of Symmetric Lattice Networks

Definition 8. Consider a tripartite network $\mathbb{T}$ having vertex-edge connections as follows:

$$\begin{aligned} V(\mathbb{T}) &= \{x, y, z, u, v, w\}, \\ E(\mathbb{T}) &= \{uv, vw, xy, yz, uy, wy, xv, zv, vy\}. \end{aligned} \quad (32)$$

In this section, we study the S-(a,0)-EAM total labelling of symmetric lattice networks $\mathbb{L}_n^1$, $\mathbb{L}_n^2$, and $\mathbb{L}_n^3$. These symmetric lattices contain n copies of the tripartite network $\mathbb{T}$.

Definition 9

- (i) For $n = 1$, $\mathbb{L}_1^1 \cong \mathbb{T}$.
- (ii) For $n \geq 2$.

The lattice network $\mathbb{L}_n^1$ is a network with order $6n$ and size $12n - 3$ defined as follows:

$$\begin{aligned} V(\mathbb{L}_n^1) &= \{x_i, y_i, c_i: 1 \leq i \leq 2n\}. \\ E(\mathbb{L}_n^1) &= \{c_i y_{i+1}, c_i x_{i+1}, c_i y_i, c_i x_i: 1 \leq i \leq 2n-1, i \equiv 1 \pmod{2}\} \\ &\cup \{c_i c_{i+1}: 2 \leq i \leq 2n-1\} \cup \{x_i x_{i+1}, y_i y_{i+1}: i \equiv 0 \pmod{2}, 2 \leq i \leq 2(n-1)\}, \\ &\cup \{c_i y_{i-1}, c_i x_{i-1}, c_i y_i, c_i x_i: i \equiv 0 \pmod{2}, 2 \leq i \leq 2n\}. \end{aligned} \quad (33)$$

Figure 5 illustrates the general formation of the lattice network $\mathbb{L}_n^1$, $\forall n \in \mathbb{N}$.

Theorem 16. For all positive integers n , the lattice network $\mathbb{L}_n^1$ is S-(a,0)-EAM total having magic constant $18n$.

Proof

- (i) For $n = 1$: the vertex labelling $\{x_1, c_1, y_1, x_2, c_2, y_2: 3, 1, 2, 5, 6, 4\}$ extends to an S-(18,0)-EAM total labelling of $\mathbb{L}_1^1$, by Lemma 1.
- (ii) For $n > 1$.

We are defining a labelling $\varphi_1: V(\mathbb{L}_n^1) \longrightarrow \{1, 2, \dots, 6n\}$ as follows:

$$\begin{aligned} \varphi_1(x_i) &= \begin{cases} 3i: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i-1: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\ \varphi_1(y_i) &= \begin{cases} 3i-1: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i-2: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\ \varphi_1(c_i) &= \begin{cases} 3i-2: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n-1, \\ 3i: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n. \end{cases} \end{aligned} \quad (34)$$

All edge sums generated by the above labelling scheme constitute a sequence of consecutive integers $3, 4, \dots, 12n-3$. So, by Lemma 1, φ_1 extends to an S-(a,0)-EAM total labelling of $\mathbb{L}_n^1$ admitting magic constant $18n$.

Definition 10

- (i) For $n = 1$, $\mathbb{L}_1^2 \cong \mathbb{T}$.
- (ii) For $n \geq 2$.

FIGURE 5: The general formation of the lattice network $\mathbb{L}_n^1, \forall n \in \mathbb{N}$.

The lattice network $\mathbb{L}_n^2$ having order $10n - 4$ and size $20n - 11$ is defined as follows:

$$\begin{aligned}
 V(\mathbb{L}_n^2) &= \{x_i, y_i: 1 \leq i \leq 2n\} \cup \{c_i: 1 \leq i \leq 2(2n - 1)\} \cup \{u_i, v_i: 1 \leq i \leq n - 1\}. \\
 E(\mathbb{L}_n^2) &= \{c_i y_{i+2/2}, c_i x_{i+2/2}, c_i y_{i/2}, c_i x_{i/2}: 2 \leq i \leq 2(2n - 1), i \equiv 2 \pmod{4}\} \\
 &\cup \{c_i y_{i+2/2}, c_i y_{i+3/2}, c_i x_{i+2/2}, c_i x_{i+3/2}: i \equiv 1 \pmod{4}, 1 \leq i \leq 4n - 3\} \\
 &\cup \{c_i c_{i+1}: 1 \leq i \leq 4n - 3, i \equiv 1 \pmod{4}\} \cup \{u_i v_i: 1 \leq i \leq n - 1\} \\
 &\cup \{x_i u_{i/2}, y_i v_{i/2}: 2 \leq i \leq 2(n - 1), i \equiv 0 \pmod{2}\} \\
 &\cup \{u_i x_{2i+1}, v_i y_{2i+1}, u_i c_{4i-1}, v_i c_{4i-1}, u_i c_{4i}, v_i c_{4i}: 1 \leq i \leq n - 1\} \\
 &\cup \{c_i c_{i+1}: i \equiv 0 \pmod{2}, 2 \leq i \leq 4(n - 1)\}.
 \end{aligned} \tag{35}$$

In Figure 6, we have presented the general formation of the lattice network $\mathbb{L}_n^2, \forall n \in \mathbb{N}$.

Theorem 17. For all positive integers n , the lattice network $\mathbb{L}_n^2$ is $S(a, 0)$ -EAM total with magic constant $30n - 12$.

Proof

(i) For $n = 1$: the vertex labelling $\{x_1, c_1, y_1, x_2, c_2, y_2: 3, 1, 2, 5, 6, 4\}$ extends to an $S(18, 0)$ -EAM total labelling of $\mathbb{L}_1^2$, by Lemma 1.

(ii) For $n > 1$.

A labelling $\varphi_2: V(\mathbb{L}_n^2) \longrightarrow \{1, 2, \dots, 10n - 4\}$ is defined as follows:

$$\begin{aligned}
 \varphi_2(x_i) &= \begin{cases} 5i - 2: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n - 1, \\ 5i - 5: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\
 \varphi_2(y_i) &= \begin{cases} 5i - 3: i \equiv 1 \pmod{2}, & 1 \leq i \leq 2n - 1, \\ 5i - 6: i \equiv 0 \pmod{2}, & 2 \leq i \leq 2n, \end{cases} \\
 \varphi_2(c_i) &= \begin{cases} \frac{1}{2}(5i - 3): 1 \leq i \leq 4n - 3, & i \equiv 1 \pmod{4}, \\ \frac{1}{2}(5i + 2): 2 \leq i \leq 4n - 2, & i \equiv 2 \pmod{4}, \\ \frac{1}{2}(5i - 1): 3 \leq i \leq 4n - 5, & i \equiv 3 \pmod{4}, \\ \frac{1}{2}(5i): 4 \leq i \leq 4(n - 1), & i \equiv 0 \pmod{4}, \end{cases} \\
 \varphi_2(u_i) &= 10i - 1, \quad 1 \leq i \leq n - 1, \\
 \varphi_2(v_i) &= 10i - 2, \quad 1 \leq i \leq n - 1.
 \end{aligned} \tag{36}$$

All edge sums generated by the above labelling scheme constitute a sequence of consecutive integers $3, 4, \dots, 20n - 9$. So, by Lemma 1, φ_2 extends to an $S(a, 0)$ -EAM total labelling of $\mathbb{L}_n^2$ having magic constant $30n - 12$.

Definition 11

(i) For $n = 1$, $\mathbb{L}_1^3 \cong \mathbb{T}$.

(ii) For $n \geq 2$.

FIGURE 6: The general formation of the lattice network $\mathbb{L}_n^2, \forall n \in \mathbb{N}$.

The lattice network $\mathbb{L}_n^3$ having order $10n - 4$ and size $20n - 11$ is defined as follows:

$$\begin{aligned}
 V(\mathbb{L}_n^3) &= \{x_i, y_i: 1 \leq i \leq 2n\} \cup \{c_i: 1 \leq i \leq 2(2n-1)\} \cup \{u_i, v_i: 1 \leq i \leq n-1\}, \\
 E(\mathbb{L}_n^3) &= \{c_i y_{i+2/2}, c_i x_{i+2/2}, c_i y_{i/2}, c_i x_{i/2}: 2 \leq i \leq 2(2n-1), i \equiv 2 \pmod{4}\} \\
 &\quad \cup \{c_i y_{i+2/2}, c_i y_{i+3/2}, c_i x_{i+1/2}, c_i x_{i+3/2}: 1 \leq i \leq 4n-3, i \equiv 1 \pmod{4}\} \\
 &\quad \cup \{x_i u_{i/2}, y_i v_{i/2}: i \equiv 0 \pmod{2}, 2 \leq i \leq 2(n-1)\} \cup \{c_i c_{i+1}: 1 \leq i \leq 4n-3\} \\
 &\quad \cup \{u_i x_{2i+1}, v_i y_{2i+1}, u_i c_{4i-1}, v_i c_{4i-1}, u_i c_{4i}, v_i c_{4i}: 1 \leq i \leq n-1\}.
 \end{aligned} \tag{37}$$

Figure 7 illustrates the general form of the lattice network $\mathbb{L}_n^3, \forall n \in \mathbb{N}$.

Theorem 18. For all positive integers n , the lattice network $\mathbb{L}_n^3$ is $S-(a, 0)$ -EAM total with magic constant $30n - 12$.

Proof. For $\mathbb{L}_n^3$, the labelling design is similar as in Theorem 17.

3. Illustration through Examples and Proposed Open Problems

3.1. Examples. The $S-(132, 0)$ -EAM total labelling of $(C_5 \circ K_{2,6}) \cup K_{1,5} \cup 2K_1$ and $S-(163, 0)$ -EAM total labelling of $(C_7 \circ K_{2,5}) \cup K_{1,7} \cup 3K_1$ are presented, respectively, in Figures 8(a) and 8(b). In Figure 8(a), the parameters are $\{n = 5, m = 6\}$ and $a = 132$, while in Figure 8(b), the parameters are $\{n = 7, m = 5\}$ and $a = 163$. These are perfect according to our depiction of the magic constants in the proofs of Theorems 2 and 3.

Similarly, Figures 9 and 10 illustrate Theorems 4 and 5 and Theorems 6 and 7, respectively, for the values of the parameters given in each. The values of a here are perfectly similar as depicted in our main findings.

Figures 11(a) and 11(b) illustrate Theorems 11 and 12, respectively, for $n = 5$.

Figure 12 reveals an example of Theorem 15 corresponding to parameter $n = 3$.

Figures 13–15 refer to the illustration of $S-(a, 0)$ -EAM total labelling of lattice networks $\mathbb{L}_5^1, \mathbb{L}_4^2$, and $\mathbb{L}_4^3$ (Theorems 16–18).

Due to facilitation of Lemma 1, edge labels are not needed to be provided in all of the above illustrative figures. As the edge sums constitute a sequence of +ve consecutive integers, assigning the remaining labels $\{q, q-1, \dots, p+2, p+1\}$ to the edges in ascending or descending order will generate $S-(a', 2)$ or $(a, 0)$ -EAM total labelling on that network, respectively, where a

(magic constant) and a' (minimum edge weight) attain some suitable values accordingly. More precisely, according to Lemma 1, this vertex labelling, consisting of consecutive integers, extends to an $S-(a, 0)$ -EAM total labelling of the networks.

3.2. Open Problems. The open problems relevant to the findings (Theorems 2–7) of Section 2 are proposed as follows:

- (i) For $2 \leq n \equiv 0 \pmod{2}$, obtain any $S-(a, 0)$ -EAM total labelling of the $(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$.
- (ii) For $2 \leq n \equiv 0 \pmod{2}$, obtain any $S-(a, 0)$ -EAM total labelling of $(C_n \circ K_{2,m}) \cup nP_2$.
- (iii) For $2 \leq n \equiv 0 \pmod{2}$, obtain any $S-(a, 0)$ -EAM total labelling of $(C_n \circ K_{2,m}) \cup P_{n+1}$.
- (iv) For $2 \leq n \equiv 0 \pmod{2}$, determine $S-(a, 0)$ -EAM total labelling of G_1, G_2, G_3, G_4, G_5 , and G_6 for any other magic constants (i.e., for any other value of a) than computed here.
- (v) For l, m and n positive integers, determine any $S-(a, 0)$ -EAM total labelling for the following networks:
  - (1) $(C_n \circ K_{2,m}) \cup K_{1,\ell}$
  - (2) $(C_n \circ K_{2,m}) \cup lP_2$
  - (3) $(C_n \circ K_{2,m}) \cup P_l$

Open problems related to Theorems 11 and 12 are given as follows:

- (i) For $2 \leq n \equiv 0 \pmod{2}$, find any $S-(a, 0)$ -EAM total labelling of $C_n \circ H_1$.
- (ii) For $2 \leq n \equiv 0 \pmod{2}$, find any $S-(a, 0)$ -EAM total labelling of $C_n \circ H_2$.
- (iii) For $3 \leq n \equiv 1 \pmod{2}$, determine some $S-(a, 0)$ -EAM total labelling of $C_n \circ H_1$ and $C_n \circ H_2$ with a different magic constant than obtained here, i.e., for any other value of a .

FIGURE 11: An S-(114,0)-EAM total labelling of $C_5 \circ H_1$ ($\rightarrow A$) and $C_5 \circ H_2$ ($\rightarrow B$).

FIGURE 12: An S-(54,0)-EAM total labelling of Γ_3 .

FIGURE 13: An S-(90,0)-EAM total labelling of the lattice network $\mathbb{L}_5^1$.

FIGURE 14: An S-(108,0)-EAM total labelling of the lattice network $\mathbb{L}_4^2$.

FIGURE 15: An S-(108,0)-EAM total labelling of the lattice network $\mathbb{L}_4^3$.

TABLE 2: Synopsis of main theorems.

Network	Parameters	a ($d=0$)	a' ($d=2$)	Planar V nonplanar
$(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$	$m \geq 2, \text{ odd } n \geq 3$	$8n + 3mn + 2$	$6n + mn + 3$	Planar
$(C_n \circ K_{2,m}) \cup nP_2$	$m \geq 2, \text{ odd } n \geq 3$	$(1/2)(17n + 6mn + 3)$	$(1/2)(13n + 2mn + 5)$	Planar
$(C_n \circ K_{2,m}) \cup P_{n+1}$	$m \geq 2, \text{ odd } n \geq 3$	$(1/2)(13n + 6mn + 7)$	$(1/2)(9n + 2mn + 9)$	Planar
$C_n \circ H_1$ and $C_n \circ H_2$	n is odd	$(1/2)(45n + 3)$	$(1/2)(17n + 5)$	Planar
Γ_n	$\forall n \in \mathbb{N}$	$18n$	$6n + 4$	Planar
$\mathbb{L}_n^1$	$\forall n \in \mathbb{N}$	$18n$	$6n + 4$	Nonplanar
$\mathbb{L}_n^2$ and $\mathbb{L}_n^3$	$\forall n \in \mathbb{N}$	$30n - 12$	$10n$	Nonplanar

FIGURE 16: Comparison of the magic constants ($\rightarrow A$) and the minimum edge weights ($\rightarrow B$) of $M_1, M_2, M_3, C_n \circ H_1, C_n \circ H_2, \Gamma_n, \mathbb{L}_n^1, \mathbb{L}_n^2$, and $\mathbb{L}_n^3$, for $m \geq 2$ and $3 \leq n \equiv 1 \pmod{2}$.

4. Synopsis and 3-D Comparison of the Magic Constants and Minimum Edge Weights

This section consists of the synopsis (Section 4.1) and 3-D graphical plots and comparison of the magic constants (a) and minimum edge weights (a') of our findings (Section 4.2).

4.1. Synopsis. Table 2 exhibits the computational results of our findings. The possible parameters for which we have determined S- $(a, 0)$ and S- $(a', 2)$ -EAM total labellings are indicated through parameters column head.

4.2. Graphical Behavior of the Magic Constants and Minimum Edge Weights. Figure 16 shows the graphical comparison of the magic constants ($\rightarrow A$) and minimum edge weights ($\rightarrow B$), in 3-D, of the networks $M_1 \cong (C_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$, $M_2 \cong (C_n \circ K_{2,m}) \cup nP_2$, $M_3 \cong (C_n \circ K_{2,m}) \cup P_{n+1}$, $C_n \circ H_1$, $C_n \circ H_2$, Γ_n , $\mathbb{L}_n^1$, $\mathbb{L}_n^2$, and $\mathbb{L}_n^3$, respectively. Moreover, Figure 16(a) shows that the most dominant layer is the one with green color. It interprets that among the magic constants of the networks discussed in this note, $(C_n \circ K_{2,m}) \cup nP_2$ attains highest values with the increase of the values of the parameters.

Figure 17 illustrates the relative 3-D comparison of the magic constants and minimum edge weights (corresponding to $d=2$) of the networks $(C_n \circ K_{2,m}) \cup K_{1,n} \cup (n-1/2)K_1$,

FIGURE 17: Continued.

FIGURE 17: Simultaneous relative comparison of the magic constants and the minimum edge weights of M_1 ($\rightarrow A$), M_2 ($\rightarrow B$), M_3 ($\rightarrow C$), $\{C_n \circ H_1, C_n \circ H_2\}$ ($\rightarrow D$), Γ_n ($\rightarrow E$), $\{\mathbb{L}_n^1, \mathbb{L}_n^2, \mathbb{L}_n^3\}$ ($\rightarrow F$), for $m \geq 2$ & $3 \leq n \equiv 1 \pmod{2}$

$(C_n \circ K_{2,m}) \cup nP_2$, $(C_n \circ K_{2,m}) \cup P_{n+1}$, $C_n \circ H_1$, $C_n \circ H_2$, Γ_n , $\mathbb{L}_n^1$, $\mathbb{L}_n^2$ and $\mathbb{L}_n^3$, for different values of the parameters.

5. Conclusion

In the present article,

- (i) We have designed S- $(a, 0)$ -EAM total labelling of the rooted product of cycle C_n and complete bipartite network $K_{2,m}$ taking its disjoint union with paths and stars. The findings are related to the open problem on $nK_{2,m}$ provided by Ngurah et al. in [48].
- (ii) We have provided S- $(a, 0)$ -EAM total labelling of rooted product of C_n and pancyclic networks H_1 and H_2 .
- (iii) We have extended the result provided in [29] by Baig et al. through exhibiting S- $(a, 0)$ -EAM total labelling of pancyclic network Γ_n involving chains of C_6 .
- (iv) We have exhibited S- $(a, 0)$ -EAM total labelling of symmetrically designed lattice networks $\mathbb{L}_n^1$, $\mathbb{L}_n^2$, and $\mathbb{L}_n^3$.
- (v) We have illustrated our findings through 3-D graphical comparison.
- (vi) For further working in this field, several research problems have also been opened.
- (vii) The obtained schemes are now all set to serve as test ready labellings for programmers, networking professionals, and engineers to avail them where they find these appropriate.

Data Availability

The data used to support the findings of this study are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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