Research Article

Results on Complex Partial b -Metric Space with an Application

Gunaseelan Mani ¹, Arul Joseph Gnanaprakasam,² Ramakrishnan Kalaichelvan,¹
and Yaé Ulrich Gaba ^{3,4}

¹Department of Mathematics, Sri Sankara Arts and Science College(Autonomous), Affiliated to Madras University, Enathur, Kanchipuram 631 561, Tamil Nadu, India

²Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar Kattankulathur, Kanchipuram, Chennai 603203, Tamil Nadu, India

³Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

⁴African Center for Advanced Studies, P.O. Box 4477, Yaounde, Cameroon

Correspondence should be addressed to Gunaseelan Mani; mathsguna@yahoo.com and Yaé Ulrich Gaba; yaeulrich.gaba@gmail.com

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In this paper, we prove a fixed point theorem in complex partial b -metric space under new contraction mapping. The proved results generalize and extend some of the well-known results in the literature. We also give some applications of our main results.

1. Introduction

Introduced in 1989 by Bakhtin [1] and Czerwik [2], the concept of b -metric spaces provided a framework to extend the results already known in the classical setting of metric spaces. About two decades later, more precisely in 2011, Azamet al. [3] came up with the notion of complex-valued metric spaces and provided some common fixed point theorems under some contractive conditions. Two years after, it was in [4] Rao et al. discussed for the first time the idea of complex-valued b -metric spaces.

It was just very recently, in 2017, that Dhivya and Marudai [5] extended all the preceding results in the setting of complex partial metric spaces making use of a rational type contraction.

This was followed by Gunaseelan [6], who introduced the concepts of complex partial b -metric spaces and discussed some results of fixed point theory for self-mappings in these new spaces.

Many authors have studied related interesting metric such as structures along with some applications. And, in this

line, significant results have been obtained and can be read in [7–23]. In this paper, under new contraction condition, we prove a fixed point theorem in complex partial b -metric space. Although there have been a significant amount of scientific contributions to the theory of partial b -metric space, very few address that of complex valued, and even less, the applicability of complex partial b -metrics in the resolution integral equations. This, however, in one the main contributions of the present work. We begin by recalling basic facts about complex partial b -metric spaces.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$. Define a partial order $<$ on $\mathbb{C}$ as follows: $\omega_1 < \omega_2$ if and only if $\text{Re}(\omega_1) \leq \text{Re}(\omega_2)$, $\text{Im}(\omega_1) \leq \text{Im}(\omega_2)$.

Consequently, one can infer that $\omega_1 < \omega_2$ if one of the following conditions is satisfied:

- (i) $\text{Re}(\omega_1) = \text{Re}(\omega_2)$ and $\text{Im}(\omega_1) < \text{Im}(\omega_2)$

- (ii) $\text{Re.part}(\omega_1) < \text{Re.part}(\omega_2)$ and $\text{Im.part}(\omega_1) = \text{Im.part}(\omega_2)$
- (iii) $\text{Re.part}(\omega_1) < \text{Re.part}(\omega_2)$ and $\text{Im.part}(\omega_1) < \text{Im.part}(\omega_2)$
- (iv) $\text{Re.part}(\omega_1) = \text{Re.part}(\omega_2)$ and $\text{Im.part}(\omega_1) = \text{Im.part}(\omega_2)$

In particular, we write $\omega_1 \preceq \omega_2$ if $\omega_1 \neq \omega_2$ and one of (i), (ii), and (iii) is satisfied, and we write $\omega_1 < \omega_2$ if only (iii) is satisfied. Notice that

- (a) If $0 < \omega_1 \preceq \omega_2$, then $|\omega_1| < |\omega_2|$
- (b) If $\omega_1 < \omega_2$ and $\omega_2 < \omega_3$, then $\omega_1 < \omega_3$
- (c) If $\tau, \gamma \in \mathbb{R}$ and $\tau \leq \gamma$, then $\tau\omega_1 < \gamma\omega_1$ for all $0 < \omega_1 \in \mathbb{C}$

Definition 1 (see [4]). Let Θ be a nonvoid set and let $s \geq 1$ be a given real number. A function $d: \Theta \times \Theta \longrightarrow \mathbb{C}$ is called a complex-valued b -metric on Θ if, for all $e, f, g \in \Theta$, the following conditions are satisfied:

- (i) $0 < d(e, f)$ and $d(e, f) = 0$ if and only if $e = f$
- (ii) $d(e, f) = d(f, g)$
- (iii) $d(e, f) < s[d(e, g) + d(g, f)]$

The pair (Θ, d) is called a complex-valued b -metric space.

Here, $\mathbb{C}^+ (= \{(e, g) | e, g \in \mathbb{R}^+\})$ and $\mathbb{R}^+ (= \{e \in \mathbb{R} | e \geq 0\})$ denote the set of nonnegative complex numbers and the set of nonnegative real numbers, respectively. We now give the complex partial metric space.

Definition 2 (see [5]). A complex partial metric on a nonvoid set Θ is a function $\xi_c: \Theta \times \Theta \longrightarrow \mathbb{C}^+$ such that, for all $\alpha, \beta, \gamma \in \Theta$,

- (i) $0 < \xi_c(\alpha, \alpha) < \xi_c(\alpha, \beta)$ (small self – distances)
- (ii) $\xi_c(\alpha, \beta) = \xi_c(\beta, \alpha)$ (symmetry)
- (iii) $\xi_c(\alpha, \alpha) = \xi_c(\alpha, \beta) = \xi_c(\beta, \beta)$ if and only if $\alpha = \beta$ (equality)
- (iv) $\xi_c(\alpha, \beta) < \xi_c(\alpha, \gamma) + \xi_c(\gamma, \beta) - \xi_c(\beta, \beta)$ (triangularity)

A complex partial metric space is a pair (Θ, ξ_c) such that Θ is a nonvoid set and ξ_c is the complex partial metric on Θ .

Definition 3 (see [6]). A complex partial b -metric on a nonvoid set Θ is a function $\gamma_{cb}: \Theta \times \Theta \longrightarrow \mathbb{C}^+$ such that, for all $\alpha, \beta, \gamma \in \Theta$,

- (i) $0 < \gamma_{cb}(\alpha, \alpha) < \gamma_{cb}(\alpha, \beta)$ (small self – distances)
- (ii) $\gamma_{cb}(\alpha, \beta) = \gamma_{cb}(\beta, \alpha)$ (symmetry)
- (iii) $\gamma_{cb}(\alpha, \alpha) = \gamma_{cb}(\alpha, \beta) = \gamma_{cb}(\beta, \beta) \Leftrightarrow \alpha = \beta$ (equality)
- (iv) $\exists$ a real number $s \geq 1$ and s is an independent of α, β, γ such that $\gamma_{cb}((\alpha, \beta) < s[\gamma_{cb}(\alpha, \gamma) + \gamma_{cb}(\gamma, \beta)] - \gamma_{cb}(\gamma, \gamma)$ (triangularity)

A complex partial b -metric space is a pair (Θ, γ_{cb}) such that Θ is a nonvoid set and γ_{cb} is the complex partial b -metric on Θ . The number s is called the coefficient of (Θ, γ_{cb}) .

Remark 1 (see [6]). In a complex partial b -metric space (Θ, γ_{cb}) if $\alpha, \beta \in \Theta$ and $\gamma_{cb}(\alpha, \beta) = 0$, then $\alpha = \beta$, but the converse may not be true.

Every complex partial b -metric γ_{cb} on a nonvoid set Θ generates a topology τ_{cb} on Θ whose base is the family of open γ_{cb} -balls $B_{\gamma_{cb}}(\alpha, \varepsilon)$, where $\tau_{cb} = \{B_{\gamma_{cb}}(\alpha, \varepsilon) : \alpha \in \Theta, \varepsilon > 0\}$ and $B_{\gamma_{cb}}(\alpha, \varepsilon) = \{\beta \in \Theta : \gamma_{cb}(\alpha, \beta) < \varepsilon + \gamma_{cb}(\alpha, \alpha)\}$. Now, we define Cauchy sequence and convergent sequence in complex partial b -metric spaces.

Definition 4 (see [6]). Let (Θ, γ_{cb}) be a complex partial b -metric space with coefficient s . Let $\{\alpha_n\}$ be any sequence in Θ and $\alpha \in \Theta$. Then,

- (i) The sequence $\{\alpha_n\}$ is said to be convergent with respect to τ_{cb} and converges to α if $\lim_{n \rightarrow \infty} \gamma_{cb}(\alpha_n, \alpha) = \gamma_{cb}(\alpha, \alpha)$
- (ii) The sequence $\{\alpha_n\}$ is said to be Cauchy sequence in (Θ, γ_{cb}) if $\lim_{n, m \rightarrow \infty} \gamma_{cb}(\alpha_n, \alpha_m)$ exists and is finite
- (iii) (Θ, γ_{cb}) is said to be a complete complex partial b -metric space if, for every Cauchy sequence $\{\alpha_n\}$ in Θ , there exists $\alpha \in \Theta$ such that $\lim_{n, m \rightarrow \infty} \gamma_{cb}(\alpha_n, \alpha_m) = \lim_{n \rightarrow \infty} \gamma_{cb}(\alpha_n, \alpha) = \gamma_{cb}(\alpha, \alpha)$
- (iv) A mapping $\Xi: \Theta \longrightarrow \Theta$ is said to be continuous at $\alpha_0 \in \Theta$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\Xi(B_{\gamma_{cb}}(\alpha_0, \delta)) \subset B_{\gamma_{cb}}(\Xi(\alpha_0), \varepsilon)$

Let (Θ, γ_{cb}) be a complex partial b -metric space and $D \subseteq \Theta$. A point $\alpha \in \Theta$ is called an interior of set D if there exists $0 < r \in \mathbb{C}$ such that $D_{\gamma_{cb}}(\alpha, r) = \{\beta \in \Theta : \gamma_{cb}(\alpha, \beta) < \gamma_{cb}(\alpha, \alpha) + r\} \subseteq D$. A subset D is called open if each point of D is an interior point of D . A point $\alpha \in \Theta$ is said to be a limit point of D , for every $0 < r \in \mathbb{C}$, $D_{\gamma_{cb}}(\alpha, r) \cap (D - \{\alpha\}) \neq \emptyset$. A subset $E \subseteq \Theta$ is called closed iff E contains all its limit points.

Example 1 (see [6]). Let $\Theta = [0, \infty)$ be endowed with complex partial b -metric $\gamma_{cb}: \Theta \times \Theta \longrightarrow \mathbb{C}^+$ with $\gamma_{cb} = (\max\{\alpha, \beta\})^2 + i(\max\{\alpha, \beta\})^2 \forall \alpha, \beta \in \Theta$.

In 2019, Gunaseelan [6] proved the following theorem.

Theorem 1 (see [6]). Let (Θ, γ_{cb}) be a complete complex partial b -metric space with coefficient $s \geq 1$ and $\Xi: \Theta \longrightarrow \Theta$ be a mapping satisfying the following condition:

$$\gamma_{cb}(\Xi\alpha, \Xi\beta) < a[\gamma_{cb}(\alpha, \Xi\alpha) + \gamma_{cb}(\beta, \Xi\beta)], \quad \forall \alpha, \beta \in \Theta, \quad (1)$$

where $a \in [0, (1/s))$. Then, Ξ has a unique fixed point $\gamma \in \Theta$ and $\gamma_{cb}(\gamma, \gamma) = 0$.

Inspired by Theorem 1, we prove a fixed point theorem on complex partial b -metric space under new contraction mapping.

In Section 3, we first prove, under new contraction mapping, a fixed point theorem on complete complex partial b -metric space. We also provide an example of the complete complex partial b -metric space and clarify that, under certain conditions, it has a unique fixed point.

3. Main Results

Theorem 2. Let (Θ, γ_{cb}) be a complete complex partial b -metric space with constant $s \geq 1$ and let Ξ be a self-mapping on Θ . Suppose that there exist functions τ_i , $i = 1, 2, 3, 4, 5$, of $\mathbb{C}^+$ into $\mathbb{C}^+$ such that

- (C₁) Each τ_i is upper semicontinuous from the right
- (C₂) $\tau_1(v) + \tau_2(v) + \tau_3(v) + \tau_4(v) + \tau_5(v) < (1/2s^2)v$, $\forall v > 0$
- (C₃) For any distinct $\alpha, \beta \in \Theta$,

$$\begin{aligned} \frac{1}{2s} \gamma_{cb}(\alpha, \Xi\alpha) &< \gamma_{cb}(\alpha, \beta) \Rightarrow \\ \gamma_{cb}(\alpha, \beta) \gamma_{cb}(\Xi\alpha, \Xi\beta) &< \tau_1(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) \\ &+ \tau_2(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \Xi\beta) \\ &+ \tau_3(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\Xi\alpha, \beta) \\ &+ \tau_4(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \Xi\alpha) \\ &+ \tau_5(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\beta, \Xi\beta). \end{aligned} \quad (2)$$

Then, Ξ has a unique fixed point.

Proof. Let us first prove that if fixed points of Ξ exists, then it is unique. Let $\alpha, \beta \in \Theta$ be two distinct fixed points of Ξ , that is, $\Xi\alpha = \alpha \neq \beta = \Xi\beta$. Therefore, $\gamma_{cb}(\alpha, \beta) > 0$. If $\gamma_{cb}(\alpha, \alpha) = 0$, we have $0 = (1/2s) \gamma_{cb}(\alpha, \alpha) = (1/2s) \gamma_{cb}(\alpha, \Xi\alpha) < \gamma_{cb}(\alpha, \beta)$. If $\gamma_{cb}(\alpha, \alpha) > 0$. By the definition of complex partial b -metric space, we obtain

$$\frac{1}{2s} \gamma_{cb}(\alpha, \Xi\alpha) = \frac{1}{2s} \gamma_{cb}(\alpha, \alpha) < \gamma_{cb}(\alpha, \beta). \quad (3)$$

From condition (C₃), we derive

$$\begin{aligned} \gamma_{cb}(\alpha, \beta)^2 &= \gamma_{cb}(\alpha, \beta) \gamma_{cb}(\Xi\alpha, \Xi\beta) \\ &\leq \tau_1(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) + \tau_2(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \Xi\beta) \\ &\quad + \tau_3(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\Xi\alpha, \beta) + \tau_4(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \Xi\alpha) \\ &\quad + \tau_5(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\beta, \Xi\beta) \\ &= \tau_1(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) + \tau_2(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) \\ &\quad + \tau_3(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) + \tau_4(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \alpha) \\ &\quad + \tau_5(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\beta, \beta) \\ &\leq \tau_1(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) + \tau_2(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) \\ &\quad + \tau_3(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) + \tau_4(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) \\ &\quad + \tau_5(\gamma_{cb}(\alpha, \beta)) \gamma_{cb}(\alpha, \beta) \\ &= (\tau_1(\gamma_{cb}(\alpha, \beta)) + \tau_2(\gamma_{cb}(\alpha, \beta)) + \tau_3(\gamma_{cb}(\alpha, \beta)) + \tau_4(\gamma_{cb}(\alpha, \beta)) \\ &\quad + \tau_5(\gamma_{cb}(\alpha, \beta))) \gamma_{cb}(\alpha, \beta) \\ &\leq \frac{1}{2s} \gamma_{cb}(\alpha, \beta) \\ &< \gamma_{cb}(\alpha, \beta), \end{aligned} \quad (4)$$

which implies that

$$|\gamma_{cb}(\alpha, \beta)|^2 < |\gamma_{cb}(\alpha, \beta)|, \quad (5)$$

which is impossible. Therefore, $\alpha = \beta$. Choose $\alpha_1 \in \Theta$. Set

$$\alpha_2 = \Xi\alpha_1, \dots, \alpha_{m+1} = \Xi\alpha_m = \Xi^{m+1}\alpha_1. \quad (6)$$

Let $\gamma_m = \gamma_{cb}(\alpha_m, \Xi\alpha_m)$. If there exists $m \in \mathbb{N}$ such that $\gamma_m = 0$, the proof is complete. So, we presume that, for every $m \in \mathbb{N}$, $0 < \gamma_m = \gamma_{cb}(\alpha_m, \Xi\alpha_m) = \gamma_{cb}(\alpha_m, \alpha_{m+1})$. Therefore, we have

$$\frac{1}{2s} \gamma_{cb}(\alpha_m, \Xi\alpha_m) < \gamma_{cb}(\alpha_m, \Xi\alpha_m), \quad \forall m \in \mathbb{N}. \quad (7)$$

From (2), we derive

$$\begin{aligned}
\Upsilon_{cb}(\alpha_m, \Xi\alpha_m)\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) &\leq \tau_1(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi^2\alpha_m) \\
&\quad + \tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_4(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_5(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
&\leq \tau_1(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
&\quad - \tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi\alpha_m) \\
&\quad + s\tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \alpha_m) \\
&\quad + s\tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad - \tau_3(\alpha_m, \Xi\alpha_m)\Upsilon_{cb}(\alpha_m, \alpha_m) \\
&\quad + \tau_4(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_5(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
&\leq \tau_1(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
&\quad + s\tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \alpha_m) \\
&\quad + s\tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_4(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + \tau_5(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
&= [\tau_1(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m)) + s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))] \\
&\quad + 2s\tau_3(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m)) + \tau_4(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))\Upsilon_{cb}(\alpha_m, \Xi\alpha_m) \\
&\quad + [s\tau_2(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m)) + \tau_5(\Upsilon_{cb}(\alpha_m, \Xi\alpha_m))]\Upsilon_{cb}(\Xi\alpha_m, \Xi^2\alpha_m),
\end{aligned} \tag{8}$$

and consequently,

$$\begin{aligned}
\Upsilon_m \Upsilon_{m+1} &< [\tau_1(\Upsilon_m) + s\tau_2(\Upsilon_m) + 2s\tau_3(\Upsilon_m) + \tau_4(\Upsilon_m)]\Upsilon_m \\
&\quad + [s\tau_2(\Upsilon_m) + \tau_5(\Upsilon_m)]\Upsilon_{m+1},
\end{aligned} \tag{9}$$

which implies that

$$\Upsilon_{m+1} < \frac{\tau_1(\Upsilon_m) + s\tau_2(\Upsilon_m) + 2s\tau_3(\Upsilon_m) + \tau_4(\Upsilon_m)}{\Upsilon_m - s\tau_2(\Upsilon_m) - \tau_5(\Upsilon_m)}\Upsilon_m. \tag{10}$$

From (C₃), we obtain

$$\begin{aligned}
\tau_1(v) + 2s\tau_2(v) + 2s\tau_3(v) + \tau_4(v) + \tau_5(v) &< 2s[\tau_1(v) + \tau_2(v) + \tau_3(v) + \tau_4(v) + \tau_5(v)] \\
< 2s \frac{1}{2s^2} v &= \frac{1}{s} v < v, \quad \forall v > 0.
\end{aligned} \tag{11}$$

Consequently, we derive

$$\frac{\tau_1(v) + s\tau_2(v) + 2s\tau_3(v) + \tau_4(v)}{v - s\tau_2(v) - \tau_5(v)} < 1, \quad \forall v > 0. \tag{12}$$

Therefore, from (10), we obtain

$$\Upsilon_{cb}(\alpha_{m+1}, \Xi\alpha_{m+1}) = \Upsilon_{m+1} < \Upsilon_m = \Upsilon_{cb}(\alpha_m, \Xi\alpha_m), \quad \forall m \in \mathbb{N}, \tag{13}$$

which implies that

$$|\Upsilon_{cb}(\alpha_{m+1}, \Xi\alpha_{m+1})| < |\Upsilon_{cb}(\alpha_m, \Xi\alpha_m)|, \quad \forall m \in \mathbb{N}. \tag{14}$$

Consequently, $\text{Re.part}(\Upsilon_{m+1}) < \text{Re.part}(\Upsilon_m)$ and $\text{Im.part}(\Upsilon_{m+1}) < \text{Im.part}(\Upsilon_m)$. Therefore, $\{\text{Re.part}(\Upsilon_m)\}$ is a decreasing sequence of real numbers which is bounded from below. So, $\{\text{Re.part}(\Upsilon_m)\}$ converges to some point $\Upsilon \in [0, \infty)$. Similarly, $\{\text{Im.part}(\Upsilon_m)\}$ converges to some point $\Upsilon \in [0, \infty)$.

∞). Hence, $\{\gamma_m = \gamma_{cb}(\alpha_m, \Xi\alpha_m)\}$ converges to some point $\gamma \in [0, \infty)$. If $\gamma > 0$, then using condition (C_1) , we obtain

$$\begin{aligned} \gamma &= \lim_{m \rightarrow \infty} \gamma_{m+1} = \lim_{m \rightarrow \infty} \sup \gamma_{m+1} \\ &\leq \lim_{m \rightarrow \infty} \sup \frac{\tau_1(\gamma_m) + s\tau_2(\gamma_m) + 2s\tau_3(\gamma_m) + \tau_4(\gamma_m)}{\gamma_m - s\tau_2(\gamma_m) - \tau_5(\gamma_m)} \gamma_m \\ &= \frac{\tau_1(\gamma) + s\tau_2(\gamma) + 2s\tau_3(\gamma) + \tau_4(\gamma)}{\gamma - s\tau_2(\gamma) - \tau_5(\gamma)} \gamma \\ &< \gamma, \end{aligned} \quad (15)$$

which implies that $|\gamma| < |\gamma|$, which is impossible. Therefore,

$$\lim_{m \rightarrow \infty} \gamma_m = \lim_{m \rightarrow \infty} \gamma_{cb}(\alpha_m, \Xi\alpha_m) = 0. \quad (16)$$

Next, we prove that

$$\lim_{m, r \rightarrow \infty} \gamma_{cb}(\alpha_m, \alpha_r) = 0. \quad (17)$$

Suppose not, we assume that there exist $\varepsilon > 0$ and sequence $\{\zeta(m)\}$ and $\{\xi(m)\}$ of natural numbers such that

$$\begin{aligned} \zeta(m) &> \xi(m) > m, \\ \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) &\geq \varepsilon, \\ \gamma_{cb}(\alpha_{\zeta(m)-1}, \alpha_{\xi(m)}) &< \varepsilon, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (18)$$

Therefore, we derive

$$\begin{aligned} \varepsilon &< \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) < s(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\zeta(m)-1}) + \gamma_{cb}(\alpha_{\zeta(m)-1}, \alpha_{\xi(m)})) \\ &\quad - \gamma_{cb}(\alpha_{\zeta(m)-1}, \alpha_{\zeta(m)-1}) \\ &< s\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\zeta(m)-1}) + s\gamma_{cb}(\alpha_{\zeta(m)-1}, \alpha_{\xi(m)}) \\ &< s\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\zeta(m)-1}) + s\varepsilon \\ &= s\gamma_{cb}(\alpha_{\zeta(m)-1}, \Xi\alpha_{\zeta(m)-1}) + s\varepsilon. \end{aligned} \quad (19)$$

Using (16), we derive

$$\varepsilon < \lim_{m \rightarrow \infty} \inf \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) < \lim_{m \rightarrow \infty} \sup \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) < s\varepsilon. \quad (20)$$

From (16), there exists $M \in \mathbb{N}$ such that $(1/2s)\gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\zeta(m)}) < \varepsilon, \forall m > M$, and using (18), we derive

$$\begin{aligned} \frac{1}{2s}\gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\zeta(m)}) &< \gamma_{cb}(\alpha_{\zeta(m)+1}, \alpha_{\xi(m)+1}) \\ &= \gamma_{cb}(\Xi\alpha_{\zeta(m)}, \Xi\alpha_{\xi(m)}), \quad \forall m > M. \end{aligned} \quad (21)$$

Therefore, from (C_3) , for every $m > M$, we obtain

$$\begin{aligned} \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \gamma_{cb}(\Xi\alpha_{\zeta(m)}, \Xi\alpha_{\xi(m)}) &\leq \tau_1(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + \tau_2(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\xi(m)}) \\ &\quad + \tau_3(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\Xi\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + \tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\zeta(m)}) \\ &\quad + \tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\xi(m)}, \Xi\alpha_{\xi(m)}) \\ &\leq \tau_1(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + \tau_2(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\xi(m)}) \\ &\quad + \tau_3(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\Xi\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\xi(m)}, \Xi\alpha_{\zeta(m)}) \\ &\quad - \tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\xi(m)}, \alpha_{\xi(m)}) \\ &\quad + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\Xi\alpha_{\xi(m)}, \alpha_{\zeta(m)}) \\ &\quad + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad - \tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\zeta(m)}) \\ &\leq \tau_1(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + \tau_2(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \Xi\alpha_{\xi(m)}) \\ &\quad + \tau_3(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\xi(m)}, \Xi\alpha_{\zeta(m)}) \\ &\quad + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\xi(m)}, \Xi\alpha_{\zeta(m)}) \\ &\quad + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\Xi\alpha_{\xi(m)}, \alpha_{\zeta(m)}) \\ &\quad + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\leq [\tau_1(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}))] \gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &\quad + [\tau_2(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) + s\tau_5(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}))] \gamma_{cb}(\Xi\alpha_{\xi(m)}, \alpha_{\zeta(m)}) \\ &\quad + [\tau_3(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})) + s\tau_4(\gamma_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}))] \gamma_{cb}(\alpha_{\xi(m)}, \Xi\alpha_{\zeta(m)}). \end{aligned} \quad (22)$$

Hence, from (C_2) , we derive

$$\begin{aligned} \Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\Xi \alpha_{\zeta(m)}, \Xi \alpha_{\xi(m)}) &\leq \frac{1}{2s^2} [\Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})]^2 \\ &+ \frac{1}{2s^2} \Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\Xi \alpha_{\zeta(m)}, \alpha_{\xi(m)}) \\ &+ \frac{1}{2s^2} \Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\alpha_{\xi(m)}, \Xi \alpha_{\xi(m)}). \end{aligned} \quad (23)$$

From (16)–(23), we obtain

$$\begin{aligned} \varepsilon^2 &< \lim_{m \rightarrow \infty} \sup [\Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\alpha_{\zeta(m)+1}, \alpha_{\xi(m)+1})] \\ &< \lim_{m \rightarrow \infty} \sup \left[\frac{1}{2s^2} [\Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)})]^2 n + \frac{1}{2s^2} \Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\Xi \alpha_{\zeta(m)}, \alpha_{\xi(m)}) n \right. \\ &\quad \left. + \frac{1}{2s^2} \Upsilon_{cb}(\alpha_{\zeta(m)}, \alpha_{\xi(m)}) \Upsilon_{cb}(\alpha_{\xi(m)}, \Xi \alpha_{\xi(m)}) \right] n \\ &< \frac{1}{2s^2} (s\varepsilon)^2 = \frac{1}{2} \varepsilon^2 < \varepsilon^2, \end{aligned} \quad (24)$$

which is impossible. Hence, $\lim_{m,r \rightarrow \infty} \Upsilon_{cb}(\alpha_m, \alpha_r) = 0$. By completeness of (Θ, Υ_{cb}) , there exists $\alpha \in \Theta$ such that

$$\Upsilon_{cb}(\alpha, \alpha) = \lim_{m \rightarrow \infty} \Upsilon_{cb}(\alpha_m, \alpha) = \lim_{m,r \rightarrow \infty} \Upsilon_{cb}(\alpha_m, \alpha_r) = 0. \quad (25)$$

We shall prove that, for every $m \in \mathbb{N}$,

$$\frac{1}{2s} \Upsilon_{cb}(\alpha_m, \Xi \alpha_m) < \Upsilon_{cb}(\alpha_m, \alpha), \quad (26)$$

or

$$\frac{1}{2s} \Upsilon_{cb}(\Xi \alpha_m, \Xi^2 \alpha_m) < \Upsilon_{cb}(\Xi \alpha_m, \alpha). \quad (27)$$

Suppose not, we assume that there exists $q \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{2s} \Upsilon_{cb}(\alpha_q, \Xi \alpha_q) &\geq \Upsilon_{cb}(\alpha_q, \alpha), \\ \frac{1}{2s} \Upsilon_{cb}(\Xi \alpha_q, \Xi^2 \alpha_q) &\geq \Upsilon_{cb}(\Xi \alpha_q, \alpha). \end{aligned} \quad (28)$$

By the definition of complex partial b -metric space and (28), we derive

$$\begin{aligned} \Upsilon_{cb}(\alpha_q, \Xi \alpha_q) &< s [\Upsilon_{cb}(\alpha_q, \alpha) + \Upsilon_{cb}(\alpha, \Xi \alpha_q)] - \Upsilon_{cb}(\alpha, \alpha) \\ &= s \Upsilon_{cb}(\alpha_q, \alpha) + s \Upsilon_{cb}(\alpha, \Xi \alpha_q) \\ &< s \frac{1}{2s} \Upsilon_{cb}(\alpha_q, \Xi \alpha_q) + s \frac{1}{2s} \Upsilon_{cb}(\Xi \alpha_q, \Xi^2 \alpha_q) \\ &< \frac{1}{2} \Upsilon_{cb}(\alpha_q, \Xi \alpha_q) \\ &\quad + \frac{1}{2} \Upsilon_{cb}(\alpha_q, \Xi \alpha_q) \\ &= \Upsilon_{cb}(\alpha_q, \Xi \alpha_q), \end{aligned} \quad (29)$$

which is impossible. Hence, (25) and (26) holds. From (25), we obtain

$$\begin{aligned} \Upsilon_{cb}(\alpha_m, \alpha) \Upsilon_{cb}(\Xi \alpha_m, \Xi \alpha) &\leq \tau_1 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \alpha) \\ &\quad + \tau_2 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \Xi \alpha) \\ &\quad + \tau_3 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\Xi \alpha_m, \alpha) \\ &\quad + \tau_4 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \Xi \alpha_m) \\ &\quad + \tau_5 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha, \Xi \alpha) \\ &\leq \tau_1 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \alpha) \\ &\quad + s \tau_2 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \Xi \alpha_m) \\ &\quad + s \tau_2 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\Xi \alpha_m, \Xi \alpha) \\ &\quad + \tau_3 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\Xi \alpha_m, \alpha) \\ &\quad + \tau_4 (\Upsilon_{cb}(\alpha_m, \alpha)) \Upsilon_{cb}(\alpha_m, \Xi \alpha_m) \end{aligned}$$

$$\begin{aligned}
& + s\tau_5(\gamma_{cb}(\alpha_m, \alpha))\gamma_{cb}(\alpha, \Xi\alpha_m) \\
& + s\tau_5((\gamma_{cb}(\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi\alpha)) \\
& \leq \tau_1(\gamma_{cb}(\alpha_m, \alpha))\gamma_{cb}(\alpha_m, \alpha) \\
& + [s\tau_2(\gamma_{cb}(\alpha_m, \alpha)) + \tau_4(\gamma_{cb}(\alpha_m, \alpha))]\gamma_{cb}(\alpha_m, \Xi\alpha_m) \\
& + [s\tau_2(\gamma_{cb}(\alpha_m, \alpha)) + s\tau_5(\gamma_{cb}(\alpha_m, \alpha))]\gamma_{cb}(\Xi\alpha_m, \Xi\alpha) \\
& + [\tau_3(\gamma_{cb}(\alpha_m, \alpha)) + s\tau_5(\gamma_{cb}(\alpha_m, \alpha))]\gamma_{cb}(\Xi\alpha_m, \alpha).
\end{aligned} \tag{30}$$

From (12), we derive

$$\begin{aligned}
\gamma_{cb}(\Xi\alpha_m, \Xi\alpha) & \leq \gamma_{cb}(\alpha_m, \alpha) + \gamma_{cb}(\alpha_m, \Xi\alpha_m) \\
& + \gamma_{cb}(\Xi\alpha_m, \Xi\alpha) + \gamma_{cb}(\Xi\alpha_m, \alpha).
\end{aligned} \tag{31}$$

Using (25) and (26), we derive

$$\lim_{m \rightarrow \infty} \gamma_{cb}(\Xi\alpha_m, \Xi\alpha) = 0. \tag{32}$$

Since

$$\begin{aligned}
\gamma_{cb}(\alpha, \Xi\alpha) & < s[\gamma_{cb}(\alpha, \Xi\alpha_m) + \gamma_{cb}(\Xi\alpha_m, \Xi\alpha)] - \gamma_{cb}(\Xi\alpha_m, \Xi\alpha_m) \\
& < s[\gamma_{cb}(\alpha, \Xi\alpha_m) + \gamma_{cb}(\Xi\alpha_m, \Xi\alpha)] \\
& = s\gamma_{cb}(\alpha, \alpha_{m+1}) + s\gamma_{cb}(\Xi\alpha_m, \Xi\alpha),
\end{aligned} \tag{33}$$

using (25) and (32), we get $\alpha = \Xi\alpha$. From (27), we obtain

$$\begin{aligned}
\gamma_{cb}(\Xi\alpha_m, \alpha)\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) & \leq \tau_1(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \alpha) \\
& + \tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi\alpha) \\
& + \tau_3(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \alpha) \\
& + \tau_4(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + \tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\alpha, \Xi\alpha) \\
& \leq \tau_1(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \alpha) \\
& + s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) \\
& - \tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi^2\alpha_m) \\
& + \tau_3(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \alpha) \\
& + \tau_4(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\alpha, \Xi^2\alpha_m) \\
& + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) \\
& \tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi^2\alpha_m) \\
& \leq \tau_1(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \alpha) \\
& + s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) \\
& + \tau_3(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \alpha) \\
& + \tau_4(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\alpha, \Xi^2\alpha_m) \\
& + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) \\
& \leq \tau_1(\gamma_{cb}(\Xi\alpha_m, \alpha))\gamma_{cb}(\Xi\alpha_m, \alpha) \\
& + [s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha)) + \tau_4(\gamma_{cb}(\Xi\alpha_m, \alpha))]\gamma_{cb}(\Xi\alpha_m, \Xi^2\alpha_m) \\
& + [s\tau_2(\gamma_{cb}(\Xi\alpha_m, \alpha)) + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))]\gamma_{cb}(\Xi^2\alpha_m, \Xi\alpha) \\
& + [\tau_3(\gamma_{cb}(\Xi\alpha_m, \alpha)) + s\tau_5(\gamma_{cb}(\Xi\alpha_m, \alpha))]\gamma_{cb}(\Xi^2\alpha_m, \alpha),
\end{aligned} \tag{34}$$

which means that

$$\begin{aligned} \gamma_{cb}(\Xi^2 \alpha_m, \Xi \alpha) &< \frac{\tau_1(\Xi \alpha_m, \alpha) \gamma_{cb}(\Xi \alpha_m, \alpha)}{\gamma_{cb}(\Xi \alpha_m, \alpha) - s \tau_2(\gamma_{cb}(\Xi \alpha_m, \alpha)) - s \tau_5(\gamma_{cb}(\Xi \alpha_m, \alpha))} \\ &+ \frac{(s \tau_2(\gamma_{cb}(\Xi \alpha_m, \alpha)) + \tau_4(\gamma_{cb}(\Xi \alpha_m, \alpha))) \gamma_{cb}(\Xi \alpha_m, \Xi^2 \alpha)}{\gamma_{cb}(\Xi \alpha_m, \alpha) - s \tau_2(\gamma_{cb}(\Xi \alpha_m, \alpha)) - s \tau_5(\gamma_{cb}(\Xi \alpha_m, \alpha))} \\ &+ \frac{(\tau_3(\gamma_{cb}(\Xi \alpha_m, \alpha)) + s \tau_5(\gamma_{cb}(\Xi \alpha_m, \alpha))) \gamma_{cb}(\Xi^2 \alpha_m, \alpha)}{\gamma_{cb}(\Xi \alpha_m, \alpha) - s \tau_2(\gamma_{cb}(\Xi \alpha_m, \alpha)) - s \tau_5(\gamma_{cb}(\Xi \alpha_m, \alpha))}. \end{aligned} \quad (35)$$

From (12) and (35), we derive that

$$0 < \gamma_{cb}(\Xi^2 \alpha_m, \Xi \alpha) < \gamma_{cb}(\Xi \alpha_m, \alpha) + \gamma_{cb}(\Xi \alpha_m, \Xi^2 \alpha) + \gamma_{cb}(\alpha, \Xi^2 \alpha_m). \quad (36)$$

Using (16) and (25), we derive

$$\lim_{m \rightarrow \infty} \gamma_{cb}(\Xi^2 \alpha_m, \Xi \alpha) = 0. \quad (37)$$

Since

$$\begin{aligned} \gamma_{cb}(\alpha, \Xi \alpha) &< s [\gamma_{cb}(\alpha, \Xi^2 \alpha_m) + \gamma_{cb}(\Xi^2 \alpha_m, \Xi \alpha)] \\ &- \gamma_{cb}(\Xi^2 \alpha_m, \Xi^2 \alpha_m) \\ &< s [\gamma_{cb}(\alpha, \Xi^2 \alpha_m) + \gamma_{cb}(\Xi^2 \alpha_m, \Xi \alpha)], \end{aligned} \quad (38)$$

using (25) and (37), we get $\alpha = \Xi \alpha$. $\square$

Example 2. Let $\Theta = \mathbb{N} \cup \{r + (1/j + 3) : r, j \in \mathbb{N}\}$. Define $\gamma_{cb} : \Theta \times \Theta \rightarrow \mathbb{C}^+$ by

$$\gamma_{cb}(r, j) = \max\{r, j\}^2 + |r - j|^2 + i(\max\{r, j\}^2 + |r - j|^2). \quad (39)$$

Define a mapping Ξ on Θ by

$$-10pt \Xi(r) = \begin{cases} 2, & r \in \mathbb{N}, \\ 8r + \frac{1}{j+3}, & r \in \left\{r + \frac{1}{j+3} : r, j \in \mathbb{N}\right\}. \end{cases} \quad (40)$$

Then, Ξ satisfies in the assumption of Theorem 2.

Proof. It is clear that (Ξ, γ_{cb}) is a complete complex partial b -metric space with coefficient $s = 2^2$ and 2 is a unique fixed point of Ξ . Let $j < r$:

$$\begin{aligned} \frac{1}{2} \gamma_{cb}\left(r + \frac{1}{j+3}, \Xi\left(r + \frac{1}{j+3}\right)\right) &= \frac{1}{2} \gamma_{cb}\left(r + \frac{1}{j+3}, 8r + \frac{1}{j+3}\right) \\ &= \frac{1}{2} \left(\max\left\{r + \frac{1}{j+3}, 8r + \frac{1}{j+3}\right\}^2 \right. \\ &\quad \left. + \left|r + \frac{1}{j+3} - 8r - \frac{1}{j+3}\right|^2 \right. \\ &\quad \left. + i \left(\max\left\{r + \frac{1}{j+3}, 8r + \frac{1}{j+3}\right\}^2 + \left|r + \frac{1}{j+3} - 8r - \frac{1}{j+3}\right|^2 \right) \right) \\ &= \frac{1}{2} \left(\left(8r + \frac{1}{j+3}\right)^2 + 49r^2 + i \left(\left(8r + \frac{1}{j+3}\right)^2 + 49r^2 \right) \right) \\ &= \frac{1}{2} \left(113r^2 + \frac{1}{(j+3)^2} + 16 \frac{r}{j+3} + i \left(113r^2 + \frac{1}{(j+3)^2} + 16 \frac{r}{j+3} \right) \right), \\ \gamma_{cb}\left(r + \frac{1}{j+3}, j + \frac{1}{j+3}\right) &= \max\left\{r + \frac{1}{j+3}, j + \frac{1}{j+3}\right\}^2 + \left|r + \frac{1}{j+3} - j - \frac{1}{j+3}\right|^2 \end{aligned}$$

$$\begin{aligned}
& + i \left(\max \left\{ r + \frac{1}{j+3}, j + \frac{1}{j+3} \right\}^2 + \left| r + \frac{1}{j+3} - j - \frac{1}{j+3} \right|^2 \right) \\
& = \left(r + \frac{1}{j+3} \right)^2 + |r - j|^2 + i \left(\left(r + \frac{1}{j+3} \right)^2 + |r - j|^2 \right) \\
& < r^2 + \frac{1}{(j+3)^2} + \frac{2r}{j+3} + (|r| - |j|)^2 \\
& \quad + i \left(r^2 + \frac{1}{(j+3)^2} + \frac{2r}{j+3} + (|r| - |j|)^2 \right) \\
& < r^2 + \frac{1}{(j+3)^2} + \frac{2r}{j+3} + (|r| + |j|)^2 \\
& \quad + i \left(r^2 + \frac{1}{(j+3)^2} + \frac{2r}{j+3} + (|r| + |j|)^2 \right). \tag{41}
\end{aligned}$$

So, for $j < r$, $(1/2)\Upsilon_{cb}(r + (1/j + 3), \Xi(r + (1/j + 3))) \neq \Upsilon_{cb}(r + (1/j + 3), j + (1/j + 3))$. Therefore, Ξ satisfies all the conditions of Theorem 2. $\square$

3.1. Nonlinear Integral Equations. In this section, we prove the existence and uniqueness of a solution for nonlinear Fredholm integral equations by using Theorem 2.

Theorem 3. Consider the nonlinear Fredholm integral equation:

$$r(t) = j(t) + \int_a^b K(t, s, r(s))ds, \tag{42}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $j: [a, b] \rightarrow \mathbb{R}$ and $K: [a, b]^2 \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous mappings. Suppose that the following condition holds:

- (a) The mapping $\Xi: C[a, b] \rightarrow C[a, b]$ defined by $(\Xi r)(t) = j(t) + \int_a^b K(t, s, r(s))ds$ for all $r \in C[a, b]$ and $t \in [a, b]$.

Then, the nonlinear integral equation (42) has a unique solution.

Proof. Let $\Theta = C[a, b]$. Clearly, Θ with the complex partial b -metric $\Upsilon_{cb}: \Theta \times \Theta \rightarrow C^+$ given by

$$\begin{aligned}
\Upsilon_{cb}(r(t), v(t)) &= \max\{r(t), v(t)\}^2 \\
&+ i \max\{r(t), v(t)\}^2, \quad t \in [a, b], \tag{43}
\end{aligned}$$

for all $r, v \in \Theta$ is a complete complex partial b -metric space. Without loss generality, we may assume that

$$r < v \Leftrightarrow r(t) \leq v(t), \quad \forall t \in [a, b]. \tag{44}$$

Now,

$$\begin{aligned}
& \frac{1}{2s} \left(\left(j(t) + \int_a^b K(t, s, v(s)) \right)^2 + i \left(\left(j(t) + \int_a^b K(t, s, v(s)) \right)^2 \right) \right) \\
& < \left(j(t) + \int_a^b K(t, s, v(s)) \right)^2 + i \left(\left(j(t) + \int_a^b K(t, s, v(s)) \right)^2 \right). \tag{45}
\end{aligned}$$

Therefore,

$$\frac{1}{2s} \Upsilon_{cb}(v(t), \Xi v(t)) < \Upsilon_{cb}(r(t), v(t)). \tag{46}$$

Hence, Ξ satisfies all the conditions of Theorem 2. $\square$

4. Conclusion

In the present work, we presented a new fixed point theorem for self-mappings defined on complex partial b -metric. We illustrated our main theorem by an example and showed, moreover, that the main theorem can be easily used to solve a nonlinear integral equation.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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