Research Article

An LMI Approach-Based Mathematical Model to Control *Aedes aegypti* Mosquitoes Population via Biological Control

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In this paper, a novel age-structured delayed mathematical model to control *Aedes aegypti* mosquitoes via Wolbachia-infected mosquitoes is introduced. To eliminate the deadly mosquito-borne diseases such as dengue, chikungunya, yellow fever, and Zika virus, the Wolbachia infection is introduced into the wild mosquito population at every stage. This method is one of the promising biological control strategies. To predict the optimal amount of Wolbachia release, the time varying delay is considered. Firstly, the positiveness of the solution and existence of both Wolbachia present and Wolbachia free equilibrium were discussed. Through linearization, construction of suitable Lyapunov–Krasovskii functional, and linear matrix inequality theory (LMI), the exponential stability is also analyzed. Finally, the simulation results are presented for the real-world data collected from the existing literature to show the effectiveness of the proposed model.

1. Introduction

Mosquito-borne diseases represent the vertical transmission of bacteria and viruses from mosquitoes to human while female mosquito taking a blood meal. Mosquito-borne diseases such as dengue, chikungunya, yellow fever, Zika virus, and Japanese encephalities cause over one million deaths per annum [1, 2]. Gubler in [3, 4] explained that the dengue and dengue hemorrhagic fever are the most common issues for public health. The primary vector for most of the mosquito-borne diseases is *Aedes aegypti*, and recently *Aedes albopictus* also add as a secondary vector [5]. In the past sixty years, the spread of mosquito-borne diseases has increased dramatically [6]. More than that, per year, dengue causes nearly 20 thousand deaths all over the world [7]. Also, nearly 112 countries are attacked by mosquito-borne diseases [8].

In recent years, there are several articles are available to control vectors by genetic modifications [9]. Moreover, some biological control methods to replace the wild mosquitoes by releasing genetically modified mosquitoes are also tried by some researchers. Those biological control methods are sterilization of male mosquitoes [10, 11] and genetic modification to reduce the reproduction and increase the life-shortening bacteria Wolbachia [12]. Furthermore, via finding the reproduction number of a mathematical model which depicts the virus transmission via human sexual contact was analyzed in [13]. In [14], the author tried some other types of control agents such as bed nets, mosquito repellents, indoor residual spray, condoms during sex,

medically treating infected human, and quarantine. However, in [15], the author tried to control mosquitoes by making modifications in feeding behaviours.

In this environment, our main aim is to control the vector population that transmits the virus to the uninfected human while taking a blood meal. There is a life-shortening bacterium called Wolbachia which will be very useful to reach our aim; in [16], Mcmeniman et al. analyzed the stable introduction of Wolbachia in *Aedes* mosquitoes. Wolbachia is a Gram-negative bacterium, and it is first reported in the tissues of the mosquito *Culex pipients* (Hertig and Wolbach, 1924) [17]. In recent results, they found that yellow fever virus can also be blocked by Wolbachia [18].

If a mosquito carries this bacterium, then the virus inside the mosquito does not get transmitted into the uninfected human. It blocks the virus inside the mosquito at salivary gland. This can be understood with Figure 1.

In Figure 1, the process of releasing Wolbachia bacteria into mosquito population is as follows:

(1) In laboratory, the Wolbachia pipients are injected into eggs, larvae, and pupae of *Aedes aegypti* via microinjection.

(2) Cytoplasmic incapability (CI): the adult Wolbachia-infected mosquitoes which are reared at the laboratory are released to the wild mosquito population of *Aedes aegypti*. Through this process, there exist three types of possibilities which are

   (i) If the Wolbachia-infected female mosquitoes mate with the Wolbachia-infected male, then the progeny should have the Wolbachia by birth which is compatible.

   (ii) If the Wolbachia-infected female crosses with Wolbachia-uninfected male, then the progeny face the same problems as in (i).

   (iii) If Wolbachia-uninfected female crosses with Wolbachia-infected male, then there is no viable progeny.

These two processes can be virtually understood by Figure 2.

In eggs, larvae, and pupae population, we can microinject the Wolbachia and release this in patches at dengue-suspected areas. This process is practically done by placing "Zancu kits" around the people living areas. And the adult mosquitoes which are reared at lab can also be released into wild mosquito population. This process is called "introgression."

Various mathematical models have been studied to understand the interplay among Wolbachia and non-Wolbachia mosquitoes. In [19], the author considered the Wolbachia bacteria as a mechanism to control arbovirus, and his experimental studies show that the spread of Zika virus among mosquitoes and human was notably reduced. The same process for dengue virus spread was studied by Segoli et al. in [20]. In [21], the author has created a mathematical model considering only adult female mosquitoes and converted the model into endoepidemic model consisting of adult female
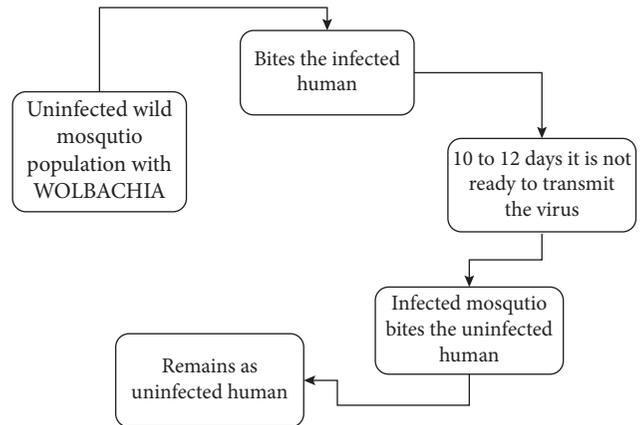

Figure 1: Dynamics of virus infection after Wolbachia introduction.

mosquitoes and human population. In [22], the authors proposed a mathematical model depicting the life stages of mosquitoes with Wolbachia and proved that Wolbachia has excellent quality to control dengue virus spread. In that work, Koiler et al. discussed the virus as well as Wolbachia in both mosquitoes and human. Also, the objective is to predict the appropriate release of this bacterium, and the basic reproduction number was analyzed. Supriatna et al. in [23] developed a mathematical model to express the dynamics of dengue virus in both human and *Aedes aegypti* mosquitoes. In that, the human vaccination and Wolbachia introduction were used as optimal control methods. In [24], authors discussed the birth and death rate impulsive model to control mosquito-borne diseases using Wolbachia via the stroboscopic map method. The integer-order mathematical model which describes the interplay among the wild and Wolbachia-infected mosquitoes was analyzed in [25]. In that work, the author divided the mosquito population into two groups: one is aquatic and another one is adult. In [26], the author proposed a mathematical model to describe the persistence of Wolbachia via two-sex stage-structured model.

Hence, with a full understanding of the interplay among the Wolbachia and non-Wolbachia mosquitoes in our work, we have created a mathematical model consisting of 10 stages to ensure the success of the proposed strategy. With reference of the practical results in [18], we can release the Wolbachia in every stages in the forms of "Zancu kits" and "Introgression." So to obtain an optimal control, it is necessary to consider each and every variable. Because we know that in mathematical modeling, each and every parameter plays an important role. Up to our knowledge, this is the first article considering the control inputs in 10 stages incorporated with time-varying delays.

By motivated by the above discussions, the main contribution of this paper is as follows:

(i) The main aim is to establish a novel mathematical model to describe the interplay among the both non-Wolbachia (wild mosquitoes) and Wolbachia-infected mosquitoes with time-varying delay.
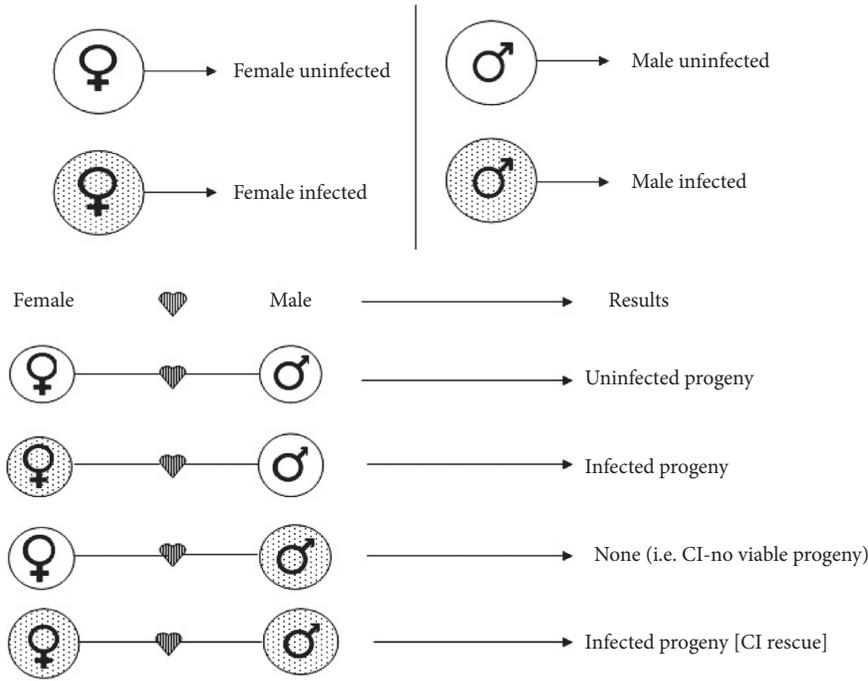
FIGURE 2: Block diagram representing the mechanism of Wolbachia infection in mosquitoes.

(ii) We found that our method will increase Wolbachia-infected mosquitoes in terms of CI rescue team and non-Wolbachia mosquitoes go to annihilation.

(iii) We have proved that the releasing of adult female Wolbachia-infected mosquitoes is more beneficial than the releasing of adult male Wolbachia-infected mosquitoes.

(iv) There exists no literature on exponential stability results of delayed Wolbachia and non-Wolbachia age-structured model. This gap is filled by our work. Finally, by using real-world data, we checked the dynamics of the proposed model by using MATLAB LMI tool box.

The rest of the paper is arranged as follows: in Section 2, the novel mathematical model which describes the interplay among the Wolbachia free and Wolbachia present mosquito population is proposed. In Section 3, the analysis of the model such as positiveness of the solution and existence of equilibrium points are discussed. In Section 4, the exponential stability results of the linearized delayed system with time-varying delay is presented. In Section 5, numerical simulation results are presented. The work is concluded in Section 6.

2. Preliminaries and Model Formulation

In this section, some basic definitions and lemmas which are used to derive our results are presented. And the interaction between wild mosquitoes and Wolbachia-infected mosquitoes is modeled.

Definition 1 (see [27]). A model is said to be exponentially stable at its equilibrium point, if there exists $\gamma > 0$ such that

$$\|y(t)\| \le \|\Phi\|e^{-2\gamma t}, \text{ for every } t > 0. \tag{1}$$

Lemma 1 (Schur Complement, see [28]). *Let us denote three $n \times n$ matrices as Ψ_1, Ψ_2, Ψ_3, where $\Psi_1 = \Psi_1^\top$ and $\Psi_2 = \Psi_2^\top > 0$. Then $\Psi_1 + \Psi_3^\top \Psi_2^{-1} \Psi_3 < 0$ if and only if $\begin{bmatrix} \Psi_1 & \Psi_3^\top \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0$ or $\begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^\top & \Psi_1 \end{bmatrix} < 0.$*

Lemma 2 (see [29]). *For any scalar $\epsilon > 0$, $E, N \in \mathbb{R}^n$, and matrix P_1, then*

$$E^\top P_1 N \le \frac{1}{2\epsilon} E^\top P_1 P_1^\top E + \frac{\epsilon}{2} N^\top N. \tag{2}$$

2.1. Modeling the Life Stages of Wild Mosquito Population. In a common environment, *Aedes aegypti* mosquito population has five important life stages such as eggs $(W_e(t))$, larvae $(W_l(t))$, pupae $(W_p(t))$, matured female mosquitoes $(W_f(t))$, and adult male mosquitoes $(W_a(t))$ (Figure 3). These life stages with respect to time t can be modeled as follows:
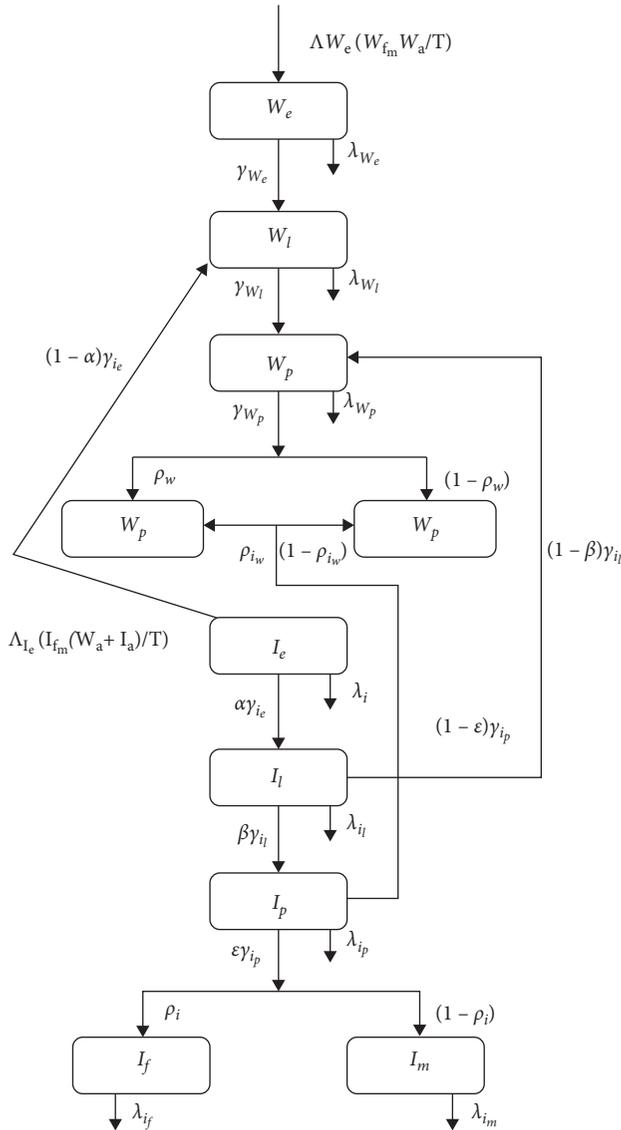
Figure 3: Schematic representation of interaction between non-Wolbachia and Wolbachia-infected mosquitoes.

$$\begin{cases} \dfrac{dW_e(t)}{dt} = \dfrac{\Lambda_{w_e} W_f W_a}{T} - \lambda_{w_e} W_e - \gamma_{w_e} W_e, \\[2mm] \dfrac{dW_l(t)}{dt} = \gamma_{w_e} W_e - \lambda_{w_l} W_l - \gamma_{w_l} W_l, \\[2mm] \dfrac{dW_p(t)}{dt} = \gamma_{w_l} W_l - \lambda_{w_p} W_p - \gamma_{w_p} W_p, \\[2mm] \dfrac{dW_f(t)}{dt} = \rho\gamma_{w_p} W_p - \lambda_{w_f} W_f, \\[2mm] \dfrac{dW_a(t)}{dt} = (1-\rho)\gamma_{w_p} W_p - \lambda_{w_a} W_a. \end{cases} \quad (3)$$

2.2. Modeling the Interaction between Wild and Wolbachia-Infected Mosquitoes. In this subsection, we modeled the release of Wolbachia-infected mosquitoes into the wild mosquitoes in the mosquito-borne disease suspected areas. The Wolbachia infection is released in both aquatic (Zanku Kits) and ariel stages. Therefore, by using the language of mathematics, we can model the interaction between wild and Wolbachia mosquitoes as follows:

$$\begin{cases} \dfrac{dW_e(t)}{dt} = \dfrac{\Lambda_{w_e} W_f W_a}{T} - \lambda_{w_e} W_e - \gamma_{w_e} W_e, \\[2mm] \dfrac{dW_l(t)}{dt} = \gamma_{w_e} W_e - \lambda_{w_l} W_l - \gamma_{w_l} W_l + (1-\alpha)\gamma_{i_e} I_e, \\[2mm] \dfrac{dW_p(t)}{dt} = \gamma_{w_l} W_l - \lambda_{w_p} W_p - \gamma_{w_p} W_p + (1-\beta)\gamma_{i_l} I_l, \\[2mm] \dfrac{dW_f(t)}{dt} = \rho\gamma_{w_p} W_p - \lambda_{w_f} W_f + (1-\varepsilon)\gamma_{i_p}\rho_{i_w} I_p, \\[2mm] \dfrac{dW_a(t)}{dt} = (1-\rho)\gamma_{w_p} W_p - \lambda_{w_a} W_a + (1-\varepsilon)\gamma_{i_p}\big(1-\rho_{i_w}\big)I_p, \\[2mm] \dfrac{dI_e(t)}{dt} = \dfrac{\Lambda_{i_e} I_f\big(W_a + I_a\big)}{T} - \lambda_{i_e} I_e - \alpha\gamma_{i_e} I_e, \\[2mm] \dfrac{dI_l(t)}{dt} = \alpha\gamma_{i_e} I_e - \lambda_{i_l} I_l - \beta\gamma_{i_l} I_l, \\[2mm] \dfrac{dI_p(t)}{dt} = \beta\gamma_{i_l} I_l - \lambda_{i_p} I_p - \varepsilon\gamma_{i_p} I_p, \\[2mm] \dfrac{dI_f(t)}{dt} = \rho_i\varepsilon\gamma_{i_p} I_p - \lambda_{i_f} I_f, \\[2mm] \dfrac{dI_a(t)}{dt} = (1-\rho_i)\varepsilon\gamma_{i_p} I_p - \lambda_{i_a} I_a. \end{cases} \quad (4)$$

To understand the system of equations in (4), refer Figure 3.

In the above mathematical model, the production of eggs can be found by the term $(\Lambda_{w_e} W_f W_a/T)$. That is, eggs without Wolbachia infection $(W_{e(t)})$ are produced by the mating between wild female $(W_{f(t)})$ and wild male $(W_{a(t)})$ mosquitoes, where T is the total population which can be calculated by the following expression:

$$\begin{aligned} T = {} & W_e(t) + W_l(t) + W_p(t) + W_f(t) + W_a(t) + I_e(t) \\ & + I_l(t) + I_p(t) + I_f(t) + I_a(t). \end{aligned} \quad (5)$$

Along with this, the terms λ_{w_e} (natural mortality rate of non-Wolbachia eggs) and γ_{w_e} (maturation rate of non-

Wolbachia eggs) denote the limitations in the growth of wild mosquito eggs. At the same time, after release of Wolbachia-infected mosquitoes (in both aquatic and ariel stages) in a common environment, the production of Wolbachia-infected mosquito eggs $I_e(t)$ depends on mating between Wolbachia-infected female $I_f(t)$ and non-Wolbachia male $W_a(t)$ and from mating between Wolbachia-infected female $I_f(t)$ and Wolbachia-infected male $I_a(t)$. This implies that the birth rate of Wolbachia-infected mosquito egg population $I_e(t)$ with the reproduction rate Λ_{i_e} is

$$\frac{\Lambda_{i_e}\left(I_f W_a + I_f I_a\right)}{T} = \frac{\Lambda_{i_e} I_f \left(W_a + I_a\right)}{T}. \tag{6}$$

Similarly, the increase in growth is limited by the natural mortality rate λ_{i_e} and the maturation rate γ_{i_e} (that is, the rate in which the corresponding compartment moved into the next stage).

Furthermore, $(1-\alpha)\gamma_{i_e} I_e$ is added to the wild mosquito larvae population. Because the terms α and $(1-\alpha)$ denote the probability of getting larvae with and without Wolbachia, respectively. Similarly, β and $(1-\beta)$ denote the probability of getting pupae with and without Wolbachia, respectively, ϵ and $(1-\epsilon)$ denote the probability rate of

having Wolbachia infection in adult mosquitoes by introgression. That is, ϵ be the probability of getting Wolbachia-infected adults (with ρ_{i_w} = probability of getting male and $(1-\rho_{i_w})$ = probability of getting female). Because of these reasons, the terms $(1-\alpha)\gamma_{i_e} I_e$, $(1-\beta)\gamma_{i_l} I_l$, $(1-\epsilon)\gamma_{i_p}\rho_{i_w} I_p$, and $(1-\epsilon)\gamma_{i_p}(1-\rho_{i_w})I_p$ are added to the corresponding stages, and similarly, the terms $\alpha\gamma_{i_e} I_e$, $\beta\gamma_{i_l} I_l$ and $\epsilon\gamma_{i_p} I_p$ are removed from the corresponding stages. And the other parameters used in this model are described in Table 1.

3. Analysis of the Model

In this section, we analyze the positivity, existence of equilibrium points, and stability of the system of equations in (4).

3.1. Positivity of Solutions

Lemma 3. *For all $t > 0$, the solutions $(W_e, W_l, W_p, W_f, W_a, I_e, I_l, I_p, I_f, I_a)$ are all nonnegative if the initial values $W_e > 0$, $W_l > 0$, $W_p > 0$, $W_f > 0$, $W_a > 0$, $I_e > 0$, $I_l > 0$, $I_p > 0$, $I_f > 0$, $I_a > 0$.*

Proof. Let us define

$$\tilde{\mu} = \sup\Big\{\mu > 0: W_e(0) > 0, W_l(0) > 0, W_p(0) > 0, W_f(0) > 0, W_a(0) > 0,$$
$$I_e(0) > 0, I_l(0) > 0, I_p(0) > 0, I_f(0) > 0, I_a(0) > 0 \in [0,\mu]\Big\}. \tag{7}$$

From (7), $\tilde{\mu} > 0$. Let us consider the first equation of model (4), that is,

$$\frac{dW_e(t)}{dt} = \frac{\Lambda_{w_e} W_f W_a}{T} - \lambda_{w_e} W_e - \gamma_{w_e} W_e,$$

$$\frac{dW_e(t)}{dt} + \left(\lambda_{w_e} + \gamma_{w_e}\right)W_e = \frac{\Lambda_{w_e} W_f W_a}{T}. \tag{8}$$

The integrating factor is

$$I.F = e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu}. \tag{9}$$

Multiply (9) with (8) on both sides

$$\frac{dW_e(t)}{dt}e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu} + \left(\lambda_{w_e}+\gamma_{w_e}\right)W_e e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu} = \frac{\Lambda_{w_e} W_f W_a}{T}e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu}. \tag{10}$$

That is,

$$\frac{d}{d\mu}\left[W_e(t)e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu}\right] = \frac{\Lambda_{w_e} W_f W_a}{T}e^{\int_0^{\tilde{\mu}}\left(\lambda_{w_e}+\gamma_{w_e}\right)d\mu}. \tag{11}$$

TABLE 1: Parameters involved in the population dynamics of *Aedes aegypti* mosquitoe.

$\Lambda_{w_e}, \Lambda_{i_e}$	Reproduction rate of non-Wolbachia mosquitoes and Wolbachia-infected mosquitoes, respectively
λ_{w_e}	The mortality rate of non-Wolbachia eggs
λ_{w_l}	The mortality rate of non-Wolbachia larvae
λ_{w_p}	The mortality rate of non-Wolbachia pupae
λ_{w_f}	The mortality rate of non-Wolbachia adult female mosquitoes
λ_{w_a}	The mortality rate of non-Wolbachia adult male mosquitoes
λ_{i_e}	The mortality rate of Wolbachia-infected eggs
λ_{i_l}	The mortality rate of Wolbachia-infected larvae
λ_{i_p}	The mortality rate of Wolbachia-infected pupae
λ_{i_f}	The mortality rate of Wolbachia-infected adult female mosquitoes
λ_{i_a}	The mortality rate of Wolbachia-infected adult male mosquitoes
γ_{w_e}	The rate at which the corresponding part of the wild mosquito eggs forms in which the larvae of wild mosquito emerge
γ_{w_l}	The rate at which the corresponding part of the wild mosquito larvae forms in which the pupae of wild mosquito emerge
γ_{w_p}	The rate at which the corresponding part of the wild mosquito pupae forms in which the immature female or male of wild mosquito emerge
γ_{i_e}	The rate at which the corresponding part of the Wolbachia-infected mosquito eggs matured into Wolbachia-infected or uninfected larvae
γ_{i_l}	The rate at which the corresponding part of the Wolbachia-infected mosquito larvae matured into Wolbachia-infected or uninfected pupae
γ_{i_p}	The rate at which the corresponding part of the Wolbachia-infected mosquito pupae matured into Wolbachia-infected or uninfected adults
ρ	The probability of having male or female mosquitoes
T	Total population

Integrating on both sides with respect to $\mu \in [0, \widetilde{\mu}]$,

$$\left[W_e(t) e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) d_t \mu} \right]_0^{\widetilde{\mu}} = \int_0^{\widetilde{\mu}} \frac{\Lambda_{w_e} W_f(s) W_a(s)}{T} e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) ds} ds,$$

$$W_e(\widetilde{\mu}) e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) d\widetilde{\mu}} - W_e(0) e^0 = \int_0^{\widetilde{\mu}} \frac{\Lambda_{w_e} W_f(s) W_a(s)}{T} e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) ds} ds,$$

$$W_e(\widetilde{\mu}) = \left[\int_0^{\widetilde{\mu}} \frac{\Lambda_{w_e} W_f(s) W_a(s)}{T} e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) ds} ds + W_e(0) \right] e^{-\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) d\widetilde{\mu}}, \qquad (12)$$

$$W_e(\widetilde{\mu}) = \left[\int_0^{\widetilde{\mu}} \frac{\Lambda_{w_e} W_f(s) W_a(s)}{T} e^{\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) ds} ds e^{-\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) d\widetilde{\mu}} \right]$$

$$+ W_e(0) e^{-\int_0^{\widetilde{\mu}} \left(\lambda_{w_e} + \gamma_{w_e} \right) d\widetilde{\mu}}.$$

This implies that $W_e(\widetilde{\mu}) > 0$.

Similarly, we can prove that for the positive initial values, the solution is positive.   □

3.2. Existence of Equilibrium Points. In this section, the possible cases of existing equilibrium points are discussed.

Furthermore, the null mosquitoes equilibrium point $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ is omitted because this case does not exist in nature.

3.2.1. Wolbachia Free Equilibrium. In this subsection, we can find Wolbachia free equilibrium by equating the system

of equations in (2) to zero and putting $I_{e_1}^* = 0$, $I_{l_1}^* = 0$, $I_{p_1}^* = 0$, $I_{f_1}^* = 0$, and $I_{a_1}^* = 0$.

That is,

$$\begin{cases} \dfrac{\Lambda_{w_e} W_{f_1}^* W_{a_1}^*}{T} - \lambda_{w_e} W_{e_1}^* - \gamma_{w_e} W_{e_1}^* = 0, \\[2mm] \gamma_{w_e} W_{e_1}^* - \lambda_{w_l} W_{l_1}^* - \gamma_{w_l} W_{l_1}^* + (1-\alpha)\gamma_{i_e} I_{e_1}^* = 0, \\[2mm] \gamma_{w_l} W_{l_1}^* - \lambda_{w_p} W_{p_1}^* - \gamma_{w_p} W_{p_1}^* + (1-\beta)\gamma_{i_l} I_{l_1}^* = 0, \\[2mm] \rho\gamma_{w_p} W_{p_1}^* - \lambda_{w_f} W_{f_1}^* + (1-\varepsilon)\gamma_{i_p}\rho_{i_w} I_{p_1}^* = 0, \\[2mm] (1-\rho)\gamma_{w_p} W_{p_1}^* - \lambda_{w_a} W_{a_1}^* + (1-\varepsilon)\gamma_{i_p}\left(1-\rho_{i_w}\right) I_{p_1}^* = 0. \end{cases}$$

(13)

By solving equations (13), we get the equilibrium point S_1 as

$$S_1 = \left\{ W_{e_1}^*, \frac{\gamma_{w_e} W_{e_1}^*}{\left(\lambda_{w_l} + \gamma_{w_l}\right)}, \frac{\gamma_{w_l}\gamma_{w_e} W_{e_1}^*}{\left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\lambda_{w_p} + \gamma_{w_p}\right)}, \frac{\rho\gamma_{w_p}\gamma_{w_l}\gamma_{w_e} W_{e_1}^*}{\lambda_{w_f}\left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\lambda_{w_p} + \gamma_{w_p}\right)} \right.$$

$$\left. \cdot \frac{(1-\rho)\gamma_{w_p}\gamma_{w_l}\gamma_{w_e} W_{e_1}^*}{\lambda_{w_a}\left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\lambda_{w_p} + \gamma_{w_p}\right)}, 0,0,0,0,0,0 \right\},$$

(14)

where

$$W_{e_1}^* = \frac{T\lambda_{w_f}\lambda_{w_a}\left(\lambda_{w_e} + \gamma_{w_e}\right)\left(\lambda_{w_l} + \gamma_{w_l}\right)^2\left(\lambda_{w_p} + \gamma_{w_p}\right)^2}{\rho(1-\rho)\Lambda_{w_e}\gamma_{w_p}^2\gamma_{w_e}^2\gamma_{w_l}^2}.$$

(15)

3.2.2. Wolbachia Present Equilibrium. In this subsection, we can find Wolbachia present equilibrium by equating the system of equations in (4) to zero. That is,

$$\begin{cases} \dfrac{\Lambda_{w_e} W_{f_n}^* W_{a_n}^*}{T} - \lambda_{w_e} W_{e_n}^* - \gamma_{w_e} W_{e_n}^* = 0, \\[2mm] \gamma_{w_e} W_{e_n}^* - \lambda_{w_l} W_{l_n}^* - \gamma_{w_l} W_{l_n}^* + (1-\alpha)\gamma_{i_e} I_{e_n}^* = 0, \\[2mm] \gamma_{w_l} W_{l_n}^* - \lambda_{w_p} W_{p_n}^* - \gamma_{w_p} W_{p_n}^* + (1-\beta)\gamma_{i_l} I_{l_n}^* = 0, \\[2mm] \rho\gamma_{w_p} W_{p_n}^* - \lambda_{w_f} W_{f_n}^* + (1-\varepsilon)\gamma_{i_p}\rho_{i_w} I_{p_n}^* = 0, \\[2mm] (1-\rho)\gamma_{w_p} W_{p_n}^* - \lambda_{w_a} W_{a_n}^* + (1-\varepsilon)\gamma_{i_p}\left(1-\rho_{i_w}\right) I_{p_n}^* = 0, \\[2mm] \dfrac{\Lambda_{i_e} I_{f_n}^*\left(W_{a_n}^* + I_{a_n}^*\right)}{T} - \lambda_{i_e} I_{e_n}^* - \alpha\gamma_{i_e} I_{e_n}^* = 0, \\[2mm] \alpha\gamma_{i_e} I_{e_n}^* - \lambda_{i_l} I_{l_n}^* - \beta\gamma_{i_l} I_{l_n}^* = 0, \\[2mm] \beta\gamma_{i_l} I_{l_n}^* - \lambda_{i_p} I_{p_n}^* - \varepsilon\gamma_{i_p} I_{p_n}^* = 0, \\[2mm] \rho_i\varepsilon\gamma_{i_p} I_{p_n}^* - \lambda_{i_f} I_{f_n}^* = 0, \\[2mm] (1-\rho_i)\varepsilon\gamma_{i_p} I_{p_n}^* - \lambda_{i_a} I_{a_n}^* = 0. \end{cases}$$

(16)

By solving the system of equations in (16), we get the following equilibrium point in terms of I_a^*:

$$S_n = \left\{ W_{e_n}^*, W_{l_n}^*, W_{p_n}^*, W_{f_n}^*, W_{a_n}^*, I_{e_n}^*, I_{l_n}^*, I_{p_n}^*, I_{f_n}^*, I_{a_n}^* \right\}, \quad n = 2, 3,$$

$$W_{e_n}^* = \left(\frac{\lambda_{w_l} + \gamma_{w_l}}{\gamma_{w_e}} \right)\left(\frac{\lambda_{w_p} + \gamma_{w_p}}{\gamma_{w_l}} \right)\left(\frac{T B_1 B_2 B_3 \lambda_{i_f} \lambda_{w_a}}{\Lambda_{i_e}(1-\rho)\rho_i \gamma_{w_p}} \right) - \frac{I_{a_n}^*}{\gamma_{w_e}}$$

$$\cdot \left[B_4\left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\frac{\lambda_{w_p} + \gamma_{w_p}}{\gamma_{w_l}} \right) + \left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\frac{(1-\beta)\gamma_{i_l}}{\gamma_{w_l}} \right)\left(\frac{\lambda_{i_a} B_1}{\beta \gamma_{i_l}(1-\rho_i)} \right) + \frac{(1-\alpha)\gamma_{i_e}\lambda_{i_a} B_1 B_2}{\alpha \gamma_{i_e}(1-\rho_i)} \right],$$

$$W_{l_n}^* = \left(\frac{\lambda_{w_p} + \gamma_{w_p}}{\gamma_{w_l}} \right)\left[\frac{T B_1 B_2 B_3 \lambda_{i_f} \lambda_{w_a}}{\Lambda_{i_e}(1-\rho)\rho_i \gamma_{w_p}} - B_4 I_{a_n}^* \right] - \left(\frac{(1-\beta)\gamma_{i_l}}{\gamma_{w_l}} \right)\left[\frac{\lambda_{i_a} B_1}{\beta \gamma_{i_l}(1-\rho_i)} \right],$$

$$W_{p_n}^* = \left[\frac{T B_1 B_2 B_3 \lambda_{i_f} \lambda_{w_a}}{\Lambda_{i_e}(1-\rho)\rho_i \gamma_{w_p}} - B_4 I_{a_n}^* \right],$$

$$W_{f_n}^* = \frac{\rho \gamma_{w_p}}{\lambda_{w_f}}\left[\frac{T B_1 B_2 B_3 \lambda_{i_f} \lambda_{w_a}}{\Lambda_{i_e}(1-\rho)\rho_i \gamma_{w_p}} - B_4 I_{a_n}^* \right] + \frac{(1-\epsilon)\rho_{i_w}\lambda_{i_a}}{\epsilon \lambda_{w_f}(1-\rho_i)} I_{a_n}^*,$$

$$W_{a_n}^* = \frac{T B_1 B_2 B_3 \lambda_{i_f}}{\rho_i \Lambda_{i_e}} - I_{a_n}^*,$$

$$I_{e_n}^* = \frac{B_1 B_2 \lambda_{i_a} I_{a_n}^*}{\alpha \gamma_{i_e}(1-\rho_i)},$$

$$I_{p_n}^* = \frac{\lambda_{i_a}}{(1-\rho_i)\epsilon \gamma_{i_p}} I_{a_n}^*,$$

$$I_{f_n}^* = \frac{\rho_i \lambda_{i_a} I_{a_n}^*}{(1-\rho_i)\lambda_{i_f}},$$

(17)

with $I_{a_3}^* > I_{a_4}^*$, both roots can be found from the quadratic equation:

$$a_1 I_a^{*2} + a_2 I_a^* + a_3 = 0, \qquad (18)$$

where

$$a_1 = \frac{\Lambda_{w_e}\rho B_4 \gamma_{w_p}}{T\lambda_{w_f}};$$

$$a_2 = \left(\frac{\lambda_{w_e} + \gamma_{w_e}}{T\lambda_{w_f}} \right)\left(\frac{\lambda_{w_e}\lambda_{i_f}\rho B_1 B_2 B_3}{\rho_i \Lambda_{i_e}\lambda_{w_f}} \right)\left(\frac{\lambda_{w_a}}{(1-\rho)} + B_4 \gamma_{w_p} \right)$$

$$\cdot \left(\frac{\left(\lambda_{w_l} + \gamma_{w_l}\right)\left(\lambda_{w_p} + \gamma_{w_p}\right) B_4}{\gamma_{w_l}} + \frac{\left(\lambda_{w_l} + \gamma_{w_l}\right)(1-\beta)\lambda_{i_a} B_1}{\gamma_{w_l}\beta(1-\rho_i)} + \frac{(1-\alpha)\lambda_{i_a} B_1 B_2}{\alpha(1-\rho_i)} \right);$$

$$a_3 = \frac{\Lambda_{w_e}\rho T B_1^2 B_2^2 B_3^2 \lambda_{w_a}}{\Lambda_{i_e}^2(1-\rho)\rho_i^2 \lambda_{w_f}}.$$

(19)

where

$$B_1 = 1 + \frac{\lambda_{i_p}}{\epsilon\gamma_{i_p}};$$

$$B_2 = 1 + \frac{\lambda_{i_l}}{\beta\gamma_{i_l}};$$

$$B_3 = 1 + \frac{\lambda_{i_e}}{\alpha\gamma_{i_e}}; \qquad (20)$$

$$B_4 = 1 + \frac{(1-\epsilon)(1-\rho_{i_w})\lambda_{i_a}}{(1-\rho)(1-\rho_i)\epsilon\gamma_{w_p}}.$$

3.2.3. Existence of Wolbachia Free and Wolbachia Present Equilibrium. The equilibrium points exist if it satisfies the following conditions:

(i) S_1 exists provided that $W_{e_1}^* \neq 0$ and $0 < \rho < 1$.

(ii) S_2 and S_3 exist provided that $I_{a_2}^*, I_{a_3}^* \neq 0$ and $(1-\rho)\epsilon\gamma_{i_p} I_p^* < \lambda_{i_a}$.

4. Stability Analysis

In this section, the stability results of the system of equations in (2) by linearization, Lyapunov–Karasovskii functional, and LMI approach. To optimize our differential model, we consider the delay terms into account as discussed in [30] i.e., where $W_e(t - \tau_1(t))$ is the produced wild egg density at time $(t - \tau_1(t))$ from which the wild larvae density at time t is produced, $W_l(t - \tau_2(t))$ is the produced wild larvae density at time $(t - \tau_2(t))$ from which the wild pupae density at time t is produced, $W_p(t - \tau_3(t))$ is the produced wild pupae density at time $(t - \tau_3(t))$ from which the wild female or male density at time t is produced, $I_e(t - \tau_6(t))$ is the produced Wolbachia-infected egg density at time $(t - \tau_1(t))$ from which the Wolbachia-infected larvae density at time t is produced, $I_l(t - \tau_7(t))$ is the produced Wolbachia-infected larvae density at time $(t - \tau_7(t))$ from which the Wolbachia-infected pupae density at time t is produced, and $I_p(t - \tau_8(t))$ is the produced Wolbachia-infected pupae density at time $(t - \tau_8(t))$ from which the Wolbachia-infected female or male density at time t is produced. These considerations make our system of differential equations to system of delay differential equations:

$$\begin{cases} \dfrac{dW_e(t)}{dt} = \dfrac{\Lambda_{w_e} W_f W_a}{T} - \lambda_{w_e} W_e - \gamma_{w_e} W_e, \\[2mm] \dfrac{dW_l(t)}{dt} = \gamma_{w_e} W_e(t - \tau_1(t)) - \lambda_{w_l} W_l - \gamma_{w_l} W_l + (1-\alpha)\gamma_{i_e} I_e, \\[2mm] \dfrac{dW_p(t)}{dt} = \gamma_{w_l} W_l(t - \tau_2(t)) - \lambda_{w_p} W_p - \gamma_{w_p} W_p + (1-\beta)\gamma_{i_l} I_l, \\[2mm] \dfrac{dW_f(t)}{dt} = \rho\gamma_{w_p} W_p(1 - \tau_3(t)) - \lambda_{w_f} W_f + (1-\varepsilon)\gamma_{i_p}\rho_{i_w} I_p, \\[2mm] \dfrac{dW_a(t)}{dt} = (1-\rho)\gamma_{w_p} W_p(1 - \tau_3(t)) - \lambda_{w_a} W_a + (1-\varepsilon)\gamma_{i_p}(1 - \rho_{i_w}) I_p, \\[2mm] \dfrac{dI_e(t)}{dt} = \dfrac{\Lambda_{i_e} I_f(W_a + I_a)}{T} - \lambda_{i_e} I_e - \alpha\gamma_{i_e} I_e, \\[2mm] \dfrac{dI_l(t)}{dt} = \alpha\gamma_{i_e} I_e(1 - \tau_6(t)) - \lambda_{i_l} I_l - \beta\gamma_{i_l} I_l, \\[2mm] \dfrac{dI_p(t)}{dt} = \beta\gamma_{i_l} I_l(1 - \tau_7(t)) - \lambda_{i_p} I_p - \varepsilon\gamma_{i_p} I_p, \\[2mm] \dfrac{dI_f(t)}{dt} = \rho_i\varepsilon\gamma_{i_p} I_p(1 - \tau_8(t)) - \lambda_{i_f} I_f, \\[2mm] \dfrac{dI_a(t)}{dt} = (1-\rho_i)\varepsilon\gamma_{i_p} I_p(1 - \tau_8(t)) - \lambda_{i_a} I_a. \end{cases} \qquad (21)$$

By linearizing, the following system describes the interaction between wild mosquitoes and Wolbachia mosquitoes as

$$\begin{cases} \dot{M}(t) = AM(t) + BM(t - \tau(t)) + g(M(t)) + Cu(t), & t \geq 0, \\ M(t) = \pi(t), & t \in [-\breve{\tau}, 0], \end{cases}$$

(22)

where

$$M(t) = \left[W_e(t) W_l(t) W_p(t) W_f(t) W_a(t) I_e(t) I_l(t) I_p(t) I_f(t) I_a(t) \right]^{\top};$$

$$M(t - \tau) = \left[W_e\left(t - \tau_1(t)\right) \quad W_l\left(t - \tau_2(t)\right) \quad W_p\left(t - \tau_3(t)\right) \quad W_f\left(t - \tau_4(t)\right) \quad W_a\left(t - \tau_5(t)\right) \quad I_e\left(t - \tau_6(t)\right) \right.$$

$$\left. \cdot \; I_l\left(t - \tau_7(t)\right) \quad I_p\left(t - \tau_8(t)\right) \quad I_f\left(t - \tau_9(t)\right) \quad I_a\left(t - \tau_{10}(t)\right) \right]^{\top},$$

$$u(t) = \left[0 \; 0 \; 0 \; 0 \; 0 \; u_1 \; u_2 \; u_3 \; u_4 \; u_5 \right]^{\top}; \quad 0 \leq \breve{\tau} \leq \hat{\tau}.$$

$$A = \begin{bmatrix} -\lambda_{w_e} - \gamma_{w_e} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_{w_l} - \gamma_{w_l} & 0 & 0 & 0 & (1-\alpha)\gamma_{i_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_{w_p} - \lambda_{w_p} & 0 & 0 & 0 & (1-\beta)\gamma_{i_l} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_{w_f} & 0 & 0 & 0 & (1-\epsilon)\gamma_{i_p}\rho_{i_w} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_{w_a} & 0 & 0 & (1-\epsilon)\gamma_{i_p}\left(1-\rho_{i_w}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_{i_e} - \alpha\gamma_{i_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{i_l} - \beta\gamma_{i_l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{i_p} - \epsilon\gamma_{i_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{i_f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{i_a} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_{w_e} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{w_l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho\gamma_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\rho)\gamma_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\gamma_{i_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta\gamma_{i_l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_i\epsilon\gamma_{i_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-\rho_i)\epsilon\gamma_{i_p} & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$g(M(t)) = \begin{bmatrix} \dfrac{\Lambda_{w_e} m_4 m_5}{T} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\Lambda_{i_e} m_9 (m_5 + m_{10})}{T} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$(23)$$

Assume that $\tau(t) = \max\{\tau_1(t), \tau_2(t), \tau_3(t), \tau_4(t), \tau_5(t), \tau_6(t), \tau_7(t), \tau_8(t), \tau_9(t), \tau_{10}(t)\}$, and it is bounded, differentiable, and nonnegative. Now, consider an equilibrium point of system (22) as

$$M^* = \left[W_e^*(t) W_l^*(t) W_p^*(t) W_f^*(t) W_a^*(t) I_e^*(t) I_l^*(t) I_p^*(t) I_f^*(t) I_a^*(t) \right]^\top,$$
$$u^* = \left[0\ 0\ 0\ 0\ 0\ u_1^*\ u_2^*\ u_3^*\ u_4^*\ u_5^* \right]^\top.$$

$$(24)$$

Let $y(t) = M(t) - M^*$, and this implies that $y(t) + M^* = M(t)$. And $U(t) = u(t) - u^*$. Then the modified system is

$$\begin{cases} \dot{y}(t) = Ay(t) + By(t - \tau(t)) + g(y(t)) + CU(t), & t \geq 0, \\ y(t) = \psi(t), & t \in [-\breve{\tau}, 0], \end{cases}$$

$$(25)$$

with $\psi(t) = \pi - M^*$ be the initial condition.

In general, stability is an important requirement to study dynamical systems [31–33]. In following theorem, we derived the sufficient for exponential stability results for system (25).

Theorem 1. *The system (25) is said to be exponentially stable if there exist positive definite matrices P_1 and P_2 and positive scalars ϵ_1 and ϵ_2 such that*

$$\Psi = \begin{bmatrix} 2\sigma P_1 + A^\top P_1 + K^\top C^\top P_1 + AP_1 + P_1 CK + BP_2 & BP_1 & P_1 & 0 & 0 \\ * & -\epsilon_1 & 0 & 0 & 0 \\ * & * & -\epsilon_2 & 0 & 0 \\ * & * & * & \epsilon_1 - (1-\eta)P_2 B & 0 \\ * & * & * & * & \epsilon_2 \end{bmatrix} > 0, \qquad (26)$$

with $\dot{\tau}(t) = \eta$. Moreover,

$$\|y(t)\| \le e^{-2\sigma t} \sqrt{\dfrac{\left[\lambda_M(P_1) + \lambda_M(P_2)\|\psi\|\left[1 - e^{2\sigma\tau(t)}/2\sigma\right]\right]\|\psi\|}{\lambda_M(P_1)}}. \qquad (27)$$

Proof. Let us consider the following Lyapunov–Krasovskii function:

$$V(y(t)) = e^{2\sigma t} y^\top(t)P_1 y(t) + \int_{t-\tau(t)}^{t} y^\top(s)Be^{2\sigma s}P_2 y(s)\mathrm{d}s. \qquad (28)$$

Now, the time derivative of the Lyapunov–Krasovskii functional $V(y(t))$ along with the trajectories of system of equation (25) is

$$\begin{aligned}
\dot{V}(y(t)) &= 2\sigma e^{2\sigma t} y^\top(t)P_1 y(t) + e^{2\sigma t}\dot{y}^\top(t)P_1 y(t) + e^{2\sigma t} y^\top(t)P_1\dot{y}(t) \\
&\quad + y^\top(t)Be^{2\sigma t}P_2 y(t) - (1-\dot{\tau}(t))y^\top(t-\tau(t))Be^{2\sigma t}P_2 y(t-\tau(t)) \\
&= e^{2\sigma t}\Big[2\sigma y^\top(t)P_1 y(t) + \dot{y}^\top(t)P_1 y(t) + y^\top(t)P_1\dot{y}(t) \\
&\quad + y^\top(t)BP_2 y(t) - (1-\eta)y^\top(t-\tau(t))P_2 By(t-\tau(t))\Big] \\
&= e^{2\sigma t}\Big[y^\top(t)2\sigma P_1 y(t) + Ay(t) + By(t-\tau(t)) + f(y(t)) + CU(t))^T P_1 y(t) \\
&\quad + y^\top(t)P_1\big(Ay(t) + By(t-\tau(t)) + f(y(t)) + CU(t)\big) + y^\top(t)BP_2 y(t) \\
&\quad - (1-\eta)y^\top(t-\tau(t))P_2 By(t-\tau(t))\Big].
\end{aligned} \qquad (29)$$

Put $U(t) = Ky(t)$, the linear feedback control,

$$\begin{aligned}
&= e^{2\sigma t}\Big[y^\top(t)2\sigma P_1 y(t) + y^\top(t)A^\top P_1 y(t) + y^\top(t-\tau(t))B^\top P_1 y(t) + f^\top(y(t))P_1 y(t) \\
&\quad + y^\top(t)K^\top C^\top P_1 y(t) + y^\top(t)AP_1 y(t) + y^\top(t)BP_1 y(t-\tau(t)) + y^\top(t)P_1 f(y(t)) \\
&\quad + y^\top(t)P_1 CKy(t) + y^\top(t)BP_2 y(t) - (1-\eta)y^\top(t-\tau(t))P_2 By(t-\tau(t))\Big] \\
&= e^{2\sigma t}\Big[y^\top(t)\big(2\sigma P_1 + A^\top P_1 + K^\top C^\top P_1 + AP_1 + P_1 CK + BP_2\big)y(t) + y^\top(t-\tau(t))B^\top P_1 y(t) \\
&\quad + y^\top(t)BP_1 y(t-\tau(t)) + f^\top(y(t))P_1 y(t) + y^\top(t)P_1 f(y(t)) - y^\top(t-\tau(t))(1-\eta)P_2 By(t-\tau(t))\Big].
\end{aligned} \qquad (30)$$

By using Lemma 2, we can get the following inequalities:

$$
\begin{aligned}
y^\top (t - \tau(t)) B^\top P_1 y(t) + y^\top(t) BP_1 y(t - \tau(t)) &\leq \epsilon_1 y^\top(t - \tau(t)) y(t - \tau(t)) \\
&\quad + \epsilon_1^{-1} y^\top(t) BP_1 B^\top p_1^\top y(t), \\
f^\top(y(t)) P_1 y(t) + y^\top(t) P_1 f(y(t)) &\leq \epsilon_2 f^\top(y(t)) f(y(t)) + \epsilon_2^{-1} y^\top(t) P_1 P_1^\top y(t) \\
&\leq e^{2\sigma t}\big[y^\top(t)(2\sigma P_1 + A^\top P_1 + K^\top C^\top P_1 + AP_1 + P_1 CK + BP_2) y(t) \\
&\quad + \epsilon_1 y^\top(t - \tau(t)) y(t - \tau(t)) + \epsilon_1^{-1} y^\top(t) BP_1 B^\top P_1^\top y(t) \\
&\quad + \epsilon_2 f^\top(y(t)) f(y(t)) + \epsilon_2^{-1} y^\top(t) P_1 P_1^\top y(t) \big] \\
&= e^{2\sigma t}\big[y^\top(t)(\ldots_2 \\
&\quad + \epsilon_1^{-1} BP_1 P_1^\top B^\top + \epsilon_2^{-1} P_1 P_1^\top) y(t) \\
&\quad + f^\top(y(t)) \epsilon_2 f(y(t)) + y^\top(t - \tau(t))(\epsilon_1 \\
&\quad - (1 - \eta) P_2 B) y(t - \tau(t))\big].
\end{aligned}
$$
(31)

$\dot{V}(y(t)) \leq e^{2\sigma t} \zeta^\top(t) \Psi \zeta(t)$, where

$$
\Psi = \begin{bmatrix} \Psi_{11} & 0 & 0 \\ 0 & \epsilon_1 - (1 - \eta) P_2 B & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix},
$$
(32)

where $\Psi_{11} = 2\sigma P_1 + A^\top P_1 + K^\top C^\top P_1 + AP_1 + P_1 CK + BP_2 + \epsilon_1^{-1} BP_1 P_1^\top B^\top + \epsilon_2^{-1} P_1 P_1^\top$.

$$
\Psi = \begin{bmatrix}
2\sigma P_1 + A^\top P_1 + K^\top C^\top P_1 + AP_1 + P_1 CK + BP_2 & BP_1 & P_1 & 0 & 0 \\
* & -\epsilon_1 & 0 & 0 & 0 \\
* & * & -\epsilon_2 & 0 & 0 \\
* & * & * & \epsilon_1 - (1 - \eta) P_2 B & 0 \\
* & * & * & * & \epsilon_2
\end{bmatrix} > 0.
$$
(33)

Hence, by Lemma 1, $\Psi > 0 \implies \dot{V}(y(t)) < 0$. Moreover, $V(y(t)) \leq V(y(0))$. Here,

$$
V(y(0)) = e^{2\sigma(0)} y^\top(0) P_1 y(0) + \int_{0 - \tau(t)}^{0} y^\top(s) Be^{2\sigma s} P_2 y(s) ds
$$

$$
\leq \lambda_M(P_1)\|\psi\| + \lambda_M(P_2)\|\psi\|^2 \int_{-\tau(t)}^{0} e^{2\sigma s} ds
$$

$$
= \lambda_M(P_1)\|\psi\| + \lambda_M(P_2)\|\psi\|^2 \left[\frac{1 - e^{2\sigma\tau(t)}}{2\sigma}\right]
$$

$$
= \left[\lambda_M(P_1) + \lambda_M(P_2)\|\psi\|\left[\frac{1 - e^{2\sigma\tau(t)}}{2\sigma}\right]\right]\|\psi\|.
$$
(34)

where $\lambda_M(\cdot)$ is the maximum eigen value of $(\cdot)$. Here,

$$
V(y(t)) \geq e^{2\sigma t} \lambda_M(p_1)\|y(t)\|^2.
$$
(35)

Now,

$$
V(y(t)) \leq V(y(0)) \leq \left[\lambda_M(P_1) + \lambda_M(P_2)\|\psi\|\left[\frac{1 - e^{2\sigma\tau(t)}}{2\sigma}\right]\right]\|\psi\|.
$$
(36)

This implies that

$$
\|y(t)\| \leq e^{-2\sigma t} \sqrt{\frac{\left[\lambda_M(P_1) + \lambda_M(P_2)\|\psi\|\left[1 - e^{2\sigma\tau(t)}/2\sigma\right]\right]\|\psi\|}{\lambda_M(P_1)}},
$$
(37)

and by Definition 1, our system (25) is exponentially stable. □

Remark 1. If the system (25) is without time-varying delay terms, then the results of Theorem 1 have changed as follows: the possible Lyapunov function is $V(t) = e^{2\sigma t} y^\top(t) P_1 y(t)$ with the sufficient conditions

$$\Psi = \begin{bmatrix} 2\sigma P_1 + A^\top P_1 + P_1 A + P_1 CK & P_1 & P_2 \\ * & -\epsilon_1 & 0 \\ * & * & -\epsilon_1 \end{bmatrix} < 0.$$

$$\|y(t)\| \le e^{-\sigma t} \sqrt{\|\psi\|}.$$

$$(38)$$

Then the system (25) is without time-varying delay is exponentially stable.

5. Numerical Simulations

The numerical simulation results of system of equations in (21), based on the real-world data, are mentioned in Table 2.

Consider the following system:

$$\begin{cases} \dot{y}(t) = Ay(t) + By(t - \tau(t)) + g(y(t)) + CU(t), & t \ge 0, \\ y(t) = \psi(t), & t \in [-\breve{\tau}, 0], \end{cases}$$

$$(39)$$

with numerical values

$$A = \begin{bmatrix} -0.2784 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2784 & 0 & 0 & 0 & 0.0300 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2784 & 0 & 0 & 0 & 0.0300 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0714 & 0 & 0 & 0 & 0.0600 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0714 & 0 & 0 & 0.0150 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.2484 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2484 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2035 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1429 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1429 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1499 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1499 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1199 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0300 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1199 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1199 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0150 & 0 & 0 \end{bmatrix};$$

$$(40)$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

with the control gain matrix

TABLE 2: The parameters used in numerical simulation.

Λ_{w_e}	Reproduction rate of non-Wolbachia mosquitoes	1.25/day [34]
$\lambda_{w_e}, \lambda_{w_l}, \lambda_{w_p}$	Non-Wolbachia aquatic death rate	(1/7.78)/day [35]
$\gamma_{w_e}, \gamma_{w_l}, \gamma_{w_p}$	Non-Wolbachia maturation rate	(1/6.67)/day [36]
$\lambda_{w_f}, \lambda_{w_a}$	Non-Wolbachia adult death rate	(1/14)/day [35]
$\lambda_{i_e}, \lambda_{i_l}, \lambda_{i_p}$	Wolbachia adult death rate	(1/7.78)/day [35]
$\lambda_{i_f}, \lambda_{i_a}$	Wolbachia adult death rate	(1/7)/day [37]
Λ_{i_e}	Wolbachia reproduction rate	(0.95 * Λ_{w_e}/day) [34]
$\gamma_{i_e}, \gamma_{i_l}, \gamma_{i_p}$	Wolbachia-infected mosquitoes maturation rate	(1/6.67)/day [37]

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1882 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.2860 & 0.1611 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0979 & 1.1765 \end{bmatrix}; \quad (41)$$

with $\epsilon_1 = 0.7875$ and $\epsilon_2 = 0.9151$.

Remark 2. The numerical simulations derived in our paper shows that the allowable upper bound of the time-varying delay $\tau(t) = 0.7899$. However, the existing literature used the classical method such as Pontryagin maximum principle and Jacobian matrix method, by finding the sign of eigen values, fractional-order mathematical models [38, 39], and through finding the reproduction number to analyze the stability of the model. In our work, we have used LMI approach. However, compared with the other algebraic methods LMI approach is less conservative.

Figures 4 and 5 depict the population dynamics of wild mosquitoes and Wolbachia-infected mosquitoes at the initial release of Wolbachia, respectively. From Figure 6, we can observe that the Wolbachia-infected population also tends to decrease in numbers. Our aim is to maintain the Wolbachia population in certain quantity to spread Wolbachia among wild mosquitoes naturally. Because of this reason, we have released another batch of Wolbachia-infected mosquitoes as an optimal control. After this release, we can observe from Figures 7 and 8 that the level of Wolbachia-infected mosquitoes is increased to certain quantity and wild mosquito population is decreased to zero.
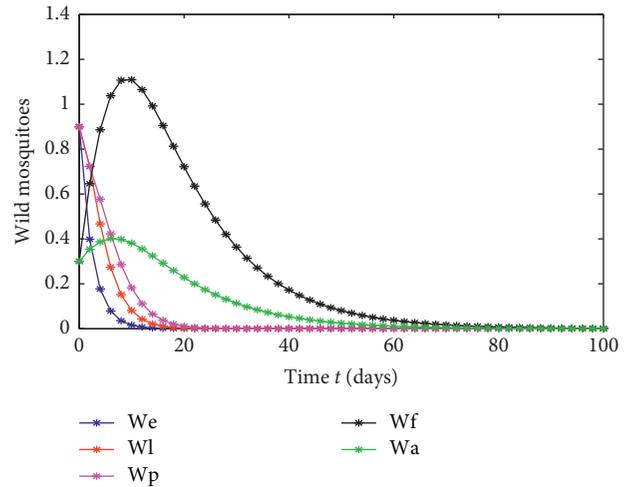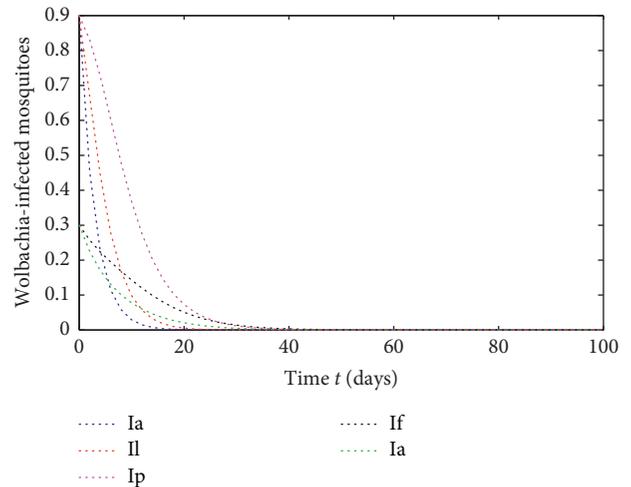

FIGURE 4: Population dynamics of wild mosquito.

FIGURE 5: Population dynamics of Wolbachia-infected mosquitoes.

Remark 3. For the optimal control to ensure the successful release of Wolbachia, the data from Table 2 are substituted in the system of equations in (21). From this we obtained the following results: Figure 4 denotes the dynamics of non-Wolbachia mosquitoes, and Figure 5 denotes the dynamics
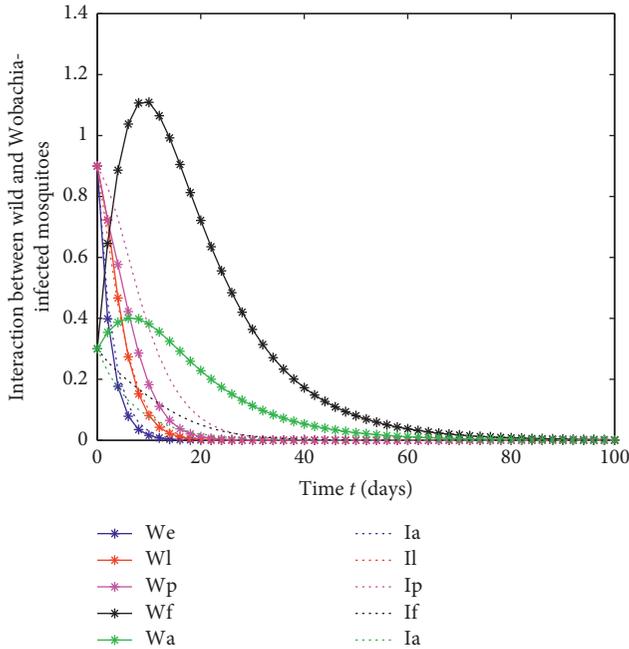
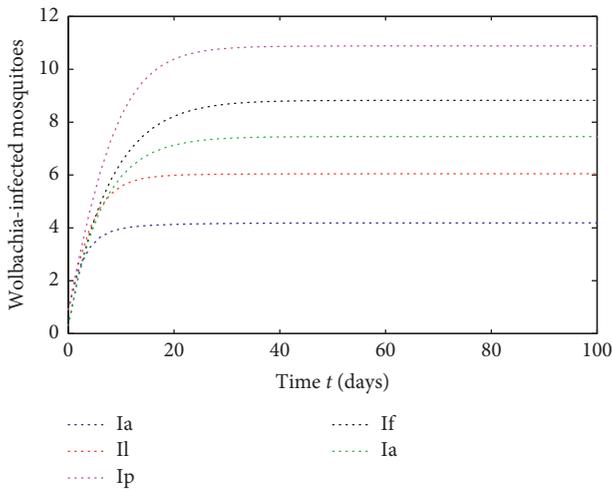Figure 6: Population dynamics of both wild and Wolbachia-infected mosquitoes.

Figure 7: Population dynamics of Wolbachia-infected mosquitoes after the linear feedback control.

of Wolbachia-infected mosquitoes. We can observe that the Wolbachia release in two main stages is not enough to control wild mosquitoes. In this, we noticed that the Wolbachia-infected mosquito population also goes to annihilation. Refer Figure 6, when comparing wild mosquitoes with Wolbachia-infected mosquitoes, it is increased and Wolbachia is not carried out to next generation. By implementing our proposed strategy, if we released Wolbachia in every stage, the Wolbachia-infected population is increased to certain level and maintained at that level (Figure 7) for the remaining period; at the same time, the Wolbachia population is annihilated without any increase (Figure 8).
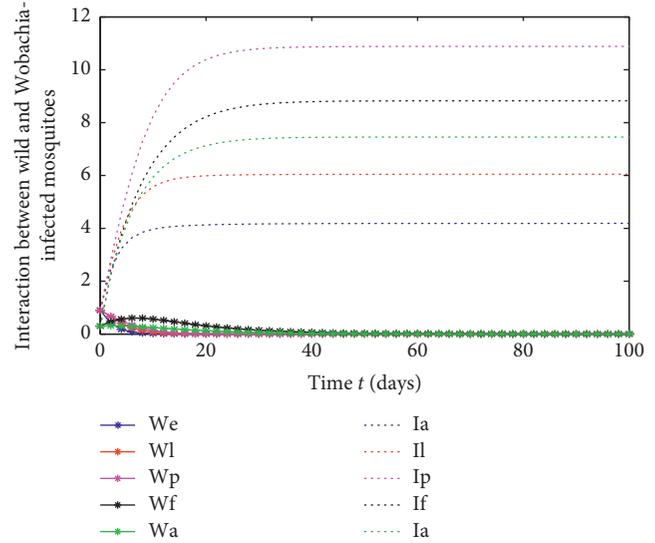

Figure 8: Population dynamics of both wild and Wolbachia-infected mosquitoes after the linear feedback control.

6. Conclusion

A novel integer-ordered age-structured delayed mathematical model to describe the interplay among the wild and Wolbachia-infected mosquitoes was proposed. The positiveness of the solution and existence of both Wolbachia present and Wolbachia free equilibriums were analyzed. By using some mathematical techniques such as linearization, Lyapunov–Krasovskii functional, and LMI theory, the exponential stability results of the proposed model were established. By using MATLAB, the numerical simulation results were presented to show the effectiveness of the created mathematical model. Our results show that the release of adult female Wolbachia-infected mosquitoes is more beneficial than the release of adult male Wolbachia-infected mosquitoes.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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