Research Article

Global Regularity Criterion for the 3D Incompressible Navier–Stokes Equations Involving the Velocity Partial Derivative

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In this paper, we study the regularity of the weak solutions for the incompressible 3D Navier–Stokes equations with the partial derivative of the velocity. By the embedded technology, we prove that the weak solution u is regular on $(0, T]$ if $\partial_3 u \in L^p(0, T; L^q(R^3))$ with $(2/p) + (3/q) = (70/37) + (15/37q)$, $(15/4) \leq q \leq \infty$, or $(2/p) + (3/q) = (34/19) + (9/19q)$, $(9/4) \leq q \leq \infty$.

1. Introduction and the Main Result

This paper focuses on the following three-dimensional incompressible Navier–Stokes (N-S) equations:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \Delta u, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1)$$

where u and p denote the velocity field and the pressure, respectively, and $u_0(x)$ is the initial fluid which satisfied $\nabla \cdot u_0 = 0$.

The existence of weak solutions of N-S equations was proved by Leray [1] and Hopf [2]. However, the existence of 3D global regular solutions is still an open question. Prodi [3] and Serrin [4] first considered the regularity of solutions. They, respectively, proved that the weak solution of 3D N-S equations is regular when the exponents p and q satisfy

$$u \in L^p(0, T; L^q(R^3)) = L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty. \quad (2)$$

In 1995, Veiga [5] generalized the result to

$$\nabla u \in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} \leq q \leq \infty. \quad (3)$$

When $\nabla u, \nabla u_3, u_3$, and the like satisfy a certain integrable condition, the weak solution is regular, and a large number of results are obtained (for details, refer to [6–16]). And Penel and Pokorný [13], Kukavica and Ziane [14], Cao [15], and Zhang [16], respectively, proved that the weak solution is regular on $(0, T]$ when the weak solution satisfies the following conditions:

$$\begin{aligned} \partial_3 u &\in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq q \leq \infty, \\ \partial_3 u &\in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{9}{4} \leq q \leq 3, \\ \partial_3 u &\in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{27}{16} \leq q \leq \frac{5}{2}, \\ \partial_3 u &\in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 1.56 \approx \frac{3\sqrt{37}}{4} - 3 \leq q \leq 3. \end{aligned} \quad (4)$$

Recently, Zhang, Yuan, and Zhou in [17] proved if

$$\partial_3 u \in L_t^p L_x^q, \quad \frac{2}{p} + \frac{3}{q} = \frac{8}{5} + \frac{3}{5q} \leq 1.75, \quad 4 \leq q \leq \infty, \quad (5)$$

or

$$\partial_3 u \in L_t^p L_x^q, \frac{2}{p} + \frac{3}{q} = \frac{14}{11} + \frac{9}{11q} \leq 1.6, \quad \frac{5}{2} \leq q \leq \infty, \quad (6)$$

the weak solution is regular on $(0, T]$.

Very recently, LI and Dong in [18] proved if

$$\partial_3 u \in L_t^p L_x^q, \frac{2}{p} + \frac{3}{q} = \frac{46}{25} + \frac{3}{25q} \leq 1.871, \quad \frac{31}{8} \leq q \leq \infty, \quad (7)$$

or

$$\partial_3 u \in L_t^p L_x^q, \frac{2}{p} + \frac{3}{q} = \frac{22}{13} + \frac{3}{13q} \leq 1.789, \quad \frac{19}{8} \leq q \leq \infty, \quad (8)$$

the weak solution is regular on $(0, T]$.

One of our main tasks is that when the value of the equation is as constant as possible, we expand the range of q to make the value of q as small as possible. Second, when the range of q does not shrink, the equation is tried to enlarge. Inspired by the texts [14, 17], we get better results than the above [17] as follows.

Theorem 1. Assume $u_0 \in L^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0, T > 0$. Let u be a weak solution of N-S equations (1) on $(0, T]$ which satisfies the initial value of u_0 . If

$$\partial_3 u \in L_t^p L_x^q, \frac{2}{p} + \frac{3}{q} = \frac{70}{37} + \frac{15}{37q} \leq 2, \quad \frac{15}{4} \leq q \leq \infty, \quad (9)$$

then the weak solution is regular on $(0, T]$.

Theorem 2. Assume $u_0 \in L^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0, T > 0$. Let u be a weak solution of N-S equations (1) on $(0, T]$ which satisfies the initial value of u_0 . If

$$\partial_3 u \in L_t^p L_x^q, \frac{2}{p} + \frac{3}{q} = \frac{34}{19} + \frac{9}{19q} \leq 2, \quad \frac{9}{4} \leq q \leq \infty, \quad (10)$$

then the weak solution is regular on $(0, T]$.

Remark 1. Comparing (9) and (7) and (10) and (8), respectively, the value of q is reduced, and within the range of q , the value of the equation is larger than that, so it is better. And when q obtains the minimum value, respectively, each equation takes the critical value of 2, so this condition is the optimal critical value.

However, we find that, in the existing results, the maximum value of q is bounded when the equality value can reach 2. The highlight of this article is that the equation value goes to 2, and the q value reaches infinity. When the value of the equation reached 2, it is a difficult problem to prove the regularity of the weak solutions to the 3D incompressible N-S equations by adding the value of q getting to infinity and getting to the current minimum of $(3\sqrt{37}/4) - 3$. We hope we can overcome this problem in the near future.

We shall give the proof of our main result in the third part. In order to facilitate reading, we will give the necessary preparatory knowledge in the following section.

2. Preliminaries

Throughout this text, C stands for a generic positive constant which may differ in value from one line to another. We use $\|\cdot\|_{L^p}$ to denote the norm of the Lebesgue space L^p ($1 \leq p \leq \infty$) as follows:

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^3} |f(x)|^p dx \right)^{(1/p)}, & 1 \leq p < \infty, \\ \text{esssup}_{x \in \mathbb{R}^3} |f(x)|, & p = \infty. \end{cases} \quad (11)$$

Definition 1 (see [19]). Assume $u_0 \in L^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0, T > 0$. The measurable function u defined on $(0, T] \times \mathbb{R}^3$ is called the weak solution of equation (1) if

- (1) $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$.
- (2) $\nabla \cdot u = 0$ and $\forall \varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$ satisfy

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \varphi dx dt = 0, \quad (12)$$

and equation (1) holds in the sense of distributions.

For $\forall \varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$ and $\nabla \cdot \varphi = 0$ satisfy

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^3} (\partial_t \varphi + (u \cdot \nabla) \varphi) \cdot u dx dt + \int_{\mathbb{R}^3} u_0 \cdot \varphi(x, 0) dx \\ &= \int_0^T \int_{\mathbb{R}^3} \nabla u : \nabla \varphi dx dt, \end{aligned} \quad (13)$$

where $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$, $A = (a_{ij}), B = (b_{ij})$.

- (3) The strong energy inequality, that is,

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u(0)\|_{L^2}^2, \quad \forall 0 < t < T. \quad (14)$$

Lemma 1 (see [13]).

$$\|p\|_{L^r} \leq C \|u\|_{L^{2r}}^2, \quad (15)$$

$$\|\partial_3 p\|_{L^r} \leq C \sum_{i,j=1}^3 \|\partial_3(u_i u_j)\|_{L^r}, \quad 1 < r < \infty. \quad (16)$$

Lemma 2 (Sobolev embedding inequality).

$$\|u\|_{L^{3r}} \leq C \|\partial_1 u\|_{L^2}^{(1/3)} \|\partial_2 u\|_{L^2}^{(1/3)} \|\partial_3 u\|_{L^r}^{(1/3)}, \quad 1 < r \leq \infty. \quad (17)$$

3. Proof of Main Results

In this part, we give the proof of main results. In order to prove Theorems 1 and 2, we thank the results in [13]. In [13], Penel and Pokorný showed that if

$$u_3 \in L^{(2s/(s-3))}(0, T; L^s(R^3)), \quad 3 < s \leq \infty, \quad (18)$$

$$\partial_3 u_1, \partial_3 u_1 \in L^{(2q/(2q-3))}(0, T; L^s(R^3)), \quad \frac{3}{2} < q \leq \infty,$$

then the weak solution of the N-S equations is regular on $(0, T]$.

For arbitrary $(15/4) \leq q \leq \infty$ and $(9/4) \leq q \leq \infty$, we have

$$L^{(37q/(35q-48))}(0, T; L^q(R^3)) \subset L^{(2q/(2q-3))}(0, T; L^q(R^3)),$$

$$L^{(19q/(17q-24))}(0, T; L^q(R^3)) \subset L^{(2q/(2q-3))}(0, T; L^q(R^3)),$$

respectively.

(19)

Hence, they just have to prove that

$$\partial_3 u \in L^p(0, T; L^q(R^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{70}{37} + \frac{15}{37q}, \quad \frac{15}{4} \leq q \leq \infty,$$

$$\partial_3 u \in L^p(0, T; L^q(R^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{34}{19} + \frac{9}{19q}, \quad \frac{9}{4} \leq q \leq \infty, \quad (20)$$

and in both cases, $u_3 \in L^{(2s/(s-3))}(0, T; L^s(R^3))$, $3 < s \leq \infty$, is established.

Taking the inner product for u_3 by $|u_3|u_3$ and integrating over R^3 , we have

$$\frac{1}{3} \frac{d}{dt} \|u_3\|_{L^3}^3 + \frac{4}{9} \|\nabla |u_3|^{(3/2)}\|_{L^2}^2 = - \int_{R^3} \partial_3 p |u_3| u_3 dx = I, \quad (21)$$

where we use $\nabla \cdot u = 0$ and the following identities:

$$\begin{aligned} \int_{R^3} u \cdot \nabla u_3 |u_3| u_3 dx &= \frac{1}{3} \int_{R^3} u \cdot \nabla |u_3|^3 dx \\ &= \frac{1}{3} \int_{R^3} \nabla \cdot u |u_3|^3 dx = 0, \\ - \int_{R^3} \Delta u_3 |u_3| u_3 dx &= \int_{R^3} (\nabla u_3) \cdot \nabla |u_3| u_3 dx + \int_{R^3} (\nabla u_3) \cdot \nabla u_3 |u_3| dx \\ &= \frac{1}{2} \int_{R^3} (\nabla |u_3|^2) \cdot \nabla |u_3| dx + \int_{R^3} |\nabla u_3|^2 |u_3| dx \\ &= \frac{4}{9} \int_{R^3} (\nabla |u_3|^{(3/2)})^2 dx + \int_{R^3} |\nabla u_3|^2 |u_3| dx \\ &\geq \frac{4}{9} \int_{R^3} (\nabla |u_3|^{(3/2)})^2 dx. \end{aligned} \quad (22)$$

Proof of Theorem 1. We estimate the right side of (21). By using the Hölder inequality, the Young inequality, (14), (16), and (17), we get that

$$\begin{aligned} |I| &\leq C \|\partial_3 p\|_{L^3} \|u_3\|_{L^3}^2 \\ &\leq C \|u \partial_3 u\|_{L^3} \|u_3\|_{L^3}^2 \\ &\leq C \|u\|_{L^{(3q/(q-3))}} \|\partial_3 u\|_{L^q} \|u_3\|_{L^3}^2 \\ &\leq C \|u\|_{L^2}^{((4q-15)/(2(3q-2)))} \|u\|_{L^{3q}}^{((q+24)/(4(3q-2)))} \|\partial_3 u\|_{L^q} \|u_3\|_{L^3}^2 \\ &\leq C \|\partial_1 u\|_{L^2}^{((q+24)/(12(3q-2)))} \|\partial_2 u\|_{L^2}^{((q+24)/(12(3q-2)))} \\ &\quad \cdot \|\partial_3 u\|_{L^q}^{((q+24)/(12(3q-2)))} \|\partial_3 u\|_{L^q} \|u_3\|_{L^3}^2 \\ &\leq C \|\nabla u\|_{L^2}^{(q+24/(6(3q-2)))} \|\partial_3 u\|_{L^q}^{(37q/(12(3q-2)))} \|u_3\|_{L^3}^2 \\ &\leq C \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(37q/(35q-48))} \right) \|u_3\|_{L^3}^2. \end{aligned} \quad (23)$$

Substituting (23) into (21), we have

$$\frac{d}{dt} \|u_3\|_{L^3}^3 \leq C \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(37q/(35q-48))} \right) \|u_3\|_{L^3}^2. \quad (24)$$

Dividing both sides by $\|u_3\|_{L^3}^2$ and integrating with respect to t imply that

$$\int_0^T d\|u_3\|_{L^3} \leq C \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(37q/(35q-48))} \right) dt. \quad (25)$$

We deduce from (9) and (14) that

$$\|u_3\|_{L^\infty(0, T; L^3(R^3))} \leq C. \quad (26)$$

It is available from type (23) that

$$\frac{4}{9} \|\nabla |u_3|^{(3/2)}\|_{L^2}^2 \leq C \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(37q/(35q-48))} \right) \|u_3\|_{L^3}^2. \quad (27)$$

So, by the embedding inequality, we get

$$\begin{aligned}\|u_3\|_{L^3(0,T;L^9(R^3))} &= \| |u_3|^{(3/2)} \|_{L^2(0,T;L^6(R^3))}^{(2/3)} \\ &\leq C \|\nabla |u_3|^{(3/2)}\|_{L^2(0,T;L^2(R^3))}^{(2/3)} \leq C.\end{aligned}\quad (28)$$

So, Theorem 1 is proved. $\square$

Proof of Theorem 2. We have another estimation for the right side of (21). By using the Hölder inequality, the Young inequality, (14), (15), and (17), we have

$$\begin{aligned}|I| &\leq C \int_{R^3} |p| |\partial_3 u_3| |u_3| dx \\ &\leq C \|p\|_{L^{(3q/(2q-3))}} \|\partial_3 u_3\|_{L^q} \|u_3\|_{L^3} \\ &\leq C \|u\|_{L^{(6q/(2q-3))}}^2 \|\partial_3 u\|_{L^q} \|u_3\|_{L^3} \\ &\leq C \|u\|_{L^2}^{((4q-9)/(3(3q-2)))} \|u\|_{L^{3q}}^{((q+12)/(2(3q-2)))} \|\partial_3 u\|_{L^q} \|u_3\|_{L^3} \\ &\leq C \|\partial_1 u\|_{L^2}^{((q+12)/(6(3q-2)))} \|\partial_2 u\|_{L^2}^{((q+12)/(6(3q-2)))} \\ &\quad \cdot \|\partial_3 u\|_{L^q}^{((q+12)/(6(3q-2)))} \|\partial_3 u\|_{L^q} \|u_3\|_{L^3} \\ &\leq C \|\nabla u\|_{L^2}^{((q+12)/(3(3q-2)))} \|\partial_3 u\|_{L^q}^{(19q/(6(3q-2)))} \|u_3\|_{L^3} \\ &\leq C \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(19q/(17q-24))} \right) \|u_3\|_{L^3}.\end{aligned}\quad (29)$$

Inserting (29) into (21), one has

$$\frac{d}{dt} \|u_3\|_{L^3}^3 \leq C \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(19q/(17q-24))} \right) \|u_3\|_{L^3}. \quad (30)$$

Dividing both sides by $\|u_3\|_{L^3}$ and integrating with respect to t imply that

$$\int_0^T \|u_3\|_{L^3} d\|u_3\|_{L^3} \leq C \int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\partial_3 u\|_{L^q}^{(19q/(17q-24))} \right) dt. \quad (31)$$

By (10) and (14), we have

$$\|u_3\|_{L^\infty(0,T;L^3(R^3))} \leq C. \quad (32)$$

The same can be proved:

$$u_3 \in L^3(0,T;L^9(R^3)). \quad (33)$$

So, Theorem 2 is proved. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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