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Research Article

Computing Exact Values for Gutman Indices of Sum Graphs under Cartesian Product

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Gutman index of a connected graph is a degree-distance-based topological index. In extremal theory of graphs, there is great interest in computing such indices because of their importance in correlating the properties of several chemical compounds. In this paper, we compute the exact formulae of the Gutman indices for the four sum graphs (S-sum, R-sum, Q-sum, and T-sum) in the terms of various indices of their factor graphs, where sum graphs are obtained under the subdivision operations and Cartesian products of graphs. We also provide specific examples of our results and draw a comparison with previously known bounds for the four sum graphs.

1. Introduction

Theory of topological indices (TIs) started when Wiener discovered a close correlation between boiling points of certain alkanes and sums of the distances among pairs of vertices. Later, this calculated number was named Wiener Index [1]. After 25 years, Gutman and Trinajstić discovered degree-based indices (first and second Zagreb indices) which they used to compute the total π-electron energy of conjugate molecules [2]. Following these discoveries, many scientists began to introduce various TIs as invariant numbers for the prediction of the certain properties of molecular structures such as boiling point, freezing point, volume, density, vaporization, and weight. Two deeper approaches, namely, quantitative structures property relationships (QSPR) and quantitative structure activity relationships (QSAR) have also been used under the subject of cheminformatics (combination of chemistry, mathematics, statistics, and information sciences) and in conjunction with TIs, to find correlation values between the physical structures and chemical properties of molecules, see [3–5].

TIs have been classified into three main classes depending upon degrees of nodes (vertices), distances among the vertices, and enumerative polynomials of the molecular graphs. Distance-based TIs are generally considered more important than the others. Some of the distance-based TIs are Wiener index [1], average distance index [6], Harary index [7, 8], degree distance index, and the Gutman index [9]. For more details, see [10–14].

In graph theory, various operations such as union, intersection, addition, and Cartesian product are used to obtain the new graphs. Yan et al. [15] defined four subdivision-related operations S, R, Q, and T. They applied these operations on a connected graph G to obtain the four new graphs $S(G)$ (subdivided graph), $R(G)$ (triangle parallel graph), $Q(G)$ (line superposition graph), and $T(G)$ (total graph), respectively. Afterwards, Das and Gutman [10] introduced the F-sum graphs using the operation of Cartesian product on the graphs $F(G_1)$ and G_2, where $F \in \{S, R, Q \text{ and } T\}$. Various hexagonal chains were later derived from these F-sum graphs, which have been found isomorphic to many chemical structures. They also

determined the Wiener indices of the following S-sum $(G_1 +_S G_2)$, R-sum $(G_1 +_R G_2)$, Q-sum $(G_1 +_Q G_2)$, and T-sum $(G_1 +_T G_2)$ graphs.

Recently, Liu et al. [16] computed the first general Zagreb indices of the F-sum graphs. Akhter and Imran [17] found out the sharp bounds of the general sum-connectivity index for F-sum graphs. Ahmad et al. [18] discovered the exact formulae of the general sum-connectivity index for F-sum graphs, by improving the bounds. An et al. [19] determined the upper bounds of the degree distance indices for all the F-sum graphs. Pattabiraman and Bhat [20] derived the upper bounds of the Gutman index for all the F-sum graphs. For more studies, we refer to [21–32].

In this paper, we obtain the exact values of the Gutman index for the F-sum $(G_1 +_S G_2)$, $(G_1 +_R G_2)$, $(G_1 +_Q G_2)$, and $(G_1 +_T G_2)$ graphs. Moreover, the results are illustrated with special classes of the F-sum graphs and a comparison is drawn between the obtained exact values and the previously known bounded values.

The sections of paper are organized as follows. Section 2 comprises of preliminaries (some important definitions and statements of related lemmas). Section 3 contains the main result consisting of statements and proofs of theorems about Gutman indices of F-sum graphs. Lastly, Section 4 covers the applications of the main result to the computation of Gutman indices of particular classes of the F-sum graphs and a comparison among exact and known bounded values.

2. Preliminaries

We give a detailed consideration of two simple graphs G_1 and G_2. The degree of a vertex x ($\deg(x)$ or $\mathrm{d}(x)$) is equal to the number of vertices connected to it. For each $(x, y) \in V(G_1 \times G_2)$, degree of the vertex (x, y) is denoted by $\deg(x, y)$. The distance $\mathrm{d}(x, y)$ between two vertices $x, y \in V(G)$ is defined as the length of the shortest path between both the vertices x and y. Further details can be found in [33, 34].

Definition 1 (see [1]). The Wiener index $W(G)$ of a connected graph G is defined as

$$W(G) = \frac{1}{2} \sum_{x,y \in V(G)} \mathrm{d}(x, y). \tag{1}$$

Definition 2 (see [11]). The degree distance index $\mathrm{DD}(G)$ of a connected graph G is defined as

$$\mathrm{DD}(G) = \frac{1}{2} \sum_{x,y \in V(G)} \{\mathrm{d}(x, y)(\deg(x) + \deg(y))\}. \tag{2}$$

Definition 3 (see [35]). The Gutman index $\mathrm{GM}(G)$ of a connected graph G is defined as

$$\mathrm{GM}(G) = \frac{1}{2} \sum_{x,y \in V(G)} \{\mathrm{d}(x, y)(\deg(x)\deg(y))\}. \tag{3}$$

Yan et al. [15] defined four special graphs derivable from a given graph G by applying the four respective operations S, R, Q, and T on the graph G as follows:

(i) Subdivided graph $S(G)$ is formed from G if distance of one between two adjacent pair of vertices is increased by two after inserting a new vertex between them. The vertices of G are named as black vertices, while new vertices are called white vertices.

(ii) Triangle parallel graph $R(G)$ is formed from G if the new vertex corresponded to each edge of G is joined with the end vertices of the each respective edge. The vertices of G are named as black vertices, while new vertices are called white vertices.

(iii) Line superposition graph $Q(G)$ is formed when two white vertices obtained from $S(G)$ are further joined if incident edges of these white vertices have one common end vertex in G.

(iv) Total graph $T(G)$ is formed from $R(G)$ by applying the further operation Q on it.

Figure 1 illustrates the graphs obtained by the operations S, R, Q, and T based on the path graph (P_6).

Definition 4. Let $F \in \{S, R, Q \text{ and } T\}$. Then, $G_1 +_F G_2$ is called a F-sum graph with vertex set $V(G_1 +_F G_2 = (V(G_1) \cup E(G_1)) \times V(G_2)$ and $(u, v), (x, y) \in V(G_1 +_F G_2)$ are adjacent such that either $u = x$ and $(v, y) \in E(G_2)$ or $v = y$ and $(u, x) \in E(F(G_1))$.

Figure 2 shows instances of S-sum $(G_1 +_S G_2)$, R-sum $(G_1 +_R G_2)$, Q-sum $(G_1 +_Q G_2)$, and T-sum $(G_1 +_T G_2)$ graphs. We now state some important lemmas which are frequently used in the main results.

Lemma 1 (see [36]). *Let G_1 and G_2 be two simple and connected graphs.*

(a) For $F \in \{S, R, Q, T\}$, if both vertices (x, y) and (w, z) are black, then $d((x, y), (w, z)|G_1 +_F G_2) = d(x, w|F(G_1)) + d(y, z|G_2)$

(b) If one vertex (x, y) is white and second (w, z) is black with $F \in \{S, R, Q, T\}$, then $d((x, y), (w, z)|G_1 +_F G_2) = d(x, w|F(G_1)) + d(y, z|G_2)$

Lemma 2 (see [36]). *Let G_1 and G_2 be two simple and connected graphs. If both vertices (x, y) and (w, z) are white vertices and $F = S$ or R, then*

$$\mathrm{d}((x, y), (w, z)|G_1 +_F G_2) = \begin{cases} 2 + \mathrm{d}(y, z|G_2)), & \text{If } x = w, \\ \mathrm{d}(x, w|F(G_1) + \mathrm{d}(y, z|G_2), & \text{If } x \neq w. \end{cases} \tag{4}$$

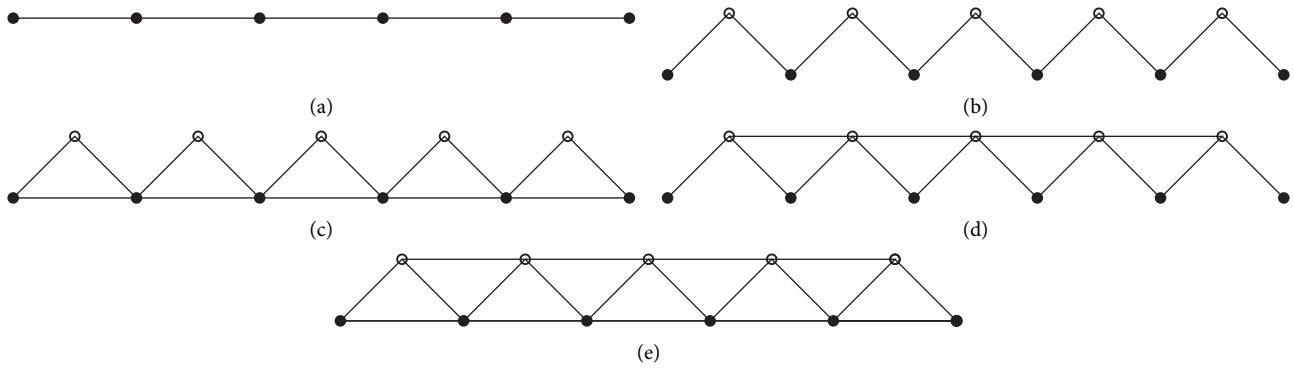

Figure 1: (a) P_6, (b) $S(P_6)$, (c) $R(P_6)$, (d) $Q(P_6)$, and (e) $T(P_6)$.

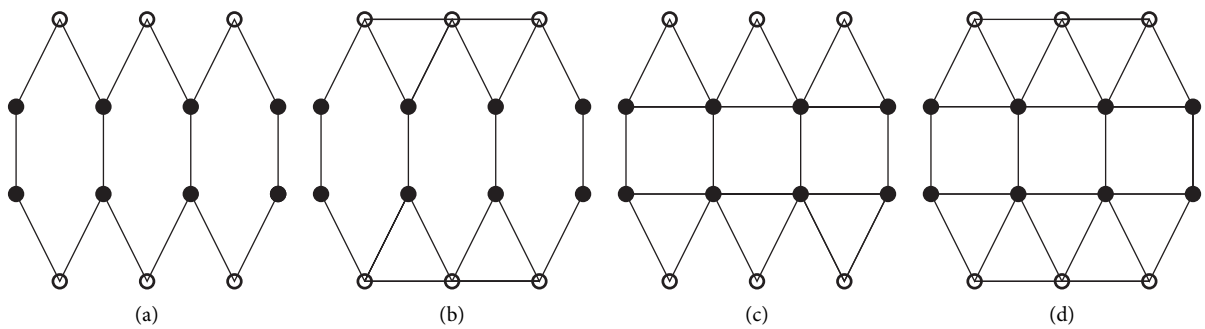

Figure 2: (a) $(P_4 +_S P_2)$. (b) $(P_4 +_R P_2)$. (c) $(P_4 +_Q P_2)$. (d) $(P_4 +_T P_2)$.

Lemma 3 (see [36]). *Let G_1 and G_2 be two simple and connected graphs. If both vertices (x, y) and (w, z) are white vertices and $F = Q$ or T, then*

$$\mathrm{d}\left((x, y), (w, z)|G_1 +_F G_2\right) = \begin{cases} 2 + \mathrm{d}\left(y, z|G_2\right)), & \text{If } x = w, \\ 1 + \mathrm{d}\left(x, w|F\left(G_1\right) + \mathrm{d}\left(y, z|G_2\right), & \text{If } x \neq w, y \neq z. \end{cases} \tag{5}$$

3. Main Results

This section is devoted to providing main theorems on the Gutman index of F-sum graphs $(G_1 +_S G_2)$, $(G_1 +_R G_2)$, $(G_1 +_Q G_2)$, and $(G_1 +_F G_2)$. Consider the set $V(G_1) = \{u_1, u_2, u_3, \ldots, u_n\}$ of black vertices with $|E(G_1)| = k$ so that $W = \{w_1, w_2, w_3, \ldots, w_k\}$ consists of white vertices and $V(G_2) = \{v_1, v_2, v_3, \ldots, v_m\}$ with $|E(G_2)| = l$. Then, $V(G_1 +_F G_2) = \{(u_i, v_j)\colon 1 \le i \le n, \quad 1 \le j \le m\} \cup \{(w_i, v_j)\colon 1 \le i \le k, 1 \le j \le m\}$, where $F \in \{S, R, Q, T\}$.

Theorem 1. *Let G_1 and G_2 be connected and simple graphs. If $(G_1 +_S G_2)$ is the S-sum graph of G_1 and G_2, then*

$$\begin{aligned} \mathrm{GM}\left(G_1 +_S G_2\right) &= m^2 \mathrm{GM}\left(S\left(G_1\right)\right) + n^2 \mathrm{GM}\left(G_2\right) + 4nk\mathrm{DD}\left(G_2\right) + 16k^2 W\left(G_2\right) + 4k\left(m^2 - m\right) \\ &\quad + lm \sum_{i,j=1}^{n} \left\{\mathrm{d}\left(u_i, u_j\right)|S\left(G_1\right)\right\}\left\{\mathrm{d}\left(u_i\right) + \mathrm{d}\left(u_j\right)\right\} + 2l^2 \sum_{i,j=1}^{n} \left\{\mathrm{d}\left(u_i, u_j\right)|S\left(G_1\right)\right\} + 4lm \sum_{r=1}^{k}\sum_{i=1}^{n} \left\{\mathrm{d}\left(w_r, u_i\right)|S\left(G_1\right)\right\}. \end{aligned} \tag{6}$$

Proof. Case 1: when both vertices are black,

$$A = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(u_i, v_p\right), \left(u_j, v_q\right)\right)\left(\deg\left(u_i, v_p\right)\deg\left(u_j, v_q\right)\right)\middle|G_1 +_s G_2\right\}, \tag{7}$$

for $i, j = 1$ to n and $p, q = 1$to m. For S-sum $\deg(u_i, v_p) = \mathrm{d}(u_i) + \mathrm{d}(v_p)$,

$$\begin{aligned}
A &= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1) + \mathrm{d}\left(v_p, v_q\right|(G_2)\right\}\left\{\mathrm{d}(u_i) + \mathrm{d}\left(v_p\right)\right\}\left\{\mathrm{d}\left(u_j\right) + \mathrm{d}\left(v_q\right)\right\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1) + \mathrm{d}\left(v_p, v_q\right|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right) + \mathrm{d}(u_i)\mathrm{d}\left(v_q\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(u_j\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right) + \mathrm{d}(u_i)\mathrm{d}\left(v_q\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(u_j\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right) + \mathrm{d}(u_i)\mathrm{d}\left(v_q\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(u_j\right) + \mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(v_q\right)\right\} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(u_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(v_q\right)\right\} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(u_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)|(G_2)\right\}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\}.
\end{aligned} \tag{8}$$

Substituting $\sum_{p,q=1}^{m}\mathrm{d}(v_q) = 2lm, \sum_{i,j=1}^{n}\mathrm{d}(u_j) = 2kn,$

$$\begin{aligned}
&= \frac{m^2}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\mathrm{d}(u_i) \\
&\quad + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}\left(u_j\right)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p\right)\mathrm{d}\left(v_q\right)\right\} \\
&= \frac{m^2}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|S(G_1)\right\}\mathrm{d}(u_i) \\
&\quad + \frac{1}{2}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + kn\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\left\{\mathrm{d}\left(v_q\right)\right\} \\
&\quad + kn\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|(G_2)\right\}\left\{\mathrm{d}\left(v_p\right)\right\} + n^2\mathrm{GM}(G_2).
\end{aligned} \tag{9}$$

Substituting $\sum_{i,j=1}^{n}\{d(u_i)d(u_j)\} = 4k^2$ and $\sum_{p,q=1}^{m}\{d(v_p)d(v_q)\} = 4l^2$,

$$= \frac{m^2}{2}\sum_{i,j=1}^{n}\{d(u_i,u_j)|S(G_1)\}\{d(u_i)d(u_j)\} + lm\sum_{i,j=1}^{n}\{d(u_i,u_j)|S(G_1)\}\{d(u_i)+d(u_j)\}$$
$$+ 2l^2\sum_{i,j=1}^{n}\{d(u_i,u_j)|S(G_1)\} + 4k^2W(G_2) + 2kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2) \tag{10}$$

Case 2: when one vertex is white and the other is black,

$$B_1 = \frac{1}{2}\sum\{d((w_r,v_p),(u_i,v_q))(\deg(w_r,v_p)\deg(u_i,v_q)|G_1 +_s G_2\}. \tag{11}$$

For $r = 1$ to k, $i = 1$ to n, and $p,q = 1$ to m and $\deg(w_r,v_p) = 2$,

$$B_1 = \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|S(G_1) + d(v_p,v_q|(G_2)\}(2)\{d(u_i)+d(v_q)\}$$
$$= \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|S(G_1)(du_i)\} + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|S(G_1)\}(dv_q)$$
$$+ \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(v_p,v_q)|G_2\}d(u_i) + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(v_p,v_q)|G_2\}d(v_q) \tag{12}$$
$$= m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|S(G_1)\}(du_i) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|S(G_1)\} + 4k^2W(G_2) + kn\mathrm{DD}(G_2).$$

The summation of the distances between vertices with different colours is twice the B_1, i.e.,

$$B = 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|S(G_1)\}(du_i) + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}$$
$$\cdot\{d(w_r,u_i)|S(G_1)\} + 8k^2W(G_2) + 2kn\mathrm{DD}(G_2). \tag{13}$$

Case 3: when both vertices are white,

$$C = \frac{1}{2}\sum\{d((w_r,v_p),(w_s,v_q))(\deg(w_r,v_p)\deg(w_s,v_q)|G_1 +_s G_2\}. \tag{14}$$

This summation consists of two parts $C = C_1 + C_2$ and $\deg(w_r, v_p) = 2$, where

$$
\begin{aligned}
C_1 &= \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_s G_2 \colon r = s, p \neq q\right\}, \\
C_2 &= \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_s G_2 \colon r \neq s\right\}, \\
C_1 &= \frac{1}{2}\sum \mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_s G_2 \\
&= \frac{1}{2}\sum\left\{\left(2 + \mathrm{d}\left(v_p, v_q\right)\right)(2)(2)|G_1 +_s G_2\right\} \\
&= 4\sum_{r,s=1, r=s}^{k}\sum_{p,q=1, p\neq q}^{m} + 2\sum_{r,s=1, r=s}^{k}\sum_{p,q=1, p\neq q}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|G_2\right\} \\
&= 4k\left(m^2 - m\right) + 4kW(G_2), \\
C_2 &= \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_s G_2, r \neq s\right\} \\
&= \frac{1}{2}\sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|S(G_1)\}\{4\} + \frac{1}{2}\sum_{p=1}^{m}\sum_{p=1}^{m}\sum_{r,s=1, r\neq s}^{k}\left\{\mathrm{d}\left(v_p, v_q\right)|G_2\right\}\{4\} \\
&= 2m^2\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)S(G_1)\} + 4\left(k^2 - k\right)W(G_2), \\
C &= 4k\left(m^2 - m\right) + 4kW(G_2) + 2m^2\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|S(G_1)\} + 4\left(k^2 - k\right)W(G_2) \\
&= 4k\left(m^2 - m\right) + 2m^2\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|S(G_1)\} + 4k^2W(G_2).
\end{aligned}
\tag{15}
$$

Now, Gutman index of $G_1 +_s G_2$ is given by

$$\mathrm{GM}(G_1 +_s G_2) = A + B + C, i.e.,$$

$$\begin{aligned}
\mathrm{GM}(G_1 +_s G_2) = {} & \frac{m^2}{2}\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\} + lm\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\}\{\mathrm{d}(u_i)+\mathrm{d}(u_j)\} \\
& + 2l^2\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\} + 4k^2W(G_2) + 2kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2) \\
& + 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|S(G_1)\}(\mathrm{d}u_i) + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|S(G_1)\} + 8k^2W(G_2) \\
& + 2kn\mathrm{DD}(G_2) + 4k(m^2-m) + 2m^2\sum_{r,s=1,r\neq s}^{k}\{\mathrm{d}(w_r,w_s)|S(G_1)\} + 4k^2W(G_2),
\end{aligned}$$

$$\begin{aligned}
\mathrm{GM}(G_1 +_s G_2) = {} & m^2\left[\frac{1}{2}\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\} + \sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)S(G_1)\}\{2\mathrm{d}(u_i)\} + 2\sum_{r,s=1,r\neq s}^{k}\{\mathrm{d}(w_r,w_s)|S(G_1)\}\right] \\
& + n^2\mathrm{GM}(G_2) + 4kn\mathrm{DD}(G_2) + 16k^2W(G_2) + 4k(m^2-m) \\
& + lm\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\}\{\mathrm{d}(u_i)+\mathrm{d}(u_j)\} + 2l^2\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\} + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|S(G_1)\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{GM}(G_1 +_s G_2) = {} & m^2\mathrm{GM}(S(G_1)) + n^2\mathrm{GM}(G_2) + 4nk\mathrm{DD}(G_2) + 16k^2W(G_2) + 4k(m^2-m) \\
& + lm\sum_{i,j=1}^{n}[\mathrm{d}(u_i,u_j)|S(G_1)]\{\mathrm{d}(u_i)+\mathrm{d}(u_j)\} + 2l^2\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|S(G_1)\} + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|S(G_1)\}.
\end{aligned} \tag{16}$$

□

Theorem 2. *Let G_1 and G_2 are connected and simple graphs. If $(G_1 +_R G_2)$ is the R-sum graph of G_1 and G_2, then*

$$\begin{aligned}
\mathrm{GM}(G_1 +_R G_2) = {} & m^2\mathrm{GM}(R(G_1)) + n^2\mathrm{GM}(G_2) + 6nk\mathrm{DD}(G_2) + 36k^2W(G_2) + 4k(m^2-m) \\
& + 2lm\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|R(G_1)\}\{\mathrm{d}(u_i)+\mathrm{d}(u_j)\} + 2l^2\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|R(G_1)\} + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|R(G_1)\}.
\end{aligned} \tag{17}$$

Proof. 1) When both vertices are black,

$$A = \frac{1}{2}\sum\left\{\mathrm{d}((u_i,v_p),(u_j,v_q))(\deg(u_i,v_p)\deg(u_j,v_q)\middle|G_1 +_R G_2\right\}, \tag{18}$$

for $i, j = 1$ to n and $p, q = 1$to m. For R-sum $\deg(u_i, v_p) = 2\mathrm{d}(u_i) + \mathrm{d}(v_p)$,

$$
\begin{aligned}
A &= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1) + \mathrm{d}(v_p,v_q|(G_2)\right\}\left\{2\mathrm{d}(u_i) + \mathrm{d}(v_p)\right\}\left\{2\mathrm{d}(u_j) + \mathrm{d}(v_q)\right\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1) + \mathrm{d}(v_p,v_q|(G_2)\right\}\left\{4\mathrm{d}(u_i)\mathrm{d}(u_j) + 2\mathrm{d}(u_i)\mathrm{d}(v_q) + 2\mathrm{d}(v_p)\mathrm{d}(u_j) + \mathrm{d}(v_p)\mathrm{d}(v_q)\right\} \\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{4\mathrm{d}(u_i)\mathrm{d}(u_j) + 2\mathrm{d}(u_i)\mathrm{d}(v_q) + 2\mathrm{d}(v_p)\mathrm{d}(u_j) + \mathrm{d}(v_p)\mathrm{d}(v_q)\right\} \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{4\mathrm{d}(u_i)\mathrm{d}(u_j) + 2\,d(u_i)\mathrm{d}(v_q) + 2\mathrm{d}(v_p)\mathrm{d}(u_j) + \mathrm{d}(v_p)\mathrm{d}(v_q)\right\} \\
&= 2\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\} + \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(v_q)\right\} \\
&\quad + \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{\mathrm{d}(v_p)\mathrm{d}(u_j)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{\mathrm{d}(v_p)\mathrm{d}(v_q)\right\} \\
&\quad + 2\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\} + \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(v_q)\right\} \\
&\quad + \sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(v_p)\mathrm{d}(u_j)\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(v_p)\mathrm{d}(v_q)\right\}.
\end{aligned}
\tag{19}
$$

Substituting $\sum_{p,q=1}^{\mathrm{m}} \mathrm{d}(v_q) = 2lm$ and $\sum_{i,j=1}^{n} \mathrm{d}\ (u_j) = 2kn$,

$$
\begin{aligned}
&= 2m^2\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\} + 2lm\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\mathrm{d}(u_i) \\
&\quad + 2lm\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\mathrm{d}(u_i) + \frac{1}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|R(G_1)\right\}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p)\mathrm{d}(v_q)\right\} \\
&\quad + 2\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\} + 2kn\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(v_q)\right\} \\
&\quad + 2kn\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\left\{\mathrm{d}(v_p)\right\} + n^2\mathrm{GM}(G_2).
\end{aligned}
\tag{20}
$$

Substituting $\sum_{i,j=1}^{n}\{d(u_i)d(u_j)\} = 4k^2$ and $\sum_{p,q=1}^{m}\{d(v_p)d(v_q)\} = 4l^2$,

$$A = 2m^2\sum_{i,j=1}^{n}\{d(u_i,u_j)|R(G_1)\}\{d(u_i)d(u_j)\} + 2lm\sum_{i,j=1}^{n}\{d(u_i,u_j)|R(G_1)\}\{d(u_i)+d(u_j)\} + 2l^2\sum_{i,j=1}^{n}\{d(u_i,u_j)|S(G_1)\} + 16k^2W(G_2) + 4kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2). \tag{21}$$

Case 2: when one vertex is white and the other is black,

$$B_1 = \frac{1}{2}\sum\left\{d((w_r,v_p),(u_i,v_q))(\deg(w_r,v_p)\deg(u_i,v_q)\middle|G_1 +_R G_2\right\}. \tag{22}$$

For $r = 1$ to k, $i = 1$ to n, and $p,q = 1$ to m and $\deg(w_r,v_p) = 2$,

$$\begin{aligned} B_1 &= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|R(G_1) + d(v_p,v_q|(G_2)\}(2)\{2d(u_i)+d(v_q)\} \\ &= 2\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|R(G_1)\}(du_i) + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|R(G_1)\}(dv_q) \\ &\quad + 2\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(v_p,v_q)|G_2\}d(u_i) + \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(v_p,v_q)|G_2\}d(v_q) \\ &= 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|R(G_1)\}(du_i) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|R(G_1)\} + 8k^2W(G_2) + kn\mathrm{DD}(G_2). \end{aligned} \tag{23}$$

The summation of the distances between vertices with different colours is twice the B_1, i.e.,

$$B = 4m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|R(G_1)\}(du_i) + 4lm\sum_{r=1}^{k}\sum_{i=1}^{n}\cdot\{d(w_r,u_i)|R(G_1)\} + 16k^2W(G_2) + 2kn\mathrm{DD}(G_2). \tag{24}$$

Case 3: when both vertices are white, C can be calculated similar to that in the previous theorem:

$$C = 4k(m^2-m) + 2m^2\sum_{r,s=1,r\neq s}^{k}\cdot\{d(w_r,w_s)|R(G_1)\} + 4k^2W(G_2). \tag{25}$$

Now, Gutman index of $(G_1 +_R G_2)$ is given by

$$\begin{aligned}
&\mathrm{GM}(G_1 +_R G_2) = A + B + C, \text{i.e.,}\\
&\mathrm{GM}(G_1 +_R G_2) = 2m^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\} + 2lm \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\}\\
&\qquad + 2l^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\} + 16k^2 W(G_2) + 4kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2)\\
&\qquad + 4m^2 \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|R(G_1)\}(\mathrm{d}u_i) + 4lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|R(G_1)\} + 16k^2 W(G_2)\\
&\qquad + 2kn\mathrm{DD}(G_2) + 4k(m^2 - m) + 2m^2 \sum_{r,s=1, r\neq s}^{k} \{\mathrm{d}(w_r, w_s)|R(G_1)\} + 4k^2 W(G_2),\\
&\mathrm{GM}(G_1 +_R G_2) = m^2\left[2\sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\} + \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|R(G_1)\}\{4\mathrm{d}(u_i)\}\right.\\
&\qquad \left. +2\sum_{r,s=1, r\neq s}^{k} \{\mathrm{d}(w_r, w_s)|R(G_1)\}\right] + n^2\mathrm{GM}(G_2) + 6kn\mathrm{DD}(G_2) + 36k^2 W(G_2) + 4k(m^2 - m)\\
&\qquad + 2lm \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\}\{\mathrm{d}(u_i) + \mathrm{d}(u_j)\} + 2l^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\} + 4lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|R(G_1)\},\\
&\mathrm{GM}(G_1 +_R G_2) = m^2\mathrm{GM}(R(G_1)) + n^2\mathrm{GM}(G_2) + 6nk\mathrm{DD}(G_2) + 36k^2 W(G_2) + 4k(m^2 - m)\\
&\qquad + 2lm \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\}\{\mathrm{d}(u_i) + \mathrm{d}(u_j)\} + 2l^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|R(G_1)\} + 4lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|R(G_1)\},
\end{aligned} \tag{26}$$

□

Theorem 3. *Let G_1 and G_2 be two simple and connected graphs. If $(G_1 +_Q G_2)$ is the Q-sum graph of G_1 and G_2, then*

$$\begin{aligned}
\mathrm{GM}(G_1 +_Q G_2) &= m^2\mathrm{GM}(Q(G_1)) + n^2\mathrm{GM}(G_2) + n(6k - t)\mathrm{DD}(G_2) + (6k - t)^2 W(G_2)\\
&\quad + \frac{1}{2}(m^2 - m)\left[(4k - t)^2 + 16k - 7t\right] + lm \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|Q(G_1)\}(\mathrm{d}(u_i) + \mathrm{d}(u_J))\\
&\quad + 2l^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|Q(G_1)\} + 2lm \sum_{r=1}^{k}\sum_{i=1}^{n} \{\mathrm{d}(w_r, u_i)|Q(G_1)\}\deg(w_r, v_p).
\end{aligned} \tag{27}$$

Proof

Case 1: when both vertices are black, A can be determined similar to that in Case 1 of Theorem 1:

$$\begin{aligned}
A &= \frac{m^2}{2} \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|Q(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\} + lm \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|Q(G_1)\}\{\mathrm{d}(u_i)\mathrm{d}(u_j)\}\\
&\quad + 2l^2 \sum_{i,j=1}^{n} \{\mathrm{d}(u_i, u_j)|Q(G_1)\} + 4k^2 W(G_2) + 2kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2).
\end{aligned} \tag{28}$$

Case 2: when one vertex is white and the other is black,

$$B_1 = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(u_i, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(u_i, v_q\right)|G_1 +_Q G_2\right\}. \tag{29}$$

For $r = 1$ to k, $i = 1$ to n and $p, q = 1$ to m,

$$\begin{aligned}
B_1 &= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(w_r, u_i\right)|S\left(G_1\right) + \mathrm{d}\left(v_p, v_q\right|\left(G_2\right)\right\}\left(\left(\deg\left(w_r, v_p\right)\left\{\mathrm{d}\left(u_i\right) + \mathrm{d}\left(v_q\right)\right\}\right.\right.\\
&= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\mathrm{d}u_i + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\mathrm{d}v_q\\
&\quad + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|\left(G_2\right)\right\}\left(\deg\left(w_r, v_p\right)\mathrm{d}\left(u_i\right) + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|\left(G_2\right)\right\}\left(\deg\left(w_r, v_p\right)\mathrm{d}\left(v_q\right)\right)\right.\\
&= \frac{m^2}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\mathrm{d}\left(u_i\right) + lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\\
&\quad + 2k\left(4k - t\right)W\left(G_2\right) + \frac{1}{2}n\left(4k - t\right)\mathrm{DD}\left(G_2\right).
\end{aligned} \tag{30}$$

The summation of the distances between vertices with different colours is twice the B_1, i.e.,

$$\begin{aligned}
B = {} & m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\mathrm{d}\left(u_i\right) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r, u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r, v_p\right)\\
& + 4k\left(4k - t\right)W\left(G_2\right) + n\left(4k - t\right)\mathrm{DD}\left(G_2\right).
\end{aligned} \tag{31}$$

Case 3: When both vertices are white,

$$C = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)|G_1 +_Q G_2\right\}. \tag{32}$$

This summation consists of three parts $C = C_1 + C_2 + C_3$, where

$$C_1 = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_Q G_2 \colon r = s, p \neq q\right\},$$

$$C_2 = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_Q G_2 \colon r \neq s, p = q\right\},$$

$$C_3 = \frac{1}{2}\sum\left\{\mathrm{d}\left(\left(w_r, v_p\right), \left(w_s, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_Q G_2 \colon r \neq s, p \neq q\right\},$$

$$C_1 = \frac{1}{2}\sum\left\{\left(2 + \mathrm{d}\left(v_p, v_q\right)\right)\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)|G_1 +_Q G_2 \colon r = s, p \neq q\right\},$$

$$C_1 = \sum_{r,s=1,r=s}^{k}\sum_{p,q=1,p\neq q}^{m}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) + \frac{1}{2}\sum_{r,s=1,r=s}^{k}\sum_{p,q=1,p\neq q}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|G_2\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right),$$

$$C_1 = \left(m^2 - m\right)[16(k-t) + 9t] + W(G_2)[16(k-t) + 9t],$$

$$\begin{aligned} C_2 &= \frac{1}{2}\sum_{p,q=1,p=q}^{m}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &= \frac{m}{2}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right), \end{aligned}$$

$$\begin{aligned} C_3 &= \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1,p\neq q}^{m}\left\{1 + d(w_r, w_s)|Q(G_1) + d\left(v_p, v_q\right)|G_2\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &= \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1,p\neq q}^{m}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) + \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1,p\neq q}^{m}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &\quad + \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\sum_{p,q=1,p\neq q}^{m}\left\{\mathrm{d}\left(v_p, v_q\right)|G_2\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &= \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 - (16k-7t)\right] + \frac{1}{2}\left(m^2 - m\right)\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &\quad + \left[(4k-t)^2 - (16k-7t)\right]W(G_2), \end{aligned}$$

$$\begin{aligned} C &= \left(m^2 - m\right)(16k-7t) + W(G_2)(16k-7t) + \frac{m}{2}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right) \\ &\quad + \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 - (16k-7t)\right] + \frac{1}{2}\left(m^2 - m\right)\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right), \end{aligned}$$

$$C = \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 + 16k - 7t\right] + (4k-t)^2 W(G_2) + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}(w_r, w_s)|Q(G_1)\right\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right). \tag{33}$$

Now, Gutman index of $(G_1 +_Q G_2)$ is given by

$$\mathrm{GM}\left(G_1 +_Q G_2\right) = A + B + C, \text{ i.e.,} \tag{34}$$

using values of A, B, and C from Case 1, Case 2, and Case 3, respectively,

$$\begin{aligned}
GM\left(G_1 +_Q G_2\right) = {} & \frac{m^2}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\}\left\{\mathrm{d}\left(u_i\right)\mathrm{d}\left(u_j\right)\right\} + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\}\left\{\mathrm{d}\left(u_i\right)+\mathrm{d}\left(u_j\right)\right\} \\
& + 2l^2\sum_{i,j=1}^{n}\left\{d\left(u_i,u_j\right)|Q\left(G_1\right)\right\} + 4k^2W\left(G_2\right) + 2kn\mathrm{DD}\left(G_2\right) + n^2\mathrm{GM}\left(G_2\right) \\
& + m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r,u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r,v_p\right)\mathrm{d}\left(u_i\right) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r,u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r,v_p\right) \\
& + 4k\left(4k-t\right)W\left(G_2\right) + n\left(4k-t\right)\mathrm{DD}\left(G_2\right) + \frac{1}{2}\left(m^2-m\right)\left[\left(4k-t\right)^2+16k-7t\right] \\
& + \left(4k-t\right)^2W\left(G_2\right) + \frac{m^2}{2}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}\left(w_r,w_s\right)|Q\left(G_1\right)\right\}\left(\deg\left(w_r,v_p\right)\deg\left(w_s,v_q\right)\right), \\
\mathrm{GM}\left(G_1 +_Q G_2\right) = {} & m^2\left[\frac{1}{2}\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\}\mathrm{d}\left(u_i\right)\mathrm{d}\left(u_j\right) + \sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r,u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r,v_p\right)\mathrm{d}\left(u_i\right)\right. \\
& \left. + \frac{1}{2}\sum_{r,s=1,r\neq s}^{k}\left\{\mathrm{d}\left(w_r,w_s\right)|Q\left(G_1\right)\right\}\left(\deg\left(w_r,v_p\right)\deg\left(w_s,v_q\right)\right)\right] + n^2\mathrm{GM}\left(G_2\right) + n\left(6k-t\right)\mathrm{DD}\left(G_2\right) \\
& + \left(W\left(G_2\right)\left[4k^2+4k\left(4k-t\right)+\left(4k-t\right)^2\right] + \frac{1}{2}\left(m^2-m\right)\left[\left(4k-t\right)^2+16k-7t\right]\right. \\
& + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\}\mathrm{d}\left(u_i\right)\mathrm{d}\left(u_j\right) + 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\} \\
& + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r,u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r,v_p\right), \\
\mathrm{GM}\left(G_1 +_Q G_2\right) = {} & m^2\mathrm{GM}\left(Q\left(G_1\right)\right) + n^2\mathrm{GM}\left(G_2\right) + n\left(6k-t\right)\mathrm{DD}\left(G_2\right) + \left(6k-t\right)^2W\left(G_2\right) \\
& + \frac{1}{2}\left(m^2-m\right)\left[\left(4k-t\right)^2+16k-7t\right] + lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\}\left(\mathrm{d}\left(u_i\right)+\mathrm{d}\left(u_j\right)\right) \\
& + 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i,u_j\right)|Q\left(G_1\right)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\left\{\mathrm{d}\left(w_r,u_i\right)|Q\left(G_1\right)\right\}\deg\left(w_r,v_p\right).
\end{aligned} \tag{35}$$

□

Theorem 4. *Let G_1 and G_2 be two simple and connected graphs. If $(G_1 +_T G_2)$ is the T-sum graph of G_1 and G_2, then*

$$\begin{aligned}\mathrm{GM}(G_1 +_T G_2) &= m^2\mathrm{GM}(T(G_1)) + n^2\mathrm{GM}(G_2) + n(8k-t)\mathrm{DD}(G_2) + (8k-t)^2 W(G_2)\\ &+ \frac{1}{2}(m^2-m)\left[(4k-t)^2 + 16k - 7t\right] + 4lm\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|T(G_1)\right\}(\mathrm{d}(u_i))\\ &+ 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|T(G_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|T(G_1)\}\deg(w_r,v_p).\end{aligned} \tag{36}$$

Proof

Case 1: when both vertices are black, A can be calculated similar to that in Case 1 of Theorem 2:

$$\begin{aligned}A &= 2m^2\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|T(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\} + 2lm\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|T(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}(u_j)\right\}\\ &+ 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}(u_i,u_j)|T(G_1)\right\} + 16k^2W(G_2) + 4kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2).\end{aligned} \tag{37}$$

Case 2: when one vertex is white and the other is black,

$$B_1 = \frac{1}{2}\sum\left\{\mathrm{d}((w_r,v_p),(u_i,v_q))(\deg(w_r,v_p)\deg(u_i,v_q)|G_1 +_T G_2\right\}. \tag{38}$$

For $r = 1$ to k, $i = 1$ to n, and $p, q = 1$ to m,

$$\begin{aligned}B_1 &= \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(w_r,u_i)|T(G_1) + \mathrm{d}(v_p,v_q)|(G_2)\right\}\deg(w_r,v_p)\left\{2\mathrm{d}(u_i) + \mathrm{d}(v_q)\right\}\\ &= \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\mathrm{d}(u_i) + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\{d(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\mathrm{d}(v_q)\\ &+ \sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\deg(w_r,v_p)\mathrm{d}(u_i) + \frac{1}{2}\sum_{r=1}^{k}\sum_{i=1}^{n}\sum_{p,q=1}^{m}\left\{\mathrm{d}(v_p,v_q)|(G_2)\right\}\deg(w_r,v_p)\mathrm{d}(v_q)\\ &= m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{d(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\mathrm{d}(u_i) + lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\\ &+ 4k(4k-t)W(G_2) + \frac{1}{2}n(4k-t)DD(G_2).\end{aligned} \tag{39}$$

The summation of the distances between vertices with different colours is twice the B_1, i.e.,

$$\begin{aligned}B &= 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\mathrm{d}(u_i) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|T(G_1)\}\deg(w_r,v_p)\\ &+ 8k(4k-t)W(G_2) + n(4k-t)\mathrm{DD}(G_2).\end{aligned} \tag{40}$$

Case 3: when both vertices are white, C can be calculated similar to that in Case 3 of Theorem 3:

$$C = \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 + 16k - 7t\right] + (4k-t)^2 W(G_2) + \frac{m^2}{2}\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|Q(G_1)\} \cdot \left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right). \tag{41}$$

Now, Gutman index of $(G_1 +_T G_2)$ is given by

$$\mathrm{GM}\left(G_1 +_T G_2\right) = A + B + C, \text{i.e.,} \tag{42}$$

using values of A, B, and C from Case 1, Case 2, and Case 3, respectively,

$$\begin{aligned}
\mathrm{GM}\left(G_1 +_T G_2\right) &= 2m^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} + 2lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\}\left\{\mathrm{d}(u_i)\mathrm{d}\left(u_j\right)\right\} \\
&\quad + 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\} + 16k^2 W(G_2) + 4kn\mathrm{DD}(G_2) + n^2\mathrm{GM}(G_2) \\
&\quad + 2m^2\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r, u_i)|T(G_1)\}\deg\left(w_r, v_p\right)\mathrm{d}(u_i) + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r, u_i)|T(G_1)\}\deg\left(w_r, v_p\right) \\
&\quad + 8k(4k-t)W(G_2) + n(4k-t)\mathrm{DD}(G_2) + \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 + 16k - 7t\right] + (4k-t)^2 W(G_2) \\
&\quad + \frac{m^2}{2}\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|T(G_1)\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right), \\
\mathrm{GM}\left(G_1 +_T G_2\right) &= m^2\left[2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\}(\mathrm{d}(u_i))\mathrm{d}\left(u_j\right) + 2\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r, u_i)|T(G_1)\}\deg\left(w_r, v_p\right)\mathrm{d}u_i\right. \\
&\quad \left. + \frac{1}{2}\sum_{r,s=1, r\neq s}^{k}\{\mathrm{d}(w_r, w_s)|T(G_1)\}\left(\deg\left(w_r, v_p\right)\deg\left(w_s, v_q\right)\right)\right] + n^2\mathrm{GM}(G_2) + n(8k-t)\mathrm{DD}(G_2) \\
&\quad + \left(W(G_2)\left[16k^2 + 8k(4k-t) + (4k-t)^2\right] + \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 + 16k - 7t\right]\right. \\
&\quad + 4lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\}(\mathrm{d}(u_i)) \\
&\quad + 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r, u_i)|T(G_1)\}\deg\left(w_r, v_p\right), \\
\mathrm{GM}\left(G_1 +_T G_2\right) &= m^2\mathrm{GM}(T(G_1)) + n^2\mathrm{GM}(G_2) + n(8k-t)\mathrm{DD}(G_2) + (8k-t)^2 W(G_2) \\
&\quad + \frac{1}{2}\left(m^2 - m\right)\left[(4k-t)^2 + 16k - 7t\right] + 4lm\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\}(\mathrm{d}(u_i)) \\
&\quad + 2l^2\sum_{i,j=1}^{n}\left\{\mathrm{d}\left(u_i, u_j\right)|T(G_1)\right\} + 2lm\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r, u_i)|T(G_1)\}\deg\left(w_r, v_p\right).
\end{aligned} \tag{43}$$

□

4. Discussion and Conclusion

In this section, we apply the main results of Section 3 by taking G_1 equal to, firstly, P_n, and secondly, C_n, while G_2 is set equal to P_m. The Wiener index degree, distance index, and Gutman index of P_n are $W(P_n) = (n(n^2-1)/6)$, $\mathrm{DD}(P_n) = ((n(n-1)(2n-1))/3)$, and $\mathrm{GM}(P_n) = (((n-1)(2n^2-4n+3))/3)$, respectively. Now, we construct Tables 1 and 2.

Then, the following results are found by using Theorems 1–4 and Tables 3 and 4:

$$\begin{aligned}
\mathrm{GM}(P_n + {}_S P_m) &= \frac{2m^2(n-1)\left(8n^2-16n+9\right)}{3} + \frac{n^2(m-1)\left(2m^2-4m+3\right)}{3} \\
&\quad + \frac{4n(n-1)m(m-1)(2m-1)}{3} + \frac{4mn(n-1)(m-1)(2m-1)}{3} + \frac{8(n-1)^2 m\left(m^2-1\right)}{3} + 4m(n-1)(m-1) \\
&\quad + \frac{4mn(m-1)(n-1)(2n-1)}{3} + \frac{8(m-1)^2 n\left(n^2-1\right)}{6} + \frac{4m(m-)n(n-1)(2n-1)}{3}, \\
\mathrm{GM}(P_{n3} + {}_R P_m) &= 6m^2 n\left(n^2-1\right) + \frac{n^2(m-1)\left(2m^2-4m+3\right)}{3} + 2nm(n-1)(m-1)(2m-1) \\
&\quad + 6(n-1)^2 m\left(m^2-1\right) + 4m(n-1)(m-1) + \frac{4mn(m-1)(n-1)(2n-1)}{3} + \frac{2(m-1)^2 n\left(n^2-1\right)}{3} \\
&\quad + \frac{4mn(m-1)\left(n^2-1\right)}{3}, \\
\mathrm{GM}\left(P_n + {}_Q P_m\right) &= 2m^2\left(3n^3-9n^2+10n-5\right) + \frac{n^2(m-1)\left(2m^2-4m+3\right)}{3} + \frac{mn(6n-8)(m-1)(2m-1)}{3} \\
&\quad + \frac{(6n-8)^2 m\left(m^2-1\right)}{6} + \frac{\left(m^2-m\right)\left[(4n-6)^2+16(n-1)-14\right]}{2} + \frac{2m(m-1)\left(2n^3+3n^2-11n+6\right)}{3} \\
&\quad + \frac{2(m-1)^2 n(n+4)(n-1)}{3} + \frac{2m(m-1)\left(4n^3-3n^2-n-6\right)}{3}, \\
\mathrm{GM}(P_n + {}_T P_m) &= \frac{m^2\left(32n^3-96n^2+115n-66\right)}{3} + \frac{n^2(m-1)\left(2m^2-4m+3\right)}{3} + \frac{mn(8n-10)(m-1)(2m-1)}{3} \\
&\quad + \frac{(8n-10)^2 m\left(m^2-1\right)}{6} + \frac{\left(m^2-m\right)\left[(4n-6)^2+16(n-1)-14\right]}{2} + \frac{4mn(m-1)(n-1)(2n-1)}{3} \\
&\quad + \frac{2(m-1)^2 n\left(n^2-1\right)}{3} + \frac{2m(m-1)\left(4n^3-3n^2-n-6\right)}{3}.
\end{aligned} \tag{44}$$

Table 1: Wiener index, degree distance, and Gutman Index of $F(P_n)$.

F	$W(F(P_n))$	$\mathrm{DD}(F(P_n))$	$\mathrm{GM}(F(P_n))$
S	$((2n(n-1)(2n-1))/3)$	$((2(n-1)(2n-1)(4n-3))/3)$	$((2(n-1)(8n^2-16n+9))/3)$
R	$(((n-1)(2n^2+2n-3))/3)$	$4(n+1)(n-1)^2$	$6n(n-1)^2$
Q	$((2n(n-1)(n+1))/3)$	$4n^3-6n^2+2n-2$	$2(3n^3-9n^2+10n-5)$
T	$((n(n-1)(4n+1))/6)$	$(2/3)(8n^3-15n^2+10n-6)$	$((32n^3-96n^2+115n-66)/3)$

Table 2: $\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|F(P_n)\}$, $\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|F(P_n)\}$, and $\sum_{i,j=1}^{N}\{\mathrm{d}(u_i,u_j)|F(P_n)\}(\mathrm{d}u_i+\mathrm{d}u_j)$.

F	$\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)\|F(P_n)\}$	$\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)\|F(P_n)\}$	$\sum_{i,j=1}^{N}\{\mathrm{d}(u_i,u_j)\|F(P_n)\}(\mathrm{d}u_i+\mathrm{d}u_j)$
S	$((2n(n-1)(n+1))/3)$	$((n(n-1)(2n-1))/3)$	$((4n(n-1)(2n-1))/3)$
R	$((n(n^2-1))/3)$	$(n(n^2-1)/3)$	$((2n(n-1)(2n-1))/3)$
Q	$((n(n-1)(n+4))/3)$	$((n(n-1)(n+1))/3)$	$((2(2n^3+3n^2-11n+6))/3)$
T	$((n(n-1)(n+1))/3)$	$((n(n-1)(n+1))/3)$	$((2n(n-1)(2n-1))/3)$

Table 3: Wiener index and degree distance of $F(C_n)$.

F	$W(F(C_n))$	$\mathrm{DD}(F(C_n))$	$\mathrm{GM}(F(C_n))$
S	n^3	$4n^3$	$4n^3$
R	$(n(n^2+2n-1)/2)$	$n(n+2)(3n-1)$	$\begin{cases}((n(9n^2+12n-4))/2) & \text{if } n \text{ is even}\\ ((n(9n^2+12n-5))/2) & \text{if } n \text{ is odd}\end{cases}$
Q	$(n(n^2+2n-1)/2)$	$n(n+2)(3n-1)$	$\begin{cases}((n(9n^2+12n-4))/2) & \text{if } n \text{ is even}\\ ((n(9n^2+12n-5))/2) & \text{if } n \text{ is odd}\end{cases}$
T	$((n^2(n+1))/2)$	$4n^2(n+1)$	$8n^2(n+1)$

Table 4: $\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)|F(P_n)\}$ and $\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)|F(P_n)\}$ of $F(C_n)$.

F	$\sum_{i,j=1}^{n}\{\mathrm{d}(u_i,u_j)\|F(C_n)\}$	$\sum_{r=1}^{k}\sum_{i=1}^{n}\{\mathrm{d}(w_r,u_i)\|F(C_n)\}$
S	$\begin{cases}(n^3/2) & \text{if } n \text{ is even}\\ ((n(n^2-1))/2) & \text{if } n \text{ is odd}\end{cases}$	$\begin{cases}(n^3/2) & \text{if } n \text{ is even}\\ ((n(n^2+1))/2) & \text{if } n \text{ is odd}\end{cases}$
R	$\begin{cases}(n^3/4) & \text{if } n \text{ is even}\\ ((n(n^2-1))/4) & \text{if } n \text{ is odd}\end{cases}$	$\begin{cases}((n^2(n+2))/4) & \text{if } n \text{ is even}\\ ((n(n+1)^2)/4) & \text{if } n \text{ is odd}\end{cases}$
Q	$\begin{cases}((n(n^2+4n-4))/4) & \text{if } n \text{ is even}\\ ((n(n^2+4n-5))/4) & \text{if } n \text{ is odd}\end{cases}$	$\begin{cases}((n^2(n+2))/4) & \text{if } n \text{ is even}\\ ((n(n+1)^2)/4) & \text{if } n \text{ is odd}\end{cases}$
T	$\begin{cases}(n^3/4) & \text{if } n \text{ is even}\\ ((n(n^2-1))/4) & \text{if } n \text{ is odd}\end{cases}$	$\begin{cases}((n^2(n+2))/4) & \text{if } n \text{ is even}\\ ((n(n+1)^2)/4) & \text{if } n \text{ is odd}\end{cases}$

If C_n is cycle of n vertices, then the Wiener index, degree distance index, and Gutman index of cycle are as follows:

$$W(C_n)=\begin{cases}\dfrac{n^3}{8}, & \text{if } n \text{ is even},\\[2ex] \dfrac{n(n^2-1)}{8}, & \text{if } n \text{ is odd},\end{cases}$$
$$\mathrm{DD}(C_n)=\mathrm{GM}(C_n)=\begin{cases}\dfrac{n^3}{2}, & \text{if } n \text{ is even},\\[2ex] \dfrac{n(n^2-1)}{2}, & \text{if } n \text{ is odd}.\end{cases} \tag{45}$$

Table 5: Comparision between exact, computed, and bounded values.

No.	F-sum graph	Exact value	Computed value	Bounded value
1	$GM(P_3 +_S P_2)$	505	505	1024
2	$GM(P_3 +_R P_2)$	689	689	2002
3	$GM(P_3 +_Q P_2)$	619	619	1393
4	$GM(P_3 +_T P_2)$	907	907	2473
5	$GM(C_3 +_S P_2)$	921	921	2832
6	$GM(C_3 +_R P_2)$	1341	1341	3614
7	$GM(C_3 +_Q P_2)$	1617	1617	4052
8	$GM(C_3 +_T P_2)$	2373	2373	6032

It is very essential to know that, in cycle, $du_i = 2$ and

$$\deg(w_r, v_p) = \begin{cases} 2 & \text{if } F = S \text{ or } R, \\ 4 & \text{if } F = Q \text{ or } T. \end{cases} \tag{46}$$

Then, the following results are found by using Theorems 1–4 and Tables 3 and 4:

$$\begin{aligned}
GM(C_n +_S P_m) &= \frac{n^2(m-1)(2m^2-4m+3)}{3} + \frac{4n^2m(m-1)(2m-1)}{3} + \frac{8n^2m(m^2-1)}{3} \\
&\quad + 4m(n-1)(m-1) + 4m^2n^3 + (m-1)\begin{cases} n^3(5m-1), & \text{if } n \text{ is even}, \\ n^3(5m-1) - n(m-1) & \text{if } n \text{ is odd}, \end{cases} \\
GM(C_n +_R P_m) &= \frac{n^2(m-1)(2m^2-4m+3)}{3} + 2n^2m(m-1)(2m-1)) + 6n^2m(m^2-1) \\
&\quad + 4m(n-1)(m-1) + \frac{m^2n(9n^2+12n-4)}{2} + (m-1)n^2\left[\frac{n(7m-1)}{2} + 2m\right] + \begin{cases} 0 & \text{if } n \text{ is even}, \\ -\dfrac{n(2m-1)^2}{2} & \text{if } n \text{ is odd}, \end{cases} \\
GM(C_n +_Q P_n) &= \frac{n^2(m-1)(2m^2-4m+3)}{3} + 2n^2m(m-1)(2m-1) + 6n^2m(m^2-1) + 8n(n+1)(m^2-m) \\
&\quad + \frac{m^2n(9n^2+12n-4)}{2} + n(m-1)\left[\frac{n^2(7m-1)}{2} + 2n(5m-1) - 6m + 2\right] + \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{-n}{2} & \text{if } n \text{ is odd}, \end{cases} \\
GM(C_n +_T P_m) &= \frac{n^2(m-1)(2m^2-4m+3)}{3} + \frac{8n^2m(m-1)(2m-1)}{3} + \frac{32n^2m(m^2-1)}{3} + +8n(n+1)(m^2-m) \\
&\quad + 8m^2n^2(n+1) + n(m-1)\left[\frac{n^2(9m-1)}{2} + 4mn\right] + \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{-n(m-1)^2}{2} & \text{if } n \text{ is odd}. \end{cases}
\end{aligned} \tag{47}$$

After deriving formulae, a comparison among the exact values, computed values, and bounded values is also drawn in Table 5.

Now, we close our discussion with the comments that the upper bounds for the Gutman indices on the F-sum (S-sum, R-sum, Q-sum, and T-sum) graphs are obtained in

[20]. In this paper, we considered for the improvement of the already existing bounds and determined the exact values of the Gutman indices for the F-sum graphs. However, the problem is still open to find the exact values of other distance-based TIs for these sum graphs.

Data Availability

All data used to support the findings of the study are included within the article and can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

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