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Research Article

On Cyclic-Vertex Connectivity of (n, k)-Star Graphs

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A vertex subset $F \subseteq V(G)$ is a cyclic vertex-cut of a connected graph G if $G - F$ is disconnected and at least two of its components contain cycles. The cyclic vertex-connectivity $\kappa_c(G)$ is denoted as the cardinality of a minimum cyclic vertex-cut. In this paper, we show that the cyclic vertex-connectivity of the (n, k)-star network $S_{n,k}$ is $\kappa_c(S_{n,k}) = n + 2k - 5$ for any integer $n \geq 4$ and $k \geq 2$.

1. Introduction

Let $G = (V, E)$ be a simple connected graph, where V and E are the vertex set and the edge set, respectively. $G[H]$ is an induced subgraph by $H \subseteq V$, whose vertex set is H and whose edge set consists of all the edges of G with both ends in H. For any vertex v, define the neighborhood $N_G(v) = \{u \in V | (u, v) \in E\}$. Let $S \subseteq V(G)$ and the set $\cup_{v \in S} N_G(v) \setminus S$ is denoted by $N_G(S)$. We use $N(v)$ to replace $N_G(v)$, $N(S)$ to replace $N_G(S)$, and $N[S]$ to replace $N_G[S]$. A graph G is said to be k-regular if $d(v) = k$ for any vertex $v \in V$. For any subset $F \subseteq V(G)$, $G \setminus F$ or $G - F$ denotes the graph obtained by removing all vertices in F from G. If there exists a nonempty subset $F \subseteq G$ such that $G \setminus F$ is disconnected, then F is called a vertex-cut of G. The connectivity $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. Let $\delta(G)$ and $g(G)$ denote the minimum degree and the girth of G, respectively. As usual, we use K_n and C_n to denote the complete graph and the cycle of order n, respectively.

In this work, we study a kind of restricted vertex-connectivity known as the cyclic vertex-connectivity. A vertex subset $F \subset V$ is a cyclic vertex-cut of G if $G - F$ has at least two components containing cycles. Not all connected graphs have a cyclic vertex-cut. The cyclic vertex-connectivity $\kappa_c(G)$ of a graph G is the cardinality of the minimum cyclic vertex-cut of G. When G has no cyclic vertex-cut, the definition of $\kappa_c(G)$ can be found in [1] using Betti number. A graph G is said to be κ_c-connected if G has a cyclic vertex-cut. Similarly, changing "edge" to "vertex," the cyclic edge-connectivity $\lambda_c(G)$ of graph G can be defined.

The definition of the cyclic vertex- (edge-) connectivity dates to Tait in attacking four color conjecture [2] and the graph colouring [2, 3]. It is used in many classic fields, such as integer flow conjectures [4] and n-extendable graphs [5, 6]. In many works, the cyclic vertex-connectivity has been studied. Cheng et al. [7] studied the cyclic vertex-connectivity of Cayley graphs generated by transposition trees. Yu et al. [8] obtained the cyclic vertex-connectivity of star graphs. For more research studies on the cyclic vertex-connectivity, see [7, 9–11] for references.

This paper focuses on the cyclic vertex-connectivity of the (n, k)-star network $S_{n,k}$. We will show that $\kappa_c(S_{n,k}) = n + 2k - 5$ for any integer $n \geq 4, k \geq 2$ and find out the minimum circle vertex-cut structure of the (n, k)-star network $S_{n,k}$.

2. Some Preliminaries

We provide the definition of the (n, k)-star graph $S_{n,k}$ and its structural properties, which are useful for the following discussion.

For convenience, let $\langle n \rangle = \{1, 2, \ldots, n\}$ and $V(n, k) = \{q_1 q_2 \cdots q_k\colon\ q_i \in \langle n \rangle, q_i \neq q_j, 1 \leq i \neq j \leq k\}$ for any integers n and k with $1 \leq k \leq n$. Clearly, $|V(n, k)| = n!/(n, k)!$.

Definition 1 (see [12]). The (n,k)-star graph, denoted by $S_{n,k}$ (see Figure 1), is a graph with the vertex-set $V(n,k)$ and the edge set defined as follows:

(1) A vertex $q_1q_2\cdots q_i\cdots q_k$ is adjacent to the vertex $q_iq_2\cdots q_1\cdots q_k$ through an edge of dimension i, where $2\le i\le k$ (i.e., exchange q_1 with q_i)

(2) A vertex $q_1q_2\cdots q_i\cdots q_k$ is adjacent to the vertex $yq_2\cdots q_i\cdots q_k$ through an edge of dimension 1, where $y\in\langle n\rangle\setminus\{q_1,q_2,\ldots,q_k\}$ (i.e., replace q_1 by y)

The edges of type (1) are referred to as i-edges, and the corresponding neighboring vertices are called i-neighbors. The edges of type (2) are referred to as 1-edges, and the corresponding neighboring vertices are called 1-neighbors. Let $S^i_{n,k}$ be induced by all the vertices having the symbol i in one of the rightmost $k-1$ positions of $S_{n,k}$. Clearly, $S_{n,k}$ can be decomposed into n subgraphs $S^i_{n,k}$ and $S^i_{n,k}\cong S_{n-1,k-1}$, where $1\le i\le n$ and $2\le k\le n$.

Lemma 1 (see [13]). *The (n,k)-star graph $S_{n,k}$ has the following properties:*

(1) $S_{n,k}$ is a graph of degree $n-1$ with $n!/(n-k)!$ vertices and $(n-1)n!/2(n-k)!$ edges.

(2) $S_{n,1}\cong K_n$, $S_{n,n-1}\cong S_n$, and $S_{n,n-2}\cong AN_n$, where K_n is a complete graph, S_n is a n-dimensional star graph, and AN_n is a n-dimensional alternating group network.

(3) $E(i,j)$ is the set of all cross edges between any two subgraphs $S^i_{n,k}$ and $S^j_{n,k}$ $(i\ne j\in\langle n\rangle)$, and $|E(i,j)|=((n-2)!/(n-k)!)$.

(4) For any two vertices u and v in $S_{n,k}$,

$$|N(u)\cap N(v)|=\begin{cases} n-k-1, & \text{if } (u,v)\in E\left(S_{n,k}\right) \text{ is } 1-\text{edge},\\ 1, & \text{if } (u,v)\notin E\left(S_{n,k}\right) \text{ and } N(u)\cap N(v)\ne\varnothing,\\ 0, & \text{otherwise}.\end{cases}\tag{1}$$

Lemma 2 (see [13]). *$S_{n,k}$ is a $(n-1)$-regular $(n-1)$-connected graph.*

Theorem 1 (see [14]). *Let F be a faulty vertex set of $S_{n,k}$ $(n\ge5,k\ge3,n-k\ge2)$ with $|F|\le n+2k-6$. Then, the survival graph $S_{n,k}\backslash F$ satisfies one of the following conditions:*

(1) $S_{n,k}\backslash F$ is connected

(2) $S_{n,k}\backslash F$ has two components, one of which has exactly one vertex or two vertices with one 1-edge

(3) $S_{n,k}\backslash F$ has three components, two of which are singletons

Lemma 3 (see [13]). *If a cycle has a length at least 6 in an $S_{n,k}$, then it contains one i-edge, $2\le i\le k$.*

Theorem 2 (see [8]). *For any integer $n\ge4$, $\kappa_c(S_n)=6(n-3)$.*

3. Main Result

By Lemma 1 and Theorem 2, we know $S_{n,n-1}\cong S_n$ if $k=n-1$ and $\kappa_c(S_n)=6(n-3)$. Thus, we determine the value of $\kappa_c(S_{n,k})$ with $n-k\ge2$ for $n\ge4,k\ge2$.

Lemma 4. *For $n\ge4$, the girth of $S_{n,2}$ is 3 and the edges of every 3-cycle are 1-edges.*

Proof. Choose any vertex u_1 from $S_{n,2}$ and make it as $u_1=q_1q_2$. Since $n\ge4$, there exist $q_i,q_j\in\langle n\rangle\backslash\{q_1,q_2\}$ and $q_i\ne q_j$. From the definition of $S_{n,2}$, we have two vertices $u_2=q_iq_2$ and $u_3=q_jq_2$ in $V(S_{n,2})$ and $\{u_1u_2,u_2u_3,u_3u_1\}\in E(S_{n,2})$. Clearly, $u_1u_2u_3u_1$ is one 3-cycle, and all the edges of it are 1-edges. Hence, the lemma holds. □

Lemma 5. *Let C be any cycle of length 3 in $S_{n,2}$ $(n\ge4)$. Then, $|N_{S_{n,2}}(C)|=n-1$.*

Proof. From Lemma 4, we can suppose $C=(q_1q_2)(q_3q_2)(q_4q_2)(q_1q_2)$. Let $S=\{q_5q_2,q_6q_2,\ldots,q_nq_2\}\subseteq V(S_{n,2})$, then $|S|=n-4$. By the definition of $S_{n,2}$, we have

$$\begin{aligned} N_{S_{n,2}}(q_1q_2)\backslash\{q_3q_2,q_4q_2\}&=S\cup\{q_2q_1\},\\ N_{S_{n,2}}(q_3q_2)\backslash\{q_1q_2,q_4q_2\}&=S\cup\{q_2q_3\},\\ N_{S_{n,2}}(q_4q_2)\backslash\{q_1q_2,q_3q_2\}&=S\cup\{q_2q_4\}.\end{aligned}\tag{2}$$

So,

$$\left|N_{S_{n,2}}(C)\right|=\left|S\cup\{q_2q_1,q_2q_3,q_2q_4\}\right|=n-4+3=n-1.\tag{3}$$

□

Lemma 6. *Let C be a 3-cycle of $S_{n,2}$ $(n\ge4)$. Then, $N_{S_{n,2}}(C)$ is a cyclic vertex-cut of $S_{n,2}$.*

Proof. Clearly, $S_{n,2}-N_{S_{n,2}}(C)$ is disconnected and contains C as a connected component. In order to prove the lemma, it suffices to show that the subgraph $H=S_{n,2}-N_{S_{n,2}}[C]$ has a cycle. In fact, we can prove a stronger property $\delta(H)\ge2$ as follows.

Suppose $C=(q_1q_2)(q_3q_2)(q_4q_2)(q_1q_2)$. By Lemma 5, $N_{S_{n,2}}(C)=S\cup\{q_2q_1,q_2q_3,q_2q_4\}$ and $S=\{q_5q_2,q_6q_2,\ldots,q_nq_2\}$. If $\delta(H)\le1$, and then there exists a vertex

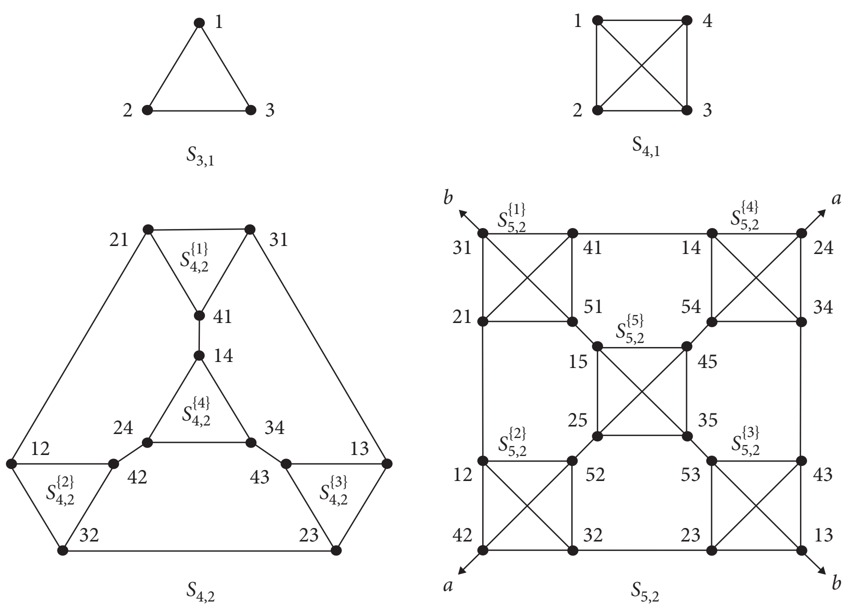

Figure 1: Some examples of the *(n, k)*-star graph $S_{3,1}$, $S_{4,1}$, $S_{4,2}$, and $S_{5,2}$

$v \in V(H)$ satisfying $N_H(v) = 1$ and $v \notin S \cup \{q_2q_1, q_2q_3, q_2q_4\}$. Since $S_{n,2}$ is $n-1$-regular, $v \in V(H)$ has at least $n - 2\ (\geq 2)$ neighbors in $N_{S_{n,2}}(C)$. Let v_1 and v_2 be two distinct vertices in $N_{S_{n,2}}(v) \cap N_{S_{n,2}}(C) \subseteq S \cup \{q_2q_1, q_2q_3, q_2q_4\}$.

If $v_1, v_2 \in S$, without loss of generality, let $v_1 = q_iq_2$, $v_2 = q_jq_2$, where $q_i, q_j \in \{q_5, q_6, \ldots, q_n\}$ and $q_i \neq q_j$. By Definition 1, all the edges in $S_{n,2}$ are i-edges or 1-edges. Since $vv_1, vv_2 \in E(S_{n,2})$, $v = q_sq_2$, and $q_s \in \{q_5, q_6, \ldots, q_n\} \setminus \{q_i, q_j\}$. It means $v \in S$, contradicting $v \notin S \cup \{q_2q_1, q_2q_3, q_2q_4\}$.

If $v_1 \in S$ and $v_2 \in \{q_2q_1, q_2q_3, q_2q_4\}$, without loss of generality, then let $v_1 = q_iq_2$ and $v_2 = q_2q_j$, where $q_i \in \{q_5, q_6, \ldots, q_n\}$ and $q_j \in \{q_1, q_3, q_4\}$. Since $vv_1, vv_2 \in E(S_{n,2})$, $v = q_jq_2$, and $q_j \in \{q_1, q_3, q_4\}$, it means $v \in C$, contradicting $v \in H = S_{n,2} - N_{S_{n,2}}[C]$.

If $v_1, v_2 \in \{q_2q_1, q_2q_3, q_2q_4\}$, clearly, if $vv_1 \in E(S_{n,2})$, then $vv_2 \notin E(S_{n,2})$. If $vv_2 \in E(S_{n,2})$, then $vv_1 \notin E(S_{n,2})$, similarly, contradicting $vv_1, vv_2 \in E(S_{n,2})$.

From the above discussion, we know that $\delta(H) \geq 2$. Furthermore, $N_{S_{n,2}}(C)$ is a cyclic vertex-cut of $S_{n,2}$.

Combining Lemma 5 and 6, we have the following theorem. □

Theorem 3. *For any integer* $n \geq 4$, $\kappa_c(S_{n,2}) = n - 1$.

Proof. Let C be a 3-cycle in $S_{n,2}$ and $F = N_{S_{n,2}}(C)$. By Lemmas 5 and 6, F is a cyclic vertex-cut of $S_{n,2}$. Hence, $\kappa_c(S_{n,2}) \leq |F| = n - 1$. By Lemma 2, $\kappa(S_{n,2}) = n - 1$. We have $\kappa_c(S_{n,2}) \geq \kappa(S_{n,2}) = n - 1$, and then $\kappa_c(S_{n,2}) = n - 1$. □

Theorem 4. *For any integer* $n \geq 5, k \geq 3, n - k \geq 2$, *and* $\kappa_c(S_{n,k}) = n + 2k - 5$.

Proof. Let F be a faulty vertex set of $S_{n,k}$ with $|F| \leq n + 2k - 6$. By Theorem 1, $S_{n,k} - F$ is connected or $S_{n,k} - F$ is disconnected, and at most one of its component contains cycles. Hence, $\kappa_c(S_{n,k}) \geq n + 2k - 5$. To prove the converse, we need to find a cyclic vertex-cut F of $S_{n,k}$ with $|F| = n + 2k - 5$.

Suppose $v = q_1q_2 \cdots q_k \in V(S_{n,k})$, we have two vertices u, w such that vu and vw are 1-edges by $n - k \geq 2$. Without loss of generality, we can assume $u = q_{k+1}q_2 \cdots q_k$ and $w = q_{k+2}q_2 \cdots q_k$. From the definition of $S_{n,k}$, we have $uw \in E(S_{n,k})$. Hence, $C_3 = uvwu$ in $S_{n,k}$. Let $A = \{q_jq_2 \cdots q_k \mid j = k+3, k+4, \ldots, n\}$, then $A = N_{S_{n,k}}(u) \cap N_{S_{n,k}}(v) \cap N_{S_{n,k}}(w)$ and $|A| = n - k - 2$. Since u, v, w have another $k - 1$ i-neighbors in $S_{n,k} \setminus A$, respectively, $|N_{S_{n,k}}(C_3)| = |A| + 3(k-1) = n - k - 2 + 3(k-1) = n + 2k - 5$. Clearly, $S_{n,k} - N_{S_{n,k}}(C_3)$ is disconnected and contains C_3 as a connected component. In order to find a cyclic vertex-cut, it suffices to show that $S_{n,k} - N_{S_{n,k}}[C_3]$ has a cycle. In fact, we can prove $\delta(S_{n,k} - N_{S_{n,2}}[C_3]) \geq 2$. Suppose that there exists one vertex x of $S_{n,k} - N_{S_{n,k}}[C_3]$ with $d_{S_{n,k} - N_{S_{n,k}}[C_3]}(x) = 1$. Since $S_{n,k}$ is $n-1$-regular $(n - 1 \geq 4)$, $|N_{S_{n,k}}(x) \cap N_{S_{n,k}}(C_3)| \geq 3$.

If x is adjacent to one neighbor vertex of u, v, and w, respectively, it means there are three vertices u', v', and w' such that $u' \in N_{S_{n,k}-C_3}(u)$, $v' \in N_{S_{n,k}-C_3}(v)$, $w' \in N_{S_{n,k}-C_3}(w)$, and $\{u', v', w'\} \subseteq N_{S_{n,k}}(x) \cap N_{S_{n,k}}(C_3)$. Then, all of uu', vv', and ww' are i-edges. Let $u' = q_sq_2 \cdots q_{k+1} \cdots q_k$, $v' = q_tq_2 \cdots q_1 \cdots q_k$, $w' = q_mq_2 \cdots q_{k+2} \cdots q_k$ and $s, t, m \in \{2, 3, \ldots, k\}$. By the definition of $S_{n,k}$, we know x is adjacent to at most one of u', v', and w', a contradiction.

If x is adjacent to two neighbor vertices of u and one neighbor vertex of v, it means there are three vertices u_1, u_2, and v_1 such that $u_1, u_2 \in N_{S_{n,k}-C_3}(u)$, $v_1 \in N_{S_{n,k}-C_3}(v)$, and $\{u_1, u_2, v_1\} \subseteq N_{S_{n,k}}(x) \cap N_{S_{n,k}}(C_3)$. From the definition of $S_{n,k}$, both uu_1 and uu_2 are 1-edges and $x = q_tq_2 \cdots q_{k+1} \cdots q_k$, $t \in \{k+3, k+4, \ldots, n\}$. Furthermore, $x \in N(C_3)$, a contradiction with $x \in S_{n,k} - N_{S_{n,k}}[C_3]$.

If x is adjacent to three neighbor vertices of u, it means there are three vertices u_1, u_2, and u_3 such that $u_1, u_2, u_3 \in N_{S_{n,k}-C_3}(u)$ and $\{u_1, u_2, u_3\} \subseteq N_{S_{n,k}}(x) \cap N_{S_{n,k}}(C_3)$.

From the definition of $S_{n,k}$, all of uu_1, uu_2, and uu_3 are 1-edges and $x = q_t q_2 \cdots q_{k+1} \cdots q_k$ and $t \in \{k+3, k+4, \ldots, n\}$. Furthermore, $x \in N(C_3)$, a contradiction with $x \in S_{n,k} - N_{S_{n,k}}[C_3]$.

From the above discussion, we know that $\delta(S_{n,k} - N_{S_{n,2}}[C_3]) \geq 2$. Then, $S_{n,k} - N_{S_{n,k}}[C_3]$ contains a cycle, and $N_{S_{n,k}}(C_3)$ is a cyclic vertex-cut of $S_{n,k}$ with $|N_{S_{n,k}}(C_3)| = n + 2k - 5$. Hence, $\kappa_c(S_{n,k}) \geq n + 2k - 5$. The theorem holds.

Combining Theorems 3 and 4, we have the following theorem. □

Theorem 5. *For any integer* $n \geq 4, k \geq 2$, *and* $\kappa_c(S_{n,k}) = n + 2k - 5$.

4. Conclusion

In this paper, we determine the cyclic vertex-connectivity of the (n, k)-star network $S_{n,k}$. We can consider the cyclic vertex-connectivity of other graphs and the cyclic edge-connectivity of the (n, k)-star network $S_{n,k}$ in our future research.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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