Research Article
A New General Decay Rate of Wave Equation with Memory-Type Boundary Control

Sheng Fan

Department of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Correspondence should be addressed to Sheng Fan; fansheng@swufe.edu.cn

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Of interest is a wave equation with memory-type boundary oscillations, in which the forced oscillations of the rod is given by a memory term at the boundary. We establish a new general decay rate to the system. And it possesses the character of damped oscillations and tends to a finite value for a large time. By assuming the resolvent kernel that is more general than those in previous papers, we establish a more general energy decay result. Hence the result improves earlier results in the literature.

1. Introduction

It is well-known that if we add a damping to a system, the amplitude of the oscillations can be reduced very fast. The memory term can be as a damping (viscoelastic damping) which is weaker than frictional damping. For viscoelastic materials, Boltzmann theory gives us that the stress-strain viscoelastic law depending on a relaxation measure, see Prüss [1] and Eden et al. [2]. Based on the Boltzmann principle, the viscoelastic stress-strain relations can be generally given by a convolution term, which can be regarded as a lower order perturbation and can also be regarded as a kind of memory effect, for instance, \( g(t) u \). And we call the function \( g(t) \) memory kernel. One can find a detail derivation on some systems with memory in [3].

To motivate our work, we start with some known results on wave equation with memory-type oscillations. The general wave equation with viscoelastic term in the internal feedback

\[
    u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = \mathcal{F}(u).
\]  

Messaoudi and Messaoudi [4, 5] studied \( \mathcal{F}(u) = 0 \) and \( \mathcal{F}(u) = |u|^\mu u \), by introducing the assumption \( g' (t) \leq -\xi(t) g(t) \), and obtained the energy decays exponentially (polynomially) as \( g \) decays exponentially (polynomially), respectively.

Lasiecka et al. [6] considered the general assumption on \( g \): \( g' (t) \leq -H (g(t)) \) to establish general decay of energy. Here \( H \), which was introduced by Alabau-Boussouira and Cannarsa [7], is strictly convex and increasing function. Cavalcanti et al. [8, 9], Lasiecka and Wang [10], Mustafa and Messaoudi [11], and Xiao and Liang [12] also used this
assumption to obtain some general decay rates of corresponding models. In recent papers [13–15], the authors investigated three classes of viscoelastic wave equation as in [4, 5] and established optimal and explicit decay results of energy by adopting the assumption on \( g \):
\[
 g'(t) \leq -\xi(t)H(g(t)).
\]

In this paper, we considered the following wave equation with boundary oscillations of memory type:
\[
\begin{cases}
 u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\
 u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\
 u + \int_0^t g(t - s) \frac{\partial u}{\partial n}(s)ds = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\
 u(x, 0) = u_0(x), \\
 u_t(x, 0) = u_t(x), & x \in \Omega,
\end{cases}
\]
where \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with smooth boundary \( \Gamma \), \( \Gamma = \Gamma_0 \cup \Gamma_1 \), and \( \Gamma_0 \) and \( \Gamma_1 \) are closed and disjoint with measure \( (\Gamma_0) > 0 \). \( n \) is the unit outward normal to \( \Gamma \).

For wave equation with memory-type boundary oscillations, it can be regarded as a wave equation with viscoelastic damping at the boundary. Santos [16] considered a one-dimensional wave equation with memory conditions at the boundary, respectively. He proved that the energy of solutions decays exponentially (polynomially) as \( k \) and \( k' \) decay exponentially (polynomially). Here \( k \) is the resolvent kernel of \( -(g' / g(0)) \). Santos et al. [17] extended the results in [16] to an n-dimensional wave equation of Kirchhoff type with memory-type boundary. They proved the global existence of solutions and obtained that the energy of solution decays uniformly with the same rate of decay \( k \) under the same conditions on \( k \) and \( k' \), which improves the results in [18] by Park et al. Santos and Junior [19] obtained a similar result for plate equation with memory-boundary type. We also mention the work of Cavalcanti et al. [20], where the authors showed the global existence and the uniform decay of solutions to a semiwave equation with memory-type boundary condition and a nonlinear boundary source. Messaoudi and Soufyane [21] considered a general assumption on \( k' \): \( k'' \geq -\xi(t)k'(t) \) and established a general decay result. Wu [22] used this assumption to study a wave Kirchhoff-type wave equation with a boundary control of memory type. For nonlinear wave equations with memory-type boundary condition, we refer to Cavalcanti and Guesmia [23], Feng [24], Feng et al. [25–27], Muñoz Rivera and Andrade [28], and Zhang [29].

Concerning the system (2), Mustafa [30], by assuming the function \( k \): \( k'' \geq -\xi(t)H(k'(t)) \), where \( k \) is the resolvent kernel of \( -(g' / g(0)) \), established a general decay of solutions of the form
\[
 E(t) \leq k_3 H_1^{-1}(k_1 t + k_2), \quad \forall t \geq 0.
\]

Here
\[
 H_1(t) = \int_0^1 \frac{1}{sH(0)(s)}ds, \quad H_0(t) = D(H(t)),
\]
and \( D \) is a positive \( C^1 \) function with \( D(0) = 0 \), and \( H_0 \) is strictly increasing and strictly convex \( C^1 \) function on \( (0, r] \). In particular, for \( H(t) = t^p \), i.e., \( k'' \geq c(-k')^p \), the author proved the energy decay holds for \( 1 \leq p < (3/2) \). Whether the range be extended to a more larger range? In this paper, we give a positive answer to study problem (2) and extend the result to get a more general decay rate. In particular, we obtain that the energy result holds for \( H(t) = t^p \) with the full admissible range \( 1 \leq p < 2 \). More exactly, by assuming the relaxation function \( k \) with minimal conditions on \( L^1_{\text{loc}}(0, \infty) \), i.e., \( k''(t) \geq \eta(t)H(-k'(t)) \), where \( H \) is linear or strictly increasing and strictly convex functions of class \( C^2(\mathbb{R}^+) \), we establish an optimal explicit and general energy decay result. In particular, the energy result holds for \( H(t) = t^p \) with the range \( p \in [1, 2] \) instead of \( p \in [1, (3/2)) \) in [30]. Hence our results extend and improve the stability results in [30] and also in [16–18, 21]. We mainly adopt the idea of [14, 15, 31] and some properties of convex function developed in [7, 32].

The remaining of the paper is organized as follows: In Section 2, we propose some preliminaries. In Section 3, main results are given. Section 4 is devoted to proving the general decay result.

2. Preliminaries

Taking the derivative of (2) with respect to \( t \), we shall see that
\[
 \frac{\partial u}{\partial t} = -\frac{1}{g(0)} \left[ u_t + g' \frac{\partial u}{\partial n} \right].
\]

We denote the resolvent kernel of \( -(g' / g(0)) \) by \( k \) satisfying for \( t \geq 0 \):
\[
 k(t) + \frac{1}{g(0)}(g' k)(t) = -\frac{1}{g(0)}g'(t).
\]

Using Volterra’s inverse operator and taking \( \alpha = (1/g(0)) \), we have
\[
 \frac{\partial u}{\partial t} = -\alpha \left[ u_t + \frac{k_2 u_t}{k_1} \right].
\]

Assume \( u_0 = 0 \) on \( \Gamma_1 \) in this paper, we get
\[
 \frac{\partial u}{\partial t} = -\alpha \left[ u_t + k(0) + k u_t \right], \quad \text{on } \Gamma_1 \times \mathbb{R}^+.
\]

In the following, we use boundary conditions (8) instead of (2).

As in [30], we consider the following assumption:
(A1) There exists a fixed point $x_0 \in \mathbb{R}^2$ and some constant $\delta_0 > 0$ such that for $m(x) = x - x_0$, 
\begin{align*}
\Gamma_0 &= \{ x \in \Gamma : m(x) \cdot \nu(x) \leq 0 \}, \\
\Gamma_1 &= \{ x \in \Gamma : m(x) \cdot \nu(x) \geq \delta_0 \}.
\end{align*}

For the kernel $k$, we assume

(A2) $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nonincreasing and twice differentiable function satisfying for any $t \geq 0$, 
\begin{align*}
k(0) &> 0, \\
k'(t) &\leq 0.
\end{align*} 

(A3) There exist a $C^1$ function $H: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with $H(0) = H'(0) = 0$, which is linear or is strictly increasing and strictly convex function of class $C^2(\mathbb{R}^+)$ on $(0, r)$, $r < -k'(0)$ such that 
\begin{align*}
k''(t) &> \eta(t)H(-k'(t)), \quad \forall t \geq 0,
\end{align*}

where $\eta(t)$ is $C^1$ nonincreasing continuous function.

Remark 2.1. If assuming further $\lim_{t \rightarrow -\infty} k(t) = 0$, then 
\begin{align*}
\lim_{t \rightarrow -\infty} (-k'(t)) &= 0.
\end{align*}

Then for some $t_1 \geq 0$,
\begin{align*}
-k'(t_1) &= r \Rightarrow -k'(t) \leq r, \quad \forall t \geq t_1.
\end{align*}

Noting that $(-k')$ is nonincreasing, $-k'(0) > 0$, and $-k'(t_1) > 0$, we have $-k'(t) > 0$ for any $t \in [0, t_1]$, and for any $t \in [0, t_1]$, 
\begin{align*}
0 &< -k'(t_1) \leq -k'(t) \leq -k'(0), \\
0 &< \eta(t_1) \leq \eta(t) \leq \eta(0).
\end{align*}

Therefore we obtain that there exist two positive constants $a$ and $b$ such that for any $t \in [0, t_1]$, 
\begin{align*}
a &\leq \eta(t)H(-k'(t)) \leq b.
\end{align*}

Then for any $t \in [0, t_1]$, 
\begin{align*}
k''(t) &\geq \eta(t)H(-k'(t)) \geq a - \frac{a}{k'(0)}k'(0) \geq \frac{a}{k'(0)}k'(t).
\end{align*}

This implies that there exists a constant $d > 0$ such that for any $t \in [0, t_1]$, 
\begin{align*}
k''(t) &\geq -dk'(t).
\end{align*}

The proof is done.

3. Main Results

The well-posedness result is given in [30] proved by using the Faedo–Galerkin method as in [17].

**Theorem 1.** Assume that (A1) and (A2) hold. Let $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$, and then problem (2) admits a unique solution $u$ satisfying 
\begin{align*}
&u \in L^\infty(0, T; H^2(\Omega) \cap V) \cap W^{1, \infty}(0, T; V) \\
&\cap W^{2, \infty}(0, T; L^2(\Omega)),
\end{align*}

where $V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0 \}$.

The total energy of the system is defined by
\begin{align*}
\mathcal{E}(s) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\|V u\|^2 + \alpha \int_r^s k''(t)\|t\|_\Gamma - \int_{\Gamma} k' u d\Gamma,
\end{align*}

where
\begin{align*}
(ku)(t) &= \int_0^t (k(t) - s)|u(t) - u(s)|^2 ds.
\end{align*}

We can get the following stability result.

**Theorem 2.** Assume $k$ satisfies (A1)–(A3) and further $\lim_{t \rightarrow -\infty} k(t) = 0$. Then there exist $\lambda_1, \lambda_2 > 0$ such that 
\begin{align*}
\mathcal{E}(t) &\leq \lambda_2 H_4^{-1}\left(\lambda_1 \int_t^{r} k^{-1}(r) \eta(s) ds\right), \quad \forall t > K^{-1}(r),
\end{align*}

where
\begin{align*}
H_4(t) &= \int_t^{r} \frac{1}{sH_0(s)} ds, \\
H_0(t) &= H'(t),
\end{align*}

and $K(t) = -k'(t)$. In particular, if $H(t) = t^p$, then for any $t > 0$, 
\begin{align*}
\mathcal{E}(t) &\leq \begin{cases} 
\frac{c_1}{c_2} - \frac{c_1}{c_2} \int_0^t \eta(s) ds, & \text{if } p = 1, \\
\frac{c_3}{c_4} \left(1 + \int_0^t \eta(s) ds \right)^{-\frac{1}{(p-1)}}, & \text{if } 1 < p < 2,
\end{cases}
\end{align*}

where $c_1, c_3$, and $c_2 \leq 1$ are positive constants.

Remark 3.1. From (23), the energy result holds for $H(t) = t^p$ with the full admissible range $p \in [1, 2]$ instead of $p \in [1, (3/2)]$. If the viscoelastic term is as internal feedback, Lasiecka and Wang [10] provided the proof for optimal decay rates of second-order systems in the full admissible range $[1, 2]$.

At last, we show two examples to illustrate explicit formulas for the decay rates of the energy, which can be found in the studies of Mustafa and Mustafa [14, 15].

**Example 1.** Take $k'(t) = e^{-qt}$ with $0 < q < 1$, we get $k''(t) = H(-k'(t))$, where $H(t) = ((q t) / ((\ln(1/t))^{1/(q-1)})$. Since
\[ H'(t) = \frac{(1 - q) + q \ln(1/t)}{[\ln(1/t)]^{(1/q)}}, \]
\[ H''(t) = \frac{(1 - q)[\ln(1/t) + (1/q)]}{[\ln(1/t)]^{((1/q)+1)}}, \]  

we can deduce that the function \( H \) satisfies (A3) on \((0, r] \) for any \( 0 < r < 1 \). Then,
\[ \mathcal{E}(t) \leq c_1 e^{-c_2 t}. \]  

Example 2. Consider \( k'(t) = (-1/((t+e)[\ln(t+e)]^p)) \) with \( p > 1 \), we get \( k''(t) = ((\ln(t+e) + p)/((t+e)^2[\ln(t+e)]^{p+1})) \). Clearly,
\[ k''(t) = \frac{[\ln(t+e) + p]}{(t+e)[\ln(t+e)]^p} \left[ -k'(t) \right]. \]  

By part 1 of (23), we get
\[ \mathcal{E}(t) \leq c_1 \exp \left( -c_2 \int_0^t \left[ \ln(t+e) + p \right] \frac{1}{(t+e)[\ln(t+e)]^p} \, ds \right) \]
\[ = \left[ (t+e)[\ln(t+e)]^p \right]^{-c_2}. \]

As \( c_2 \leq 1 \), this is slower rate than \(-[k'(t)]\). In addition,
\[ k'''(t) = \frac{[\ln(t+e) + p]}{(t+e)^{(1-(1/p))}} (t+e)^{(1+(1/p))}. \]  

From part 2 of (23), we infer that for large \( t \)
\[ \mathcal{E}(t) \leq c_3 \left( 1 + \int_0^t \left[ \ln(t+e) + p \right] \frac{1}{(t+e)[\ln(t+e)]^p} \, ds \right) \]
\[ \leq \frac{c_3}{(t+e)[\ln(t+e)]^p}. \]  

which is the same rate as \(-k'(t)\).

4. Proof of Main Result

To prove Theorem 2, we need the following lemmas.

4.1. Technical Lemmas

Lemma 1. The total energy functional \( E(t) \) satisfies for any \( t \geq 0 \),
\[ \mathcal{E}'(t) \leq -\frac{\alpha}{2} \left\| u_t \right\|^2_{\Gamma_1} + \int_{\Gamma_1} k'' \nu u \, d\Gamma \leq 0. \]  

Proof. See [30].

As in [31], for \( 0 < \delta < 1 \), we introduce
\[ C_\delta = \int_0^\infty \left[ k'(s) \right]^2 \, ds, \]
\[ h(t) = k''(s) - \delta k'(s). \]

Lemma 2. Define the functional \( \Phi(t) \) by
\[ \Phi(t) = \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t \, dx. \]  

Then we can get for any \( t \geq t_1 \),
\[ \Phi'(t) \leq -\left\| u_t \right\|^2 - \frac{1}{2} \left\| \nabla u \right\|^2 + c \left\| u_t \right\|^2_{\Gamma_1} - \int_{\Gamma_1} h \cdot u \, d\Gamma = 0. \]  

Proof. From the same arguments as in the study of Mustafa [30], we can obtain
\[ \Phi'(t) \leq -\left\| u_t \right\|^2 - \left\| \nabla u \right\|^2 - \delta_0\left\| \nabla u \right\|^2_{\Gamma_1} + \int_{\Gamma_1} (m \cdot \nabla) \left| u_t \right|^2 \, d\Gamma \]
\[ + \int_{\Omega} (2m \cdot \nabla u) \frac{\partial u}{\partial y} \, dy + (n-1) \int_{\Gamma_1} u \frac{\partial u}{\partial y} \, d\Gamma \]
\[ \leq \delta_0\left\| \nabla u \right\|^2_{\Gamma_1} + c \left\| u_t \right\|^2_{\Gamma_1} + \frac{\left\| u \right\|^2_{\Gamma_1}}{\left\| \nabla u \right\|_{\Gamma_1}}. \]  

Recalling \( k' \ast u = (-k' \ast u) + [k(t) - k(0)]u_t \) where \( k \ast u = \int_0^t k(t-s)(u(t)-u(s)) \, ds \); then we have from (8),
\[ \frac{\partial u}{\partial y}(t) = -\alpha \left[ u_t(t) + k(t)u(t) + (-k' \ast u)(t) \right]. \]  

By using Young’s inequality, we obtain
\[ \left\| \frac{\partial u}{\partial y}(t) \right\|_{L^2_{\Gamma_1}}^2 \leq 4\alpha^2 \left[ \left\| u_t \right\|_{L^2_{\Gamma_1}}^2 + k^2(t) \left\| u \right\|_{H^1_{\Gamma_1}}^2 + \int_{\Gamma_1} (-k' \odot u)^2 \, dl' \right]. \tag{37} \]

Hölder's inequality implies

\[
(-k' \odot u)^2 = \left( \int_0^t (-k'(t-s)) (u(t) - u(s)) \, ds \right)^2
\]

\[
= \left( \int_0^t \frac{-k'(t-s)}{\sqrt{k''(t-s) - \delta k'(t-s)}} \sqrt{k''(t-s) - \delta k'(t-s)} (u(t) - u(s)) \, ds \right)^2
\]

\[
\leq \left( \int_0^t \frac{[k'(s)]^2}{k'(s) - \delta k'(s)} \, ds \right) \int_0^t (k''(t-s) - \delta k'(t-s)) (u(t) - u(s))^2 \, ds
\]

\[
\leq C_\delta (h \ast u),
\]

which, together with (37), gives us that

\[
\left\| \frac{\partial u}{\partial y}(t) \right\|_{L^2_{\Gamma_1}}^2 \leq 4\alpha^2 \left[ \left\| u_t \right\|_{L^2_{\Gamma_1}}^2 + k^2(t) \left\| u \right\|_{H^1_{\Gamma_1}}^2 + C_\delta \int_{\Gamma_1} (h \ast u) \, dl' \right]. \tag{39} \]

Inserting (39) into (35), we obtain for any \( \epsilon > 0, \)

\[
\int_{\Gamma_1} (2m \cdot \nabla u) \frac{\partial u}{\partial y} \, dl' + (n-1) \int_{\Gamma_1} u \frac{\partial u}{\partial y} \, dl'
\]

\[
\leq \delta_0 \left\| \nabla u \right\|_{L^2_{\Gamma_1}}^2 + (\epsilon + 4\alpha^2 k^2(t)) \left\| u \right\|_{H^1_{\Gamma_1}}^2
\]

\[
+ 4\alpha^2 \epsilon \left\| u \right\|_{L^2_{\Gamma_1}}^2 + C_\delta \int_{\Gamma_1} (h \ast u) \, dl'. \tag{40} \]

Noting that

\[
\left\| u \right\|_{L^2_{\Gamma_1}}^2 \leq \epsilon \left\| \nabla u \right\|_{L^2_{\Gamma_1}}^2,
\]

using \( \lim_{t \to -\infty} k(t) = 0 \) and taking \( \epsilon > 0 \) small enough, we can get (33) from (34) and (40). The proof is done. \hfill \Box

To get the optimal energy decay, we need the following estimate.

\[
2 \int_{\Gamma_1} u(t) \int_0^t k'(t-s) \left[ u(s) - u(t) \sqrt{a^2 + b^2} \right] \, ds \, dl'.
\]

\[
\leq 2k(0) \int_{\Gamma_1} u^2(t) \, dl' + \frac{1}{2k(0)} \int_{\Gamma_1} \left( \int_0^t \sqrt{-k'(t-s) \sqrt{-k'(t-s)} [u(s) - u(t)]} \, ds \right)^2 \, dl'. \tag{45} \]

\[
\leq 2k(0) \left\| u(t) \right\|_{L^2_{\Gamma_1}}^2 + \frac{\int_0^t k'(s) \, ds}{2k(0)} \int_0^t k'(t-s) \left\| u(s) - u(t) \right\|_{L^2_{\Gamma_1}}^2 \, ds.
\]
Then we can get (43) following from the fact
\[ k(t) \leq k(0), \]
\[ \int_0^1 k'(s) ds \leq \frac{1}{2k(0)} \geq -\frac{1}{2} \]  
(46)

The proof is complete. \[ \square \]

4.2. Proof of Theorem 2

Proof. Define the functional \( L(t) \) by
\[ L(t) := N^\varepsilon(g) + \Phi(t), \]
(47)
where \( N > 0 \) is a constant that will be taken later. Clearly we can take \( N \) a large value to get
\[ L(t) \sim \varepsilon(t). \]
(48)

Recalling \( k'' = \delta k' + h \), combining (30) and (33), we conclude that for any \( t > t_1 \),
\[ L' (t) \leq -\left( \frac{\alpha}{2} N - c \right) ||u_t||^2 \leq \frac{1}{2} ||u||^2, \]
(49)

\[ -\frac{\alpha}{2} N \int_{\Gamma_1} k'' u d\Gamma - \left( \frac{\alpha}{2} N - c C_\delta \right) \int_{\Gamma_1} h^2 u d\Gamma. \]

Noting \(-k' > 0 \) and \( k'' > 0 \), for each \( s \in [0, \infty) \), we shall see below,
\[ \lim_{\delta \to 0} \frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)} ds = 0, \]  
(50)
\[ \frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)} < - k'(s). \]

It follows from Lebesgue dominated convergence theorem that
\[ \lim_{\delta \to 0} \delta C_\delta = \lim_{\delta \to 0} \int_0^{\infty} \frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)} ds = 0. \]  
(51)

Therefore there exist \( 0 < \gamma < 1 \) such that if \( \delta < \gamma \), then
\[ \delta C_\delta < \frac{1}{4c} \]  
(52)

And then we choose \( N \) a larger value that
\[ \frac{\alpha}{2} N - c > 4k(0), \]  
(53)
and take \( \delta > 0 \) satisfying
\[ \delta = \frac{1}{2anN} \gamma. \]  
(54)

This implies
\[ \frac{\alpha}{2} N - c C_\delta > 0. \]  
(55)

Then there exists a positive constant \( \beta \) such that for large \( t_1 > 0 \),
\[ L' (t) \leq - \beta \left( ||u_t||^2 + ||u||^2 \right) - 4k(0) ||u_t||_{L_1}, \]
(56)

\[ -\frac{1}{4} \int_{\Gamma_1} k'' u d\Gamma \leq \forall t \geq t_1. \]

By (17) and (30), we get
\[ \int_0^{t_1} (-k'(s)) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds \leq - c \varepsilon'(t). \]
(57)

Then from (56), we infer that there exists a constant \( \chi > 0 \) such that
\[ L' (t) \leq - \chi \varepsilon(t) - c \int_0^{t_1} k'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds \]
\[ \leq - \chi \varepsilon(t) - c \int_0^{t_1} k'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \]  
(58)

Denoting \( F(t) = L(t) + \varepsilon(t) \sim E(t) \), and using (58), we know that
\[ F' (t) \leq - \chi \varepsilon(t) - c \int_0^{t_1} k'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \]  
(59)

In the sequel, we consider two cases.

Case 1. The particular case \( H(t) = t^p \).

(I) \( p = 1 \).

Multiplying (59) by \( \eta(t) \), and using (19) and (A2)-(A3), we have
\[ \eta(t) F'(t) \leq - \chi \eta(t) \varepsilon(t) - c \varepsilon'(t), \quad \forall t \geq t_1. \]  
(60)

Since \( \eta(t) \) is a nonincreasing continuous function and \( \eta'(t) \leq 0 \) for a.e. \( t \), then
\[ \left( \eta F + c \varepsilon \right)' (t) \leq \eta(t) F'(t) + c \varepsilon'(t) \]
\[ \leq - m \eta(t) \varepsilon(t), \quad \text{a.e. } t \geq t_1. \]  
(61)

In view of \( \eta F + c \varepsilon \sim \varepsilon \), we obtain that there exist two positive constants \( c_1, c_2 > 0 \),
\[ \varepsilon(t) \leq c_1 e^{-c_2 \int_0^t \eta(s) ds}. \]  
(62)

(II) \( 1 < p < 2 \).

Define \( \varepsilon(t) \) by
\[ \varepsilon (t) = L(t) + \Psi(t). \]  
(63)

It follows from (43) and (56) that \( \varepsilon(t) \geq 0 \), and for any \( t \geq t_1 \),
\[ G'(t) \leq -\beta \left( \|u_t\|^2 + \|\nabla u\|^2 \right) - k(0)\|u_t\|^2 + \frac{1}{4} \int_{\Gamma_s} k^\prime \omega d\Gamma. \]  

(64)

Then there exists a certain constant \( \beta_0 > 0, \)

\[ G'(t) \leq -\beta_0 G(t), \quad \forall t \geq t_1. \]  

(65)

This gives us

\[ \beta_0 \int_{t_1}^t E(s)ds \leq G(t_1) - G(t) \leq G(t_1). \]  

(66)

Hence

\[ \int_0^\infty G'(s)ds < \infty. \]  

(67)

Define

\[ I(t) = \int_0^t \int_{\Gamma_s} [u(t) - u(t - s)]^2 d\Gamma ds, \]  

(68)

we know that

\[ I(t) \leq c \int_0^t G(s)ds. \]  

(69)

Without loss of generality assuming \( t_1 \) so large that \( I(t_1) > 0 \), then

\[ 0 < I(t_1) \leq I(t) < \infty, \quad \forall t \geq t_1. \]  

(70)

Using Jensen’s inequality and by (30) and (A2)-(A3), we can derive from (56) that for some constant \( q > 0, \)

\[ L'(t) \leq -qG(t) + \frac{cI(t)}{I(t)} \int_{\Gamma_s} \left[ \left( -k' \right)^{(1/p)} \circ u \right] d\Gamma \]

\[ \leq -qG(t) + cI(t) \int_{\Gamma_s} \left[ \left( -k' \right)^{(1/p)} \circ u \right] d\Gamma \]

\[ \leq -qG(t) + cL^{-1/(p-1)}(t) \int_{\Gamma_s} \left[ k'' \circ u \right] d\Gamma \]

\[ \leq -qG(t) + \frac{c}{\eta(t)^{(1/p)}} \int_{\Gamma_s} \left[ k'' \circ u \right] d\Gamma \]

\[ \leq -qG(t) + \frac{c}{\eta(t)^{(1/p)}} \left[ \left( -G'(t) \right)^{(1/p)} \right]. \]  

(71)

We multiply (71) by \( G^{p-1}(t) \) and use (19) to deduce

\[ \left( L G^{p-1} \right)'(t) \leq L'(t) G^{p-1}(t) \leq -qG^p(t) \]

\[ + \frac{c}{\eta(t)} \left[ \left( G'(t) \right)^{(1/p)} \right]. \]  

(72)

By Young’s inequality, we have for any \( \varepsilon_1 > 0, \)

\[ \left( L G^{p-1} \right)'(t) \leq -qG^p(t) + \varepsilon_1 G^p(t) + \frac{c}{\varepsilon_1} \left[ \left( G'(t) \right)^{(1/p)} \right]. \]  

(73)

Taking \( \varepsilon_1 < (1/2)q, \) we conclude

\[ \left( L G^{p-1} \right)'(t) \leq -\frac{q}{2} G^p(t) - \frac{c}{\eta(t)} \left[ \left( G'(t) \right)^{(1/p)} \right]. \]  

(74)

Define \( F(t) = \eta L G^{p-1} + cG \sim G. \) Multiplying (74) by \( \eta(t), \) we have

\[ F'(t) \leq -\frac{q}{2} \eta(t) G^p(t). \]  

(75)

Then there exists a certain constant \( q_0 > 0 \) such that

\[ F'(t) \leq -q_0 \eta(t) F^p(t), \]  

(76)

from which we obtain

\[ G(t) \leq c_3 \left( 1 + \int_0^t \eta(s)ds \right)^{-1/(1(p-1))}, \]  

(77)

where \( c_3 \) is a positive constant.

Combining (I) and (II) and using the boundedness of \( \eta(t) \) and \( G(t) \), we can get (23).

Case 2. The general case.

Define

\[ I(t) = \int_{t_1}^t \int_{\Gamma_s} [u(t) - u(t - s)]^2 d\Gamma ds. \]  

(78)

In view of (67), we can take \( 0 < q < 1 \) such that

\[ I(t) < 1, \quad \forall t \geq t_1. \]  

(79)

Without loss of generality, we assume that \( I(t) > 0 \) for all \( t \geq t_1 \). On the other hand, we define

\[ \pi(t) = \int_{t_1}^t k''(s) \int_{\Gamma_s} [u(t) - u(t - s)]^2 d\Gamma ds. \]  

(80)

From (30), we can easily get \( \pi(t) \leq -cE(t) \). As \( H(t) \) is strictly convex on \((0, t)\) and \( H(0) = 0 \), we see that
\[ H(\lambda x) \leq \lambda H(x), \quad i = 1, 2, 0 \leq \lambda \leq 1, x \in (0, r]. \quad (81) \]

It follows from Jensen’s inequality and (11) and (79) that

\[
\pi_i(t) = \frac{1}{qI(t)} \int_{t_i}^{t} I(t)(k''(s))q \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \\
\geq \frac{1}{qI(t)} \int_{t_i}^{t} I(t) \eta(s) H(-k'(s))q \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \\
\geq \frac{\eta(t)}{qI(t)} \int_{t_i}^{t} H(I(t)(-k'(s)))q \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \\
\geq \frac{\eta(t)}{q} H \left( \frac{1}{I(t)} \int_{t_i}^{t} I(t)(-k'(s))q \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \right) \\
= \frac{\eta(t)}{q} H \left( q \int_{t_i}^{t} (-k'(s)) \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \right) \\
= \frac{\eta(t)}{q} \mathcal{H}' \left( q \int_{t_i}^{t} (-k'(s)) \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \right), \tag{82}
\]

where \( \mathcal{H} \), which is strictly convex and increasing function on \((0, \infty)\) of class \(C^2\), is called the extension of \( H \). We infer from (82) that

\[
\int_{t_i}^{t} (-k'(s)) \int_{\Gamma_i} [u(t) - u(t - s)]^2 \, ds \leq \frac{1}{q} H^{-1} \left( \frac{q \pi(t)}{\eta(t)} \right). \tag{83}
\]

Then we can get from (59) that for any \( t \geq t_i \),

\[
F'(t) \leq -\chi \mathcal{E}(t) + c\mathcal{H}^{-1} \left( \frac{q \pi(t)}{\eta(t)} \right). \tag{84}
\]

Denote

\[
H_0(t) = \mathcal{H}'(t). \tag{85}
\]

For \( r_0 < r \), we define \( \mathcal{K}_1(t) \) by

\[
\mathcal{K}_1(t) = H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{G}(0)} \right) F(t) + \mathcal{E}(t) - \mathcal{E}(t). \tag{86}
\]

Since \( E'(t) \leq 0, \mathcal{H} > 0 \), and \( \mathcal{H}'' > 0 \), we get from (84) that

\[
\mathcal{K}_1'(t) = r_0 \frac{\mathcal{E}'(t)}{\mathcal{G}(0)} H_0' \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{G}(0)} \right) F(t) + H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{G}(0)} \right) F'(t) + \mathcal{E}'(t) \\
\leq -m \mathcal{E}(t) H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{G}(0)} \right) + c H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{G}(0)} \right) \mathcal{H}^{-1} \left( \frac{q \pi(t)}{\eta(t)} \right). \tag{87}
\]

We denote by \( \mathcal{H}' \) the conjugate function of the convex function \( \mathcal{H} \) (see Arnold [33]), and then

\[
\mathcal{H}'(s) = s(\mathcal{H})^{-1}(s) - \mathcal{H}'(\mathcal{H})^{-1}(s) \tag{88}
\]

satisfies Young’s inequality,

\[
AB \leq \mathcal{H}'(A) + \mathcal{H}(B). \tag{89}
\]
Taking \( A = \mathcal{H}^{-1}_0 (r_0 (E(t) / E(0))) \) and \( B = \mathcal{H}^{-1} ((q \pi(t)) / \eta(t)) \), and using \( \mathcal{H}^t(s) \leq s (\mathcal{H}^t)^{-1}(s) \) and (87), we have

\[
\mathcal{X}'_1(t) \leq - \chi \mathcal{E}'(t) H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c \mathcal{H} \left( H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q \pi(t)}{\eta(t)} 
\]

\[
\leq - \chi \mathcal{E}'(t) H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) (\mathcal{H}^t)^{-1} \left( H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q \pi(t)}{\eta(t)}
\]

\[
\leq - \chi \mathcal{E}'(t) H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) (\mathcal{H}^t)^{-1} \left( H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right) + c \frac{q \pi(t)}{\eta(t)}
\]

\[
\leq - (\chi \mathcal{E}(0) - cr_0) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c q \pi(t)
\]

We multiply (90) by \( \eta(t) \) to arrive at

\[
\eta(t) \mathcal{X}'_1(t) \leq - (\chi \mathcal{E}(0) - cr_0) \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + c q \pi(t)
\]

\[
\leq - (\chi \mathcal{E}(0) - cr_0) \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) - c \mathcal{E}'(t).
\]

The functional \( \mathcal{X}_2(t) \) is defined by

\[
\mathcal{X}_2(t) = \eta(t) \mathcal{X}_1(t) + c \mathcal{E}(t).
\]

Then we can easily obtain that there exist constants \( \beta_2 > 0 \) and \( \beta_3 > 0 \) such that

\[
\beta_2 \frac{\mathcal{X}_2(t)}{\eta(t)} \leq E(t) \leq \beta_3 \frac{\mathcal{X}_2(t)}{\eta(t)}.
\]

Choosing a suitable \( r_0 > 0 \), and defining \( H_3(t) = t H_0 (r_0 t) \), from (91), we infer that for a constant \( \gamma > 0 \),

\[
\mathcal{X}_2(t) \leq - \gamma \eta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0 \left( r_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) = - \gamma \eta(t) H_3 \left( \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right).
\]

(94)

It follows from \( 0 \leq r_0 (\mathcal{E}(t)/\mathcal{E}(0)) < r \) that for any \( t > 0 \),

\[
\int_{t_1}^{t} \frac{-R'(s)}{H_4 (R(s))} \, ds \geq \gamma_1 \int_{t_1}^{t} \eta(s) \, ds \Rightarrow \int_{t_1}^{t} \frac{R'(s)}{H_4 (R(s))} \, ds \geq \gamma_1 \int_{t_1}^{t} \eta(s) \, ds.
\]

Define

\[
H_4(t) = \int_{t}^{s} \frac{1}{s H_0(s)} \, ds.
\]

(99)

It is to verify that \( H_4 \) is strictly decreasing on \( (0, r] \) and \( \lim_{t \to 0} H_4(t) = +\infty \). It follows that

\[
R(t) \leq \frac{1}{r_0 H_4 \left( \lim_{t \to 0} H_4(t) \right)} \int_{t_1}^{t} \eta(s) \, ds.
\]

(100)
Combining (96) and (100), we can obtain (21). This finishes the proof of Theorem 2

Data Availability

No data were used during this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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