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Research Article

A New General Decay Rate of Wave Equation with Memory-Type Boundary Control

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Of interest is a wave equation with memory-type boundary oscillations, in which the forced oscillations of the rod is given by a memory term at the boundary. We establish a new general decay rate to the system. And it possesses the character of damped oscillations and tends to a finite value for a large time. By assuming the resolvent kernel that is more general than those in previous papers, we establish a more general energy decay result. Hence the result improves earlier results in the literature.

1. Introduction

It is well-known that if we add a damping to a system, the amplitude of the oscillations can be reduced very fast. The memory term can be as a damping (viscoelastic damping) which is weaker than frictional damping. For viscoelastic materials, Boltzmann theory gives us that the stress-strain viscoelastic law depending on a relaxation measure, see Prüss [1] and Eden et al. [2]. Based on the Boltzmann principle, the viscoelastic stress-strain relations can be generally given by a convolution term, which can be regarded as a lower order perturbation and can also be regarded as a kind of memory effect, for instance, $g^* u$. And we call the function $g(t)$ memory kernel. One can find a detail derivation on some systems with memory in [3].

To motivate our work, we start with some known results on wave equation with memory-type oscillations. The general wave equation with viscoelastic term in the internal feedback

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = \mathscr{F}(u). \tag{1}$$

Messaoudi and Messaoudi [4, 5] studied $\mathscr{F}(u) = 0$ and $\mathscr{F}(u) = |u|^\rho u$, by introducing the assumption $g'(t) \leq -\xi(t)g(t)$, and obtained the energy decays exponentially (polynomially) as g decays exponentially (polynomially), respectively.

Lasiecka et al. [6] considered the general assumption on g: $g'(t) \leq -H(g(t))$ to establish general decay of energy. Here H, which was introduced by Alabau-Boussouira and Cannarsa [7], is strictly convex and increasing function. Cavalcanti et al. [8, 9], Lasiecka and Wang [10], Mustafa and Messaoudi [11], and Xiao and Liang [12] also used this assumption to obtain some general decay rates of corresponding models. In recent papers [13–15], the authors investigated three classes of viscoelastic wave equation as in [4, 5] and established optimal and explicit decay results of energy by adopting the assumption on g: $g'(t) \leq -\xi(t)H(g(t))$.

In this paper, we considered the following wave equation with boundary oscillations of memory type:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ u + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & \\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary Γ, $\Gamma = \Gamma_0 \cup \Gamma_1$, and Γ_0 and Γ_1 are closed and disjoint with measure $(\Gamma_0) > 0$. ν is the unit outward normal to Γ.

For wave equation with memory-type boundary oscillations, it can be regarded as a wave equation with viscoelastic damping at the boundary. Santos [16] considered a one-dimensional wave equation with memory conditions at the boundary, respectively. He proved that the energy of solutions decays exponentially (polynomially) as k and k' decay exponentially (polynomially). Here k is the resolvent kernel of $(-g'/g(0))$. Santos et al. [17] extended the results in [16] to an n-dimensional wave equation of Kirchhoff type with memory-type boundary. They proved the global existence of solutions and obtained that the energy of solution decays uniformly with the same rate of decay k under the same conditions on k and k', which improves the results in [18] by Park et al. Santos and Junior [19] obtained a similar result for plate equation with memory-type boundary. We also mention the work of Cavalcanti et al. [20], where the authors showed the global existence and the uniform decay of solutions to a semilinear wave equation with memory-type boundary condition and a nonlinear boundary source. Messaoudi and Soufyane [21] considered a general assumption on k': $k'' \geq -\xi(t)k'(t)$ and established a general decay result. Wu [22] used this assumption to study a wave Kirchhoff-type wave equation with a boundary control of memory type. For nonlinear wave equations with memory-type boundary condition, we refer to Cavalcanti and Guesmia [23], Feng [24], Feng et al. [25–27], Muñoz Rivera and Andrade [28], and Zhang [29].

Concerning the system (2), Mustafa [30], by assuming the function k: $k''(t) \geq H(-k'(t))$, where k is the resolvent kernel of $(-g'/g(0))$, established a general decay of solutions of the form

$$E(t) \leq k_3 H_1^{-1}(k_1 t + k_2), \quad \forall t \geq 0. \tag{3}$$

Here

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)}ds, \qquad H_0(t) = H(D(t)), \tag{4}$$

and D is a positive C^1 function with $D(0) = 0$, and H_0 is strictly increasing and strictly convex C^2 function on $(0,r]$. In particular, for $H(t) = t^p$, i.e., $k'' \geq c(-k')^p$, the author proved the energy decay holds for $1 \leq p < (3/2)$. Whether can the range be extended to a more larger range? In this paper, we give a positive answer to study problem (2) and extend the result to get a more general decay rate. In particular, we obtain that the energy result holds for $H(t) = t^p$ with the full admissible range $1 \leq p < 2$. More exactly, by assuming the relaxation function k with minimal conditions on $L^1(0,\infty)$, i.e., $k''(t) \geq \eta(t)H(-k'(t))$, where H is linear or strictly increasing and strictly convex functions of class $C^2(\mathbb{R}^+)$, we establish an optimal explicit and general energy decay result. In particular, the energy result holds for $H(t) = t^p$ with the range $p \in [1,2)$ instead of $p \in [1,(3/2))$ in [30]. Hence our results extend and improve the stability results in [30] and also in [16–18, 21]. We mainly adopt the idea of [14, 15, 31] and some properties of convex function developed in [7, 32].

The remaining of the paper is organized as follows: in Section 2, we propose some preliminaries. In Section 3, main results are given. Section 4 is devoted to proving the general decay result.

2. Preliminaries

Taking the derivative of (2) with respect to t, we shall see that

$$\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)}\left[u_t + g_2' * \frac{\partial u}{\partial \nu}\right]. \tag{5}$$

We denote the resolvent kernel of $(-g'/g(0))$ by k satisfying for $t \geq 0$:

$$k(t) + \frac{1}{g(0)}\left(g'^{*}k\right)(t) = -\frac{1}{g(0)}g'(t). \tag{6}$$

Using Volterra's inverse operator and taking $\alpha = (1/g(0))$, we have

$$\frac{\partial u}{\partial \nu} = -\alpha[u_t + k_2 * u_t]. \tag{7}$$

Assume $u_0 = 0$ on Γ_1 in this paper, we get

$$\frac{\partial u}{\partial \nu} = -\alpha[u_t + k(0) + k * u], \quad \text{on } \Gamma_1 \times \mathbb{R}^+. \tag{8}$$

In the following, we use boundary conditions (8) instead of (2).

As in [30], we consider the following assumption:

(A1) There exists a fixed point $x_0 \in \mathbb{R}^2$ and some constant $\delta_0 > 0$ such that for $m(x) = x - x_0$,

$$\Gamma_0 = \{x \in \Gamma: m(x) \cdot \nu(x) \le 0\},\quad \Gamma_1 = \{x \in \Gamma: m(x) \cdot \nu(x) \ge \delta_0\}. \tag{9}$$

For the kernel k, we assume

(A2) k: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is nonincreasing and twice differentiable function satisfying for any $t \ge 0$,

$$k(0) > 0,\quad k'(t) \le 0. \tag{10}$$

(A3) There exist a C^1 function H: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $H(0) = H'(0) = 0$, which is linear or is strictly increasing and strictly convex function of class $C^2(\mathbb{R}^+)$ on $(0, r]$, $r \le -k'(0)$ such that

$$k''(t) \ge \eta(t) H(-k'(t)), \quad \forall t \ge 0, \tag{11}$$

where $\eta(t)$ is C^1 nonincreasing continuous function.

Remark 2.1. If assuming further $\lim_{t\longrightarrow\infty} k(t) = 0$, since $\lim_{t\longrightarrow\infty} k(t) = 0$ and $(-k'(t))$ is nonincreasing and nonnegative, we can get

$$\lim_{t\longrightarrow\infty} (-k'(t)) = 0. \tag{12}$$

Then for some $t_1 \ge 0$ large,

$$-k'(t_1) = r \Rightarrow -k'(t) \le r, \quad \forall t \ge t_1. \tag{13}$$

Noting that $(-k')$ is nonincreasing, $-k'(0) > 0$, and $-k'(t_1) > 0$, we have $-k'(t_1) > 0$ for any $t \in [0, t_1]$, and for any $t \in [0, t_1]$,

$$0 < -k'(t_1) \le -k'(t) \le -k'(0),\quad 0 < \eta(t_1) \le \eta(t) \le \eta(0). \tag{14}$$

Therefore we obtain that there exist two positive constants a and b such that for any $t \in [0, t_1]$,

$$a \le \eta(t) H(-k'(t)) \le b. \tag{15}$$

Then for any $t \in [0, t_1]$,

$$k''(t) \ge \eta(t) H(-k'(t)) \ge a = \frac{a}{k'(0)} k'(0) \ge \frac{a}{k'(0)} k'(t). \tag{16}$$

This implies that there exists a constant $d > 0$ such that for any $t \in [0, t_1]$,

$$k''(t) \ge -d k'(t). \tag{17}$$

The proof is done.

3. Main Results

The well-posedness result is given in [30] proved by using the Faedo–Galerkin method as in [17].

Theorem 1. *Assume that (A1) and (A2) hold. Let $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$, and then problem (2) admits a unique solution u satisfying*

$$u \in L^\infty\left(0, T; H^2(\Omega) \cap V\right) \cap W^{1,\infty}(0, T; V) \cap W^{2,\infty}\left(0, T; L^2(\Omega)\right), \tag{18}$$

where $V = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_0\}$.

The total energy of the system is defined by

$$\mathscr{E}(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\alpha}{2}\left[k(t)\|u\|^2_{\Gamma_1} - \int_{\Gamma_1} k' \circ u \, d\Gamma\right], \tag{19}$$

where

$$(k \circ u)(t) = \int_0^t k(t-s)[u(t) - u(s)]^2 ds. \tag{20}$$

We can get the following stability result.

Theorem 2. *Assume k satisfies (A1)–(A3) and further $\lim_{t\longrightarrow\infty} k(t) = 0$. Then there exist $\lambda_1, \lambda_2 > 0$ such that*

$$\mathscr{E}(t) \le \lambda_2 H_4^{-1}\left(\lambda_1 \int_{K^{-1}(r)}^t \eta(s) ds\right), \quad \forall t > K^{-1}(r), \tag{21}$$

where

$$H_4(t) = \int_t^r \frac{1}{s H_0(s)} ds,\quad H_0(t) = H'(t), \tag{22}$$

and $K(t) = -k'(t)$. In particular, if $H(t) = t^p$, then for any $t > 0$,

$$\mathscr{E}(t) \le \begin{cases} c_1 e^{-c_2 \int_0^t \eta(s) ds}, & \text{if } p = 1, \\ c_3\left(1 + \int_0^t \eta(s) ds\right)^{-(1/(p-1))}, & \text{if } 1 < p < 2, \end{cases} \tag{23}$$

where c_1, c_3, and $c_2 \le 1$ are positive constants.

Remark 3.1. From (23), the energy result holds for $H(t) = t^p$ with the full admissible range $p \in [1, 2)$ instead of $p \in [1, (3/2))$. If the viscoelastic term is as internal feedback, Lasiecka and Wang [10] provided the proof for optimal decay rates of second-order systems in the full admissible range $[1, 2)$.

At last, we show two examples to illustrate explicit formulas for the decay rates of the energy, which can be found in the studies of Mustafa and Mustafa [14, 15].

Example 1. Take $k'(t) = -e^{-t^q}$ with $0 < q < 1$, we get $k''(t) = H(-k'(t))$, where $H(t) = ((qt)/([\ln(1/t)]^{(1/q)-1}))$. Since

$$H'(t) = \frac{(1-q) + q\ln(1/t)}{[\ln(1/t)]^{(1/q)}},$$
$$H''(t) = \frac{(1-q)[\ln(1/t) + (1/q)]}{[\ln(1/t)]^{((1/q)+1)}}, \tag{24}$$

we can deduce that the function H satisfies (A3) on $(0, r]$ for any $0 < r < 1$. Then,

$$\mathscr{E}(t) \le c_1 e^{-c_2 t^q}. \tag{25}$$

Example 2. Consider $k'(t) = (-1/((t+e)[\ln(t+e)]^p))$ with $p > 1$, we get $k''(t) = ([\ln(t+e) + p]/((t+e)^2[\ln(t+e)]^{p+1}))$. Clearly,

$$k''(t) = \frac{[\ln(t+e) + p]}{(t+e)\ln(t+e)}[-k'(t)]. \tag{26}$$

By part 1 of (23), we get

$$\mathscr{E}(t) \le c_1 \exp\left(-c_2 \int_0^t \frac{[\ln(t+e) + p]}{(t+e)\ln(t+e)} ds\right)$$
$$= \frac{c_1}{[(t+e)(\ln(t+e))^p]^{c_2}}. \tag{27}$$

As $c_2 \le 1$, this is slower rate than $[-k'(t)]$. In addition,

$$k''(t) = \frac{[\ln(t+e) + p]}{(t+e)^{(1-(1/p))}}(-k'(t))^{(1+(1/p))}. \tag{28}$$

From part 2 of (23), we infer that for large t

$$\mathscr{E}(t)c_3\left(1 + \int_0^t \frac{\ln(t+e) + p}{(t+e)^{1-(1/p)}} ds\right)^{-p} \le \frac{c_3}{(t+e)[\ln(t+e)]^p}, \tag{29}$$

which is the same rate as $[-k'(t)]$.

4. Proof of Main Result

To prove Theorem 2, we need the following lemmas.

4.1. Technical Lemmas

Lemma 1. *The total energy functional $E(t)$ satisfies for any $t \ge 0$,*

$$\mathscr{E}'(t) \le -\frac{\alpha}{2}\left(\|u_t\|_{\Gamma_1}^2 + \int_{\Gamma_1} k'' \circ u d\Gamma\right) \le 0. \tag{30}$$

Proof. See [30]. □

As in [31], for $0 < \delta < 1$, we introduce

$$C_\delta = \int_0^\infty \frac{[k'(s)]^2}{k''(s) - \delta k'(s)} ds,$$
$$h(t) = k''(s) - \delta k'(s). \tag{31}$$

Lemma 2. *Define the functional $\Phi(t)$ by*

$$\Phi(t) := \int_\Omega [2m \cdot \nabla u + (n-1)u]u_t dx. \tag{32}$$

Then we can get for any $t \ge t_1$,

$$\Phi'(t) \le -\|u_t\|^2 - \frac{1}{2}\|\nabla u\|^2 + c\|u_t\|_{\Gamma_1}^2 + C_\delta \int_{\Gamma_1} h \circ u d\Gamma. \tag{33}$$

Proof. From the same arguments as in the study of Mustafa [30], we can obtain

$$\Phi'(t) \le -\|u_t\|^2 - \|\nabla u\|^2 - \delta_0\|\nabla u\|_{\Gamma_1}^2 + \int_{\Gamma_1} (m \cdot \nu)|u_t|^2 d\Gamma$$
$$+ \int_{\Gamma_1} (2m \cdot \nabla u)\frac{\partial u}{\partial \nu} d\Gamma + (n-1)\int_{\Gamma_1} u \frac{\partial u}{\partial \nu} d\Gamma. \tag{34}$$

It follows from Young's inequality that for any $\varepsilon > 0$,

$$\int_{\Gamma_1} (2m \cdot \nabla u)\frac{\partial u}{\partial \nu} d\Gamma + (n-1)\int_{\Gamma_1} u \frac{\partial u}{\partial \nu} d\Gamma$$
$$\le \delta_0\|\nabla u\|_{\Gamma_1}^2 + \varepsilon\|u\|_{\Gamma_1}^2 + c\left\|\frac{\partial u}{\partial \nu}\right\|_{\Gamma_1}^2. \tag{35}$$

Recalling $k' * u = (-k' \odot u) + [k(t) - k(0)]u$, where $k \odot u = \int_0^t k(t-s)(u(t) - u(s))ds$; then we have from (8),

$$\frac{\partial u}{\partial \nu}(t) = -\alpha[u_t(t) + k(t)u(t) + (-k' \odot u)(t)]. \tag{36}$$

By using Young's inequality, we obtain

$$\left\|\frac{\partial u}{\partial \nu}(t)\right\|_{\Gamma_1}^2 \leq 4\alpha^2\left[\|u_t\|_{\Gamma_1}^2 + k^2(t)\|u\|_{\Gamma_1}^2 + \int_{\Gamma_1}(-k'\odot u)^2 \mathrm{d}\Gamma\right]. \tag{37}$$

Hölder's inequality implies

$$\begin{aligned}(-k'\odot u)^2 &= \left(\int_0^t (-k'(t-s))(u(t)-u(s))\mathrm{d}s\right)^2\\ &= \left(\int_0^t \frac{-k'(t-s)}{\sqrt{k''(t-s)-\delta k'(t-s)}}\sqrt{k''(t-s)-\delta k'(t-s)}\,(u(t)-u(s))\mathrm{d}s\right)^2\\ &\leq \left(\int_0^t \frac{[k'(s)]^2}{k''(s)-\delta k'(s)}\mathrm{d}s\right)\int_0^t (k''(t-s)-\delta k'(t-s))(u(t)-u(s))^2\mathrm{d}s\\ &\leq C_\delta(h\circ u),\end{aligned} \tag{38}$$

which, together with (37), gives us that

$$\left\|\frac{\partial u}{\partial \nu}(t)\right\|_{\Gamma_1}^2 \leq 4\alpha^2\left[\|u_t\|_{\Gamma_1}^2 + k^2(t)\|u\|_{\Gamma_1}^2 + C_\delta\int_{\Gamma_1}(h\circ u)\mathrm{d}\Gamma\right]. \tag{39}$$

Inserting (39) into (35), we obtain for any $\varepsilon > 0$,

$$\begin{aligned}&\int_{\Gamma_1}(2m\cdot\nabla u)\frac{\partial u}{\partial \nu}\mathrm{d}\Gamma + (n-1)\int_{\Gamma_1} u\frac{\partial u}{\partial \nu}\mathrm{d}\Gamma\\ &\leq \delta_0\|\nabla u\|_{\Gamma_1}^2 + \left(\varepsilon + 4\alpha^2 k^2(t)\right)\|u\|_{\Gamma_1}^2\\ &\quad + 4\alpha^2 c\|u_t\|_{\Gamma_1}^2 + C_\delta\int_{\Gamma_1}(h\circ u)\mathrm{d}\Gamma.\end{aligned} \tag{40}$$

Noting that

$$\|u\|_{\Gamma_1}^2 \leq c\|\nabla u\|^2, \tag{41}$$

using $\lim_{t\longrightarrow\infty} k(t) = 0$ and taking $\varepsilon > 0$ small enough, we can get (33) from (34) and (40). The proof is done. □

To get the optimal energy decay, we need the following estimate.

Lemma 3. *The functional* $\Psi(t)$ *is defined by*

$$\Psi(t) := \int_0^t k(t-s)\|u(s)\|_{\Gamma_1}^2\mathrm{d}s, \tag{42}$$

which satisfies for any $t > 0$,

$$\Psi'(t) \leq \frac{1}{2}\int_{\Gamma_1} k'\circ u\mathrm{d}\Gamma + 3k(0)\|u(t)\|_{\Gamma_1}^2. \tag{43}$$

Proof. Differentiating $\Psi(t)$ with respect to t, we get

$$\begin{aligned}\Psi'(t) &= k_2(0)\|u(t)\|_{\Gamma_1}^2 + \int_0^t k_2'(t-s)\|u(s)\|_{\Gamma_1}^2\mathrm{d}s\\ &= \int_0^t k'(t-s)\int_{\Gamma_1}[u(s)-u(t)]^2\mathrm{d}\Gamma\mathrm{d}s + k(t)\|u(t)\|_{\Gamma_2}^2\\ &\quad + 2\int_{\Gamma_1} u(t)\int_0^t k'(t-s)[u(s)-u(t)]\mathrm{d}s\mathrm{d}\Gamma.\end{aligned} \tag{44}$$

In view of Young's and Hölder's inequalities, we obtain

$$\begin{aligned}&2\int_{\Gamma_1} u(t)\int_0^t k'(t-s)\left[u(s)-u(t)\sqrt{a^2+b^2}\right]\mathrm{d}s\mathrm{d}\Gamma\\ &\leq 2k(0)\int_{\Gamma_1} u^2(t)\mathrm{d}\Gamma + \frac{1}{2k(0)}\int_{\Gamma_1}\left(\int_0^t \sqrt{-k'(t-s)}\sqrt{-k'(t-s)}\,[u(s)-u(t)]\mathrm{d}s\right)^2\mathrm{d}\Gamma\\ &\leq 2k(0)\|u(t)\|_{\Gamma_1}^2 + \frac{\int_0^t k'(s)\mathrm{d}s}{2k(0)}\int_0^t k'(t-s)\|u(s)-u(t)\|_{\Gamma_1}^2\mathrm{d}s.\end{aligned} \tag{45}$$

Then we can get (43) following from the fact

$$k(t) \le k(0),$$
$$\frac{\int_0^t k'(s)ds}{2k(0)} \ge -\frac{1}{2}. \quad (46)$$

The proof is complete. □

4.2. Proof of Theorem 2

Proof. Define the functional $L(t)$ by

$$L(t) := N\mathscr{E}(t) + \Phi(t), \quad (47)$$

where $N > 0$ is a constant that will be taken later. Clearly we can take N a large value to get

$$L(t) \sim \mathscr{E}(t). \quad (48)$$

Recalling $k'' = \delta k' + h$, combining (30) and (33), we conclude that for any $t > t_1$,

$$L'(t) \le -\left(\frac{\alpha}{2}N - c\right)\|u_t\|_{\Gamma_1}^2 - \|u_t\|^2 - \frac{1}{2}\|\nabla u\|^2$$
$$\cdot -\frac{\alpha}{2}N\delta\int_{\Gamma_1} k'\circ u d\Gamma - \left(\frac{\alpha}{2}N - cC_\delta\right)\int_{\Gamma_1} h\circ u d\Gamma. \quad (49)$$

Noting $-k' > 0$ and $k'' > 0$, for each $s \in [0,\infty)$, we shall see below,

$$\lim_{\delta\to 0}\frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)}ds = 0,$$
$$\frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)} < -k'(s). \quad (50)$$

It follows from Lebesgue dominated convergence theorem that

$$\lim_{\delta\to 0}\delta C_\delta = \lim_{\delta\to 0} l\int_0^\infty \frac{\delta[k'(s)]^2}{k''(s) - \delta k'(s)}ds = 0. \quad (51)$$

Therefore there exist $0 < \gamma < 1$ such that if $\delta < \gamma$, then

$$\delta C_\delta < \frac{1}{4c}. \quad (52)$$

And then we choose N a larger value that

$$\frac{\alpha}{2}N - c > 4k(0), \quad (53)$$

and take $\delta > 0$ satisfying

$$\delta = \frac{1}{2\alpha N} < \gamma. \quad (54)$$

This implies

$$\frac{\alpha}{2}N - cC_\delta > 0. \quad (55)$$

Then there exists a positive constant β such that for large $t_1 > 0$,

$$L'(t) \le -\beta\left(\|u_t\|^2 + \|\nabla u\|^2\right) - 4k(0)\|u_t\|_{\Gamma_1}^2$$
$$-\frac{1}{4}\int_{\Gamma_1} k'\circ u d\Gamma, \quad \forall t \ge t_1. \quad (56)$$

By (17) and (30), we get

$$\int_0^{t_1}(-k'(s))\int_{\Gamma_1}[u(t) - u(t-s)]^2 d\Gamma ds$$
$$\le \frac{1}{d}\int_0^{t_1} k''(s)\int_{\Gamma_1}[u(t) - u(t-s)]^2 d\Gamma ds \le -c\mathscr{E}'(t). \quad (57)$$

Then from (56), we infer that there exists a constant $\chi > 0$ such that

$$L'(t) \le -\chi\mathscr{E}(t) - c\int_0^t k'(s)\int_{\Gamma_1}[u(t) - u(t-s)]^2 d\Gamma ds$$
$$\le -\chi\mathscr{E}(t) - c\mathscr{E}'(t) - c\int_{t_1}^t k'(s)\int_{\Gamma_1}[u(t) - u(t-s)]^2 d\Gamma ds. \quad (58)$$

Denoting $F(t) := L(t) + c\mathscr{E}(t) \sim E(t)$, and using (58), we know that

$$F'(t) \le -\chi\mathscr{E}(t) - c\int_{t_1}^t k_2'(s)\int_{\Gamma_1}[u(t) - u(t-s)]^2 d\Gamma ds. \quad (59)$$

In the sequel, we consider two cases.

Case 1. The particular case $H(t) = t^p$.

(I) $p = 1$.

Multiplying (59) by $\eta(t)$, and using (19) and (A2)-(A3), we have

$$\eta(t)F'(t) \le -\chi\eta(t)\mathscr{E}(t) - c\mathscr{E}'(t), \quad \forall t \ge t_1. \quad (60)$$

Since $\eta(t)$ is a nonincreasing continuous function and $\eta'(t) \le 0$ for a.e. t, then

$$(\eta F + c\mathscr{E})'(t) \le \eta(t)F'(t) + c\mathscr{E}'(t)$$
$$\le -m\eta(t)\mathscr{E}(t), \quad \text{a.e. } t \ge t_1. \quad (61)$$

In view of $\eta F + c\mathscr{E} \sim \mathscr{E}$, we obtain that there exist two positive constants $c_1, c_2 > 0$,

$$\mathscr{E}(t) \le c_1 e^{-c_2\int_{t_1}^t \eta(s)ds}. \quad (62)$$

(II) $1 < p < 2$.

Define $\mathscr{G}(t)$ by

$$\mathscr{G}(t) = L(t) + \Psi(t). \quad (63)$$

It follows from (43) and (56) that $\mathscr{E}(t) \ge 0$, and for any $t \ge t_1$,

$$\mathscr{G}'(t) \le -\beta\left(\|u_t\|^2 + \|\nabla u\|^2\right) - k(0)\|u_t\|_{\Gamma_1}^2 + \frac{1}{4}\int_{\Gamma_1} k' \circ u d\Gamma. \tag{64}$$

Then there exists a certain constant $\beta_1 > 0$,

$$\mathscr{G}'(t) \le -\beta_1 \mathscr{E}(t), \quad \forall t \ge t_1. \tag{65}$$

This gives us

$$\beta_1 \int_{t_1}^t E(s) ds \le \mathscr{G}(t_1) - \mathscr{G}(t) \le \mathscr{G}(t_1). \tag{66}$$

Hence

$$\int_0^\infty \mathscr{E}(s) ds < \infty. \tag{67}$$

Define

$$I(t) = \int_0^t \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds, \tag{68}$$

we know that

$$I(t) \le c \int_0^t \mathscr{E}(s) ds. \tag{69}$$

Without loss of generality assuming t_1 so large that $I(t_1) > 0$, then

$$0 < I(t_1) \le I(t) < \infty, \quad \forall t \ge t_1. \tag{70}$$

Using Jensen's inequality and by (30) and (A2)-(A3), we can derive from (56) that for some constant $q > 0$,

$$\begin{aligned} L'(t) &\le -q\mathscr{E}(t) + \frac{cI(t)}{I(t)} \int_{\Gamma_1} \left[(-k')^{p\cdot(1/p)} \circ u \right] d\Gamma \\ &\le -q\mathscr{E}(t) + cI(t) \left[\frac{1}{I(t)} \int_{\Gamma_1} (-k')^p \circ u d\Gamma \right]^{(1/p)} \\ &\le -q\mathscr{E}(t) + cI^{1-(1/p)}(t) \left[\int_{\Gamma_1} \frac{k''}{\eta} \circ u d\Gamma \right]^{(1/p)} \\ &\le -q\mathscr{E}(t) + \frac{c}{[\eta(t)]^{(1/p)}} \left[\int_{\Gamma_1} k'' \circ u d\Gamma \right]^{(1/p)} \\ &\le -q\mathscr{E}(t) + \frac{c}{[\eta(t)]^{(1/p)}} \left[-\mathscr{E}'(t) \right]^{(1/p)}. \end{aligned} \tag{71}$$

We multiply (71) by $\mathscr{E}^{p-1}(t)$ and use (19) to deduce

$$\begin{aligned} \left(L\mathscr{E}^{p-1}\right)'(t) \le L'(t)\mathscr{E}^{p-1}(t) &\le -q\mathscr{E}^p(t) \\ &+ c\left[-\frac{\mathscr{E}'(t)}{\eta(t)} \right]^{(1/p)} \mathscr{E}^{p-1}(t). \end{aligned} \tag{72}$$

By Young's inequality, we have for any $\varepsilon_1 > 0$,

$$\left(L\mathscr{E}^{p-1}\right)'(t) \le -q\mathscr{E}^p(t) + \varepsilon_1 \mathscr{E}^p(t) + \frac{c}{\varepsilon_1}\left[-\frac{\mathscr{E}'(t)}{\eta(t)} \right]. \tag{73}$$

Taking $\varepsilon_1 < (1/2)q$, we conclude

$$\left(L\mathscr{E}^{p-1}\right)'(t) \le -\frac{q}{2}\mathscr{E}^p(t) - c\frac{\mathscr{E}'(t)}{\eta(t)}. \tag{74}$$

Define $F(t) = \eta L\mathscr{E}^{p-1} + c\mathscr{E} \sim \mathscr{E}$. Multiplying (74) by $\eta(t)$, we have

$$F'(t) \le -\frac{q}{2}\eta(t)\mathscr{E}^p(t). \tag{75}$$

Then there exists a certain constant $q_0 > 0$ such that

$$F'(t) \le -q_0 \eta(t) F^p(t), \tag{76}$$

from which we obtain

$$\mathscr{E}(t) \le c_3 \left(1 + \int_0^t \eta(s) ds \right)^{-(1/(p-1))}, \tag{77}$$

where c_3 is a positive constant.

Combining (**I**) and (**II**) and using the boundedness of $\eta(t)$ and $\mathscr{E}(t)$, we can get (23).

Case 2. The general case.

Define

$$I(t) := q \int_{t_1}^t \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds. \tag{78}$$

In view of (67), we can take $0 < q < 1$ such that

$$I(t) < 1, \quad \forall t \ge t_1. \tag{79}$$

Without loss of generality, we assume that $I(t) > 0$ for all $t \ge t_1$. On the other hand, we define

$$\pi(t) := \int_{t_1}^t k''(s) \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds. \tag{80}$$

From (30), we can easily get $\pi(t) \le -c\mathrm{E}'(t)$. As $H(t)$ is strictly convex on $(0, r]$ and $H(0) = 0$, we see that

$$H(\lambda x) \leq \lambda H(x), \quad i = 1, 2, 0 \leq \lambda \leq 1, x \in (0, r]. \tag{81}$$

It follows from Jensen's inequality and (11) and (79) that

$$\begin{aligned}
\pi_1(t) &= \frac{1}{qI(t)} \int_{t_1}^{t} I(t)(k''(s))q\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds \\
&\geq \frac{1}{qI(t)} \int_{t_1}^{t} I(t)\eta(s)H(-k'(s))q\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds \\
&\geq \frac{\eta(t)}{qI(t)} \int_{t_1}^{t} H(I(t)(-k'(s)))q\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds \\
&\geq \frac{\eta(t)}{q} H\left(\frac{1}{I(t)} \int_{t_1}^{t} I(t)(-k'(s))q\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds\right) \\
&= \frac{\eta(t)}{q} H\left(q\int_{t_1}^{t} (-k')(s)\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds\right) \\
&= \frac{\eta(t)}{q} \overline{H}\left(q\int_{t_1}^{t} (-k'(s))\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds\right),
\end{aligned} \tag{82}$$

where $\overline{H}$, which is strictly convex and increasing function on $(0, \infty)$ of class C^2, is called the extension of H. We infer from (82) that

$$\int_{t_1}^{t} (-k'(s))\int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds \leq \frac{1}{q}\overline{H}^{-1}\left(\frac{q\pi(t)}{\eta(t)}\right). \tag{83}$$

Then we can get from (59) that for any $t \geq t_1$,

$$F'(t) \leq -\chi\mathscr{E}(t) + c\overline{H}^{-1}\left(\frac{q\pi(t)}{\eta(t)}\right). \tag{84}$$

Denote

$$H_0(t) = \overline{H}'(t). \tag{85}$$

For $r_0 < r$, we define $\mathscr{K}_1(t)$ by

$$\mathscr{K}_1(t) = H_0\left(r_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)F(t) + \mathscr{E}(t) \sim \mathscr{E}(t). \tag{86}$$

Since $E'(t) \leq 0$, $\overline{H}' > 0$, and $\overline{H}'' > 0$, we get from (84) that

$$\begin{aligned}
\mathscr{K}_1'(t) &= r_0\frac{\mathscr{E}'(t)}{\mathscr{E}(0)}H_0'\left(r_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)F(t) + H_0\left(r_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)F'(t) + \mathscr{E}'(t) \\
&\leq -m\mathscr{E}(t)H_0\left(r_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + cH_0\left(r_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\overline{H}^{-1}\left(\frac{q\pi(t)}{\eta(t)}\right).
\end{aligned} \tag{87}$$

We denote by $\overline{H}^*$ the conjugate function of the convex function $\overline{H}$ (see Arnold [33]), and then

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}\left[(\overline{H}')^{-1}(s)\right] \tag{88}$$

satisfies Young's inequality,

$$AB \leq \overline{H}^*(A) + \overline{H}(B). \tag{89}$$

Taking $A=\overline{H}_0'(r_0(\mathrm{E}(t)/\mathrm{E}(0)))$ and $B=\overline{H}^{-1}((q\pi(t))/\eta(t))$, and using $\overline{H}^*(s)\le s(\overline{H}')^{-1}(s)$ and (87), we have

$$\begin{aligned}\mathcal{K}_1'(t)&\le-\chi\mathcal{E}(t)H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+c\overline{H}^*\left(H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\right)+c\frac{q\pi(t)}{\eta(t)}\\&\le-\chi\mathcal{E}(t)H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+cH_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)(\overline{H}')^{-1}\left(H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\right)+c\frac{q\pi(t)}{\eta(t)}\\&\le-\chi\mathcal{E}(t)H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+cH_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)(\overline{H}')^{-1}\left(\overline{H}'\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\right)+c\frac{q\pi(t)}{\eta(t)}\\&\le-(\chi\mathcal{E}(0)-cr_0)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+cq\frac{\pi(t)}{\eta_1(t)}.\end{aligned}\tag{90}$$

We multiply (90) by $\eta(t)$ to arrive at

$$\begin{aligned}\eta(t)\mathcal{K}_1'(t)&\le-(\chi\mathcal{E}(0)-cr_0)\eta(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+cq\pi(t)\\&\le-(\chi\mathcal{E}(0)-cr_0)\eta(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)-c\mathcal{E}'(t).\end{aligned}\tag{91}$$

The functional $\mathcal{K}_2(t)$ is defined by

$$\mathcal{K}_2(t)=\eta(t)\mathcal{K}_1(t)+c\mathcal{E}(t).\tag{92}$$

Then we can easily obtain that there exist constants $\beta_5>0$ and $\beta_6>0$ such that

$$\beta_5\mathcal{K}_2(t)\le\mathrm{E}(t)\le\beta_6\mathcal{K}_2(t).\tag{93}$$

Choosing a suitable $r_0>0$, and defining $H_3(t)=tH_0(r_0t)$, from (91), we infer that for a constant $\gamma>0$,

$$\mathcal{K}_2'(t)\le-\gamma\eta(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right):=-\gamma\eta(t)H_3\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right).\tag{94}$$

It follows from $0\le r_0(\mathcal{E}(t)/\mathcal{E}(0))<r$ that for any $t>0$,

$$H_0\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)=\overline{H}'\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)=H'\left(r_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right).\tag{95}$$

Using (93), we have

$$R(t):=\frac{\beta_5\mathcal{K}_2(t)}{\mathcal{E}(0)}\sim\mathcal{E}(t).\tag{96}$$

Since $H_3'(t)=H_0(r_0t)+r_0tH_0'(r_0t)$, then, noting the strict convexity of H_0 on $(0,r]$, we know $H_3'(t),H_3(t)>0$ on $(0,1]$. By (94), we conclude that there exists $\gamma_1>0$ such that for any $t\ge t_1$,

$$R'(t)\le-\gamma_1\eta(t)H_3(R(t)).\tag{97}$$

Integrating (97) over (t_1,t), we see that

$$\int_{t_1}^{t}\frac{-R'(s)}{H_3(R(s))}\mathrm{d}s\ge\gamma_1\int_{t_1}^{t}\eta(s)\mathrm{d}s\Rightarrow\int_{r_0R(t)}^{r_0R(t_1)}\frac{1}{sH_0(s)}\mathrm{d}s\ge\gamma_1\int_{t_1}^{t}\eta(s)\mathrm{d}s.\tag{98}$$

Define

$$H_4(t)=\int_t^r\frac{1}{sH_0(s)}\mathrm{d}s.\tag{99}$$

It is to verify that H_4 is strictly decreasing on $(0,r]$ and $\lim_{t\longrightarrow0}H_4(t)=+\infty$. It follows that

$$R(t)\le\frac{1}{r_0}H_4^{-1}\left(\zeta_1\int_{t_1}^{t}\eta(s)\mathrm{d}s\right).\tag{100}$$

Combining (96) and (100), we can obtain (21). This finishes the proof of Theorem 2

Data Availability

No data were used during this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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