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Research Article

On Some Algebraic Properties of *n*-Refined Neutrosophic Elements and *n*-Refined Neutrosophic Linear Equations

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This paper studies the problem of determining invertible elements (units) in any n-refined neutrosophic ring. It presents the necessary and sufficient condition for any n-refined neutrosophic element to be invertible, idempotent, and nilpotent. Also, this work introduces some of the elementary algebraic properties of n-refined neutrosophic matrices with a direct application in solving n-refined neutrosophic algebraic equations.

1. Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [1]. It becomes a useful tool in many areas of science such as number theory [2, 3], solving equations [4], and medical studies [5, 6]. Also, we find many applications of neutrosophic structures in statistics [7], optimization [8], topology [9], and decision making [10, 11].

On the other hand, neutrosophic algebra began in [12], where Smarandache and Kandasamy defined concepts such as neutrosophic groups and neutrosophic rings. These notions were handled widely by Agboola et al. in [13, 14], where homomorphisms and AH-substructures were studied [15].

Recently, there is an increasing interest by the generalizations of neutrosophic algebraic structures. Smarandache and Abobala proposed n-refined neutrosophic rings [16], modules [17, 18], and spaces [19–22].

Neutrosophic algebraic equations are useful in many scientific areas; there is a full description of their solutions in neutrosophic fields and refined neutrosophic fields [23]. In particular, the relations between neutrosophic matrices and equations are defined in [24].

From this point of view, we are motivated to generalize the previous studies so that we study some of the algebraic properties of n-refined neutrosophic elements such as invertibility, nilpotency, and idempotency. Also, we study elementary properties of n-refined neutrosophic matrices and their application in solving the n-refined neutrosophic linear system of equations as a new generalization of previous efforts in [23–25].

2. Preliminaries

Definition 1 (see [16])

Let $(R, +, \times)$ be a ring and $I_k, 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1 I + \cdots + a_n I_n;\ a_i \in R\}$ to be n-refined neutrosophic ring. If $n = 2$, we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Addition and multiplication on $R_n(I)$ are defined as follows:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i, \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j, \quad (1)$$

where $\times$ is the multiplication defined on the ring R and $x I_0 = x$ for all $x \in R$ $I_j I_i = I_i I_j = I_{\min(i,j)}, I_0 I_j = I_j$.

It is easy to see that $R_n(I)$ is a ring in the classical concept and contains a proper ring R.

Definition 2 (see [16])

Let $R_n(I)$ be an n-refined neutrosophic ring, and it is said to be commutative if $xy = yx$ for each $x, y \in R_n(I)$; if there is $1 \in R_n(I)$ such $1 \cdot x = x \cdot 1 = x$, then it is called an n-refined neutrosophic ring with unity.

Theorem 1 (see [16]). *Let $R_n(I)$ be an n-refined neutrosophic ring. Then,*

(a) *R is commutative if and only if $R_n(I)$ is commutative*
(b) *R has unity if and only if $R_n(I)$ has unity*
(c) $R_n(I) = \sum_{i=0}^{n} RI_i = \{\sum_{i=0}^{n} x_i I_i:\ x_i \in R\}$

Definition 3 (see [16])

(a) Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n:\ a_i \in P_i\}$ where P_i is a subset of R; we define P to be an AH-subring if P_i is a subring of R for all i; AHS-subring is defined by the condition $P_i = P_j$ for all i, j.
(b) P is an AH-ideal if P_i is a two-sided ideal of R for all i, and the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j.
(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Definition 4 (see [16])

Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$ be an AH-ideal; we define AH-factor $R(I)/P = \sum_{i=0}^{n} (R/P_i) I_i = \sum_{i=0}^{n} (x_i + P_i) I_i$; $x_i \in R$.

Theorem 2 (see [16])

Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$ be an AH-ideal;

$R_n(I)/P$ is a ring with the following two binary operations:

$$\begin{aligned}
&\sum_{i=0}^{n} (x_i + P_i) I_i + \sum_{i=0}^{n} (y_i + P_i) I_i \\
&\quad = \sum_{i=0}^{n} (x_i + y_i + P_i) I_i, \sum_{i=0}^{n} (x_i + P_i) I_i \times \sum_{i=0}^{n} (y_i + P_i) I_i \\
&\quad = \sum_{i=0}^{n} (x_i \times y_i + P_i) I_i.
\end{aligned} \tag{2}$$

3. Main Discussion

In this section, we study the invertibility of any element in any n-refined neutrosophic ring, and we show the conditions of idempotency and nilpotency in these rings. All rings in this section are considered with unity 1.

Definition 5 Let $X = A_0 + A_1 I_1 + \cdots + A_n I_n$ be an n-refined neutrosophic element; we define its canonical sequence as follows:

$$\begin{aligned}
&M_0 = A_0, \\
&M_j = A_0 + A_j + A_{j+1} + \cdots + A_n, \quad 1 \le j \le n.
\end{aligned} \tag{3}$$

For example,

$$M_3 = A_0 + A_3 + A_4 + \cdots + A_n. \tag{4}$$

Remark 1

The multiplication operation between two n-refined neutrosophic elements can be represented by the following equation:

$$\begin{aligned}
&(A_0 + A_1 I_1 + \cdots + A_n I_n)(B_0 + B_1 I_1 + \cdots + B_n I_n) = M_0 N_0 \\
&\quad + (M_n N_n - M_0 N_0) I_n + \sum_{i=1}^{n-1} (M_i N_i - M_{i+1} N_{i+1}) I_i,
\end{aligned} \tag{5}$$

where M_i and N_i are the canonical sequences of $A_0 + A_1 I_1 + \cdots + A_n I_n$ and $B_0 + B_1 I_1 + \cdots + B_n I_n$, respectively.

Proof. For $n = 0$, the statement is true easily. Suppose that it is true for $n = k$, we must prove it for $n = k + 1$. We compute the multiplication $L = (A_0 + A_1 I_1 + \cdots + A_{k+1} I_{k+1})(B_0 + B_1 I_1 + \cdots + B_{k+1} I_{k+1})$.

$$\begin{aligned}
&(A_0 + A_1 I_1 + \cdots + A_{k+1} I_{k+1})(B_0 + B_1 I_1 + \cdots + B_{k+1} I_{k+1}) = (A_0 + A_1 I_1 + \cdots + A_k I_k)(B_0 + B_1 I_1 + \cdots + B_k I_k) \\
&\quad + A_{k+1} I_{k+1} (B_0 + B_1 I_1 + \cdots + B_k I_k) + (A_0 + A_1 I_1 + \cdots + A_k I_k) B_{k+1} I_{k+1} + A_{k+1} I_{k+1} B_{k+1} I_{k+1} \\
&= M_0 N_0 + (M_k N_k - M_0 N_0) I_k + \sum_{i=1}^{k} (M_i N_i - M_{i+1} N_{i+1}) I_i + I_1 [A_{k+1} B_1 + A_1 B_{k+1}] \\
&\quad + I_2 [A_{k+1} B_2 + A_2 B_{k+1}] + \cdots + I_k [A_{k+1} B_k + A_k B_{k+1}] + I_{k+1} [A_0 B_{k+1} + A_{k+1} B_0 + A_{k+1} B_{k+1}].
\end{aligned} \tag{6}$$

Thus, the coefficient of I_{k+1} is $A_0 B_{k+1} + A_{k+1} B_0 + A_{k+1} B_{k+1} = (A_{k+1} + A_0)(B_{k+1} + B_0) - (A_0)(B_0) = M_{k+1} N_{k+1} - M_0 N_0$. Also, the coefficient of I_i, $1 \le i \le k$ is $M_i N_i - M_{i+1} N_{i+1} + A_{k+1} B_i + A_i B_{k+1} = (A_0 + A_i + A_{i+1} + \cdots + A_k)(B_0 + B_i + B_{i+1} + \cdots + B_k) - (A_0 + A_{i+1} + A_{i+2} + \cdots + A_k)(B_0 + B_{i+1} + B_{i+2} + \cdots + B_k) + A_{k+1} B_i + A_i B_{k+1} = (A_0 + A_i + A_{i+}$

$1+ \cdots + A_k + A_{k+1})$ $(B_0 + B_i + B_{i+1} + \cdots + B_k + B_{k+1}) - (A_0 + A_{i+1} + A_{i+2} + \cdots + A_k + A_{k+1})$ $(B_0 + B_{i+1} + B_{i+2} + \cdots + B_k + B_{k+1}) = M_i N_i - M_{i+1} N_{i+1}$, where $1 \le i \le k+1$. Hence, our proof is completed by induction.

Theorem 3

Let $X = A_0 + A_1 I_1 + \cdots + A_n I_n$ be an n-refined neutrosophic element, then it is invertible if and only if $M_j, 0 \le j \le n$ are invertible. The inverse of X is $X^{-1} = (M_0)^{-1} + (M_n^{-1} - M_0^{-1}) I_n + \sum_{j=1}^{n-1} (M_j^{-1} - M_{j+1}^{-1}) I_j = (A_0)^{-1} + ((A_0 + A_1 + \cdots + A_n)^{-1} - (A_0 + A_2 + \cdots + A_n)^{-1}) I_1 + ((A_0 + A_2 + \cdots + A_n)^{-1} - (A_0 + A_3 + \cdots + A_n)^{-1}) I_2 + ((A_0 + A_3 + \cdots + A_n)^{-1} - (A_0 + A_4 + \cdots + A_n)^{-1}) I_3 + \cdots + ((A_0 + A_n)^{-1} - (A_0)^{-1}) I_n$.

Proof. X is invertible if and only if there exists $Y = B_0 + B_1 I_1 + \cdots + B_n I_n$, where $XY = YX = 1$. By using Remark 14, we can write the following:

$M_0 N_0 + (M_n N_n - M_0 N_0) I_n + \sum_{i=1}^{n-1} (M_i N_i - M_{i+1} N_{i+1}) I_i = 1$. This implies that $M_0 N_0 = 1$ and $M_i N_i - M_{i+1} N_{i+1} = 0$ for all i, where 0 is the zero element. Hence, we get $M_i N_i = M_{i+1} N_{i+1} = M_0 N_0 = 1$. So, M_j, $0 \le j \le n$ are invertible.

On the other hand, we put $X^{-1} = (M_0)^{-1} + (M_n^{-1} - M_0^{-1}) I_n + \sum_{j=1}^{n-1} (M_j^{-1} - M_{j+1}^{-1}) I_j$, and now we compute XX^{-1} as follows:

$$XX^{-1} = M_0 M_0^{-1} + \left(M_1 M_1^{-1} - M_2 M_2^{-1}\right) I_1 + \left(M_2 M_2^{-1} - M_3 M_3^{-1}\right) I_2 + \cdots + \left(M_n M_n^{-1} - M_0 M_0^{-1}\right) I_n = 1. \tag{7}$$

Example 1

Considering $Z(I) = \{a + bI_1 + cI_2;\ a, b, c \in Z_2\}$ the 2-refined neutrosophic ring of integers, the set of invertible elements in Z_2 is $\{-1, 1\}$. Hence, the set of all invertible elements in the corresponding 2-refined neutrosophic ring is $\{1, -1, 1 - 2I_2, -1 + 2I_2, 1 - 2I_1, -1 + 2I_1, 1 + 2I_1 - 2I_2, -1 - 2I_1 + 2I_2\}$.

Theorem 4

Let $X = A_0 + A_1 I_1 + \cdots + A_n I_n$ be an n-refined neutrosophic element, and we have the following:

(a) X is nilpotent if and only if M_j for all j are nilpotent

(b) X is idempotent if and only if M_j for all j are idempotent

Proof

(a) First of all we will prove that $X^r = M_0^r + I_n [(M_n)^r - (M_0)^r] + \sum_{i=1}^{n-1} ((M_i^r) - (M_{i+1}^r)) I_i$.

We use the induction, for $r = 1$ it is clear. Suppose that it is true for $r = k$, we prove it for $k + 1$.

$$\begin{aligned}
X^{k+1} &= X^k X = \left(M_0^k + I_n\left[(M_n)^k - (M_0)^k\right] + \sum_{i=1}^{n-1}\left(\left(M_i^k\right) - \left(M_{i+1}^k\right)\right) I_i \right)(A_0 + A_1 I_1 + \cdots + A_n I_n) \\
&= \left(M_0^k + I_n\left[(M_n)^k - (M_0)^k\right] + \sum_{i=1}^{n-1}\left(\left(M_i^k\right) - \left(M_{i+1}^k\right)\right) I_i \right)\left(M_0 + (M_n - M_0) I_n + \sum_{i=1}^{n-1} (M_i - M_{i+1}) \right) \\
&= M_0^k M_0 + I_n\left[(M_n)^k M_n - M_0^k M_0\right] + \sum_{i=1}^{n-1}\left(\left(M_i^k M_i\right) - \left(M_{i+1}^k M_{i+1}\right)\right) I_i \\
&= M_0^{k+1} + I_n\left[(M_n)^{k+1} - (M_0)^{k+1}\right] + \sum_{i=1}^{n-1}\left(\left(M_i^{k+1}\right) - \left(M_{i+1}^{k+1}\right)\right) I_i.
\end{aligned} \tag{8}$$

X is nilpotent if there is a positive integer r such that $X^r = 0$. This is equivalent to

$$\begin{aligned}
M_0^r &= (M_n)^k \\
&= \left(M_j\right)^k \\
&= 0 \quad \text{for all } j, \text{which implies the proof.}
\end{aligned} \tag{9}$$

(b) The proof is similar to (a).

4. *n*-Refined Neutrosophic Linear Algebraic Equations

This section is dedicated to introduce an algorithm to solve n-refined neutrosophic linear equations over any n-refined neutrosophic field by turning them into classical systems of numbers.

Also, we discuss some elementary properties of n-refined neutrosophic matrices.

Definition 6

Let $\mathbf{F_n}(\mathbf{I})$ be any n-refined neutrosophic field. The n-refined linear neutrosophic equation with one variable over $\mathbf{F_n}(\mathbf{I})$ is defined as follows:

$$\begin{aligned} &\mathbf{AX}+\mathbf{B}=0,\\ &\mathbf{A},\mathbf{B},\mathbf{X}\in \mathbf{F}_n(I), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mathbf{A}&=\mathbf{a}_0+\mathbf{a}_1\mathbf{I}_1+\cdots+\mathbf{a_n}\mathbf{I_n},\\ B&=\mathbf{b}_0+\mathbf{b}_1\mathbf{I}_1+\cdots+\mathbf{b_n}\mathbf{I_n},\\ \mathbf{X}&=\mathbf{x}_0+\mathbf{x}_1\mathbf{I}_1+\cdots+\mathbf{x_n}\mathbf{I_n}. \end{aligned} \tag{11}$$

Theorem 5

Let $\mathbf{F_n}(\mathbf{I})$ be any n-refined neutrosophic field and $(*)\mathbf{AX}+\mathbf{B}=0$ be any n-refined linear neutrosophic equation over $\mathbf{F_n}(\mathbf{I})$. Then, $(*)$ is solvable over $\mathbf{F_n}(\mathbf{I})$ if and only if the following classical system is solvable over the classical field F:

(1) $\mathbf{a}_0\mathbf{x}_0+\mathbf{b}_0=0$

(2) $(\mathbf{a}_0+\mathbf{a_n})(\mathbf{x}_0+\mathbf{x_n})+(\mathbf{b}_0+\mathbf{b_n})=0$

(3) $(\mathbf{a}_0+\mathbf{a_n}+\mathbf{a_{n-1}})(\mathbf{x}_0+\mathbf{x_n}+\mathbf{x_{n-1}})+(\mathbf{b}_0+\mathbf{b_n}+\mathbf{b_{n-1}})=0$

.

.

(n+1−) $(\mathbf{a}_0+\mathbf{a}_1+\cdots+\mathbf{a_n})(\mathbf{x}_0+\mathbf{x}_1+\cdots+\mathbf{x_n})+(\mathbf{b}_0+\mathbf{b}_1+\cdots+\mathbf{b_n})=0$

Proof. We will show that Equation (18) is equivalent to the previous classical system of equations.

We compute Equation (18) by using the canonical form, and we get

$$\mathbf{M}_0\mathbf{N}_0+(\mathbf{M_n}\mathbf{N_n}-\mathbf{M}_0\mathbf{N}_0)\mathbf{I_n}+\sum_{\mathbf{i}=1}^{\mathbf{n}-1}(\mathbf{M_i}\mathbf{N_i}-\mathbf{M_{i+1}}\mathbf{N_{i+1}})\mathbf{I_i}=-\mathbf{b}_0-\mathbf{b}_1\mathbf{I}_1-\cdots-\mathbf{b_n}\mathbf{I_n}, \tag{12}$$

where $\mathbf{M_i}$ and $\mathbf{N_i}$ are the canonical forms of A and X, respectively.

From (12), we get the following classical system:

$$\begin{aligned} \mathbf{M}_0\mathbf{N}_0&=-\mathbf{b}_0,\\ \mathbf{M_n}\mathbf{N_n}-\mathbf{M}_0\mathbf{N}_0&=-\mathbf{b_n},\\ \mathbf{M_i}\mathbf{N_i}-\mathbf{M_{i+1}}\mathbf{N_{i+1}}&=-\mathbf{b_i},\quad \textbf{for all } 1\le \mathbf{i}\le \mathbf{n}-1. \end{aligned} \tag{13}$$

The equation $\mathbf{M}_0\mathbf{N}_0=-\mathbf{b}_0$ equivalents $\mathbf{a}_0\mathbf{x}_0+\mathbf{b}_0=0$. The equation $\mathbf{M_n}\mathbf{N_n}-\mathbf{M}_0\mathbf{N}_0=-\mathbf{b_n}$ equivalents $(\mathbf{a}_0+\mathbf{a_n})(\mathbf{x}_0+\mathbf{x_n})+(\mathbf{b}_0+\mathbf{b_n})=0$.

Also, any equation with form $\mathbf{M_i}\mathbf{N_i}-\mathbf{M_{i+1}}\mathbf{N_{i+1}}=-\mathbf{b_i}$ **for all** $1\le \mathbf{i}\le \mathbf{n}-1$ equivalents $(\mathbf{a}_0+\mathbf{a_n}+\mathbf{a_{n-1}}+\cdots+\mathbf{a_i})(\mathbf{x}_0+\mathbf{x_n}+\mathbf{x_{n-1}}+\cdots+\mathbf{x_i})+(\mathbf{b}_0+\mathbf{b_n}+\mathbf{b_{n-1}}+\cdots+\mathbf{b_i})=0$ by mathematical induction; thus, our proof is complete.

Now, we can apply the previous theorem to solve n-refined neutrosophic linear equations, and we illustrate an example.

Example 2

Let R be the real field and $\mathbf{R}_3(\mathbf{I})$ be its corresponding 3-refined neutrosophic field. Consider the following 3-refined neutrosophic Equation (18) $(1+\mathbf{I}_2+\mathbf{I}_3)\mathbf{X}+(\mathbf{I}_1+2\mathbf{I}_2)=0$. To solve it, we turn it into the classical equivalent system.

(1) $1\cdot \mathbf{x}_0+0=0$; its solution $\mathbf{x}_0=0$.

(2) $(1+1)(\mathbf{x}_0+\mathbf{x}_3)+(0+0)=0$; its solution is $\mathbf{x}_0+\mathbf{x}_3=0$; **thus** $\mathbf{x}_3=0$.

(3) $(1+1+1)(\mathbf{x}_0+\mathbf{x}_3+\mathbf{x}_2)+(0+0+2)=0$; its solution is $3(\mathbf{x}_0+\mathbf{x}_3+\mathbf{x}_2)=-2$; **thus** $\mathbf{x}_2=-2/3$.

(4) $(1+1+1+0)(\mathbf{x}_0+\mathbf{x}_3+\mathbf{x}_2+\mathbf{x}_1)+(0+1+2+0)=0$; its solution is $\mathbf{x}_0+\mathbf{x}_3+\mathbf{x}_2+\mathbf{x}_1=-1$; **thus** $\mathbf{x}_1=-1/3$.

Hence, the solution of Equation (18) is

$$X=-\frac{2}{3}I_2-\frac{1}{3}I_1. \tag{14}$$

Definition 7

Let $A=\begin{pmatrix} a_{11} & \cdots & a_{1m}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nm}\end{pmatrix}$ be an $m\times n$ matrix; if $a_{ij}=x+yI_1+zI_2+\cdots+tI_n\in R_n(I)$, then it is called an n-refined neutrosophic matrix, where $R_n(I)$ is an n-refined neutrosophic ring.

Remark 2

If A is an $m\times n$ matrix, then it can be represented as an element of the n-refined neutrosophic ring of matrices like the following: $A=B+CI_1+DI_2+\cdots+KI_n$ where $D,B,C,\ldots,K$ are classical matrices with elements from the ring R and from size $m\times n$.

For example, $A=\begin{pmatrix} 2+I_1+3I_2-I_3 & 1-I_1-I_2\\ 3+4I_2+2I_3 & 1+I_1\end{pmatrix}=\begin{pmatrix} 2 & 1\\ 3 & 1\end{pmatrix}+\begin{pmatrix} 1 & -1\\ 0 & 1\end{pmatrix}I_1+\begin{pmatrix} 3 & -1\\ 4 & 0\end{pmatrix}I_2+I_3\begin{pmatrix} -1 & 0\\ 2 & 0\end{pmatrix}$ is a 3-refined neutrosophic matrix.

Remark 3

The identity with respect to multiplication is the normal unitary matrix.

Definition 8

Let A be a square $\mathbf{m}\times\mathbf{m}$ n-refined neutrosophic matrix, then it is called invertible if there exists an n-refined square $\mathbf{m}\times\mathbf{m}$ neutrosophic matrix B such that $\mathbf{AB}=\mathbf{U_{m\times m}}$ where $\mathbf{U_{m\times m}}$ is the unitary classical matrix.

Remark 4

Let $X=A_0+A_1I_1+\cdots+A_nI_n$ be a square $m\times m$ n-refined neutrosophic matrix, then it is invertible if and only if M_j, $0\le j\le n$ are invertible. The inverse of X is

$$\begin{aligned}X^{-1} &= (M_0)^{-1} + \left(M_n^{-1} - M_0^{-1}\right)I_n + \sum_{j=1}^{n-1}\left(M_j^{-1} - M_{j+1}^{-1}\right)I_j \\ &= (A_0)^{-1} + \left((A_0 + A_1 + \cdots + A_n)^{-1} - (A_0 + A_2 + \cdots + A_n)^{-1}\right)I_1 \\ &\quad + \left((A_0 + A_2 + \cdots + A_n)^{-1} - (A_0 + A_3 + \cdots + A_n)^{-1}\right)I_2 \\ &\quad + \left((A_0 + A_3 + \cdots + A_n)^{-1} - (A_0 + A_4 + \cdots + A_n)^{-1}\right)I_3 + \cdots + \left((A_0 + A_n)^{-1} - (A_0)^{-1}\right)I_n.\end{aligned} \quad (15)$$

The proof holds directly as a special case of Theorem 3.

Definition 9

We defined the determinant of a square $m \times m$ n-refined neutrosophic matrix as

$$\begin{aligned}\det X &= \det A_0 + [\det(A_0 + A_1 + \cdots + A_n) - \det(A_0 + A_2 + \cdots + A_n)]I_1 \\ &\quad + [\det(A_0 + A_2 + \cdots + A_n) - \det(A_0 + A_3 + \cdots + A_n)]I_2 + \cdots \\ &\quad + [\det(A_0 + A_n) - \det(A_0)]I_n = \det(M_0) + (\det(M_n) - \det(M_0))I_n + \sum_{i=1}^{n-1}(\det(M_i) - \det(M_{i+1}))I_i.\end{aligned} \quad (16)$$

This definition is supported by the condition of invertibility.

Theorem 6

Let $X = A_0 + A_1 I_1 + \cdots + A_n I_n$ be a square $m \times m$ n-refined neutrosophic matrix, and we have the following:

(a) X is invertible if and only if $\det X \neq 0$

(b) If $Y = B_0 + B_1 I_1 + \cdots + B_n I_n$ is a square $m \times m$ n-refined neutrosophic matrix, then $\det XY = \det X \det Y$

(c) $\det X^{-1} = (\det X)^{-1}$

Proof.

(a) If $\det X \neq 0$, this will be equivalent to $\det M_j \neq 0$ for all j, i.e., M_j are invertible; thus, X is invertible according to Theorem 3.

(b) $XY = M_0 N_0 + (M_n N_n - M_0 N_0)I_n + \sum_{i=1}^{n-1}(M_i N_i - M_{i+1} N_{i+1})I_i$. Hence,

$$\begin{aligned}\det XY &= \det(M_0 N_0) + I_n[\det(M_n N_n) - \det(M_0 N_0)] \\ &\quad + \sum_{i=1}^{n-1}[(\det(M_i N_i) - \det(M_{i+1} N_{i+1}))I_i] \\ &= \det M_0 \det N_0 + I_n[\det(M_n)\det(N_n) - \det(M_0)\det(N_0)] \\ &\quad + \sum_{i=1}^{n-1}(\det(M_i)\det(N_i) - \det(M_{i+1})\det(N_{i+1}))I_i \\ &= \left[\det(M_0) + (\det(M_n) - \det(M_0))I_n + \sum_{i=1}^{n-1}(\det(M_i) - \det(M_{i+1}))I_i\right] \\ &\quad \cdot \left[\det(N_0) + (\det(N_n) - \det(N_0))I_n + \sum_{i=1}^{n-1}(\det(N_i) - \det(N_{i+1}))I_i\right] = \det X \det Y.\end{aligned} \quad (17)$$

(c) It holds directly from (b).

Now, we can find an easy algorithm to solve a linear system of n-refined neutrosophic algebraic equations over any n-refined neutrosophic field by using the inverse matrix method.

We construct an example.

Example 3

Consider the following system of 2-refined neutrosophic linear equations:

$$(2 + I_1 + 3I_2)X + (1 - I_1 - I_2)Y = -I_1, \quad (18)$$

$$(3 + 4I_2)X + (1 + I_1)Y = I_2. \quad (19)$$

The corresponding refined neutrosophic matrix is $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix}$.

We have the following:

(a) $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$ where $B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$. $B + D = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$, $B + C + D = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$.

(b) $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$, $(B + D)^{-1} = \begin{pmatrix} 1/5 & 0 \\ -7/5 & 1 \end{pmatrix}$, $(B + C + D)^{-1} = \begin{pmatrix} 2/19 & 1/19 \\ -7/19 & 6/19 \end{pmatrix}$.

(c) $A^{-1} = B^{-1} + I_1 \quad [(B + C + D)^{-1} - (B + D)^{-1}] \quad + I_2 [(B + D)^{-1} - B^{-1}] = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} + I_1 \begin{pmatrix} -9/95 & 1/19 \\ 98/95 & -13/19 \end{pmatrix} + I_2 \begin{pmatrix} 6/5 & -1 \\ -22/5 & 3 \end{pmatrix} = \begin{pmatrix} -1 - (9/95)I_1 + (6/5)I_2 & 1 + (1/19)I_1 - I_2 \\ 3 + (98/95)I_1 - (22/5)I_2 & -2 - (13/19)I_1 + 3I_2 \end{pmatrix}$.

It is easy to find that $A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\det B = -1$, $\det (B + D) = 5$, $\det (B + C + D) = 19$, $\det A = -1 + I_1 [19 - 5] + I_2 [5 - (-1)] = -1 + 14I_1 + 6I_2$.

Since A is invertible, we get the solution of the previous system of the 2-refined linear system by computing the product:

$$A^{-1} \begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} -1 - \frac{9}{95}I_1 + \frac{6}{5}I_2 & 1 + \frac{1}{19}I_1 - I_2 \\ 3 + \frac{98}{95}I_1 - \frac{22}{5}I_2 & -2 - \frac{13}{19}I_1 + 3I_2 \end{pmatrix},$$

$$\begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} I_1 \left[1 + \frac{9}{95} - \frac{6}{5} + \frac{1}{19}\right] \\ I_1 \left[-3 - \frac{98}{95} + \frac{22}{5} - \frac{13}{19}\right] + I_2 [-2 + 3] \end{pmatrix},$$

$$= \begin{pmatrix} -I_1 \frac{1}{19} \\ -\frac{6}{19}I_1 + I_2 \end{pmatrix}. \tag{20}$$

Thus,

$$X = -\frac{1}{19}I_1, \qquad Y = -\frac{6}{19}I_1 + I_2. \tag{21}$$

5. Conclusion

In this paper, we have determined the necessary and sufficient conditions for the invertibility, nilpotency, and idempotency of elements in a refined neutrosophic ring. In particular, we have studied some of algebraic properties of n-refined neutrosophic matrices such as determinants and inverses with an application solving the n-refined neutrosophic linear algebraic system of equations.

As a future research direction, we aim to study n-refined neutrosophic Diophantine equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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