Research Article

A Third-Order Shear Deformation Theory for Bending Behaviors of Rotating FGM Beams Resting on Elastic Foundation with Geometrical Imperfections in Thermal Environments

Nguyen Van Dang

University of Transport Technology, 54 Trieu Khuc, Thanh Xuan, Hanoi, Vietnam

Correspondence should be addressed to Nguyen Van Dang; dangnv@utt.edu.vn

Received 12 February 2021; Revised 9 March 2021; Accepted 16 March 2021; Published 15 April 2021

Academic Editor: Chiara Bedon

Copyright © 2021 Nguyen Van Dang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Beam-shaped components in large mechanical structures such as propellers, gas turbine blades, engine turbines, rotating railway bridges, and so on, when operating, usually engage in rotational movement around the fixed axis. Studying the mechanical behavior of these structures has great significance in engineering practice. Therefore, this paper is the first investigation on the static bending of rotating functionally graded material (FGM) beams with initial geometrical imperfections in thermal environments, where the higher-order shear deformation theory and the finite element method (FEM) are exercised. The material properties of beams are assumed to be varied only in the thickness direction and changed by the temperature effect, which increases the correctness and proximity to technical reality. The numerical results of this work are compared with those of other published papers to evaluate the accuracy of the proposed theory and mechanical model used in this paper. A series of parameter studies is carried out such as geometrical and material properties, especially the rotational speed and temperature, to evaluate their influences on the bending responses of structures.

1. Introduction

Functionally graded materials (FGM) are made from two or more different materials, where the mechanical and physical properties vary smoothly from one surface to the other one. One common type of FG material is fabricated from ceramic and metal, where the mechanical properties change in one [1–4] or multi directions of the structures [5]. These materials are widely used in some important fields such as aerospace, nuclear plants, and so on. Due to the novel advantage of these materials, scientists across the world have focused on studying the mechanical behaviors of FGM structures. Giunta et al. [3] used several axiomatic refined beam theories to investigate the static bending of FGM beams based on a Navier solution. Shi-Rong et al. [6] introduced an exact solution for the static bending of FGM beams, where Timoshenko and Euler–Bernoulli beam theories were employed. Hui-Shen and Zhen-Xin [7] employed

a higher-order shear deformation beam theory to find the solution of the nonlinear bending, vibration, and buckling of FGM beams resting on an elastic foundation, in which the effect of temperature was taken into calculations. Aldousari [8] used the finite element method to establish the finite element formulations for the static deflection and bending stresses analysis of FGM beams based on Euler–Bernoulli beam theory. Shan et al. [9] analyzed the nonlinear bending of FGM nano-beams in a thermal environment based on a high-order shear deformation theory and perturbation method. Katili and Katili [10] presented a new two-node element with the Hermitian interpolation functions to study the static and free vibration response of FGM beams. Çelik and Artan [11] employed Euler–Bernoulli beam theory to carry out the static bending analysis of bi-direction FGM nano-beams. Magnucki et al. [12] used both analytical approach and finite element method to figure out the three-point bending of beams, where the mechanical

characteristics of materials were assumed to be varied in the thickness direction. Sahmani and Safaei [13] used a refined exponential shear deformation model to explore the nonlinear bending and post-buckling responses of FGM beams. Demirbas et al. [14] investigated the stress evaluations of FGM beams based on a refined beam theory and the Carrera unified formulation. Garga et al. [15] used a fourth-order zigzag theory and finite element model to research the bending behavior of sandwich FGM beams. Hadji [16] studied static analysis of functionally graded beams with considering porosities using a new higher-order shear deformation theory. Pei and Li [17] used Navier's solution to find an exact solution for the static deflection responses of FG curved beams. Bagheri et al. [18] used the first-order shear deformation plate theory to investigate the non-axisymmetric buckling behavior of isotropic homogeneous annular plates subjected to simultaneous effects of uniform temperature rise and constant angular speed.

In technical operations, some mechanical components can be involved rotational movements such as rotor blades, turbine blades, rotating railway bridges, etc. Therefore, mechanical response investigations of these elements with rotational movements also play a very important role in computational design, which attracted scientists worldwide. Some plentiful publications can be counted as follows. Pradhan and Murmu [19] presented the results of the mechanical behavior of rotating nano-beam using the differential quadrature method. Li et al. [20] considered the effect of the bending and stretching on the free vibration behavior of a rotating FGM beam based on a dynamic model. Das [21] used the Ritz method and Timoshenko theory to calculate the in-plane and out-of-plane mechanical behavior of rotating FGM beams. Dejin et al. [22] introduced the results of vibration analysis of rotating micro-beams considering the initial geometrical imperfection. Khosravi et al. [23] employed the Timoshenko beam theory and von Kármán type of kinematic assumptions to examine the influence of uniform temperature elevation on the vibration of rotating composite beams reinforced with carbon nanotubes which may lead to instability. Timoshenko beam theory and von Kármán type of kinematic assumptions were also used by Khosravi and his coworkers [24] to study the thermal buckling of rotating carbon nanotube reinforced composite beams. Kiani et al. [25] investigated the buckling behavior of an isotropic homogeneous rotating annular plate subjected to uniform compression on both inner and outer edges. Arvin et al. [26] researched the free vibration treatment of prebuckled and postbuckled rotating functionally graded beams using the Euler–Bernoulli beam theory. Hosseini et al. [27] analyzed the buckling and postbuckling of rotating clamped-clamped functionally graded beams due to interaction between thermal environment and rotation based on The Euler–Bernoulli beam theory.

In engineering practice, imperfection occurs as a result of the manufacturing, transporting, and handling processes. The imperfection appearance will affect the work of the structure. Research on the mechanical behavior of structures that account for initial geometrical imperfections has also been studied by scientists worldwide, for example, the problem of nonlinear static and dynamic buckling and vibration behavior of structures. The plate and shell structures made of FGM material taking into account the initial geometrical imperfections were also presented by Duc et al. in the works [28–33], [34].

Based on the above review, it can be seen that there are no any publications dealing with the bending analysis of rotating (around one fixed axis) FG beams resting on elastic foundation in thermal environments. Therefore, this is truly a novel exploration, which has a significant meaning in engineering practices. As a result, this work aims to focus on the vibration response of the mentioned structures based on the higher-order shear deformation theory of Reddy and the finite element method.

The body of this paper is structured as follows. Section 2 presents fully the finite element formulations for the bending problem of rotating FGM beams resting on elastic foundations, in which the effects of the initial geometrical imperfection and temperature are considered. Verification problems are carried out in Section 3. Section 4 presents numerical explorations and discussions on the bending behavior of rotating FGM beams. Novel investigation points are concluded in Section 5.

2. Finite Element Model of Rotating FGM Beam in a Thermal Environment

Consider an FGM beam with the length L , width b , and thickness h as shown in Figure 1. The beam has an initial geometric imperfection $w_{\text{imp}}(x)$ in the z -direction, and the structure is resting on a two-parameter elastic foundation with k_w and k_s coefficients. The whole mechanical system is rotating around one fixed axis Δ with the speed ϕ ; one side of the beam is a distance from the rotating axis r .

Assume that the beam is made from ceramic (denoted by c) and metal (denoted by m), where their volume proportions (V_c and V_m) are changed according to the thickness direction based on the following function [1, 2, 7, 35–39]:

$$V_c = \left(\frac{z}{h} + \frac{1}{2} \right)^n, \quad (1)$$

$$V_m = 1 - V_c, \text{ with } n \geq 0,$$

where z is the thickness coordinate variable with $-(h/2) \leq z \leq (h/2)$ and n is the volume fraction gradient index and its variation in which Young's modulus E , the density ρ , Poisson's ratio ν , and the coefficient of the thermal expansion α are functions of the power-law distribution as [1, 2, 7]

FIGURE 1: The model of a rotating FGM beam resting on a two-parameter elastic foundation.

$$\begin{aligned}
 E(z) &= E_m + (E_c - E_m) \left(\frac{1}{2} + \frac{z}{h} \right)^n, \\
 \rho(z) &= \rho_m + (\rho_c - \rho_m) \left(\frac{1}{2} + \frac{z}{h} \right)^n, \\
 \nu(z) &= \nu_m + (\nu_c - \nu_m) \left(\frac{1}{2} + \frac{z}{h} \right)^n, \\
 \alpha(z) &= \alpha_m + (\alpha_c - \alpha_m) \left(\frac{1}{2} + \frac{z}{h} \right)^n.
 \end{aligned} \quad (2)$$

The beam is placed in a thermal environment; therefore, the material properties are changed by the temperature as follows [35]:

$$P = P_0 (P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3), \quad (3)$$

in which $T = T_0 + \Delta T$, ΔT is the temperature increment, $T_0 = 300K$ is the room temperature (this temperature also

does not generate the stress of the material), and $P_0, P_{-1}, P_1, P_2, P_3$ are constants, which depend on different materials.

To describe exactly the mechanical responses of FGM beams, this work uses the third-order shear deformation theory of Reddy; thus, the displacements u and w in the x - and z -directions at any point with the coordinate (x, z) are expressed as follows:

$$\begin{cases} u(x, z) = u_0(x, 0) + z\varphi_x - \frac{4}{3h^2} z^3 \left(\varphi_x + \frac{\partial w_0}{\partial x} \right), \\ w(x, z) = w_0(x, 0) + w_{\text{imp}}(x), \end{cases} \quad (4)$$

in which $w_0(x, 0)$ is the displacement of the point with x coordinate in the neutral axis.

The longitudinal and shear strains of the beam are calculated as follows:

$$\begin{aligned}
 \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \underbrace{\frac{\partial u_0}{\partial x}}_{\varepsilon_{0x}} + z \underbrace{\frac{\partial \varphi_x}{\partial x}}_{\varepsilon_{1x}} + z^3 \underbrace{\frac{-4}{3h^2} \left(\frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right)}_{\varepsilon_{3x}} + \underbrace{\frac{\partial w_0}{\partial x} \frac{dw_{\text{imp}}}{dx}}_{\varepsilon_{\text{imp}}} = \varepsilon_{0x} + z\varepsilon_{1x} + z^3\varepsilon_{3x} + \varepsilon_{\text{imp}}, \\
 \gamma_{xz} &= \varphi_x + \frac{\partial w_0}{\partial x} + z^2 \frac{-4}{h^2} \left(\varphi_x + \frac{\partial w_0}{\partial x} \right) = \gamma_{0xz} + z^2 \gamma_{2xz}.
 \end{aligned} \quad (5)$$

The normal and shear stresses are expressed as follows:

$$\begin{cases} \sigma_{xx} = E(\varepsilon_{xx} - \varepsilon_{xT}), \\ \tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz}, \end{cases} \quad (6)$$

where the stress relating to the temperature is $\varepsilon_{xT} = \alpha(z)(T - T_0)$.

The energy of the FGM beam has the following expression:

$$\begin{aligned} U^E &= \frac{1}{2} \int_V (\sigma_{xx}^T \varepsilon_{xx} + \tau_{xz}^T \gamma_{xz}) dV \\ &= \frac{1}{2} \int_V \left(\begin{Bmatrix} \varepsilon_{0x} & z\varepsilon_{1x} & z^3\varepsilon_{3x} & \varepsilon_{imp} & -\varepsilon_{xT} \end{Bmatrix}^T E \begin{Bmatrix} \varepsilon_{0x} \\ z\varepsilon_{1x} \\ z^3\varepsilon_{3x} \\ \varepsilon_{imp} \\ -\varepsilon_{xT} \end{Bmatrix} \right) dV \\ &\quad + \frac{1}{2} \int_V \left(\begin{Bmatrix} \gamma_{0xz} & z^2\gamma_{2xz} \end{Bmatrix}^T \frac{E}{2(1+\nu)} \begin{Bmatrix} \gamma_{0xz} \\ z^2\gamma_{2xz} \end{Bmatrix} \right) dV \\ &= \frac{1}{2} \int_V \left(\begin{aligned} &\varepsilon_{0x}^T E \varepsilon_{0x} + \varepsilon_{0x}^T z E \varepsilon_{1x} + \varepsilon_{0x}^T z^3 E \varepsilon_{3x} + \varepsilon_{0x}^T E \varepsilon_{imp} - \varepsilon_{0x}^T E \varepsilon_{xT} \\ &\varepsilon_{1x}^T E z \varepsilon_{0x} + \varepsilon_{1x}^T E z^2 \varepsilon_{1x} + \varepsilon_{1x}^T E z^4 \varepsilon_{3x} + \varepsilon_{1x}^T E z \varepsilon_{imp} - \varepsilon_{1x}^T E z \varepsilon_{xT} \\ &\varepsilon_{3x}^T E z^3 \varepsilon_{0x} + \varepsilon_{3x}^T E z^4 \varepsilon_{1x} + \varepsilon_{3x}^T E z^6 \varepsilon_{3x} + \varepsilon_{3x}^T E z^3 \varepsilon_{imp} - \varepsilon_{3x}^T E z^3 \varepsilon_{xT} \\ &\varepsilon_{imp}^T E \varepsilon_{0x} + \varepsilon_{imp}^T E z \varepsilon_{1x} + \varepsilon_{imp}^T E z^3 \varepsilon_{3x} + \varepsilon_{imp}^T E \varepsilon_{imp} - \varepsilon_{imp}^T E \varepsilon_{xT} \\ &-\varepsilon_{xT}^T E \varepsilon_{0x} - \varepsilon_{xT}^T E z \varepsilon_{1x} - \varepsilon_{xT}^T E z^3 \varepsilon_{3x} - \varepsilon_{xT}^T E \varepsilon_{imp} + \varepsilon_{xT}^T E \varepsilon_{xT} \end{aligned} \right) dV \\ &\quad + \frac{1}{2} \int_V \left(\gamma_{0xz}^T \frac{E}{2(1+\nu)} \gamma_{0xz} + \gamma_{0xz}^T \frac{E z^2}{2(1+\nu)} \gamma_{2xz} + \gamma_{2xz}^T \frac{E z^2}{2(1+\nu)} \gamma_{0xz} + \gamma_{2xz}^T \frac{E z^4}{2(1+\nu)} \gamma_{2xz} \right) dV. \end{aligned} \quad (7)$$

The energy of the elastic foundation has the following form:

$$U^F = \frac{1}{2} b \int_L \left(k_w w_0^2 + k_s \left(\frac{\partial w_0}{\partial x} \right)^2 \right) dx, \quad (8)$$

where k_w and k_s are the two coefficients of the elastic foundation.

For the FGM beam rotating around one axis Δ with the speed ϕ , the potential energy of this beam generated by the rotational movement is calculated as [22, 40]

$$U^R = \frac{1}{2} \int_L \left(Q_\phi(x) \left(\frac{\partial w_0}{\partial x} \right)^2 \right) dx, \quad (9)$$

with the centrifugal force Q_ϕ [22]:

$$Q_\phi = \frac{1}{2} \int_S \left(\rho \phi^2 \left[r(L-x) + \frac{1}{2}(L^2 - x^2) \right] \right) dS, \quad (10)$$

where ρ is the density of the material.

The work done by external uniformly distributed load P_0 acting on the FGM beam is calculated as

$$W^{\text{For}} = b \int_L (w_0^T P_0) dx. \quad (11)$$

The beam is balanced by the external forces; the following equation is obtained by the minimizing the potential energy as

$$\delta(U^E + U^F + U^R) - \delta W^{\text{For}} = 0. \quad (12)$$

Herein, a two-node beam element is used, where each node has four degrees of freedom:

$$\mathbf{q}_e = \sum_{i=1}^2 \left\{ \begin{array}{c} u_{0i} \\ \varphi_{xi} \\ w_{0i} \\ \left(\frac{\partial w_0}{\partial x} \right)_i \end{array} \right\}, \quad (13)$$

in which the displacement components at each point within the beam element are calculated through Lagrange and Hermit interpolation functions N_i and H_i .

$$\left\{ \begin{array}{l} u_0 = \sum_{i=1}^2 N_i u_{0i} = N_u \mathbf{q}_e, \\ \varphi_x = \sum_{i=1}^2 N_i \varphi_{xi} = N_\varphi \mathbf{q}_e, \\ w_0 = \sum_{i=1}^2 \left\{ H_i w_{0i} + H_{i+1} \left(\frac{\partial w_0}{\partial x} \right)_i \right\} = H \mathbf{q}_e, \\ \frac{\partial w_0}{\partial x} = \sum_{i=1}^2 \left\{ \frac{\partial H_i}{\partial x} w_{0i} + \frac{\partial H_{i+1}}{\partial x} \left(\frac{\partial w_0}{\partial x} \right)_i \right\} = H_1 \mathbf{q}_e, \\ \frac{\partial^2 w_0}{\partial x^2} = \sum_{i=1}^2 \left\{ \frac{\partial^2 H_i}{\partial x^2} w_{0i} + \frac{\partial^2 H_{i+1}}{\partial x^2} \left(\frac{\partial w_0}{\partial x} \right)_i \right\} = H_2 \mathbf{q}_e. \end{array} \right. \quad (14)$$

Equation (14) can be rewritten in the matrix form as

$$u = \left\{ \begin{array}{c} u_0 \\ \varphi_x \\ w_0 \\ \left(\frac{\partial w_0}{\partial x} \right) \end{array} \right\} = \left[\begin{array}{c} N_u \\ N_\varphi \\ H \\ H_1 \end{array} \right] \cdot \mathbf{q}_e = H_q \cdot \mathbf{q}_e. \quad (15)$$

Then, strain components are written according to the nodal displacement as

$$\left\{ \begin{array}{l} \varepsilon_{0x} = \frac{\partial u_0}{\partial x} = \frac{\partial N_u}{\partial x} \mathbf{q}_e = B_{0x} \mathbf{q}_e, \\ \varepsilon_{1x} = \frac{\partial \varphi_x}{\partial x} = \frac{\partial N_\varphi}{\partial x} \mathbf{q}_e = B_{1x} \mathbf{q}_e, \\ \varepsilon_{3x} = \frac{-4z^3}{3h^2} \left(\frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) = \frac{-4z^3}{3h^2} \left(\frac{\partial N_\varphi}{\partial x} + H_2 \right) \mathbf{q}_e = B_{3x} \mathbf{q}_e, \\ \varepsilon_{\text{imp}} = \frac{\partial w_0}{\partial x} \frac{dw_{\text{imp}}}{dx} = \frac{dw_{\text{imp}}}{dx} H_1 \mathbf{q}_e = B_{\text{imp}} \mathbf{q}_e, \\ \gamma_{0xz} = \varphi_x + \frac{\partial w_0}{\partial x} = N_\varphi \mathbf{q}_e + H_1 \mathbf{q}_e = (N_\varphi + H_1) \mathbf{q}_e = B_{0\gamma} \mathbf{q}_e, \\ \gamma_{2xz} = \frac{-4}{h^2} \left(\varphi_x + \frac{\partial w_0}{\partial x} \right) = \frac{-4}{h^2} (N_\varphi \mathbf{q}_e + H_1 \mathbf{q}_e) = \frac{-4}{h^2} (N_\varphi + H_1) \mathbf{q}_e = B_{2\gamma} \mathbf{q}_e. \end{array} \right. \quad (16)$$

Therefore, the energy of the element FGM beam is expressed as follows:

$$\begin{aligned}
 U_e^E &= \frac{1}{2} \mathbf{q}_e^T \int_V \begin{pmatrix} \mathbf{B}_{0x}^T E \mathbf{B}_{0x} + \mathbf{B}_{0x}^T z E \mathbf{B}_{1x} + \mathbf{B}_{0x}^T z^3 E \mathbf{B}_{3x} + \mathbf{B}_{0x}^T E \mathbf{B}_{imp} - \mathbf{B}_{0x}^T E \varepsilon_{xT} \\ \mathbf{B}_{1x}^T E z \mathbf{B}_{0x} + \mathbf{B}_{1x}^T E z^2 \mathbf{B}_{1x} + \mathbf{B}_{1x}^T E z^4 \mathbf{B}_{3x} + \mathbf{B}_{1x}^T E z \mathbf{B}_{imp} - \mathbf{B}_{1x}^T E z \varepsilon_{xT} \\ \mathbf{B}_{3x}^T E z^3 \mathbf{B}_{0x} + \mathbf{B}_{3x}^T E z^4 \mathbf{B}_{1x} + \mathbf{B}_{3x}^T E z^6 \mathbf{B}_{3x} + \mathbf{B}_{3x}^T E z^3 \mathbf{B}_{imp} - \mathbf{B}_{3x}^T E z^3 \varepsilon_{xT} \\ \mathbf{B}_{imp}^T E \mathbf{B}_{0x} + \mathbf{B}_{imp}^T E z \mathbf{B}_{1x} + \mathbf{B}_{imp}^T E z^3 \mathbf{B}_{3x} + \mathbf{B}_{imp}^T E \mathbf{B}_{imp} - \mathbf{B}_{imp}^T E \varepsilon_{xT} \\ - \varepsilon_{xT}^T E \mathbf{B}_{0x} - \varepsilon_{xT}^T E z \mathbf{B}_{1x} - \varepsilon_{xT}^T E z^3 \mathbf{B}_{3x} - \varepsilon_{xT}^T E \mathbf{B}_{imp} + \varepsilon_{xT}^T E \varepsilon_{xT} \end{pmatrix} dV \mathbf{q}_e, \\
 &+ \frac{1}{2} \mathbf{q}_e^T \int_V \begin{pmatrix} \mathbf{B}_{0xz}^T \frac{E}{2(1+\nu)} \mathbf{B}_{0xz} + \mathbf{B}_{0xz}^T \frac{E z^2}{2(1+\nu)} \mathbf{B}_{2xz} \\ + \mathbf{B}_{2xz}^T \frac{E z^2}{2(1+\nu)} \mathbf{B}_{0xz} + \mathbf{B}_{2xz}^T \frac{E z^4}{2(1+\nu)} \mathbf{B}_{2xz} \end{pmatrix} dV \mathbf{q}_e \\
 &= \frac{1}{2} \mathbf{q}_e^T \int_V \begin{pmatrix} \mathbf{B}_{0x}^T E \mathbf{B}_{0x} + \mathbf{B}_{0x}^T z E \mathbf{B}_{1x} + \mathbf{B}_{0x}^T z^3 E \mathbf{B}_{3x} + \mathbf{B}_{0x}^T E \mathbf{B}_{imp} \\ \mathbf{B}_{1x}^T E z \mathbf{B}_{0x} + \mathbf{B}_{1x}^T E z^2 \mathbf{B}_{1x} + \mathbf{B}_{1x}^T E z^4 \mathbf{B}_{3x} + \mathbf{B}_{1x}^T E z \mathbf{B}_{imp} \\ \mathbf{B}_{3x}^T E z^3 \mathbf{B}_{0x} + \mathbf{B}_{3x}^T E z^4 \mathbf{B}_{1x} + \mathbf{B}_{3x}^T E z^6 \mathbf{B}_{3x} + \mathbf{B}_{3x}^T E z^3 \mathbf{B}_{imp} \\ \mathbf{B}_{imp}^T E \mathbf{B}_{0x} + \mathbf{B}_{imp}^T E z \mathbf{B}_{1x} + \mathbf{B}_{imp}^T E z^3 \mathbf{B}_{3x} + \mathbf{B}_{imp}^T E \mathbf{B}_{imp} \end{pmatrix} dV \mathbf{q}_e \\
 &+ \frac{1}{2} \mathbf{q}_e^T \int_V \left(\mathbf{B}_{0xz}^T \frac{E}{2(1+\nu)} \mathbf{B}_{0xz} + \mathbf{B}_{0xz}^T \frac{E z^2}{2(1+\nu)} \mathbf{B}_{2xz} + \mathbf{B}_{2xz}^T \frac{E z^2}{2(1+\nu)} \mathbf{B}_{0xz} + \mathbf{B}_{2xz}^T \frac{E z^4}{2(1+\nu)} \mathbf{B}_{2xz} \right) dV \mathbf{q}_e \\
 &- \mathbf{q}_e^T \int_V \left(\mathbf{B}_{0x}^T E \varepsilon_{xT} + \mathbf{B}_{1x}^T E z \varepsilon_{xT} + \mathbf{B}_{3x}^T E z^3 \varepsilon_{xT} + \mathbf{B}_{imp}^T E \varepsilon_{xT} \right) dV + \frac{1}{2} \int_V \left(\varepsilon_{xT}^T E \varepsilon_{xT} \right) dV.
 \end{aligned} \tag{17}$$

Equation (17) can be also rewritten in the matrix form as

$$U_e^E = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e^E \mathbf{q}_e - \mathbf{q}_e^T \mathbf{P}_{Te} + \frac{1}{2} C_{Te}, \tag{18}$$

where $C_{Te} = 1/2 \int_V (\varepsilon_{xT}^T E \varepsilon_{xT}) dV$ is a constant dependent only on the thermal strain, which does not depend on the element nodal displacement vector.

The energy of the elastic foundation and centrifugal inertia force has the following expression:

$$U_e^F = \frac{1}{2} \mathbf{q}_e^T \left(b \int_L (k_1 H^T H + k_2 H_1^T H_1) dx \right) \mathbf{q}_e = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e^F \mathbf{q}_e,$$

$$U_e^R = \frac{1}{2} \mathbf{q}_e^T \left(\int_L (Q_\phi(x) H_1^T H_1) dx \right) \mathbf{q}_e = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e^R \mathbf{q}_e. \tag{19}$$

The work done by uniformly distributed load P_0 acting on the FGM beam element is calculated as

$$W_e^{\text{For}} = \mathbf{q}_e^T \left(b \int_L (H^T P_0) dx \right) = \mathbf{q}_e^T \mathbf{P}_e. \tag{20}$$

Substituting equations (18)–(20) into (12), one gets the equilibrium equation of the FGM beam taking into account the temperature as

$$\sum_e (\mathbf{K}_e^E + \mathbf{K}_e^F + \mathbf{K}_e^R) \mathbf{q}_e = \sum_e (\mathbf{P}_e + \mathbf{P}_{Te}). \tag{21}$$

One can see that the stiffness matrix of the element FGM beam includes the components, which are related to the rotational speed ϕ , elastic foundation coefficients k_w , k_s , and the geometrical imperfection coefficient w_0 of the FGM beam.

TABLE 1: Nondimensional maximum deflections $\bar{w}$ of the FGM beam under uniformly distributed load, S-S, $L/h = 16$.

n	This work					Ritz method [33]
	6 elements	8 elements	10 elements	12 elements	14 elements	
0	1.0094	1.0094	1.0094	1.0094	1.0094	1.00975
0.2	0.7561	0.7563	0.7564	0.7564	0.7564	0.75737
0.5	0.6397	0.6398	0.6399	0.6399	0.6399	0.64065
1	0.5661	0.5663	0.5664	0.5664	0.5664	0.56699
2	0.5073	0.5074	0.5075	0.5075	0.5075	0.50780
5	0.4441	0.4442	0.4443	0.4443	0.4443	0.44442

TABLE 2: Nondimensional maximum deflections $\bar{w} = (EI/P_0 L^4)w_{\max}$ of the FGM beam under uniformly distributed load resting on the two-parameter elastic foundation, S-S.

Foundation parameters		$L/h = 120$			
$\bar{K}_w$	$\bar{K}_s$	DQM [41]	Exact [41]	Exact [42]	This work
0	0	1.302290	1.302290	1.3033	1.301692
	10	0.644827	0.644827	0.6457	0.644679
	25	0.366111	0.366111	0.3671	0.366063
10	0	1.180567	1.180567	1.1814	1.180075
	10	0.613325	0.613326	0.6141	0.613192
	25	0.355668	0.355668	0.3566	0.355622
100	0	0.640074	0.640074	0.6403	0.639927
	10	0.425582	0.425582	0.4261	0.425517
	25	0.282846	0.282846	0.2836	0.282817

TABLE 3: Nondimensional maximum deflections $\bar{w} = (EI/P_0 L^4)w_{\max}$ of the FGM beam under uniformly distributed load resting on the two-parameter elastic foundation, C-C.

Foundation parameters		$L/h = 120$		
$\bar{K}_w$	$\bar{K}_s$	DQM [41]	Exact [42]	This work
0	0	0.26064	0.2616	0.26033
	10	0.20862	0.2095	0.20840
	25	0.16081	0.1617	0.16066
10	0	0.25547	0.2565	0.25518
	10	0.20528	0.2062	0.20507
	25	0.15880	0.1597	0.15865
100	0	0.21670	0.2174	0.21649
	10	0.17935	0.1800	0.17919
	25	0.14273	0.1435	0.14261

TABLE 4: The dependence of material characteristics on the temperature [35].

Materials	P_0	P_{-1}	P_1	P_2	P_3	P (300 K)
<i>Ceramic zirconium oxide (ZrO₂)</i>						
E (Pa)	244.27e9	0	-1.371e-3	1.214e-6	-3.681e-10	168.06e9
α (1/K)	12.766e-6	0	-1.491e-3	1.006e-5	-6.778e-11	18.591e-6
ν	0.288	0	1.133e-4	—	0	0.298
ρ (kg/m ³)	3657	0	0	—	0	3657
<i>Ceramic silicon nitride (Si₃N₄)</i>						
E (Pa)	348.43e9	0	-3.070e-4	2.160e-7	-8.946e-11	322.27e9
α (1/K)	5.8723e-6	0	9.095e-4	0	0	7.475e-6
ν	0.24	0	0	0	0	0.240
ρ (kg/m ³)	2370	0	0	0	0	2370
<i>Metal stainless steel SUS304</i>						
E (Pa)	201.04e9	0	3.079e-4	-6.534e-7	0	207.79e9
α (1/K)	12.330e-6	0	8.086e-4	0	0	15.321e-6
ν	0.326	0	-2.002e-4	3.797e-7	0	0.318
ρ (kg/m ³)	8166	0	0	0	0	8166

FIGURE 2: The dependence of Young modulus of ceramic and metal on the temperature.

FIGURE 3: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on temperature and boundary conditions, $(r)/(L)=1$, $\omega = 10$. (a) C-F. (b) S-S. (c) C-C.

FIGURE 4: The dependence of the nondimensional maximum deflection of the $\text{ZrO}_2/\text{SUS304}$ beam on temperature and boundary conditions, $(r)/(L)=1$, $\omega = 10$. (a) C-F. (b) S-S. (c) C-C.

The common boundary conditions are used in this paper as follows:

(i) Simply supported (denoted as S):

$$\begin{aligned} u_0 &= 0, \\ w &= 0. \end{aligned} \quad (22)$$

(ii) Clamped (denoted as C):

$$\begin{aligned} u_0 &= 0, \\ \varphi_x &= 0, \\ w &= 0, \\ \frac{\partial w}{\partial x} &= 0. \end{aligned} \quad (23)$$

and F represents free boundary condition. This work calculates for the FGM beam with three boundary conditions as follows:

- (iii) One side is clamped; the other side is free: C-F
- (iv) Fully simply supported beam: S-S
- (v) Fully clamped beam: C-C

3. Verification Study

To verify the theory and mechanical model used in this paper, this section carries out two examples to compare the results of the maximum deflection of the FGM beam.

Example 1. This example considers the bending response of the S-S functionally graded Al/ ZrO_2 beam. The geometrical and material properties of the beam are $L/h=16$, $E_m=70$ GPa, $E_c=200$ GPa, and $\nu_c = \nu_m=0.3$, where the top surface is metal (Al) and bottom surface is ceramic (ZrO_2). The beam is subjected to uniformly distributed load q_0 , and the nondimensional maximum deflection of the beam is defined as $\bar{w} = (w_{\max}/(5q_0L^4/384E_{Al}I))$ (with $I=bh^3/12$). Table 1 presents nondimensional maximum deflections $\bar{w}$ obtained from this work and the Ritz method [33], in which

FIGURE 5: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on temperature, volume fraction exponent (n), and boundary conditions, (r)/(L) = 1, $\omega = 10$. (a) C-F. (b) S-S. (c) C-C.

incremental mesh size is used in this example. It can be observed that the 10-element mesh ensures the required accuracy. Therefore, the following investigations will employ this mesh.

Example 2. This example continues to compare the results of nondimensional maximum deflections of the FGM beam resting on the two-parameter elastic foundation. The geometrical and material parameters of the structure are the length L , width b , thickness h , $\bar{K}_w = (k_w L^4/EI)$, and $\bar{K}_s = k_s L^2/EI$ (with $I = bh^3/12$). The beam is under uniformly distributed load q_0 . Tables 2 and 3 present nondimensional maximum deflections $\bar{w} = (EI/P_0 L^4)w_{\max}$ obtained from this work, the differential quadrature method (DQM) [41], and exact solutions [41, 42].

4. Numerical Results

Now, the bending analysis of the rotating FGM beam with initial geometrical imperfection resting on the two-parameter elastic foundation in the thermal environment is carried out in this section. Two types ($\text{ZrO}_2/\text{SUS304}$ and $\text{Si}_3\text{N}_4/\text{SUS304}$) of FGM material are considered, where the material properties are shown in Table 4. Figure 2 presents the dependence of the Young modulus of ceramic and metal on the temperature. The imperfection of beam is $w_{\text{imp}}(x) = Y_0 \sin(\pi x)$, in which Y_0 is the amplitude of the imperfection, and the imperfection ratio is $\lambda = (Y_0/L)$.

Nondimensional maximum deflection of the FGM beam and other parameters are defined as

FIGURE 6: The dependence of the nondimensional maximum deflection of the $\text{ZrO}_2/\text{SUS304}$ beam on temperature, volume fraction exponent (n), and boundary conditions, (r)/(L) = 1, $\omega = 10$. (a) C-F. (b) S-S. (c) C-C.

$$w^* = 100 \frac{E_0 h_0^3}{q_0 L^4} w_{\max},$$

$$K_w^* = \frac{k_w L^4}{D_0},$$

$$K_s^* = \frac{k_s L^2}{D_0},$$

$$\omega = L^2 \phi \sqrt{\frac{12 \rho_0}{E_0 h^2}},$$

$$D_0 = \frac{E_0 h^3}{12},$$

with $h_0 = L/20$ and $E_0 = 168.06 \cdot 10^9 \text{ N/m}^2$.

4.1. Bending Analysis of FGM Beam in a Thermal Environment

4.1.1. Influence of the Temperature. Consider an FGM beam structure with $L = 5$ m, $L/h = 20$, geometrical imperfection coefficient $\lambda = 0.001$, $K_w^* = 20$, $K_s^* = 2$, distance ratio $r/L = 1$, and rotational speed = 10. The active temperature is increased gradually from 300 K to 1400 K, and the volume fraction exponent n gets the values from 0 to 10. The dependences of the nondimensional maximum deflection of $\text{Si}_3\text{N}_4/\text{SUS304}$ and $\text{ZrO}_2/\text{SUS304}$ beams on the temperature are presented in Figures 3–7. The numerical results point out the following.

FIGURE 7: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on temperature and boundary conditions, $(r)/(L)=1$, $(n)=0.2$, $\omega = 10$. (a) C-F. (b) S-S. (c) C-C.

When increasing the temperature, the nondimensional maximum deflection of FGM beams depends on both material and boundary conditions. For the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam, in the cases of S-S and C-C boundaries, when increasing the temperature, the nondimensional maximum deflection increases for all values of n . However, for the C-F boundary, the nondimensional maximum deflection increases when $n \leq 1$; when n is greater than 1, the nondimensional maximum deflection decreases in the cases where the temperature is greater than 1100 K. For the $\text{ZrO}_2/\text{SUS304}$ beam, with all boundary conditions, the nondimensional maximum deflection increases when increasing the temperature. It is explained that when the temperature increases, the elastic modulus of both metal and ceramic materials

decreases, when the temperature is higher than 1100 K, the elastic modulus of SUS304 decreases stronger than that of Si_3N_4 , and the elastic modulus of ZrO_2 is greater than that of the metal SUS304.

On the other hand, due to the rotational movement of the beam, therefore, the nondimensional maximum deflection depends strongly on boundary conditions. This novel point is very different from the case of the without rotational movement phenomenon.

For the $\text{ZrO}_2/\text{SUS304}$ beam, when the temperature increases, the deflection shape of the beam w significantly changes. Besides, in the cases of S-S and C-C boundaries, normally, the deflection shape is symmetrical; however, due to the rotational movement phenomenon, the deflection shape of the beam is no longer symmetrical

FIGURE 8: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on the rotational speed and boundary conditions, $(r)/(L) = 1$, $(T) = 500$ (K). (a) C-F. (b) S-S. (c) C-C.

FIGURE 9: Continued.

FIGURE 9: The dependence of the nondimensional maximum deflection of the $\text{ZrO}_2/\text{SUS304}$ beam on the rotational speed and boundary conditions, $(r)/(L) = 1$, $(T) = 500$ (K). (a) C-F. (b) S-S. (c) C-C.

FIGURE 10: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on the rotational speed, volume fraction exponent (n), and boundary conditions, $(r)/(L) = 1$, $(T) = 500$ K. (a) C-F. (b) S-S. (c) C-C.

FIGURE 11: The dependence of the nondimensional maximum deflection of the $\text{ZrO}_2/\text{SUS304}$ beam on the rotational speed, volume fraction exponent (n), and boundary conditions, (r/L) = 1, (T) = 500 K. (a) C-F. (b) S-S. (c) C-C.

through position $x = L/2$. This means that the thermal environment has significantly changed the nondimensional maximum deflection as well as the deflection shape of the rotating FGM beam.

4.1.2. Effect of the Rotational Speed. Consider a rotating FGM beam with $L/h = 20$, imperfect coefficient $\lambda = 0.001$, $K_w^* = 20$, $K_s^* = 2$, distance ratio $r/L = 1$, and $T = 500 \text{ K}$. Change the value of the rotational speed ϕ so that the coefficient ω varies in a range of 0 to 20, while the volume fraction exponent n gets the value from 0 to 10. The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ and $\text{ZrO}_2/\text{SUS304}$ beams on

the rotational speed and the volume fraction exponent are presented in Figures 8–12. The following can be seen.

When the rotational speed increases, due to the influence of centrifugal force, the nondimensional maximum deflection of the beam decreases. However, for different materials, the response of the beam is also different, which is the novel point in comparison with the case without rotational movement phenomenon. For the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam, there is a value of the rotational speed for maximum deflection of this beam unchanged for all values of volume exponent n , where this speed depends on boundary conditions. And for the C-C beam, to keep the maximum deflection unchanged for all cases of the volume exponent n , the rotational speed needs to be higher than those of

FIGURE 12: The dependence of the nondimensional maximum deflection of the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam on the rotational speed and boundary conditions, $(r)/(L) = 1$, $(n) = 0.2$, $(T) = 500 \text{ K}$. (a) C-F. (b) S-S. (c) C-C.

FIGURE 13: The dependence of the nondimensional maximum deflection of the FGM beam on λ , $(r)/(L) = 1$, $\omega = 10$, $(T) = 1000 \text{ K}$.

FIGURE 14: The dependence of the nondimensional maximum deflection of the beam on the thickness (h) and the geometrical imperfection coefficient λ .

other boundaries (see Figure 10). However, this phenomenon does not happen for the $\text{ZrO}_2/\text{SUS304}$ beam (see Figure 11).

Figure 12 presents the dependence of the deflection shape of the beam on the rotational speed in the case of $n = 0.2$ and $T = 500$ K. It is easy to see that when the rotational speed increases, the deflection shape along the longitudinal direction of the beam changes significantly, especially for beams with symmetrical structures as S-S and C-C. The maximum deflection position is no longer located in the middle position ($x = L/2$) of the beam, which is due to the influence of centrifugal force; this position tends to move to the right side of position $x = L/2$. Thus, the rotational speed not only affects the deflection shape of the beam but also affects the maximum deflection of this beam.

4.1.3. Effect of the Initial Geometrical Imperfection. Consider an FGM beam resting on a two-parameter elastic foundation with $L/h = 20$, $K_w^* = 20$, $K_s^* = 2$, distance ratio $r/L = 1$, $T = 1000$ K, and $\omega = 10$. Change the geometrical imperfection coefficient λ so that its value ranges from 0 to 2.10^{-3} ; the nondimensional maximum deflection of the C-F functionally graded beams is presented in Figure 13. Figure 14 presents the dependence of the nondimensional maximum deflection of the beam on the thickness h in the cases of four different values of λ . One can see that when increasing the geometrical imperfection coefficient λ , the nondimensional maximum deflection decreases. This is because the stiffness of the beam increases as λ increases; however, the value of the nondimensional maximum deflection does not change much when the value of λ changes.

5. Conclusions

Based on the finite element method combining with the third-order shear deformation theory, finite element formulations are derived to carry out the static bending of the rotating FGM beams with initial geometrical imperfection resting on two-

parameter elastic foundations in thermal environments, in which material properties are assumed to be varied by temperature. The novel explorations of this work can be drawn as follows:

- (i) When increasing the value of temperature, material properties are changed; therefore, the nondimensional maximum deflection depends significantly on boundary conditions and the volume fraction exponent. Besides, the deflection shape of the beam is also affected by these parameters.
- (ii) When the rotational speed increases, the nondimensional maximum deflection decreases. For the $\text{Si}_3\text{N}_4/\text{SUS304}$ beam, there is a value of the rotational speed so that the nondimensional maximum deflection remains in all cases of the volume fraction exponent. Besides, the rotational speed has a strong effect on the deflection shape of the beam.
- (iii) Finally, when the initial geometrical imperfection coefficient increases, the nondimensional maximum deflection decreases; however, this change is not significant.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the University of Transport Technology Foundation for Science and Technology Development (Grant no. 1139/QD-DHCNGTVT).

References

- [1] H.-N. Nguyen, T. C. Tan, D. T. Luat, V.-D. Phan, D. V. Thom, and P. V. Minh, "Research on the buckling behavior of functionally graded plates with stiffeners based on the third-order shear deformation theory," *Materials*, vol. 12, no. 8, p. 1262, 2019.
- [2] V. T. Do, D. H. Doan, N. D. Duc, and T. Q. Bui, "Phase-field thermal buckling analysis for cracked functionally graded composite plates considering neutral surface," *Composite Structures*, vol. 182, pp. 542–548, 2017.
- [3] G. Giunta, S. Belouettar, and E. Carrera, "Analysis of FGM beams by means of classical and advanced theories," *Mechanics of Advanced Materials and Structures*, vol. 17, no. 8, pp. 622–635, 2010.
- [4] J. Murin, M. Aminbaghai, and V. Kutíš, "Exact solution of the bending vibration problem of FGM beams with variation of material properties," *Engineering Structures*, vol. 32, no. 6, pp. 1631–1640, 2010.
- [5] T. V. Do, D. K. Nguyen, N. D. Duc, D. H. Doan, and T. Q. Bui, "Analysis of bi-directional functionally graded plates by FEM and a new third-order shear deformation plate theory," *Thin-Walled Structures*, vol. 119, pp. 687–699, 2017.
- [6] L. Shi-Rong, C. Da-Fu, and W. Ze-Qing, "Bending solutions of FGM Timoshenko beams from those of the homogenous Euler–Bernoulli beams," *Applied Mathematical Modelling*, vol. 37, no. 10–11, pp. 7077–7085, 2013.
- [7] H.-S. Shen and Z.-X. Wang, "Nonlinear analysis of shear deformable FGM beams resting on elastic foundations in thermal environments," *International Journal of Mechanical Sciences*, vol. 81, pp. 195–206, 2014.
- [8] S. M. Aldousari, "Bending analysis of different material distributions of functionally graded beam," *Applied Physics A*, vol. 123, no. 1–9, p. 296, 2017.
- [9] W. Shan, B. Li, S. Qin, and H. Mo, "Nonlinear bending and vibration analyses of FG nanobeams considering thermal effects," *Materials Research Express*, vol. 7, no. 12, Article ID 125007, 2020.
- [10] A. M. Katili and I. Katili, "A simplified UI element using third-order Hermitian displacement field for static and free vibration analysis of FGM beam," *Composite Structures*, vol. 250, Article ID 112565, 2020.
- [11] M. Çelik and R. Artan, "An investigation of static bending of a bi-directional strain-gradient Euler–Bernoulli nano-beams with the method of initial values," *Microsystem Technologies*, vol. 26, no. 9, pp. 2921–2929, 2020.
- [12] K. Magnucki, J. Lewinski, and E. Magnucka-Blandzi, "An improved shear deformation theory for bending beams with symmetrically varying mechanical properties in the depth direction," *Acta Mechanica*, vol. 231, no. 10, pp. 4381–4395, 2020.
- [13] S. Sahmani and B. Safaei, "Influence of homogenization models on size-dependent nonlinear bending and post-buckling of bi-directional functionally graded micro/nano-beams," *Applied Mathematical Modelling*, vol. 82, pp. 336–358, 2020.
- [14] M. D. Demirbas, U. Caliskan, X. Xu, and M. Filippi, "Evaluation of the bending response of compact and thin-walled FG beams with CUF," *Mechanics of Advanced Materials and Structures*, vol. 145, 2020.
- [15] A. Garg, H. D. Chalak, and A. Chakrabarti, "Comparative study on the bending of sandwich FGM beams made up of different material variation laws using refined layerwise theory," *Mechanics of Materials*, vol. 151, Article ID 103634, 2020.
- [16] L. Hadji, "Influence of the distribution shape of porosity on the bending of FGM beam using a new higher order shear deformation model," *Smart Structures and Systems*, vol. 26, no. 2, pp. 253–262, 2020.
- [17] Y. L. Pei and L. X. Li, "A simplified theory of FG curved beams," *European Journal of Mechanics-A/Solids*, vol. 85, Article ID 104126, 2021.
- [18] H. Bagheri, Y. Kiani, and M. R. Eslami, "Asymmetric thermo-inertial buckling of annular plates," *Acta Mechanica*, vol. 228, no. 4, pp. 1493–1509, 2017.
- [19] S. C. Pradhan and T. Murmu, "Application of nonlocal elasticity and DQM in the flapwise bending vibration of a rotating nanocantilever," *Physica E: Low-Dimensional Systems and Nanostructures*, vol. 42, no. 7, pp. 1944–1949, 2010.
- [20] L. Li, D. G. Zhang, and W. D. Zhu, "Free vibration analysis of a rotating hub-functionally graded material beam system with the dynamic stiffening effect," *Journal of Sound and Vibration*, vol. 333, no. 5, pp. 1526–1541, 2014.
- [21] D. Das, "Free vibration and buckling analyses of geometrically non-linear and shear-deformable FGM beam fixed to the inside of a rotating rim," *Composite Structures*, vol. 179, pp. 628–645, 2017.
- [22] C. Dejin, F. Kai, and Z. Shijie, "Flapwise vibration analysis of rotating composite laminated Timoshenko microbeams with geometric imperfection based on a re-modified couple stress theory and isogeometric analysis," *European Journal of Mechanics - A/Solids*, vol. 76, pp. 25–35, 2019.
- [23] S. Khosravi, H. Arvin, and Y. Kiani, "Vibration analysis of rotating composite beams reinforced with carbon nanotubes in thermal environment," *International Journal of Mechanical Sciences*, vol. 164, Article ID 105187, 2019.
- [24] S. Khosravi, H. Arvin, and Y. Kiani, "Interactive thermal and inertial buckling of rotating temperature-dependent FG-CNT reinforced composite beams," *Composites Part B: Engineering*, vol. 175, Article ID 107178, 2019.
- [25] Y. Kiani, H. Bagheri, and M. R. Eslami, "Asymmetric compressive stability of rotating annular plates," *European Journal of Computational Mechanics*, vol. 28, no. 4, pp. 325–350, 2019.
- [26] H. Arvin, S. M. H. Hosseini, and Y. Kiani, "Free vibration analysis of pre/post buckled rotating functionally graded beams subjected to uniform temperature rise," *Thin-Walled Structures*, vol. 158, Article ID 107187, 2021.
- [27] S. M. H. Hosseini, H. Arvin, and Y. Kiani, "On buckling and post-buckling of rotating clamped-clamped functionally graded beams in thermal environment," *Mechanics Based Design of Structures and Machines*, vol. 66, pp. 1–16, 2020.
- [28] N. D. Duc, *Nonlinear Static and Dynamic Stability of Functionally Graded Plates and Shells*, Vietnam National University Press, Hanoi, Vietnam, 2014.
- [29] N. D. Duc, T. Q. Quan, and N. D. Khoa, "New approach to investigate nonlinear dynamic response and vibration of imperfect functionally graded carbon nanotube reinforced composite double curved shallow shells subjected to blast load and temperature," *Journal Aerospace Science and Technology*, vol. 71, pp. 360–372, 2017.
- [30] N. D. Duc and T. Q. Quan, "Nonlinear dynamic analysis of imperfect functionally graded material double curved thin shallow shells with temperature-dependent properties on elastic foundation," *Journal of Vibration and Control*, vol. 21, no. 7, pp. 1340–1362, 2015.
- [31] N. D. Duc, "Nonlinear dynamic response of imperfect eccentrically stiffened FGM double curved shallow shells on

- elastic foundation,” *Composite Structures*, vol. 99, pp. 88–96, 2013.
- [32] N. D. Duc and T. Q. Quan, “Nonlinear postbuckling of imperfect eccentrically stiffened P-FGM double curved thin shallow shells on elastic foundations in thermal environments,” *Composite Structures*, vol. 106, pp. 590–600, 2013.
 - [33] N. D. Duc, T. Q. Quan, and V. D. Luat, “Nonlinear dynamic analysis and vibration of shear deformable piezoelectric FGM double curved shallow shells under damping-thermo-electro-mechanical loads,” *Composite Structures*, vol. 125, pp. 29–40, 2015.
 - [34] M. Simsek, “Static analysis of a functionally graded beam under a uniformly distributed load by Ritz method,” *International Journal of Engineering and Applied Sciences (IJEAS)*, vol. 1, no. 3, pp. 1–11, 2009.
 - [35] Q. B. Tinh, V. D. Thom, H. T. T. Lan et al., “On the high temperature mechanical behaviors analysis of heated functionally graded plates using FEM and a new third-order shear deformation plate theory,” *Composite Part B*, vol. 92, pp. 218–241, 2016.
 - [36] N. V. Hoai, D. H. Doan, N. M. Khoa, T. V. Do, and H. T. Tran, “Phase-field buckling analysis of cracked stiffened functionally graded plates,” *Composite Structures*, vol. 217, pp. 50–59, 2019.
 - [37] T. Yu, T. Q. Bui, S. Yin et al., “On the thermal buckling analysis of functionally graded plates with internal defects using extended isogeometric analysis,” *Composite Structures*, vol. 136, pp. 684–695, 2016.
 - [38] P. T. Dat, D. V. Thom, and D. T. Luat, “Free vibration of functionally graded sandwich plates with stiffeners based on the third-order shear deformation theory,” *Vietnam Journal of Mechanics*, vol. 38, no. 2, pp. 103–122, 2016.
 - [39] N. D. Duc, T. D. Trinh, T. Van Do, and D. H. Doan, “On the buckling behavior of multi-cracked FGM plates,” *Proceedings of the International Conference on Advances in Computational Mechanics 2017*, vol. 28, pp. 29–45, 2018.
 - [40] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, Dover Publications, New York, NY, USA, 1989.
 - [41] W. Q. Chen, C. F. Lü, and Z. G. Bian, “A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation,” *Applied Mathematical Modelling*, vol. 28, no. 10, pp. 877–890, 2004.
 - [42] C. M. Wang, K. Y. Lam, and X. Q. He, “Exact solutions for Timoshenko beams on elastic foundations using green’s Functions*,” *Mechanics of Structures and Machines*, vol. 26, no. 1, pp. 101–113, 1998.