

Retraction

Retracted: Some Topological Approaches for Generalized Rough Sets via Ideals

Mathematical Problems in Engineering

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

- [1] T. M. Al-shami, H. Işık, A. S. Nawar, and R. A. Hosny, "Some Topological Approaches for Generalized Rough Sets via Ideals," *Mathematical Problems in Engineering*, vol. 2021, Article ID 5642982, 11 pages, 2021.

Research Article

Some Topological Approaches for Generalized Rough Sets via Ideals

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The idea of neighborhood systems is induced from the geometric idea of “near,” and it is primitive in the topological structures. Now, the idea of neighborhood systems has been extensively applied in rough set theory. The master contribution of this manuscript is to generate various topologies by means of the concepts of j -adhesion neighborhoods and ideals. Then, we define a new rough set model derived from these topologies and discussed main features. We show that these topologies are finer than those given in the previous ones under arbitrary binary relations. In addition, we elucidate that these topologies are finer than those topologies initiated based on different neighborhoods and ideals under reflexive relations. Several examples are provided to validate that our model is better than the previous ones.

1. Introduction

The idea of rough set theory (RST) was first put forth by Pawlak [1], where imperfect information gives rise to indiscernibility of objects. RST proves its adequacy to treat and model a lot of real-life issues that were constructed as a method to overcome imperfectness and ambiguity of information. The classical RST was characterized by a pair of approximation operators called lower and upper approximations which are established using equivalence classes. But at times, equivalence relations are tricky to be acquired in real-life issues due to the incompleteness of human information. So, a lot of ideas and articles have been introduced to generalizing the classical theory of RSs, for further specifics, see reference [2]. For interpretation of the granules, both Lin [3] and Yao [4] studied the RSs utilizing neighborhood systems. In fact, they freed RST from an equivalence relation which is a very inflexible obligation that restricts the real-life implementation scope of rough sets philosophy.

Quite recently, Abd El-Monsef et al. [5] applied j -neighborhoods to generalize the classical RST. Allam et al. presented the concepts of minimal right neighborhoods and minimal left neighborhoods in [6, 7], respectively. Dai et al. [8] presented new rough set models using maximal neighborhoods induced from similarity relations. Al-shami [9] defined containment neighborhoods and applied to protect medical staff from infected diseases. Also, Al-shami et al. [10] initiated several types of lower and upper approximations using N_j -neighborhoods. In [11], El-Bably and Fleifel investigated new topological structures by relations. El-Bably et al. [12] introduced some closure operators using arbitrary binary relation and generated some topologies from any binary relation without using sub-base or base.

Study of the rough set theory via topology is an enjoyable topic that received the attention of many researchers, see, for example [2, 13–19]. An ideal \mathcal{I} [20] on a nonempty finite set (universe) \mathcal{U} is a nonempty family of subsets of \mathcal{U} with

heredity property, as well as it is closed under finite unions. Some interesting papers studied ideals via rough set theory such as [15, 21]. Hosny [22] replaced Pawlak's approximations (lower and upper) by topological operators (interior and closure). Then, she [23] presented different methods to establish new rough set models using ideals. In 2020, Kandil et al. [24] offered the collection $\langle \mathcal{S}, \tilde{\mathcal{S}} \rangle$ generated by two ideals \mathcal{S} and $\tilde{\mathcal{S}}$ and proved that it is also an ideal on a universe.

From topology's view, our work discusses the notions of rough sets based on the topological spaces generated by j -adhesion neighborhoods and ideals. That is, it studies the concepts of lower and upper approximations in terms of j -adhesion neighborhoods via ideals. This approach minimizes the vagueness of uncertainty regions at their borders by increasing the lower approximation and decreasing the upper approximation which automatically implies increasing the accuracy measure of the uncertainty regions.

The layout of this paper is as follows. In Section 2, we recall three classes of neighborhood systems called N_j -neighborhoods, \mathcal{E}_j -neighborhoods, and \mathcal{P}_j -neighborhoods as well as the results that show the way of generating topologies using these classes of neighborhoods. In Section 3, we apply the notions of ideal and \mathcal{P}_j -neighborhood systems to establish new types of topologies. Based on these topologies, we propose new kind of rough set models and compare them with the previous ones under reflexive and arbitrary relations. We devote Section 4 to discuss the approximations and accuracy measures generated by \mathcal{E}_j -neighborhoods and \mathcal{P}_j -neighborhoods with ideals. Eventually, Section 5 gives conclusions and some directions for future works.

2. Preliminaries

In this part, we present the j -neighborhood space, which depends on a finite number of various kinds of arbitrary binary relations. So, we produce eight diverse topologies and investigate the relationships among these topologies. Then, we get eight techniques to find the lower and upper approximations of rough sets. Comparisons between the accuracy of these types of new approximations are attained.

Henceforward, we consider $j \in \{r, l, \langle r \rangle, \langle l \rangle, i, u, \langle i \rangle, \langle u \rangle\}$, unless otherwise specified.

2.1. Approximations and Topologies Generated by Different j -Neighborhoods

Definition 1 (see [5]). Let R be a binary relation on \mathcal{U} . The eight sorts of j -neighborhoods of any point $h \in \mathcal{U}$, say $N_j(h)$, are given by

- (1) $N_r(h) = \{z \in \mathcal{U} : hRz\}$.
- (2) $N_l(h) = \{z \in \mathcal{U} : zRh\}$.
- (3) $N_i(h) = N_r(h) \cap N_l(h)$.
- (4) $N_u(h) = N_r(h) \cup N_l(h)$.

- (5) $N_{\langle r \rangle}(h) = \cap \{N_r(z) : h \in N_r(z)\}$; if $N_r(z)$ does not exist there such that $h \in N_r(z)$, then $N_{\langle r \rangle}(h) = \emptyset$.
- (6) $N_{\langle l \rangle}(h) = \cap \{N_l(z) : h \in N_l(z)\}$; if $N_l(z)$ does not exist there such that $h \in N_l(z)$, then $N_{\langle l \rangle}(h) = \emptyset$.
- (7) $N_{\langle i \rangle}(h) = N_{\langle r \rangle}(h) \cap N_{\langle l \rangle}(h)$.
- (8) $N_{\langle u \rangle}(h) = N_{\langle r \rangle}(h) \cup N_{\langle l \rangle}(h)$.

Definition 2 (see [5]). The triple (\mathcal{U}, R, ψ_j) is called a j -neighborhood space (in short, N_jS), where ψ_j is a mapping from \mathcal{U} to $P(\mathcal{U})$ which assigns for each point in \mathcal{U} its j -neighborhood.

Proposition 1 (see [25]). Let (\mathcal{U}, R, ψ_j) be a N_jS and $h \in \mathcal{U}$. Then,

- (1) The reflexivity of R implies that $N_j(h)$ is a nonempty set.
- (2) The reflexivity of R implies that $N_{\langle j \rangle}(h)$ is a subset of $N_j(h)$, for every $j \in \{r, l, i, u\}$.
- (3) If R is a transitive relation, then $N_j(h)$ is a subset of $N_{\langle j \rangle}(h)$, for every $j \in \{r, l, i, u\}$.
- (4) The symmetry of R implies that $N_r(h) = N_l(h) = N_i(h) = N_u(h)$ and $N_{\langle r \rangle}(h) = N_{\langle l \rangle}(h) = N_{\langle i \rangle}(h) = N_{\langle u \rangle}(h)$.

Theorem 1 (see [5]). Let (\mathcal{U}, R, ψ_j) be a N_jS . Then, the family $\tau_j = \{M \subseteq \mathcal{U} : \forall h \in M, N_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .

Definition 3 (see [5]). Let (\mathcal{U}, R, ψ_j) be a N_jS . Then, we called a subset $M \subseteq \mathcal{U}$ a j -open set if $M \in \tau_j$, and we called its complement a j -closed set. The class Γ_j of all j -closed sets of a j -neighborhood space is given by $\Gamma_j = \{F \subseteq \mathcal{U} : F^c \in \tau_j\}$.

Definition 4 (see [5]). Let τ_j be a topology induced from j -neighborhoods. The j -lower, j -upper approximations, and j -accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\underline{R}_j(M) = \cup \{O \in \tau_j : O \subseteq M\} = \text{int}_j(M)$.
- (2) $\overline{R}_j(M) = \cap \{F \in \Gamma_j : M \subseteq F\} = \text{cl}_j(M)$.
- (3) $\alpha_j(M) = (|\underline{R}_j(M)| / |\overline{R}_j(M)|)$, where $|\overline{R}_j(M)| \neq 0$.

2.2. Approximations and Topologies Generated by Different \mathcal{E}_j -Neighborhoods

Definition 5 (see [26]). Let R be a binary relation on \mathcal{U} . The \mathcal{E}_j -neighborhood of a point $h \in \mathcal{U}$ (briefly, $\mathcal{E}_j(h)$) is formulated as

- (1) $\mathcal{E}_r(h) = \{z \in \mathcal{U} : \text{the intersection of } N_r(z) \text{ and } N_r(h) \text{ is nonempty}\}$.
- (2) $\mathcal{E}_l(h) = \{z \in \mathcal{U} : \text{the intersection of } N_l(z) \text{ and } N_l(h) \text{ is nonempty}\}$.
- (3) $\mathcal{E}_i(h) = \mathcal{E}_r(h) \cap \mathcal{E}_l(h)$.
- (4) $\mathcal{E}_u(h) = \mathcal{E}_r(h) \cup \mathcal{E}_l(h)$.

- (5) $\mathcal{E}_{\langle r \rangle}(h) = \{z \in \mathcal{U} : \text{the intersection of } N_{\langle r \rangle}(z) \text{ and } N_{\langle r \rangle}(h) \text{ is nonempty}\}$.
- (6) $\mathcal{E}_{\langle l \rangle}(h) = \{z \in \mathcal{U} : \text{the intersection of } N_{\langle l \rangle}(z) \text{ and } N_{\langle l \rangle}(h) \text{ is nonempty}\}$.
- (7) $\mathcal{E}_{\langle i \rangle}(h) = \mathcal{E}_{\langle r \rangle}(h) \cap \mathcal{E}_{\langle l \rangle}(h)$.
- (8) $\mathcal{E}_{\langle u \rangle}(h) = \mathcal{E}_{\langle r \rangle}(h) \cup \mathcal{E}_{\langle l \rangle}(h)$.

Theorem 2 (see [26]). *Let R be a binary relation on \mathcal{U} and $h \in \mathcal{U}$. Then,*

- (1) $h \in \mathcal{E}_j(z)$ iff $z \in \mathcal{E}_j(h)$.
- (2) *The reflexivity of \mathcal{R} implies that $\mathcal{E}_{\langle j \rangle}(h)$ is a subset of $\mathcal{E}_j(h)$ and $\mathcal{N}_j(h)$ is a subset of $\mathcal{E}_j(h)$ for every j .*
- (3) *The symmetry of \mathcal{R} implies that $\mathcal{E}_r(h)$, $\mathcal{E}_l(h)$, $\mathcal{E}_i(h)$, and $\mathcal{E}_u(h)$ are equal, and $\mathcal{E}_{\langle r \rangle}(h)$, $\mathcal{E}_{\langle l \rangle}(h)$, $\mathcal{E}_{\langle i \rangle}(h)$, and $\mathcal{E}_{\langle u \rangle}(h)$ are equal.*
- (4) *The transitivity of \mathcal{R} implies that $\mathcal{E}_j(h)$ is a subset of $\mathcal{E}_{\langle j \rangle}(h)$ for each $j \in \{u, i, r, l\}$.*
- (5) *If \mathcal{R} is transitive and symmetric, then $\mathcal{E}_j(h) = \mathcal{N}_j(h)$ and $\mathcal{E}_j(h) \subseteq \mathcal{E}_j(z)$ (if $h \in \mathcal{E}_j(z)$), for each j .*
- (6) *If \mathcal{R} is preorder, then $\mathcal{E}_j(h) = \mathcal{E}_{\langle j \rangle}(h)$, for all $j \in \{r, l, i, u\}$.*
- (7) *If \mathcal{R} is an equivalence, then $\mathcal{E}_j(h)$ are equal for each j , and $\mathcal{E}_j(h) = \mathcal{N}_j(h)$.*

Theorem 3 (see [26]). *For each j , the family $\tau_{\mathcal{E}_j} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{E}_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .*

Definition 6 (see [26]). Let (\mathcal{U}, R, ψ_j) be an \mathcal{E}_j S. We called a set $M \subseteq \mathcal{U}$ an \mathcal{E}_j -open set if $M \in \tau_{\mathcal{E}_j}$, and we called its complement an \mathcal{E}_j -closed set. The class $\Gamma_{\mathcal{E}_j}$ of all \mathcal{E}_j -closed sets is given by $\Gamma_{\mathcal{E}_j} = \{K \subseteq \mathcal{U} : K^c \in \tau_{\mathcal{E}_j}\}$.

Definition 7 (see [26]). Consider $\tau_{\mathcal{E}_j}$ as a topology induced by \mathcal{E}_j neighborhoods. For each j , the \mathcal{E}_j lower, \mathcal{E}_j upper approximations, and \mathcal{E}_j accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $L_j^\circ(M) = \bigcup \{O \in \tau_{\mathcal{E}_j} : O \subseteq M\} = \text{int}_{\mathcal{E}_j}(M)$.
- (2) $U_j^\circ(M) = \bigcap \{F \in \Gamma_{\mathcal{E}_j} : M \subseteq F\} = \text{cl}_{\mathcal{E}_j}(M)$.
- (3) $\mu_j(M) = (|L_j^\circ(M)|/|U_j^\circ(M)|)$, where $|U_j^\circ(M)| \neq 0$.

2.3. Approximations and Topologies Generated by Different j -Adhesion Neighborhoods

Definition 8 (see [27]). Let R be a binary relation on \mathcal{U} . The j -adhesion (\mathcal{P}_j) neighborhood of any point $h \in \mathcal{U}$ (denoted by $\mathcal{P}_j(h)$) is defined as

- (1) r -adhesion neighborhood: $\mathcal{P}_r(h) = \{z \in \mathcal{U} : N_r(z) \text{ is equal to } N_r(h)\}$.
- (2) l -adhesion neighborhood: $\mathcal{P}_l(h) = \{z \in \mathcal{U} : N_l(z) \text{ is equal to } N_l(h)\}$.
- (3) i -adhesion neighborhood: $\mathcal{P}_i(h) = \mathcal{P}_r(h) \cap \mathcal{P}_l(h)$.

- (4) u -adhesion neighborhood: $\mathcal{P}_u(h) = \mathcal{P}_r(h) \cup \mathcal{P}_l(h)$.
- (5) $\langle r \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle r \rangle}(h) = \{z \in \mathcal{U} : N_{\langle r \rangle}(z) \text{ is equal to } N_{\langle r \rangle}(h)\}$.
- (6) $\langle l \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle l \rangle}(h) = \{z \in \mathcal{U} : N_{\langle l \rangle}(z) \text{ is equal to } N_{\langle l \rangle}(h)\}$.
- (7) $\langle i \rangle$ -adhesion neighborhood $\mathcal{P}_{\langle i \rangle}(h) = \mathcal{P}_{\langle r \rangle}(h) \cap \mathcal{P}_{\langle l \rangle}(h)$.
- (8) $\langle u \rangle$ -adhesion neighborhood: $\mathcal{P}_{\langle u \rangle}(h) = \mathcal{P}_{\langle r \rangle}(h) \cup \mathcal{P}_{\langle l \rangle}(h)$.

Remark 1. It should be noted that the concept of \mathcal{P}_j -adhesion neighborhood of any point in \mathcal{U} in [27] is the same as the notion of core of neighborhood systems induced by R in [28].

Proposition 2 (see [25]). *Let R be a binary relation on \mathcal{U} and $h \in \mathcal{U}$. Then, \mathcal{P}_j -neighborhoods have the next properties:*

- (1) $\mathcal{P}_{\langle r \rangle}(h) = \mathcal{P}_l(h)$, $\mathcal{P}_{\langle l \rangle}(h) = \mathcal{P}_r(h)$.
- (2) $\mathcal{P}_{\langle i \rangle}(h) = \mathcal{P}_i(h)$, $\mathcal{P}_{\langle u \rangle}(h) = \mathcal{P}_u(h)$.

Lemma 1 (see [25]). *Let R be a binary relation on \mathcal{U} and $h, z \in \mathcal{U}$. Then,*

- (1) $h \in \mathcal{P}_j(h)$ for all j .
- (2) $z \in \mathcal{P}_j(h)$ iff $\mathcal{P}_j(z) = \mathcal{P}_j(h)$ for every $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$.

Corollary 1 (see [25]). *Let R be a binary relation on \mathcal{U} . The class $\wp(\mathcal{U}) = \{\mathcal{P}_j(y) | y \in \mathcal{U}\}$ forms a partition for \mathcal{U} in the cases of $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$.*

Proposition 3 (see [25]). *Consider R as a reflexive relation on \mathcal{U} and $h \in \mathcal{U}$. Then, $\mathcal{P}_j(h) \subseteq N_j(h) \subseteq E_j(h)$ for each j .*

Proposition 4 (see [25]). *Consider R as an equivalence relation on \mathcal{U} and $h \in \mathcal{U}$. Then, $\mathcal{P}_j(h) = N_j(h) = E_j(h)$ for each j .*

Theorem 4 (see [25]). *Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space $(\mathcal{P}_j$ S). Then, for each $j \in \{r, l, i, u\}$, the class $\tau_{\mathcal{P}_j} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{P}_j(h) \subseteq M\}$ forms a topology on \mathcal{U} .*

Definition 9 (see [25]). Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space. We called a set $M \subseteq \mathcal{U}$ a j -adhesion open set if $M \in \tau_{\mathcal{P}_j}$, and we called its complement a j -adhesion closed set. The family $\Gamma_{\mathcal{P}_j}$ of all j -adhesion closed sets of a j -neighborhood space is defined by $\Gamma_{\mathcal{P}_j} = \{K \subseteq \mathcal{U} : K^c \in \tau_{\mathcal{P}_j}\}$.

Remark 2 (see [25]). Let (\mathcal{U}, R, ψ_j) be a j -adhesion neighborhood space. Then,

- (1) The collection $\tau_{\mathcal{P}_j}$ is a quasidiscrete (clopen) topology on \mathcal{U} , $\forall j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$.
- (2) For each j , $\tau_j \subseteq \tau_{\mathcal{P}_j}$ provided that R is reflexive.

Definition 10 (see [25]). Let (\mathcal{U}, R, ψ_j) be a \mathcal{P}_j S and $\tau_{\mathcal{P}_j}$ be a topology generated by j -adhesion neighborhoods. The j -adhesion lower, j -adhesion upper approximations, and j -adhesion accuracy of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\mathfrak{L}_j(M) = \cup \left\{ O \in \tau_{\mathcal{P}_j} : O \subseteq M \right\} = \text{int}_{\mathcal{P}_j}(M)$.
- (2) $\mathfrak{U}_j(M) = \cap \left\{ F \in \Gamma_{\mathcal{P}_j} : M \subseteq F \right\} = \text{cl}_{\mathcal{P}_j}(M)$.
- (3) $\theta_j(M) = (|\mathfrak{L}_j(M)|/|\mathfrak{U}_j(M)|)$, where $|\mathfrak{U}_j(M)| \neq 0$.

2.4. Approximations and Topologies Generated by Different j -Neighborhoods and Ideals. In [23], Hosny presented an idea depending on generating different topologies by using j -neighborhoods and ideals and studied some of their properties.

Theorem 5 (see [23]). Consider (\mathcal{U}, R, ψ_j) as a N_j S and \mathcal{I} as an ideal on \mathcal{U} . Then, the family $\tau_j^{\mathcal{I}} = \{M \subseteq \mathcal{U} : \forall h \in M, N_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Theorem 6 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . Then, $\tau_j \subseteq \tau_j^{\mathcal{I}}$ for each j .

Definition 11 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . We called a set $M \subseteq \mathcal{U}$ an \mathcal{I} -open set if $M \in \tau_j^{\mathcal{I}}$, and we called its complement an \mathcal{I} -closed set. The family $\Gamma_j^{\mathcal{I}}$ of all \mathcal{I} -closed sets of a j -neighborhood space is given by $\Gamma_j^{\mathcal{I}} = \{K \subseteq \mathcal{U} : K^c \in \tau_j^{\mathcal{I}}\}$.

Definition 12 (see [23]). Let (\mathcal{U}, R, ψ_j) be a N_j S and \mathcal{I} be an ideal on \mathcal{U} . The \mathcal{I} -lower, \mathcal{I} -upper approximations, and \mathcal{I} -accuracy of the approximation of $M \subseteq \mathcal{U}$ are respectively given by

- (1) $\underline{R}_j^{\mathcal{I}}(M) = \cup \left\{ O \in \tau_j^{\mathcal{I}} : O \subseteq M \right\} = \text{int}_j^{\mathcal{I}}(M)$, where $\text{int}_j^{\mathcal{I}}(M)$ is an \mathcal{I} -interior of M .
- (2) $\overline{R}_j^{\mathcal{I}}(M) = \cap \left\{ F \in \Gamma_j^{\mathcal{I}} : M \subseteq F \right\} = \text{cl}_j^{\mathcal{I}}(M)$, where $\text{cl}_j^{\mathcal{I}}(M)$ is an \mathcal{I} -closure of M .
- (3) $\alpha_j^{\mathcal{I}}(M) = (|\underline{R}_j^{\mathcal{I}}(M)|/|\overline{R}_j^{\mathcal{I}}(M)|)$, where $|\overline{R}_j^{\mathcal{I}}(M)| \neq 0$.

2.5. Approximations and Topologies Generated by Different \mathcal{E}_j -Neighborhoods and Ideals. For any binary relation, Hosny et al. [29] utilized the concepts of \mathcal{E}_j -neighborhoods and ideal \mathcal{I} to output various topologies $\zeta_j^{\mathcal{I}}$ which are finer than the previous one generated by \mathcal{E}_j -neighborhoods due to [26].

Theorem 7 (see [29]). The class $\zeta_j^{\mathcal{I}} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{E}_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Definition 13 (see [29]). Let (\mathcal{U}, R, ψ_j) be a E_j S and \mathcal{I} be an ideal on \mathcal{U} . We called a set $M \subseteq \mathcal{U}$ a $\zeta_j^{\mathcal{I}}$ -open set if $M \in \zeta_j^{\mathcal{I}}$, and we called its complement a $\zeta_j^{\mathcal{I}}$ -closed set. The class $\Pi_j^{\mathcal{I}}$ of all $\zeta_j^{\mathcal{I}}$ -closed sets is given by $\Pi_j^{\mathcal{I}} = \{K \subseteq \mathcal{U} : K^c \in \zeta_j^{\mathcal{I}}\}$.

Theorem 8 (see [29]). Consider (\mathcal{U}, R, ψ_j) as an E_j S and \mathcal{I} as an ideal on \mathcal{U} . Then,

- (1) $\tau_{\mathcal{E}_j} \subseteq \zeta_j^{\mathcal{I}}$.

(2) The reflexivity of \mathcal{R} implies that $\zeta_j^{\mathcal{I}} \subseteq \zeta_{\langle j \rangle}^{\mathcal{I}}$ for each $j \in \{i, u, r, l\}$.

(3) If \mathcal{R} is a symmetric, then $\zeta_r^{\mathcal{I}} = \zeta_l^{\mathcal{I}} = \zeta_i^{\mathcal{I}} = \zeta_u^{\mathcal{I}}$ and $\zeta_{\langle r \rangle}^{\mathcal{I}} = \zeta_{\langle l \rangle}^{\mathcal{I}} = \zeta_{\langle i \rangle}^{\mathcal{I}} = \zeta_{\langle u \rangle}^{\mathcal{I}}$.

(4) The transitivity of \mathcal{R} implies that $\zeta_{\langle j \rangle}^{\mathcal{I}} \subseteq \zeta_j^{\mathcal{I}}$ for each $j \in \{i, u, r, l\}$.

(5) If \mathcal{R} is a preorder, then $\zeta_{\langle j \rangle}^{\mathcal{I}} = \zeta_j^{\mathcal{I}}$ for each $j \in \{i, u, r, l\}$.

(6) If \mathcal{R} is an equivalence, then all $\zeta_j^{\mathcal{I}}$ are equal, and $\tau_j^{\mathcal{I}} = \zeta_j^{\mathcal{I}}$ for each j .

Lemma 2 (see [29]). For any binary relation \mathcal{R} on \mathcal{U} , we have $\mathcal{N}_{\langle j \rangle}(h) \subseteq E_{\langle j \rangle}(h)$ for each $h \in \mathcal{U}$ and $j \in \{i, u, r, l\}$.

Definition 14 (see [29]). Consider $\zeta_j^{\mathcal{I}}$ as a topology generated by E_j -neighborhoods and ideal \mathcal{I} . Then, for each j , \mathcal{F}_{E_j} -lower, \mathcal{F}_{E_j} -upper approximations, and \mathcal{F}_{E_j} -accuracy of a subset $M \subseteq \mathcal{U}$ are defined respectively as follows:

- (1) $L_j^{\mathcal{F}}(M) = \text{int}_{E_j}^{\mathcal{F}}(M)$, where $\text{int}_{E_j}^{\mathcal{F}}(M)$ forms the interior points of M in $\zeta_j^{\mathcal{I}}$.
- (2) $U_j^{\mathcal{F}}(M) = \text{cl}_{E_j}^{\mathcal{F}}(M)$, where $\text{cl}_{E_j}^{\mathcal{F}}(M)$ forms the closure points of M in $\zeta_j^{\mathcal{I}}$.
- (3) $\sigma_j^{\mathcal{F}}(M) = (|L_j^{\mathcal{F}}(M)|/|U_j^{\mathcal{F}}(M)|)$, where $|U_j^{\mathcal{F}}(M)| \neq 0$.

3. Novel Topologies Generated from j -Adhesion Neighborhoods via Ideals

Now, we deal with ideals and four different j -adhesion neighborhoods generated from eight different j -neighborhoods via the same binary relation. By using them, new topologies are generated that generalize these topologies generated by j -adhesion neighborhoods. Several properties and relationships between these topologies are obtained.

Throughout this paper, a \mathcal{P}_j S with ideal \mathcal{I} $(\mathcal{U}, R, \mathcal{I}, \psi_j)$ is denoted by $\mathcal{P}_j^{\mathcal{I}}$ S.

Theorem 9. Let $(\mathcal{U}, R, \mathcal{I}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{I}}$ S. Then, the collection $\rho_j^{\mathcal{I}} = \{M \subseteq \mathcal{U} : \forall h \in M, \mathcal{P}_j(h) - M \in \mathcal{I}\}$ forms a topology on \mathcal{U} for each j .

Proof. First, let $M_\alpha \in \rho_j^{\mathcal{I}}$, $\alpha \in \Delta$, and $z \in \cup_{\alpha \in \Delta} M_\alpha$. Then, there is an $\alpha_0 \in \Delta$ s.t. $z \in M_{\alpha_0}$. Therefore, $[\mathcal{P}_j(z) - M_{\alpha_0}] \in \mathcal{I}$. Since $-(\cup_{\alpha \in \Delta} M_\alpha) \subseteq -M_{\alpha_0}$, $[\mathcal{P}_j(z) - (\cup_{\alpha \in \Delta} M_\alpha)] \in \mathcal{I}$, i.e., $\cup_{\alpha \in \Delta} M_\alpha \in \rho_j^{\mathcal{I}}$. Second, let M_1, M_2 be members of $\rho_j^{\mathcal{I}}$ and z belong to the intersection of M_1 and M_2 . Then, $[\mathcal{P}_j(z) - M_1] \in \mathcal{I}$ and $[\mathcal{P}_j(z) - M_2] \in \mathcal{I}$. According to the definition of \mathcal{I} , we obtain $[\mathcal{P}_j(z) - M_1] \cup [\mathcal{P}_j(z) - M_2] \in \mathcal{I}$. Hence, $[\mathcal{P}_j(z) - (M_1 \cap M_2)] \in \mathcal{I}$. This means that $M_1 \cap M_2 \in \rho_j^{\mathcal{I}}$. Finally, it is easy to see that $\emptyset, \mathcal{U} \in \rho_j^{\mathcal{I}}$, $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$. Consequently, $\rho_j^{\mathcal{I}}$ is a topology on \mathcal{U} . \square

Lemma 3. If \mathcal{I}, \mathcal{J} are ideals on a \mathcal{P}_j S (\mathcal{U}, R, ψ_j) such that \mathcal{I} is a subset of \mathcal{J} , then $\rho_j^{\mathcal{I}} \subseteq \rho_j^{\mathcal{J}}$.

Proof. Direct to prove.

The fundamental goal of the next results is to deduce the relations between the topologies generated by j -adhesion neighborhoods, topologies generated by j -neighborhoods and ideals, and topologies generated by j -adhesion neighborhoods and ideals, as it is shown in the following theorems.

Our new types of topologies, which were generated by j -adhesion neighborhoods and ideals, are finer than the previous one generated by j -adhesion neighborhoods due to [28] for any relation. \square

Theorem 10. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. If $j \in \{r, l, i, u\}$, then $\tau_{\mathcal{F}_j} \subseteq \rho_j^{\mathcal{F}}$.

Proof. Straightforward. \square

Example 1. Let $\mathcal{U} = \{a, b, c, d\}$ and $R = \{(c, a), (c, b), (c, d), (d, a), (d, c), (d, d)\}$. Then, we obtain the next topologies

$$\begin{aligned}
 \tau_r &= \{\{a\}, \{b\}, \{a, b\}, \emptyset, \mathcal{U}\}, \\
 \tau_l &= \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_i &= \{\{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_u &= \{\emptyset, \mathcal{U}\}, \\
 \tau_{\langle r \rangle} &= \tau_{\langle u \rangle} = \{\{a, d\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle l \rangle} &= \tau_{\langle i \rangle} = P(U), \\
 \tau_{\mathcal{F}_r} &= \tau_{\mathcal{F}_{\langle l \rangle}} = \{\{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\mathcal{F}_l} &= \tau_{\mathcal{F}_{\langle r \rangle}} = \{\{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\mathcal{F}_i} &= \tau_{\mathcal{F}_{\langle i \rangle}} = P(U), \\
 \tau_{\mathcal{F}_u} &= \tau_{\mathcal{F}_{\langle u \rangle}} = \{\{c\}, \{a, b, d\}, \emptyset, \mathcal{U}\}.
 \end{aligned} \tag{1}$$

If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\begin{aligned}
 \tau_r^{\mathcal{F}} &= \{\{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_l^{\mathcal{F}} &= \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_i^{\mathcal{F}} &= \{\{a\}, \{b\}, \{c, d\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_u^{\mathcal{F}} &= \{\{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle r \rangle}^{\mathcal{F}} &= \tau_{\langle u \rangle}^{\mathcal{F}} = \{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \tau_{\langle l \rangle}^{\mathcal{F}} &= \tau_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
 \rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \rho_i^{\mathcal{F}} &= \rho_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
 \rho_u^{\mathcal{F}} &= \rho_{\langle u \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}.
 \end{aligned} \tag{2}$$

If $\widehat{\mathcal{F}} = \{\emptyset, \{d\}\}$, then

$$\begin{aligned}
 \widehat{\tau}_r^{\widehat{\mathcal{F}}} &= \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_l^{\widehat{\mathcal{F}}} &= \{\{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_i^{\widehat{\mathcal{F}}} &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_u^{\widehat{\mathcal{F}}} &= \{\{a, b, c\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_{\langle r \rangle}^{\widehat{\mathcal{F}}} &= \widehat{\tau}_{\langle u \rangle}^{\widehat{\mathcal{F}}} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \emptyset, \mathcal{U}\}, \\
 \widehat{\tau}_{\langle l \rangle}^{\widehat{\mathcal{F}}} &= \widehat{\tau}_{\langle i \rangle}^{\widehat{\mathcal{F}}} \\
 \widehat{\rho}_r^{\widehat{\mathcal{F}}} &= \widehat{\rho}_{\langle l \rangle}^{\widehat{\mathcal{F}}} = \{\{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \emptyset, \mathcal{U}\},
 \end{aligned}$$

$$\begin{aligned}
\widehat{\rho}_l^{\mathcal{F}} &= \widehat{\rho}_{\langle r \rangle}^{\mathcal{F}} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \emptyset, \mathcal{U}\}, \\
\widehat{\rho}_i^{\mathcal{F}} &= \widehat{\rho}_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
\widehat{\rho}_u^{\mathcal{F}} &= \widehat{\rho}_{\langle u \rangle}^{\mathcal{F}} = \{\{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \emptyset, \mathcal{U}\}.
\end{aligned} \tag{3}$$

If $\mathcal{F} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$, then

$$\begin{aligned}
\mathbb{T}_r^{\mathcal{F}} &= \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\mathbb{T}_l^{\mathcal{F}} &= \{\{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\mathbb{T}_i^{\mathcal{F}} &= \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\mathbb{T}_u^{\mathcal{F}} &= \{\{b, c\}, \{a, b, c\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\mathbb{T}_{\langle r \rangle}^{\mathcal{F}} &= \mathbb{T}_{\langle u \rangle}^{\mathcal{F}} = \mathbb{T}_{\langle l \rangle}^{\mathcal{F}} = \mathbb{T}_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
\rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \emptyset, \mathcal{U}\}, \\
\rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = P(U).
\end{aligned} \tag{4}$$

Remark 3. If $\mathcal{F} = \{\emptyset, \{a\}\}$ in Example 1, then

- (1) $\rho_r^{\mathcal{F}} \neq \rho_i^{\mathcal{F}}$ and $\rho_l^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$.
- (2) $\rho_r^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$ and $\rho_l^{\mathcal{F}} \neq \rho_i^{\mathcal{F}}$.
- (3) $\rho_i^{\mathcal{F}} \neq \rho_u^{\mathcal{F}}$.

Remark 4. In view of Example 1,

- (1) In general, the collection $\rho_j^{\mathcal{F}}$ need not be a quasi-discrete (clopen) topology on \mathcal{U} , although $\mathbb{T}_{\mathcal{F}_j}$ is quasidiscrete topology on \mathcal{U} , $\forall j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$.
- (2) If $\forall j \in \{u, \langle u \rangle\}$, then $\rho_j^{\mathcal{F}}$ is not quasidiscrete topology on \mathcal{U} .
- (3) $\rho_r^{\mathcal{F}}$ need not be dual topology to $\rho_l^{\mathcal{F}}$.
- (4) If R is any relation, then $\mathbb{T}_j^{\mathcal{F}}, \rho_j^{\mathcal{F}}$ are not comparable, for each j .
- (5) If $\mathcal{F} = \{\emptyset\}$ in Theorem 9, then the present generated topologies coincide with the previous one in

Proposition 4.1 in [28]. So, the current work is considered as a generalization of work of [25, 27, 28].

Theorem 11 illustrates that the reflexivity condition is necessary to create a relationship between the topologies $\mathbb{T}_j^{\mathcal{F}}$ and $\rho_j^{\mathcal{F}}$.

Theorem 11. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$, where R is reflexive. Then, $\mathbb{T}_j^{\mathcal{F}} \subseteq \rho_j^{\mathcal{F}}$, for any ideal \mathcal{F} .

Proof. Let \mathcal{F} be any ideal on \mathcal{U} . According to Proposition 3, $\mathbb{T}_j^{\mathcal{F}} \subseteq \rho_j^{\mathcal{F}}$. \square

Remark 5. The new forms of topologies, which were generated by j -adhesion neighborhoods and ideals, are finer than the topologies generated by j -neighborhoods and ideals due to [23] for reflexive relation.

Example 2. Let $\mathcal{U} = \{a, b, c, d\}$ and $R = \blacktriangle \cup \{(b, c), (b, d), (c, a), (d, b), (c, d), (d, c)\}$.

If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\begin{aligned}
\mathbb{T}_r^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}, \\
\mathbb{T}_l^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
\mathbb{T}_i^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}, \\
\mathbb{T}_u^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
\mathbb{T}_{\langle r \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
\mathbb{T}_{\langle l \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}\},
\end{aligned}$$

$$\begin{aligned}
 \tau_{\langle i \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
 \tau_{\langle u \rangle}^{\mathcal{F}} &= \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\
 \rho_r^{\mathcal{F}} &= \rho_{\langle l \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}, \\
 \rho_l^{\mathcal{F}} &= \rho_{\langle r \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \\
 \rho_i^{\mathcal{F}} &= \rho_{\langle i \rangle}^{\mathcal{F}} = P(U), \\
 \rho_u^{\mathcal{F}} &= \rho_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\}.
 \end{aligned} \tag{5}$$

Theorem 12. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and R be a symmetric relation on \mathcal{U} . Then, $\rho_r^{\mathcal{F}} = \rho_l^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$.

Proof. Since R is a symmetric relation on \mathcal{U} , then all j -adhesion neighborhoods coincide. So, $\rho_r^{\mathcal{F}} = \rho_l^{\mathcal{F}} = \rho_i^{\mathcal{F}} = \rho_u^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$. \square

Proposition 5. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. Then, the following statements hold:

- (1) $\rho_r^{\mathcal{F}} = \rho_{\langle l \rangle}^{\mathcal{F}}, \rho_l^{\mathcal{F}} = \rho_{\langle r \rangle}^{\mathcal{F}}$.
- (2) $\rho_i^{\mathcal{F}} = \rho_{\langle i \rangle}^{\mathcal{F}}, \rho_u^{\mathcal{F}} = \rho_{\langle u \rangle}^{\mathcal{F}}$.
- (3) $\rho_u^{\mathcal{F}} \subseteq \rho_r^{\mathcal{F}} \cap \rho_l^{\mathcal{F}}$.
- (4) $\rho_r^{\mathcal{F}} \cup \rho_l^{\mathcal{F}} \subseteq \rho_i^{\mathcal{F}}$.
- (5) $\rho_{\langle u \rangle}^{\mathcal{F}} \subseteq \rho_{\langle r \rangle}^{\mathcal{F}} \cap \rho_{\langle l \rangle}^{\mathcal{F}}$.
- (6) $\rho_{\langle r \rangle}^{\mathcal{F}} \cup \rho_{\langle l \rangle}^{\mathcal{F}} \subseteq \rho_{\langle i \rangle}^{\mathcal{F}}$.

Proof. We prove (3) and one can prove the other cases in a similar way. Let $M \in \rho_u^{\mathcal{F}}$, then $[\mathcal{P}_u(z) - M] \in \mathcal{F}, \forall z \in M$. Hence, $[(\mathcal{P}_r(z) \cup \mathcal{P}_l(z)) - M] \in \mathcal{F}, \forall z \in M$. So, $[\mathcal{P}_r(z) - M] \in \mathcal{F}$ and $[\mathcal{P}_l(z) - M] \in \mathcal{F}, \forall z \in M$. Consequently, $M \in \rho_r^{\mathcal{F}} \cap \rho_l^{\mathcal{F}}$. \square

Definition 15. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$. A set $M \subseteq \mathcal{U}$ is called $\rho_j^{\mathcal{F}}$ -open set if $M \in \rho_j^{\mathcal{F}}$, and its complement is called $\rho_j^{\mathcal{F}}$ -closed set. The family $\Upsilon_j^{\mathcal{F}}$ of all $\rho_j^{\mathcal{F}}$ -closed sets is defined as $\Upsilon_j^{\mathcal{F}} = \{K \subseteq \mathcal{U} : K^c \in \rho_j^{\mathcal{F}}\}$.

Definition 16. Consider $\rho_j^{\mathcal{F}}$ as a topology generated by j -adhesion neighborhoods and ideals. The \mathcal{F}_{P_j} -lower, \mathcal{F}_{P_j} -upper approximations, \mathcal{F}_{P_j} -boundary regions, and \mathcal{F}_{P_j} -accuracy of $M \subseteq \mathcal{U}$ are respectively given as follows:

- (1) $\mathfrak{R}_j^{\mathcal{F}}(M) = \cup \{O \in \rho_j^{\mathcal{F}} : O \subseteq M\} = \text{int}_{\mathcal{F}_j}^{\mathcal{F}}(M)$, where $\text{int}_{\mathcal{F}_j}^{\mathcal{F}}(M)$ represents interior of M wrt $\rho_j^{\mathcal{F}}$.
- (2) $\mathfrak{U}_j^{\mathcal{F}}(M) = \cap \{F \in \Upsilon_j^{\mathcal{F}} : M \subseteq F\} = \text{cl}_{\mathcal{F}_j}^{\mathcal{F}}(M)$, where $\text{cl}_{\mathcal{F}_j}^{\mathcal{F}}(M)$ represents closure of M wrt $\rho_j^{\mathcal{F}}$.
- (3) $\mathfrak{B}_j^{\mathcal{F}}(M) = \mathfrak{U}_j^{\mathcal{F}}(M) - \mathfrak{R}_j^{\mathcal{F}}(M)$.
- (4) $\theta_j^{\mathcal{F}}(M) = (|\mathfrak{R}_j^{\mathcal{F}}(M)| / |\mathfrak{U}_j^{\mathcal{F}}(M)|)$, where $|\mathfrak{U}_j^{\mathcal{F}}(M)| \neq 0$.

Remark 6. Table 1 displays the comparison between j -approximations and j -accuracy for $j = r$ depending on Definitions 10, 12, and 16 by using any relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 1.

Remark 7. Table 2 offers the comparison between j -approximations and j -accuracy for $j = l$ depending on Definitions 12 and 16 by using a reflexive relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

The next proposition that demonstrates the fundamental properties of $\mathfrak{R}_j^{\mathcal{F}}, \mathfrak{U}_j^{\mathcal{F}}$ operators are understandable by observing that $\text{int}_{\mathcal{F}_j}^{\mathcal{F}}, \text{cl}_{\mathcal{F}_j}^{\mathcal{F}}$ fulfill all properties of the topological interior and closure operators, respectively.

Proposition 6. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^{\mathcal{F}}S$ and $j \in \{i, u, r, l\}$. If $M, M \subseteq \mathcal{U}$, then the following conditions hold:

- (L₁) $\mathfrak{R}_j^{\mathcal{F}}(M) = (\mathfrak{U}_j^{\mathcal{F}}(M^c))^c$.
- (U₁) $\mathfrak{U}_j^{\mathcal{F}}(M) = (\mathfrak{R}_j^{\mathcal{F}}(M^c))^c$.
- (L₂) $\mathfrak{R}_j^{\mathcal{F}}(\mathcal{U}) = \mathcal{U}$.
- (U₂) $\mathfrak{U}_j^{\mathcal{F}}(\emptyset) = \emptyset$.
- (L₃) if $M \subseteq M$, then $\mathfrak{R}_j^{\mathcal{F}}(M) \subseteq \mathfrak{R}_j^{\mathcal{F}}(M)$.
- (U₃) if $M \subseteq M$, then $\mathfrak{U}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₄) $\mathfrak{R}_j^{\mathcal{F}}(M \cap M) = \mathfrak{R}_j^{\mathcal{F}}(M) \cap \mathfrak{R}_j^{\mathcal{F}}(M)$.
- (U₄) $\mathfrak{U}_j^{\mathcal{F}}(M \cup M) = \mathfrak{U}_j^{\mathcal{F}}(M) \cup \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₅) $\mathfrak{R}_j^{\mathcal{F}}(M \cup M) \supseteq \mathfrak{R}_j^{\mathcal{F}}(M) \cup \mathfrak{R}_j^{\mathcal{F}}(M)$.
- (U₅) $\mathfrak{U}_j^{\mathcal{F}}(M \cap M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(M) \cap \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₆) $\mathfrak{R}_j^{\mathcal{F}}(\emptyset) = \emptyset$.
- (U₆) $\mathfrak{U}_j^{\mathcal{F}}(\mathcal{U}) = \mathcal{U}$.
- (L₇) $\mathfrak{R}_j^{\mathcal{F}}(M) \subseteq M$.
- (U₇) $M \subseteq \mathfrak{U}_j^{\mathcal{F}}(M)$.
- (L₈) $\mathfrak{R}_j^{\mathcal{F}}(M) = \mathfrak{R}_j^{\mathcal{F}}(\mathfrak{R}_j^{\mathcal{F}}(M))$.
- (U₈) $\mathfrak{U}_j^{\mathcal{F}}(M) = \mathfrak{U}_j^{\mathcal{F}}(\mathfrak{U}_j^{\mathcal{F}}(M))$.
- (L₉) $\mathfrak{R}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j^{\mathcal{F}}(\mathfrak{R}_j^{\mathcal{F}}(M))$.
- (U₉) $\mathfrak{U}_j^{\mathcal{F}}(M) \supseteq \mathfrak{R}_j^{\mathcal{F}}(\mathfrak{U}_j^{\mathcal{F}}(M))$.

Remark 8. If $\mathcal{F} = \{\emptyset, \{a\}\}$ and $j = r$, then Table 1 shows that the converse of (L₃), (L₅), (L₇), (L₉) does not hold.

TABLE 1: Comparison between j -approximations and j -accuracy for each $j = r$ depending on Definitions 10, 12, and 16 by using any relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 1.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 10			Definition 12		
	$\mathfrak{R}_r^\mathcal{F}(M)$	$\mathfrak{U}_r^\mathcal{F}(M)$	$\theta_r^\mathcal{F}(M)$	$\mathfrak{R}_r(M)$	$\mathfrak{U}_r(M)$	$\theta_r(M)$	$\underline{\mathfrak{R}}_r^\mathcal{F}(M)$	$\overline{\mathfrak{R}}_r^\mathcal{F}(M)$	$\alpha_r^\mathcal{F}(M)$
{a}	\emptyset	{a}	0	\emptyset	{a, b}	0	{a}	{a}	1
{b}	{b}	{a, b}	1/2	\emptyset	{a, b}	0	{b}	{b, c, d}	1/3
{c}	{c}	{c}	1	{c}	{c}	1	\emptyset	{c, d}	0
{d}	{d}	{d}	1	{d}	{d}	1	\emptyset	{c, d}	0
{a, b}	{a, b}	{a, b}	1	{a, b}	{a, b}	1	{a, b}	\mathcal{U}	1/2
{a, c}	{c}	{a, c}	1/2	{c}	{a, b, c}	1/3	{a}	{a, c, d}	1/3
{a, d}	{d}	{a, d}	1/2	{d}	{a, b, d}	1/3	{a}	{a, c, d}	1/3
{b, c}	{b, c}	{a, b, c}	2/3	{c}	{a, b, c}	1/3	{b}	{b, c, d}	1/3
{b, d}	{b, d}	{a, b, d}	2/3	{d}	{a, b, d}	1/3	{b}	{b, c, d}	1/3
{c, d}	{c, d}	{c, d}	1	{c, d}	{c, d}	1	\emptyset	{c, d}	0
{a, b, c}	{a, b, c}	{a, b, c}	1	{a, b, c}	{a, b, c}	1	{a, b}	\mathcal{U}	1/2
{a, b, d}	{a, b, d}	{a, b, d}	1	{a, b, d}	{a, b, d}	1	{a, b}	\mathcal{U}	1/2
{a, c, d}	{c, d}	{a, c, d}	2/3	{c, d}	\mathcal{U}	1/2	{a}	{a, c, d}	1/3
{b, c, d}	{b, c, d}	\mathcal{U}	3/4	{c, d}	\mathcal{U}	1/2	{b, c, d}	{b, c, d}	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

TABLE 2: Comparison between j -approximations and j -accuracy for each $j = l$ depending on Definitions 12 and 16 by using a reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 12		
	$\mathfrak{R}_l^\mathcal{F}(M)$	$\mathfrak{U}_l^\mathcal{F}(M)$	$\theta_l^\mathcal{F}(M)$	$\underline{\mathfrak{R}}_l^\mathcal{F}(M)$	$\overline{\mathfrak{R}}_l^\mathcal{F}(M)$	$\alpha_l^\mathcal{F}(M)$
{a}	{a}	{a}	1	\emptyset	{a}	0
{b}	{b}	{b}	1	\emptyset	\mathcal{U}	0
{c}	\emptyset	{c, d}	0	\emptyset	\mathcal{U}	0
{d}	\emptyset	{c, d}	0	\emptyset	\mathcal{U}	0
{a, b}	{a, b}	{a, b}	1	\emptyset	\mathcal{U}	0
{a, c}	{a}	{a, c, d}	1/3	\emptyset	\mathcal{U}	0
{a, d}	{a}	{a, c, d}	1/3	\emptyset	\mathcal{U}	0
{b, c}	{b}	{b, c, d}	1/3	\emptyset	\mathcal{U}	0
{b, d}	{b}	{b, c, d}	1/3	\emptyset	\mathcal{U}	0
{c, d}	{c, d}	{c, d}	1	\emptyset	\mathcal{U}	0
{a, b, c}	{a, b}	\mathcal{U}	1/2	\emptyset	\mathcal{U}	0
{a, b, d}	{a, b}	\mathcal{U}	1/2	\emptyset	\mathcal{U}	0
{a, c, d}	{a, c, d}	{a, c, d}	1	\emptyset	\mathcal{U}	0
{b, c, d}	{b, c, d}	{b, c, d}	1	{b, c, d}	\mathcal{U}	3/4
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

- (1) If $M = \{a\}$, $M = \{b\}$, then $\mathfrak{R}_r^\mathcal{F}(M) \subseteq \mathfrak{R}_r^\mathcal{F}(M)$, but $M \not\subseteq M$. Consequently, the converse of (L_3) is not true.
- (2) If $M = \{a\}$, $M = \{b\}$, then $\mathfrak{R}_r^\mathcal{F}(M \cup M) = \{a, b\} \neq \{b\} = \mathfrak{R}_r^\mathcal{F}(M) \cup \mathfrak{R}_r^\mathcal{F}(M)$. Consequently, the converse of (L_5) is not true.
- (3) If $M = \{a, c\}$, then the converse of (L_7) is not true.
- (4) The equality of properties (L_9), (U_9) does not hold wrt the topology $\rho_j^\mathcal{F}$, although they are true according to the topology $\tau_{\mathcal{F}}$. Here, if $M = \{b\}$, then $\mathfrak{R}_j^\mathcal{F}(M) = \{b\}$ and $\mathfrak{U}_j^\mathcal{F}(\mathfrak{R}_j^\mathcal{F}(M)) = \{a, b\}$. Consequently, the converse of (L_9) does not hold.

By using any binary relation and without totting any supplement conditions as Proposition 6 claims, one might observe that our approximation approach satisfies all features of the classical theory of rough set [1]. Thus, we can say

that our presented method exemplifies a noteworthy generalization to RST.

Proposition 7. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^\mathcal{F}S$ and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following conditions hold: $\mathfrak{R}_j^\mathcal{F}(M) = M = \mathfrak{U}_j^\mathcal{F}(M)$, $\forall M \in \wp(\mathcal{U})$.

Proposition 8. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^\mathcal{F}S$. If R is a reflexive relation on \mathcal{U} , then $\mathfrak{R}_j^\mathcal{F}(M) = \mathfrak{U}_j^\mathcal{F}(\mathfrak{R}_j^\mathcal{F}(M))$ and $\mathfrak{U}_j^\mathcal{F}(M) = \mathfrak{R}_j^\mathcal{F}(\mathfrak{U}_j^\mathcal{F}(M))$.

The following propositions are obvious and the proof is omitted.

Proposition 9. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^\mathcal{F}S$ and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following statements hold:

- (1) $\mathfrak{R}_u^\mathcal{F}(M) \subseteq \mathfrak{R}_r^\mathcal{F}(M) \subseteq \mathfrak{R}_i^\mathcal{F}(M)$ and $\mathfrak{R}_u^\mathcal{F}(M) \subseteq \mathfrak{R}_i^\mathcal{F}(M) \subseteq \mathfrak{R}_r^\mathcal{F}(M)$.
- (2) $\mathfrak{U}_i^\mathcal{F}(M) \subseteq \mathfrak{U}_r^\mathcal{F}(M) \subseteq \mathfrak{U}_u^\mathcal{F}(M)$ and $\mathfrak{U}_i^\mathcal{F}(M) \subseteq \mathfrak{U}_u^\mathcal{F}(M) \subseteq \mathfrak{U}_r^\mathcal{F}(M)$.
- (3) $\theta_u^\mathcal{F}(M) \leq \theta_r^\mathcal{F}(M) \leq \theta_i^\mathcal{F}(M)$ and $\theta_u^\mathcal{F}(M) \leq \theta_i^\mathcal{F}(M) \leq \theta_r^\mathcal{F}(M)$.

Proposition 10. Let \mathcal{F}, \mathcal{F} be two ideals on a $\mathcal{P}_j^\mathcal{F}S(\mathcal{U}, R, \psi_j)$ and $M \subseteq \mathcal{U}$. If $\mathcal{F} \subseteq \mathcal{F}$, then the following statements hold:

- (1) $\mathfrak{R}_j^\mathcal{F}(M) \subseteq L_j^\mathcal{F}(M)$.
- (2) $\mathfrak{U}_j^\mathcal{F}(M) \supseteq U_j^\mathcal{F}(M)$.
- (3) $\theta_j^\mathcal{F}(M) \leq \theta_j^\mathcal{F}(M)$.

In Tables 3 and 4, we make comparisons between j -approximations and j -accuracy for each $j \in \{i, u, r, l\}$.

Definition 17. Let $(\mathcal{U}, R, \mathcal{F}, \psi_j)$ be a $\mathcal{P}_j^\mathcal{F}S$ and $j \in \{i, \langle i \rangle, r, \langle r \rangle, l, \langle l \rangle\}$. A subset M of \mathcal{U} is called

- (1) Totally \mathcal{F}_j definable, if $\mathfrak{R}_j^\mathcal{F}(M) = M = \mathfrak{U}_j^\mathcal{F}(M)$.

TABLE 3: Comparison between j -approximations and j -accuracy for each $j \in \{i, u, r, l\}$ depending on Definition 16 by using any relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$.

$M \subseteq \mathcal{U}$	$\mathfrak{Q}_r^{\mathcal{F}}(M)$	$\mathfrak{U}_r^{\mathcal{F}}(M)$	$\theta_r^{\mathcal{F}}(M)$	$\mathfrak{Q}_l^{\mathcal{F}}(M)$	$\mathfrak{U}_l^{\mathcal{F}}(M)$	$\theta_l^{\mathcal{F}}(M)$	$\mathfrak{Q}_i^{\mathcal{F}}(M)$	$\mathfrak{U}_i^{\mathcal{F}}(M)$	$\theta_i^{\mathcal{F}}(M)$	$\mathfrak{Q}_u^{\mathcal{F}}(M)$	$\mathfrak{U}_u^{\mathcal{F}}(M)$	$\theta_u^{\mathcal{F}}(M)$
$\{a\}$	\emptyset	$\{a\}$	0	\emptyset	$\{a\}$	0	$\{a\}$	$\{a\}$	1	\emptyset	$\{a\}$	0
$\{b\}$	$\{b\}$	$\{a, b\}$	1/2	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{a, b\}$	1/2
$\{c\}$	$\{c\}$	$\{c\}$	1									
$\{d\}$	$\{d\}$	$\{d\}$	1	$\{d\}$	$\{a, d\}$	1/2	$\{d\}$	$\{d\}$	1	$\{d\}$	$\{a, d\}$	1/2
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	$\{b\}$	$\{a, b\}$	1/2	$\{a, b\}$	$\{a, b\}$	1	$\{b\}$	$\{a, b\}$	1/2
$\{a, c\}$	$\{c\}$	$\{a, c\}$	1/2	$\{c\}$	$\{a, c\}$	1/2	$\{a, c\}$	$\{a, c\}$	1	$\{c\}$	$\{a, c\}$	1/2
$\{a, d\}$	$\{d\}$	$\{a, d\}$	1/2	$\{a, d\}$	$\{a, d\}$	1	$\{a, d\}$	$\{a, d\}$	1	$\{d\}$	$\{a, d\}$	1/2
$\{b, c\}$	$\{b, c\}$	$\{a, b, c\}$	2/3	$\{b, c\}$	$\{b, c\}$	1	$\{b, c\}$	$\{b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3
$\{b, d\}$	$\{b, d\}$	$\{a, b, d\}$	2/3	$\{b, d\}$	$\{a, b, d\}$	2/3	$\{b, d\}$	$\{b, d\}$	1	$\{b, d\}$	$\{a, b, d\}$	2/3
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{b, c\}$	$\{a, b, c\}$	2/3
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1
$\{a, c, d\}$	$\{c, d\}$	$\{a, c, d\}$	2/3	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{c, d\}$	$\{a, c, d\}$	2/3
$\{b, c, d\}$	$\{b, c, d\}$	\mathcal{U}	3/4	$\{b, c, d\}$	\mathcal{U}	3/4	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	\mathcal{U}	3/4
\mathcal{U}	\mathcal{U}	\mathcal{U}	1									

TABLE 4: Comparison between j -approximations and j -accuracy for each $j \in \{r, l, i, u\}$ depending on Definition 16 by using reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$.

$M \subseteq \mathcal{U}$	$\mathfrak{Q}_r^{\mathcal{F}}(M)$	$\mathfrak{U}_r^{\mathcal{F}}(M)$	$\theta_r^{\mathcal{F}}(M)$	$\mathfrak{Q}_l^{\mathcal{F}}(M)$	$\mathfrak{U}_l^{\mathcal{F}}(M)$	$\theta_l^{\mathcal{F}}(M)$	$\mathfrak{Q}_i^{\mathcal{F}}(M)$	$\mathfrak{U}_i^{\mathcal{F}}(M)$	$\theta_i^{\mathcal{F}}(M)$	$\mathfrak{Q}_u^{\mathcal{F}}(M)$	$\mathfrak{U}_u^{\mathcal{F}}(M)$	$\theta_u^{\mathcal{F}}(M)$
$\{a\}$	$\{a\}$	$\{a\}$	1									
$\{b\}$	\emptyset	$\{b, d\}$	0	$\{b\}$	$\{b\}$	1	$\{b\}$	$\{b\}$	1	\emptyset	$\{b, c, d\}$	0
$\{c\}$	$\{c\}$	$\{c\}$	1	\emptyset	$\{c, d\}$	0	$\{c\}$	$\{c\}$	1	\emptyset	$\{b, c, d\}$	0
$\{d\}$	\emptyset	$\{b, d\}$	0	\emptyset	$\{c, d\}$	0	$\{d\}$	$\{d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{a, b\}$	$\{a\}$	$\{a, b, d\}$	1/3	$\{a, b\}$	$\{a, b\}$	1	$\{a, b\}$	$\{a, b\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	1	$\{a\}$	$\{a, c, d\}$	1/3	$\{a, c\}$	$\{a, c\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, d\}$	$\{a\}$	$\{a, b, d\}$	1/3	$\{a\}$	$\{a, c, d\}$	1/3	$\{a, d\}$	$\{a, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{b, c\}$	$\{c\}$	$\{b, c, d\}$	1/3	$\{b\}$	$\{b, c, d\}$	1/3	$\{b, c\}$	$\{b, c\}$	1	\emptyset	$\{b, c, d\}$	0
$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	1	$\{b\}$	$\{b, c, d\}$	1/3	$\{b, d\}$	$\{b, d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{c, d\}$	$\{c\}$	$\{b, c, d\}$	1/3	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{c, d\}$	1	\emptyset	$\{b, c, d\}$	0
$\{a, b, c\}$	$\{a, c\}$	\mathcal{U}	1/2	$\{a, b\}$	\mathcal{U}	1/2	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b\}$	\mathcal{U}	1/2	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{a, c, d\}$	$\{a, c\}$	\mathcal{U}	1/2	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a\}$	\mathcal{U}	1/4
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1									

- (2) Internally \mathcal{F}_j definable, if $\mathfrak{Q}_j^{\mathcal{F}}(M) = M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) \neq M$.
- (3) Externally \mathcal{F}_j definable, if $\mathfrak{Q}_j^{\mathcal{F}}(M) \neq M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) = M$.
- (4) \mathcal{F}_j rough set, if $\mathfrak{Q}_j^{\mathcal{F}}(M) \neq M$ and $\mathfrak{U}_j^{\mathcal{F}}(M) \neq M$.

Remark 9. According to Table 3, $\{c\}$ is totally \mathcal{F}_r -definable, $\{d\}$ is internally \mathcal{F}_l -definable, and $\{a\}$ is externally \mathcal{F}_u -definable.

4. Comparison of Approximations and Accuracy Measures Generated by \mathcal{E}_j -Neighborhoods and \mathcal{P}_j -Neighborhoods with Ideals

In this part, the comparison of approximations and accuracy measures generated by \mathcal{E}_j -neighborhoods and \mathcal{P}_j -neighborhoods with ideals is discussed.

In view of Example 1, we conclude that $\zeta_j^{\mathcal{F}}$ and $\rho_j^{\mathcal{F}}$ are not comparable, for each j , as we see in the next example.

Example 3. Continued from Example 1, if $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\begin{aligned} \zeta_r^{\mathcal{F}} &= \zeta_i^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}, \\ \zeta_l^{\mathcal{F}} &= \zeta_u^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{c, d\}, \{b, c, d\}\}, \\ \zeta_{\langle r \rangle}^{\mathcal{F}} &= \zeta_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{b, c, d\}\}, \\ \zeta_{\langle l \rangle}^{\mathcal{F}} &= \zeta_{\langle i \rangle}^{\mathcal{F}} = P(\mathcal{U}). \end{aligned} \tag{6}$$

According to Proposition 3, our new sorts of topologies, which were generated by j -adhesion neighborhoods and ideal, are finer than the topologies generated by \mathcal{E}_j -neighborhoods and ideals due to [29] for reflexive relation, as we see in the next theorem.

Theorem 13. Let R be any arbitrary binary relation on \mathcal{U} . Then, for any ideal \mathcal{F} , the following statements are true:

- (1) If R is a reflexive relation on \mathcal{U} , then $\zeta_j^{\mathcal{F}} \subseteq \tau_{\mathcal{P}_j} \subseteq \rho_j^{\mathcal{F}}$, for each j .

TABLE 5: Comparison between j -approximations and j -accuracy for each $j = \langle i \rangle$ depending on Definitions 14 and 16 by using a reflexive relation R and $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

$M \subseteq \mathcal{U}$	Our method, Definition 16			Definition 14		
	$\mathfrak{L}_{\langle i \rangle}^{\mathcal{F}}(M)$	$\mathfrak{U}_{\langle i \rangle}^{\mathcal{F}}(M)$	$\theta_{\langle i \rangle}^{\mathcal{F}}(M)$	$L_{\langle i \rangle}^{\mathcal{F}\ominus}(M)$	$U_{\langle i \rangle}^{\mathcal{F}\oplus}(M)$	$\sigma_{\langle i \rangle}^{\mathcal{F}}(M)$
$\{a\}$	$\{a\}$	$\{a\}$	1	$\{a\}$	$\{a\}$	1
$\{b\}$	$\{b\}$	$\{b\}$	1	\emptyset	$\{b, d\}$	0
$\{c\}$	$\{c\}$	$\{c\}$	1	$\{c\}$	$\{c\}$	1
$\{d\}$	$\{d\}$	$\{d\}$	1	\emptyset	$\{b, d\}$	0
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	1	$\{a\}$	$\{a, b, d\}$	1/3
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	1	$\{a, c\}$	$\{a, c\}$	1
$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	1	$\{a\}$	$\{a, b, d\}$	1/3
$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	1	$\{c\}$	$\{b, c, d\}$	1/3
$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	1	$\{b, d\}$	$\{b, d\}$	1
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c\}$	$\{b, c, d\}$	1/3
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	1	$\{a, c\}$	\mathcal{U}	1/2
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	$\{a, b, d\}$	$\{a, b, d\}$	1
$\{a, c, d\}$	$\{a, c, d\}$	$\{a, c, d\}$	1	$\{a, c\}$	\mathcal{U}	1/2
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1
\mathcal{U}	\mathcal{U}	\mathcal{U}	1	\mathcal{U}	\mathcal{U}	1

(2) If R is an equivalence relation on \mathcal{U} , then $\rho_j^{\mathcal{F}} = \tau_j^{\mathcal{F}} = \zeta_j^{\mathcal{F}}$, for each j .

(1) $L_j^{\mathcal{F}\ominus}(M) \subseteq \mathfrak{L}_j(M) \subseteq \mathfrak{L}_j^{\mathcal{F}}(M)$.

(2) $\mathfrak{U}_j^{\mathcal{F}}(M) \subseteq \mathfrak{U}_j(M) \subseteq U_j^{\mathcal{F}\oplus}(M)$.

(3) $\sigma_j^{\mathcal{F}}(M) \leq \theta_j(M) \leq \theta_j^{\mathcal{F}}(M)$.

Proof. Direct to prove. \square

Proposition 11. Let R (respectively \mathcal{F}) be a reflexive binary relation (respectively an ideal) on \mathcal{U} and $M \subseteq \mathcal{U}$. If $j \in \{r, l, i, \langle r \rangle, \langle l \rangle, \langle i \rangle\}$, then the following statements hold:

To uphold the acquired outcomes, we consider a reflexive relation R on \mathcal{U} given in Example 2. If $\mathcal{F} = \{\emptyset, \{a\}\}$, then

$$\zeta_r^{\mathcal{F}} = \zeta_u^{\mathcal{F}} = \zeta_{\langle u \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{b, c, d\}\},$$

$$\zeta_l^{\mathcal{F}} = \zeta_i^{\mathcal{F}} = \zeta_{\langle r \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{a\}, \{b, c, d\}\},$$

$$\zeta_{\langle l \rangle}^{\mathcal{F}} = \{\emptyset, \mathcal{U}, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}\},$$

$$\zeta_{\langle i \rangle}^{\mathcal{F}} = \{\{\emptyset, \mathcal{U}, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}\}.$$

(7)

Remark 10. Table 5 offers the comparison between j -approximations and j -accuracy for $j = \langle i \rangle$ depending on Definitions 14 and 16 by using a reflexive relation R and ideal $\mathcal{F} = \{\emptyset, \{a\}\}$ of Example 2.

5. Conclusions and Future Works

To deal with uncertainty issues, Pawlak [1] proposed a non-statistical approach called rough set. Its idea is based on collection of the elements that have the same values according the required attribute. Accuracy measures and approximations represent the essential ideas in rough set theory which give details of the boundary region in terms of size and structure.

In this paper, we have suggested new techniques to generate new types of accuracy measures and approximations for a rough set under any arbitrary relation. These techniques are induced from hybridization of topological structures and ideals. In addition to studying their basic

properties, we have compared between them. Then, we have compared them with last methods to show their importance to improve the approximations and maximize the accuracy measures.

In the forthcoming articles, we are going to introduce further methods that help to obtain better approximation and higher accuracy measures than those given in the literature. Also, we search how these methods can be applied to model real-life issues.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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