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Research Article

Group Generalized q-Rung Orthopair Fuzzy Soft Sets: New Aggregation Operators and Their Applications

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In recent years, q-rung orthopair fuzzy sets have been appeared to deal with an increase in the value of $q > 1$, which allows obtaining membership and nonmembership grades from a larger area. Practically, it covers those membership and nonmembership grades, which are not in the range of intuitionistic fuzzy sets. The hybrid form of q-rung orthopair fuzzy sets with soft sets have emerged as a useful framework in fuzzy mathematics and decision-makings. In this paper, we presented group generalized q-rung orthopair fuzzy soft sets (GGq-ROFSSs) by using the combination of q-rung orthopair fuzzy soft sets and q-rung orthopair fuzzy sets. We investigated some basic operations on GGq-ROFSSs. Notably, we initiated new averaging and geometric aggregation operators on GGq-ROFSSs and investigated their underlying properties. A multicriteria decision-making (MCDM) framework is presented and validated through a numerical example. Finally, we showed the interconnection of our methodology with other existing methods.

1. Introduction

Zadeh originated the fuzzy set (FS) as an enlargement of the standard sets by the concept of inclusion of vague human judgements in computing situations [1]. The FS is indicated by the fuzzy information μ, which gives values from the unit close interval $[0, 1]$ for each prospector $x \in \mathcal{X}$. The idea of the FS plays an important role in the domain of soft computing, which manages vagueness, robustness, and partial truth. In some real-world difficulties where humanoid though attains reliable and unreliable information, the FS may not be sufficient to deal with underlying uncertainties.

In 1986, another shape of the FS called intuitionistic fuzzy sets (IFSs) was authorized by Atanassov, which provide a reliable grade $\mu(x)$ and unreliable grade $\nu(x)$ for all x in the universe of discourse X. The IFSs are characterized by the sum $\mu(x) + \nu(x) \leq 1$ and the degree of indeterminacy $\pi(x) = 1 - \mu(x) - \nu(x)$ [2]. Xu and Yager [3, 4] discussed the intuitionistic fuzzy value (IFV), which is an ordered pair of reliable and unreliable information for a component in the IFS on any x. Different rudiments of IFSs have been established such as aggregation operators [4], similarity and distance function [5, 6], and multicriteria decision-makings (MCDM) [7]. The aggregation operators are imperious in the MCDM process, which attains a shape of the measurable information by the accumulation of big data [8–10].

The IFSs enhance FSs in a meaningful approach, which is more capable of overcoming uncertainties, sharpless boundaries caused by the hesitation, and lack of assurance in human cognition. Xu and Zhao [11] extended a meaningful and insightful view on the information synthesis for MCDM using IFSs. To deal with real-life cases of reliable and unreliable information, which do not satisfy inequality $\mu(x) + \nu(x) \leq 1$, Yager initiated Pythagorean fuzzy sets (PFSs) [12, 13] and q-rung orthopair fuzzy sets (q-ROFSs) [14], which are crucial generalizations of IFSs. The q-ROFSs possess overall anticipation of symmetry of reliable and unreliable information in a larger space [15], that is, $(\mu(x))^q + (\nu(x))^q \leq 1$. A q-ROFS appears as an IFS (PFS) when $q = 1\,(q = 2)$. The fundamental score function and operators for q-ROFSs were investigated by Liu and Wang [16]. Several basic properties of PFSs and q-ROFSs can be seen in the literature [17–21]. Notably, researchers around the planet check out hybrid MCDM methods of PFSs and q-ROFSs using TODIM [22], TOPSIS [23, 24], MULTIMOORA [25], MABAC method [26], aggregation operators [17, 27–33], entropy measures [34], and distance measures [35].

A general parametrization model called soft set theory initiated by Molodtsov [36] has a great tendency to cop uncertainties. The soft set is free of inadequacy as it is a classical tool for coping parameters. It is further connected with usual mathematical operations on sets by Maji et al. [37] and Ali et al. [38]. A combination of soft sets and FSs known as fuzzy soft sets was introduced by Maji et al. [39], and it has been applied in various fields [40–46]. An extended form of fuzzy soft sets, known as intuitionistic fuzzy soft sets (IFSSs), was initiated by Maji et al. [47]. Recently, q-rung orthopair fuzzy soft sets (q-ROFSSs) have been introduced by Hamid et al. [48]. The model of q-ROFSS is a valuable tool to deal with vagueness by means of the label of parameters along with reliable and unreliable grades in the larger space [49]. Hussain et al. [50] presented MCDM techniques using averaging operators on q-ROFSSs. The generalized IFSSs (GIFSSs) were investigated by Agarwal et al. [51], and it possesses an important opinion with the model IFSS. A different scenario that overcomes the adequacies [52] of the original concept of GIFSSs was given by Feng et al. [53]. Both the ideas of GIFSSs were extended by several researchers Garg and Arora [54], Hayat et al. [55, 56], and Khan et al. [57, 58]. GGIFSSs produce a deep and meaningful insight in the MCDM problem by merging aggregation operators [56]. Another aspect of GGIFSS-based operators has been investigated by Hayat et al. [59], which handle information in a collected form. On the prospect of group-based GIFSSs (GGIFSSs) [56, 59], it is required to develop underlying operators, which can handle MCDM problems in different scenarios of combinations of information. More importantly, the extended space of q-ROFSs is the general form to deal with any implicit information.

On this prospect, there is a huge capacity to exercise another view of GGIFSS aggregation operators because the q-ROFSs relays the ambiguous information in higher productive ways than the GGIFSSs. Another important and fundamental point is to develop a different study to GGIFSSs that aggregate information concerning attributes until final ranking appears. Thus, we developed the group-based generalized q-ROFSSs (GGq-ROFSSs) and new aggregation operators through entire components in GGq-ROFSSs. By motivations of the above discussion, the purpose and aim of this article are given as follows:

(1) To initiate a different form of aggregation operators for GGq-ROFSSs that do not abandon the importance of attributes initially and do not quickly fascinate alternatives

(2) To develop an internal mechanism that gently addresses the importance of parameters in aggregation operators for GGq-ROFSSs

(3) To address the higher range of reliable and unreliable information in GGq-ROFSSs for possible values of q

(4) To develop the MCDM method for GGq-ROFSSs environment

In Section 2, we recall basic ideas of IFSs, PFSs, q-ROFSs, soft sets, and q-ROFSSs. In Sections 3 and 4, we discuss the notions of GGq-ROFSSs and their operations. In Section 5, we define new aggregations operators on GGq-ROFSSs. In Section 6, we give a new method of MCDM and a numerical example of real-life applications. Section 7 gives comparisons with other existing methods, and the last section concludes the paper.

2. Preliminaries

In this section, we will recall the concepts of IFSs, PFSs, q-ROFSs, soft sets, and q-ROFSSs. Throughout this section, $\mathscr{X}$ will represent the collection of alternatives.

2.1. Intuitionistic Fuzzy Sets and Pythagorean Fuzzy Sets. A FS is a mapping $\mu\colon \mathscr{X} \longrightarrow [0,1]$, where μ is membership grade for an element $g \in \mathscr{X}$ [1]. In several real-life situations, reliable and unreliable information rectifies the proper signification of uncertainties. FSs were not sufficient in such situations; therefore, the concept of IFS was introduced.

Definition 1 (see [2]). An IFS $\mathscr{F}$ is expressed as follows:

$$\mathscr{F} = \{(g, \mu(g), \nu(g)) | g \in \mathscr{X}\}. \tag{1}$$

With the functions $\mu\colon \mathscr{X} \longrightarrow [0,1]$, $\nu\colon \mathscr{X} \longrightarrow [0,1]$ called reliable and unreliable grades of an element g of $\mathscr{X}$ under the condition that the following inequality holds:

$$0 \leq \mu(g) + \nu(g) \leq 1. \tag{2}$$

For an element $g \in \mathscr{X}$, $\langle \mu, \nu \rangle$ is called IF value (IFV) in $\mathscr{F}$. In an IFS, the hesitancy of an IFV $\langle \mu, \nu \rangle$ to $\mathscr{F}$ is given by

$$\pi_{\mathscr{F}} = 1 - \mu(g) - \nu(g). \tag{3}$$

The hesitancy of IFV is also called indeterminacy of the $g \in \mathscr{X}$ in $\mathscr{F}$.

Definition 2 (see [12, 13]). A PFS $\mathscr{F}$ is expressed as follows:

$$\mathscr{P} = \{(g, \mu(g), \nu(g)) | g \in \mathscr{X}\}. \tag{4}$$

With the functions $\mu: \mathscr{X} \longrightarrow [0,1]$, $\nu: \mathscr{X} \longrightarrow [0,1]$ called reliable and unreliable grades of an element g of $\mathscr{X}$ under the condition that the following inequality holds:

$$0 \leq (\mu(g))^2 + (\nu(g))^2 \leq 1. \quad (5)$$

For an element $g \in \mathscr{X}$, $\langle \mu, \nu \rangle$ is called PF value (PFV) in $\mathscr{P}$. In an PFS, the hesitancy (or indeterminacy) of an PFV $\langle \mu, \nu \rangle$ to $\mathscr{P}$ is given by

$$\pi_{\mathscr{P}} = \sqrt{1 - (\mu(g))^2 - (\nu(g))^2}. \quad (6)$$

2.2. q-Rung Orthopair Fuzzy Sets. In 2016, Yager extended the range of double-graded fuzzy models in higher space. It is defined as

Definition 3 (see [14]). A q-RFS $\mathscr{R}$ is defined as

$$\mathscr{R} = \{(g, \mu(g), \nu(g)) | g \in \mathscr{X}\}. \quad (7)$$

With the functions $\mu: \mathscr{X} \longrightarrow [0,1]$, $\nu: \mathscr{X} \longrightarrow [0,1]$ called reliable and unreliable grades of an element g of $\mathscr{X}$ under the condition that the following inequality holds:

$$0 \leq (\mu(g))^q + (\nu(g))^q \leq 1. \quad (8)$$

Particularly, the hesitancy degree for q-ROFS is given as

$$\pi_{\mathscr{R}} = \sqrt[q]{1 - (\mu(g))^2 - (\nu(g))^2}. \quad (9)$$

The pair $\langle \mu, \nu \rangle$ is called q-rung orthopair fuzzy value (q-ROFV) for an object $g \in \mathscr{X}$. Let $\mathscr{R}_1 = \{(g, \mu(g), \nu(g)) | g \in \mathscr{X}\}$ and $\mathscr{R}_2 = \{(g, s(g), t(g)) | g \in \mathscr{X}\}$ be two q-ROFSs; then,

$$\begin{aligned}
&\mathscr{R}_1 \cup \mathscr{R}_2 = \{(g, \max\{\mu(g), s(g)\}, \min\{\nu(g), t(g)\}) | g \in \mathscr{X}\},\\
&\mathscr{R}_1 \cap \mathscr{R}_2 = \{(g, \min\{\mu(g), s(g)\}, \max\{\nu(g), t(g)\}) | g \in \mathscr{X}\},\\
&\mathscr{R}_1 \subseteq \mathscr{R}_2 \Leftrightarrow \{\mu(g) \leq s(g), \nu(g) \leq t(g) \, \forall g \in \mathscr{X}\},\\
&(\mathscr{R}_1)^c = \{(g, \nu(g), \mu(g)) | g \in \mathscr{X}\}.
\end{aligned} \quad (10)$$

The study on q-ROFS is extended by Liu and Wang [16] in the following crucial notions:

Definition 4 (see [16]). Let $b = \langle \mu, \nu \rangle$ be q-ROFN; then, the score function is defined as follows:

$$S_{\text{liu}}(b) = \mu^q - \nu^q. \quad (11)$$

This notion is effective when we have to transfer a q-ROFN to a real value in interval $[-1, 1]$ and therefore we can compare two or more q-ROFNs on their score functions. Consider a special case when $b_1 = \langle \mu_1, \nu_1 \rangle$, $b_2 = \langle \mu_2, \nu_2 \rangle$ with the condition $\mu_i = \nu_i \, \forall i = 1, 2$. Then, $s(b_1) = 0 = s(b_2)$; thus, S_{liu} has an inadequacy for such a case. Therefore, in this study, when S_{liu} fails, we will use the following.

Definition 5 (see [32]). Let $b = \langle \mu, \nu \rangle$ be q-ROFN; then,

$$S_{px}(b) = \left(\frac{e^{\mu^q - \nu^q}}{e^{\mu^q - \nu^q} + 1} - \frac{1}{2}\right)\pi^q + \mu^q - \nu^q. \quad (12)$$

Definition 6 (see [16]). Let us take the collection of q-ROFNs $b_{k'} = (\mu_{k'}, \nu_{k'})$, $k' = 1, 2, 3, \ldots, n$ and weight vector $[\lambda_1, \lambda_2, \ldots, \lambda_n]$; then, the q-rung orthopair fuzzy weighted averaging operator (q-ROFWA) is indicated as

$$\text{q} - \text{ROFWA}(b_1, b_2, \ldots, b_n) = \left\langle \sqrt[q]{1 - \prod_{k'=1}^{n} \left(1 - \mu_{k'}^q\right)^{\lambda_{k'}}}, \prod_{k'=1}^{n} \nu_{k'}^{\lambda_{k'}} \right\rangle. \quad (13)$$

Definition 7 (see [16]). Let us take the collection of q-ROFNs $b_{k'} = (\mu_{k'}, \nu_{k'})$, $k' = 1, 2, 3, \ldots, n$ and weight vector $[\lambda_1, \lambda_2, \ldots, \lambda_n]$; then, the q-rung orthopair fuzzy weighted geometric operator (q-ROFWG) is indicated as

$$\text{q} - \text{ROFWG}(b_1, b_2, \ldots, b_n) = \left\langle \prod_{k'=1}^{n} \mu_{k'}^{\lambda_{k'}}, \sqrt[q]{1 - \prod_{k'=1}^{n} \left(1 - \nu_{k'}^q\right)^{\lambda_{k'}}} \right\rangle. \quad (14)$$

The q-ROFWA and q-ROFWG are effective to aggregate data involving large number of q-ROFNs to a single q-ROFN.

2.3. q-Rung Orthopair Fuzzy Soft Sets. Jointly the approach of soft sets [36] with q-ROFSs is known as the q-rung orthopair fuzzy soft set, which is handy in MCDM problems. The notion of soft set is described as follows.

Definition 8 (see [36]). Let $\mathscr{X}$ be a fixed set and $P(\mathscr{X})$ be the set of all subsets of $\mathscr{X}$. Consider the set of parameters E and $\mathscr{A}$ be the subset of E. Let us define a function $\mathscr{S}$ as

$$\mathscr{S}: \mathscr{A} \longrightarrow P(\mathscr{X}). \quad (15)$$

Then, pair $(\mathscr{S}, \mathscr{A}) = \{(g, \mathscr{S}(g)) | g \in \mathscr{A}, \mathscr{S}(g) \in P(\mathscr{X})\}$ is called the soft set.

Definition 9 (see [49]). Let $(\mathscr{X}, E)$ be a soft universe and $\mathscr{A} \subseteq E$. Define a mapping $\mathscr{Q}: \mathscr{A} \longrightarrow \text{q} - \text{ROFS}(\mathscr{X})$; then, pair $(\mathscr{Q}, \mathscr{A})$ is called the q-rung orthopair fuzzy soft set (q-ROFSS) over $\mathscr{X}$, where $\text{q} - \text{ROFS}(\mathscr{X})$ denotes the collection of all q-ROFSs over $\mathscr{X}$. The q-ROFSS $(\mathscr{Q}, \mathscr{A})$ can be described as

$$\begin{aligned}
&(\mathscr{Q}, \mathscr{A}) = \{(e, \{\rho, \langle \mu(\rho), \nu(\rho) \rangle\}) | e \in \mathscr{A}, \rho \in \mathscr{X}\},\\
&\text{or } (\mathscr{Q}, \mathscr{A}) = \left\{ \left(e, \left\{\frac{\rho}{\langle \mu(\rho), \nu(\rho) \rangle}\right\}\right) | e \in \mathscr{A}, \rho \in \mathscr{X} \right\},
\end{aligned} \quad (16)$$

where $\mu(\rho): \mathscr{X} \longrightarrow [0,1]$ and $\nu(\rho): \mathscr{X} \longrightarrow [0,1]$ are reliable and nonreliable grades of ρ fulfilling inequality $0 \leq \mu^q(\rho) + \nu^q(\rho) \leq 1$. Take $\mu_{ij} = \mu_{e_j}(\rho_i)$, $\nu_{ij} = \nu_{e_j}(\rho_i)$, where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Then, the tabular form of q-ROFSS $(\mathscr{Q}, \mathscr{A})$ is given in Table 1.

Table 1: q-ROFSS $(\mathcal{Q}, \mathcal{A})$.

$\mathcal{X}\|\mathcal{A}$	e_1	e_2	$\cdots$	e_n
ρ_1	$\langle\mu_{11}, \nu_{11}\rangle$	$\langle\mu_{12}, \nu_{12}\rangle$	$\cdots$	$\langle\mu_{1n}, \nu_{1n}\rangle$
ρ_2	$\langle\mu_{21}, \nu_{21}\rangle$	$\langle\mu_{22}, \nu_{22}\rangle$	$\cdots$	$\langle\mu_{2n}, \nu_{2n}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_m	$\langle\mu_{m1}, \nu_{m1}\rangle$	$\langle\mu_{m2}, \nu_{m2}\rangle$	$\cdots$	$\langle\mu_{mn}, \nu_{mn}\rangle$

For simplicity, we will represent $(\mathcal{Q}, \mathcal{A})$ by $\mathcal{Q}_{\mathcal{A}}$.

Definition 10 (see [49]). Consider two q-ROFSSs $\mathcal{Q}_{\mathcal{A}_1}$ and $\mathcal{Q}_{\mathcal{A}_2}{}'$ on $\mathcal{X}$. We say $\mathcal{Q}_{\mathcal{A}_1} \subseteq \mathcal{Q}_{\mathcal{A}_2}{}'$ if and only if

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$

(ii) $\mathcal{Q}(e)$ is q-ROF subset of $\mathcal{Q}'(e)$ for all $e \in \mathcal{A}_1 \subseteq \mathcal{A}_2$

Definition 11 (see [49]). Consider two q-ROFSSs $\mathcal{Q}_{\mathcal{A}_1}{}'$ and $\mathcal{Q}_{\mathcal{A}_2}{}''$ on $\mathcal{X}$. Then, $\mathcal{Q}_{\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2} = \mathcal{Q}_{\mathcal{A}_1}{}' \cup \mathcal{Q}_{\mathcal{A}_2}{}''$ denotes the union of $\mathcal{Q}_{\mathcal{A}_1}{}'$ and $\mathcal{Q}_{\mathcal{A}_2}{}''$ on $\mathcal{X}$, such that

$$\mathcal{Q}(e) = \begin{cases} \mathcal{Q}'(e), & \text{if } e \in \mathcal{A}_1 - \mathcal{A}_2, \\ \mathcal{Q}''(e), & \text{if } e \in \mathcal{A}_2 - \mathcal{A}_1, \\ \mathcal{Q}'(e) \cup \mathcal{Q}''(e), & \text{if } e \in \mathcal{A}_1 \cap \mathcal{A}_2, \end{cases} \tag{17}$$

where $\mathcal{Q}'(e) \cup \mathcal{Q}''(e)$ is the union of q-ROFSs $\mathcal{Q}'(e), \mathcal{Q}''(e)$.

Definition 12 (see [49]). Consider two q-ROFSSs $\mathcal{Q}_{\mathcal{A}_1}{}'$ and $\mathcal{Q}_{\mathcal{A}_2}{}''$ on $\mathcal{X}$. Then, $\mathcal{Q}_{\mathcal{A}=\mathcal{A}_1\cap\mathcal{A}_2} = \mathcal{Q}_{\mathcal{A}_1}{}' \cap \mathcal{Q}_{\mathcal{A}_2}{}''$ denotes the intersection of $\mathcal{Q}_{\mathcal{A}_1}{}'$ and $\mathcal{Q}_{\mathcal{A}_2}{}''$ on $\mathcal{X}$, such that $\mathcal{Q}(e) = \mathcal{Q}'(e) \cap \mathcal{Q}''(e)$, where $e \in \mathcal{A}_1 \cap \mathcal{A}_2 \neq \varnothing$, where $\mathcal{Q}'(e) \cap \mathcal{Q}''(e)$ is the intersection of q-ROFSs $\mathcal{Q}'(e), \mathcal{Q}''(e)$.

3. Group Generalized q-Rung Orthopair Fuzzy Soft Sets

In this section, we will define generalized q-Rung orthopair fuzzy soft sets (Gq-ROFSSs) and group-based generalized q-Rung orthopair fuzzy soft sets (GGq-ROFSSs). First Gq-ROFSS is described as follows.

Definition 13. Consider a soft universe $(\mathcal{X}, E)$ and $\mathcal{A}$ contained in E. A triple $(\mathcal{Q}, \mathcal{A}, \alpha)$ is called Gq-ROFSS over $\mathcal{X}$ if $(\mathcal{Q}, \mathcal{A})$ is a q-ROFSS over $\mathcal{X}$ and α is a q-ROFS over $\mathcal{A}$.

In Gq-ROFSS, only one extra opinion α appears, but in many real-life satisfactions, more than one crucial additional opinions are needed. Thus, we define a greater prospect of Gq-ROFSS as GGq-ROFSS as follows.

Definition 14. Consider a soft universe $(\mathcal{X}, E)$ and $\mathcal{A}$ contained in E. A triple $(\mathcal{Q}, \mathcal{A}, \eta)$ is called GGq-ROFSS over $\mathcal{X}$ if $(\mathcal{Q}, \mathcal{A})$ is a q-ROFSS over $\mathcal{X}$ and $\eta = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the parametrized q-ROFSs (Pq-ROFSs) over $\mathcal{A}$. In other words, η is the group Pq-ROFSs considered by "p" number of senior experts/moderators.

Remark 1. If $p = 1$ in GGq-ROFSSs, then it is Gq-ROFSS. Therefore, Gq-ROFSS is a special case of GGq-ROFSS, and thus, generally, we will focus on GGq-ROFSSs.

A broaden tabular form of GGq-ROFSS (in Definition 14) is given in Table 2.

In Table 2, the light gray part represents q-ROFSS and the brown part represents the group of q-ROFSs in GGq-ROFSS.

4. Operations on Group-Based Generalized q-Rung Orthopair Fuzzy Soft Sets

In this section, we will define subset, union, intersection, and complement of GGq-ROFSSs.

Definition 15. Consider a soft universe $(\mathcal{X}, E)$ and $\mathcal{A}, \mathcal{B}$ contained in E. Let two GGq-ROFSSs $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ on $\mathcal{X}$, where $\mathcal{A} \subseteq \mathcal{B}$, $\eta_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, $\eta_2 = \{\beta_1, \beta_2, \ldots, \beta_p\}$, respectively, are group Pq-ROFSs related to "p" number of senior experts/moderators. The η_1 is group q-ROF subset of η_1 if and only if $s_{\alpha_1}(e_k) \leq s_{\beta_1}(e_k)$, $s_{\alpha_2}(e_k) \leq s_{\beta_2}(e_k)$, $\ldots$, $s_{\alpha_p}(e_k) \leq s_{\beta_p}(e_k)$, and $t_{\alpha_1}(e_k) \leq t_{\beta_1}(e_k), t_{\alpha_2}(e_k) \leq t_{\beta_2}(e_k), \ldots, t_{\alpha_p}(e_k) \leq t_{\beta_p}(e_k)$ and for each $e_k \in \mathcal{A}$. It is denoted as $\eta_1 \Subset \eta_2$.

In the prospect of Definition 15, we introduce GGq-ROFS subsets.

Definition 16. Let two GGq-ROFSSs $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ on $\mathcal{X}$, where $\mathcal{A}, \mathcal{B} \subset E$. Then, $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ is a GGq-ROFS subset of $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ if

(i) $(\mathcal{Q}_1, \mathcal{A}_1) \subseteq (\mathcal{Q}_2, \mathcal{A}_2)$

(ii) $\eta_1 \Subset \eta_2$

Definition 17. Let two GGq-ROFSSs $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ on $\mathcal{X}$, where $\mathcal{A}, \mathcal{B} \subset E$ such that $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{C}$ and $\eta_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, $\eta_2 = \{\beta_1, \beta_2, \ldots, \beta_p\}$. The extended union of $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ is investigated in the form of GGq-ROFSS. which is given as follows:

$$(\mathcal{Q}, \mathcal{C}, \eta) = (\mathcal{Q}_1, \mathcal{A}_1, \eta_1) \cup_e (\mathcal{Q}_2, \mathcal{A}_2, \eta_2), \tag{18}$$

such that

(i) $(\mathcal{Q}, \mathcal{C}) = (\mathcal{Q}_1, \mathcal{A}_1) \cup_e (\mathcal{Q}_2, \mathcal{A}_2)$.

(ii) For each $e \in \mathcal{C}$, $\left\langle \mu_{\gamma_{i'}}, \nu_{\gamma_{i'}} \right\rangle i' = 1, 2, \ldots, p$ is defined as

$$\begin{aligned} \mu_{\gamma_{i'}}(e) &= \begin{cases} \mu_{\alpha_{i'}}(e), & \text{if } e \in \mathcal{A}_1 - \mathcal{A}_2; \\ \mu_{\beta_{i'}}(e), & \text{if } e \in \mathcal{A}_2 - \mathcal{A}_1; \\ \max\left\{\mu_{\alpha_{i'}}(e), \mu_{\beta_{i'}}(e)\right\}, & \text{if } e \in \mathcal{A}_1 \cap \mathcal{A}_2, \end{cases} \\ \nu_{\gamma_{i'}}(e) &= \begin{cases} \nu_{\alpha_{i'}}(e), & \text{if } e \in \mathcal{A}_1 - \mathcal{A}_2; \\ \nu_{\beta_{i'}}(e), & \text{if } e \in \mathcal{A}_2 - \mathcal{A}_1; \\ \min\left\{\nu_{\alpha_{i'}}(e), \nu_{\beta_{i'}}(e)\right\}, & \text{if } e \in \mathcal{A}_1 \cap \mathcal{A}_2, \end{cases} \end{aligned} \tag{19}$$

where $\eta = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$.

Table 2: GGq-ROFSS $(\mathcal{Q}, \mathcal{A}, \eta)$.

$\chi \vert \mathcal{A}$	e_1	e_2	$\cdots$	e_n
ρ_1	$\langle \mu_{11}, \nu_{11} \rangle$	$\langle \mu_{12}, \nu_{12} \rangle$	$\cdots$	$\langle \mu_{1n}, \nu_{1n} \rangle$
ρ_2	$\langle \mu_{21}, \nu_{21} \rangle$	$\langle \mu_{22}, \nu_{22} \rangle$	$\cdots$	$\langle \mu_{2n}, \nu_{2n} \rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_m	$\langle \mu_{m1}, \nu_{m1} \rangle$	$\langle \mu_{m2}, \nu_{m2} \rangle$	$\cdots$	$\langle \mu_{mn}, \nu_{mn} \rangle$
α_1	$\langle s_{11}, t_{11} \rangle$	$\langle s_{12}, t_{12} \rangle$	$\cdots$	$\langle s_{1n}, t_{1n} \rangle$
α_2	$\langle s_{21}, t_{21} \rangle$	$\langle s_{22}, t_{22} \rangle$	$\cdots$	$\langle s_{2n}, t_{2n} \rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_p	$\langle s_{p1}, t_{p1} \rangle$	$\langle s_{p2}, t_{p2} \rangle$	$\cdots$	$\langle s_{pn}, t_{pn} \rangle$

Definition 18. Let two GGq-ROFSSs be $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ on $\mathcal{X}$, where $\mathcal{A}, \mathcal{B} \subset E$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{C}$ and $\eta_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, $\eta_2 = \{\beta_1, \beta_2, \ldots, \beta_p\}$. The restricted intersection of $(\mathcal{Q}_1, \mathcal{A}_1, \eta_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2, \eta_2)$ is investigated in the form of GGq-ROFSS, which is given as

$$(\mathcal{Q}, \mathcal{C}, \eta) = (\mathcal{Q}_1, \mathcal{A}_1, \eta_1) \cap_r (\mathcal{Q}_2, \mathcal{A}_2, \eta_2), \tag{20}$$

such that

(i) $(\mathcal{Q}, \mathcal{C}) = (\mathcal{Q}_1, \mathcal{A}_1) \cap_r (\mathcal{Q}_2, \mathcal{A}_2)$

(ii) For each $e \in \mathcal{C}$, $\left\langle \mu_{\gamma_{i'}}, \nu_{\gamma_{i'}} \right\rangle$ $i' = 1, 2, \ldots, p$ is defined as $\mu_{\gamma_{i'}}(e) = \min\left\{\mu_{\alpha_{i'}}(e), \mu_{\beta_{i'}}(e)\right\}$ and $\nu_{\gamma_{i'}}(e) = \max\left\{\nu_{\alpha_{i'}}(e), \nu_{\beta_{i'}}(e)\right\}$ for all $e \in \mathcal{A}_1 \cap \mathcal{A}_2$, where $\eta = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$

Now, we will take an example of GGq-ROFSS to clarify the above concepts.

Example 1. Let $\mathcal{X} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be the universal set consisting of five different kinds of face masks available in market, and the set of attributes E is given as $E = \{e_1, e_2, e_3, e_4, e_5\}$, where each e_i, respectively, stands for affordable price, good fabrication, effective comfortable design, capable of stopping viruses and bacteria, and comfortable breathing while wearing the mask. Let the two buyers $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, have the following preferences while buying the face mask:

$$\begin{aligned} \mathcal{A} &= \{e_1, e_4\}, \\ \mathcal{B} &= \{e_2, e_4, e_5\} \subset E. \end{aligned} \tag{21}$$

The GGq-ROFSSs $(\mathcal{Q}, \mathcal{A}, \eta)$ and $(\mathcal{Q}\prime, \mathcal{B}, \eta\prime)$ for buyers $\mathcal{T}_1$ and $\mathcal{T}_2$ are interpreted in Tables 3 and 4, where two senior persons h_1 and h_2 provide their opinions on q-ROFSSs (given in the light gray parts of Tables 3 and 4). The extra inputs as the group of q-ROFSs of h_1 and h_2 are interpreted in the brown parts of both the tables. The union and intersection of $(\mathcal{Q}, \mathcal{A}, \eta)$ and $(\mathcal{Q}', \mathcal{B}, \eta')$ are computed in Tables 5 and 6 respectively.

Definition 19. Consider a soft universe $(\mathcal{X}, E)$ and $\mathcal{A}$ contained in E. Let GGq-ROFSS be $(\mathcal{Q}, \mathcal{A}, \eta)$ on $\mathcal{X}$ where $\eta = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$. The complement of $(\mathcal{Q}, \mathcal{A}, \eta)$ is denoted by $(\mathcal{Q}^c, \mathcal{A}, \eta^c)$, where $(\mathcal{Q}^c, \mathcal{A})$ is the complement of q-ROFSS $(\mathcal{Q}, \mathcal{A})$ and $\eta^c = \{\alpha_1^c, \alpha_2^c, \ldots, \alpha_p^c\}$.

Definition 20. Let GGq-ROFSS be $(\mathcal{Q}, \mathcal{A}, \eta)$ given in Table 2.

(1) If $>\mu_{jk}, \nu_{jk}<\, = \,>1,0<$ and $>s_{i'k}, t_{i'k}<\, = \,>1,0<$ for all $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$ and $i' = 1, 2, \ldots, p$, then $(\mathcal{Q}, \mathcal{A}, \eta)$ is called whole GGq-ROFSS. It is denoted by $\mathcal{W}_{\mathcal{A}}$.

(2) If $>\mu_{jk}, \nu_{jk}<\, = \,>0,1<$ and $>s_{i'k}, t_{i'k}<\, = \,>0,1<$ for all $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$ and $i' = 1, 2, \ldots, p$, then $(\mathcal{Q}, \mathcal{A}, \eta)$ is called null GGq-ROFSS. It is denoted by $\mathcal{N}_{\mathcal{A}}$.

Proposition 1. *Let $\mathcal{Q}_{\mathcal{A}}^{\eta} = (\mathcal{Q}, \mathcal{A}, \eta)$ be a GGq-ROFSS over $\mathcal{X}$. Then,*

(1) $\mathcal{Q}_{\mathcal{A}}^{\eta} \cup \mathcal{Q}_{\mathcal{A}}^{\eta} = \mathcal{Q}_{\mathcal{A}}^{\eta}$, $\mathcal{Q}_{\mathcal{A}}^{\eta} \cap \mathcal{Q}_{\mathcal{A}}^{\eta} = \mathcal{Q}_{\mathcal{A}}^{\eta}$

(2) $\mathcal{Q}_{\mathcal{A}}^{\eta} \cup \mathcal{W}_{\mathcal{A}} = \mathcal{W}_{\mathcal{A}}$, $\mathcal{Q}_{\mathcal{A}}^{\eta} \cap \mathcal{W}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}^{\eta}$

(3) $\mathcal{Q}_{\mathcal{A}}^{\eta} \cap \mathcal{N}_{\mathcal{A}} = \mathcal{N}_{\mathcal{A}}$, $\mathcal{Q}_{\mathcal{A}}^{\eta} \cup \mathcal{N}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}^{\eta}$

5. Aggregation Operators on GGq-ROFSS

To attain substantial effect, there is an immense need to define better aggregation operators that accurately deal with all components of GGq-ROFSSs.

5.1. GWq-ROFW Operators

Definition 21. Consider Definition 14, where GGq-ROFSS $(\mathcal{Q}, \mathcal{A}, \eta)$ is given in Table 2, where the light gray part represents q-ROFSS and the brown part describes the group of q-ROFSs of "p" number of senior moderators. Let $\left\langle \mu_{0k}^j, \nu_{0k}^j \right\rangle = \left\langle \mu_{jk}, \nu_{jk} \right\rangle$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$, that is to say $\left\langle \mu_{0k}^j, \nu_{0k}^j \right\rangle$ are q-ROFVs in the light gray part of Table 2. Moreover, $\left\langle \mu_{i'k}^j, \nu_{i'k}^j \right\rangle = \langle s_{i'k}, t_{i'k} \rangle$, $i' = 1, 2, \ldots, p$, that is to say $\left\langle \mu_{i'k}^j, \nu_{i'k}^j \right\rangle$ are q-ROFVs in the brown part of Table 2. Assume that $i = 0, 1, 2, \ldots, p$. A symbolization is given by

$$\left\langle \mu_{ik}^j, \nu_{ik}^j \right\rangle = \begin{cases} \left\langle \mu_{jk}, \nu_{jk} \right\rangle, & \text{if } i = 0; \\ \left\langle s_{i'k}, t_{i'k} \right\rangle, & \text{if } i > 0. \end{cases} \tag{22}$$

On the above fundamental and crucial symbolic notion of the GGq-ROFSS, we contemplate novel averaging aggregation operators.

Definition 22. Consider Definition 14, where GGq-ROFSS $(\mathcal{Q}, \mathcal{A}, \eta)$ is given in Table 2. Let $\Psi = [\varpi_1, \varpi_2, \ldots, \varpi_n]^T$ be the weighted vector over $\mathcal{A}$, such that $\sum_{k=1}^n \varpi_k = 1$ and $\varpi_k > 0$. Also take weighted vector $W = [\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_p]^T$ such that $\sum_{i=1}^p \varpi_i = 1$ and $\varpi_i > 0$, where $\varphi_1, \ldots, \varphi_p$ are weights for the judgements of the "p" number of senior moderators and φ_0 is the weight for each q-ROFV in the light gray part of Table 2. In other words, φ_0 is the weight of whole data in

Table 3: GGq-ROFSS $(\mathscr{Q}, \mathscr{A}, \eta)$ for $\mathscr{T}_1$.

$\chi \mid \mathscr{A}$	e_1	e_2
ρ_1	$\langle 0.8, 0.8\rangle$	$\langle 0.8, 0.6\rangle$
ρ_2	$\langle 0.5, 0.7\rangle$	$\langle 0.4, 0.3\rangle$
ρ_3	$\langle 0.6, 0.8\rangle$	$\langle 0.2, 0.2\rangle$
ρ_4	$\langle 0.5, 0.6\rangle$	$\langle 0.6, 0.1\rangle$
ρ_5	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.4\rangle$
α_{h_1}	$\langle 0.4, 0.4\rangle$	$\langle 0.7, 0.7\rangle$
α_{h_2}	$\langle 0.8, 0.6\rangle$	$\langle 0.6, 0.6\rangle$

Table 4: GGq-ROFSS $(\mathscr{Q}\prime, \mathscr{B}, \eta\prime)$ for $\mathscr{T}_2$.

$\chi \mid \mathscr{B}$	e_2	e_4	e_5
ρ_1	$\langle 0.8, 0.8\rangle$	$\langle 0.8, 0.6\rangle$	$\langle 0.8, 0.8\rangle$
ρ_2	$\langle 0.5, 0.3\rangle$	$\langle 0.5, 0.7\rangle$	$\langle 0.4, 0.3\rangle$
ρ_3	$\langle 0.6, 0.8\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.6, 0.3\rangle$
ρ_4	$\langle 0.8, 0.3\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.6, 0.1\rangle$
ρ_5	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.5, 0.5\rangle$
β'_{h_1}	$\langle 0.4, 0.4\rangle$	$\langle 0.8, 0.7\rangle$	$\langle 0.7, 0.7\rangle$
β'_{h_2}	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.4\rangle$	$\langle 0.6, 0.6\rangle$

Table 5: Union of GGq-ROFSSs $(\mathscr{R}, \mathscr{A} \cup \mathscr{B}, \delta)$.

$\chi \mid \mathscr{A} \cup \mathscr{B}$	e_1	e_2	e_4	e_5
ρ_1	$\langle 0.8, 0.8\rangle$	$\langle 0.8, 0.8\rangle$	$\langle 0.8, 0.6\rangle$	$\langle 0.8, 0.8\rangle$
ρ_2	$\langle 0.5, 0.7\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.4, 0.3\rangle$
ρ_3	$\langle 0.6, 0.8\rangle$	$\langle 0.6, 0.8\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.6, 0.3\rangle$
ρ_4	$\langle 0.5, 0.6\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.6, 0.1\rangle$
ρ_5	$\langle 0.7, 0.3\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.5, 0.5\rangle$
β_{h_1}	$\langle 0.4, 0.4\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.8, 0.7\rangle$	$\langle 0.7, 0.7\rangle$
β_{h_2}	$\langle 0.8, 0.6\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.4\rangle$	$\langle 0.6, 0.6\rangle$

Table 6: Intersection of GGq-ROFSSs $(\mathscr{R}, \mathscr{A} \cap \mathscr{B}, \delta)$.

$\chi \mid \mathscr{A} \cap \mathscr{B}$	e_4
ρ_1	$\langle 0.8, 0.6\rangle$
ρ_2	$\langle 0.4, 0.7\rangle$
ρ_3	$\langle 0.2, 0.5\rangle$
ρ_4	$\langle 0.5, 0.1\rangle$
ρ_5	$\langle 0.6, 0.5\rangle$
β_{h_1}	$\langle 0.7, 0.7\rangle$
β_{h_2}	$\langle 0.6, 0.6\rangle$

q-ROFSS (see the light gray part of Table 2). Assume $i = 0, 1, 2, \ldots, p$. Then, the generalized weighted q-Rung orthopair fuzzy averaging operator (GWq-ROFA) undertaken by GGq-ROFSS is contemplated as follows:

$$\begin{aligned}
\Theta_j &= \mathrm{GWq-ROFA}\left(\left\langle \mu_{j1}, \nu_{j1}\right\rangle, \left\langle \mu_{j2}, \nu_{j2}\right\rangle, \ldots, \left\langle \mu_{jn}, \nu_{jn}\right\rangle\right) \\
&= \mathrm{q-ROFWA}_k \begin{pmatrix} \mathrm{q-ROFWA}_i\left(\left\langle \mu^j_{01}, \nu^j_{01}\right\rangle, \left\langle \mu^j_{11}, \nu^j_{11}\right\rangle, \ldots, \left\langle \mu^j_{p1}, \nu^j_{p1}\right\rangle\right) \\ \mathrm{q-ROFWA}_i\left(\left\langle \mu^j_{02}, \nu^j_{02}\right\rangle, \left\langle \mu^j_{12}, \nu^j_{12}\right\rangle, \ldots, \left\langle \mu^j_{p2}, \nu^j_{p2}\right\rangle\right) \\ \cdots \\ \mathrm{q-ROFWA}_i\left(\left\langle \mu^j_{0n}, \nu^j_{0n}\right\rangle, \left\langle \mu^j_{1n}, \nu^j_{1n}\right\rangle, \ldots, \left\langle \mu^j_{pn}, \nu^j_{pn}\right\rangle\right) \end{pmatrix} \\
&= \mathrm{q-ROFWA}_k \begin{pmatrix} \mathrm{q-ROFWA}_i\left(\left\langle \mu_{j1}, \nu_{j1}\right\rangle, \left\langle s_{11}, t_{11}\right\rangle, \left\langle s_{21}, t_{21}\right\rangle, \ldots, \left\langle s^j_{p1}, \nu^j_{p1}\right\rangle\right) \\ \mathrm{q-ROFWA}_i\left(\left\langle \mu_{j2}, \nu_{j2}\right\rangle, \left\langle s_{12}, t_{12}\right\rangle, \left\langle s_{22}, t_{22}\right\rangle, \ldots, \left\langle s_{p2}, t_{p2}\right\rangle\right) \\ \cdots \\ \mathrm{q-ROFWA}_i\left(\left\langle \mu_{jn}, \nu_{jn}\right\rangle, \left\langle s_{1n}, t_{1n}\right\rangle, \left\langle s_{2n}, t_{2n}\right\rangle, \ldots, \left\langle s_{pn}, t_{pn}\right\rangle\right) \end{pmatrix},
\end{aligned} \tag{23}$$

where $\mathrm{q-ROFWA}_k$ is the q-rung orthopair fuzzy weighted averaging operator and it operates over the set of criteria, and $\mathrm{q-ROFWA}_i$ is the q-rung orthopair fuzzy weighted averaging operator and it operates mutually over q-ROFSS and on the set of senior moderators.

The set of all GWq-ROFA operators for m number of alternatives is indicated as $\Delta = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}$. The above novel GWq-ROFA operators are realistic instruments for linear and entire aggregations of q-ROFVs in GGq-ROFSS.

The GWq-ROFA operators have a specific way of incorporating each component of GGq-ROFSS. They entirely compel q-ROFVs in a linear way towards attributes until the final q-ROFV appears.

Example 2. Consider GGq-ROFSS indicated in Table 4 in Example 20. Given $q-\mathrm{ROFV}(\rho_1) = (\langle 0.8, 0.8\rangle, \langle 0.8, 0.6\rangle, \langle 0.8, 0.8\rangle)$. We have to calculate the GWq-ROFA operator for ρ_1. The q-ROFVs are depicted as

$$\begin{aligned} q-\mathrm{ROFWA}_{e_2}(\langle \mu_{11}, \nu_{11}\rangle, \langle s_{11}, t_{11}\rangle, \langle s_{21}, t_{21}\rangle) &= (\langle 0.8, 0.8\rangle, \langle 0.4, 0.4\rangle, \langle 0.8, 0.2\rangle),\\ q-\mathrm{ROFWA}_{e_4}(\langle \mu_{22}, \nu_{22}\rangle, \langle s_{12}, t_{12}\rangle, \langle s_{22}, t_{22}\rangle) &= (\langle 0.8, 0.6\rangle, \langle 0.8, 0.7\rangle, \langle 0.7, 0.4\rangle),\\ q-\mathrm{ROFWA}_{e_5}(\langle \mu_{23}, \nu_{23}\rangle, \langle s_{13}, t_{13}\rangle, \langle s_{23}, t_{23}\rangle) &= (\langle 0.8, 0.8\rangle, \langle 0.7, 0.7\rangle, \langle 0.6, 0.6\rangle). \end{aligned} \tag{24}$$

Let $\Psi = [\varpi_1, \varpi_2, \varpi_3]^T = [(0.3/e_2), (0.4/e_4), (0.3/e_5)]^T$ be the weighted vector over $\mathscr{A}$. Also take weighted vector $W = [\varphi_0, \varphi_1, \varphi_2]^T = [0.40, 0.25, 0.35]^T$ for q-ROFSS and judgements of senior person/moderators. For $q = 4$, we have

$$\begin{aligned} q-\mathrm{ROFWA}(\langle 0.8, 0.8\rangle, \langle 0.4, 0.4\rangle, \langle 0.8, 0.2\rangle) &= \langle 0.7584, 0.4141\rangle,\\ q-\mathrm{ROFWA}(\langle 0.8, 0.6\rangle, \langle 0.8, 0.7\rangle, \langle 0.7, 0.4\rangle) &= \langle 0.7719, 0.5411\rangle,\\ q-\mathrm{ROFWA}(\langle 0.8, 0.8\rangle, \langle 0.7, 0.7\rangle, \langle 0.6, 0.6\rangle) &= \langle 0.7272, 0.6996\rangle. \end{aligned} \tag{25}$$

Now, GWq-ROFA is given by $\Theta_1 = \mathrm{GWq-ROFA}(\langle 0.8, 0.8\rangle, \langle 0.8, 0.6\rangle, \langle 0.8, 0.8\rangle) = q-\mathrm{ROFWA}(\langle 0.7584, 0.4141\rangle, \langle 0.7719, 0.5411\rangle, \langle 0.7272, 0.6996\rangle) = \langle 0.7555, 0.5394\rangle$. Similarly, GWq-ROFA operators $\Theta_2, \Theta_3, \Theta_4$, and Θ_5 can be obtained.

Theorem 1. *Let $a_{jk} = \langle \mu_j k, \nu_j k\rangle$ and $d_{i'k} = \langle s_{i'j}, t_{i'j}\rangle$ be the q-ROFVs in GGq-ROFSS in Table 2, where $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ and $i' = 1, 2, \ldots, p$. If we consider $i = 0, 1, 2, \ldots, p$, then the GWq-ROFA operator is given by*

$$\begin{aligned} &\mathrm{GWq-ROFA}\left(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{jn}, \nu_{jn}\rangle\right)\\ &= \left\langle \sqrt[q]{1 - \prod_{k=1}^{n}\left(1 - \left(1 - \prod_{i=0}^{p}\left(1 - \left(\mu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{n}\left(\prod_{i=0}^{p}\left(\left(\nu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)^{\varpi_k}\right\rangle. \end{aligned} \tag{26}$$

Proof. Assume that $p = 1$ and $n = 2$. Take $j = 1$; we apply mathematical induction on n. By the definition of GGq-ROFSS,

$$\begin{aligned} \Theta_j &= \mathrm{GWq-ROFWA}(\langle \mu_{11}, \nu_{11}\rangle, \langle \mu_{12}, \nu_{12}\rangle)\\ &= \mathrm{IFWA}_j\begin{pmatrix} \mathrm{IWA}_i\left(\langle \mu_{01}^1, \nu_{01}^1\rangle, \langle \mu_{11}^1, \nu_{11}^1\rangle\right)\\ \mathrm{IWA}_i\left(\langle \mu_{02}^1, \nu_{02}^1\rangle, \langle \mu_{12}^1, \nu_{12}^1\rangle\right) \end{pmatrix}\\ &= \mathrm{IFWA}_j\begin{pmatrix} \mathrm{IWA}_i(\langle \mu_{11}, \nu_{11}\rangle, \langle s_{11}, t_{11}\rangle)\\ \mathrm{IWA}_i(\langle \mu_{12}, \nu_{12}\rangle, \langle s_{12}, t_{12}\rangle) \end{pmatrix}. \end{aligned} \tag{27}$$

Now, $\mathrm{IWA}_i(\langle \mu_{11}, \nu_{11}\rangle, \langle s_{11}, t_{11}\rangle) = \langle \sqrt[q]{(1 - (1 - (\mu_{11})^q))^{\varphi_0} \cdot (1 - (s_{11})^q)^{\varphi_1}}, \nu_{11}^{\varphi_0} \cdot t_{11}^{\varphi_1}\rangle$, $\mathrm{IWA}_i(\langle \mu_{12}, \nu_{12}\rangle, \langle s_{12}, t_{12}\rangle) = \langle \sqrt[q]{(1 - (1 - (\mu_{12})^q))^{\varphi_0} \cdot (1 - (s_{12})^q)^{\varphi_1}}, \nu_{12}^{\varphi_0} \cdot t_{12}^{\varphi_1}\rangle$. Thus,

$$\begin{aligned}
\Theta_j &= \mathrm{GWq-ROFWA}\left(\langle \mu_{11}, \nu_{11}\rangle, \langle \mu_{12}, \nu_{12}\rangle\right)\\
&= \mathrm{IFWA}_j\left(\left\langle \sqrt[q]{1-(1-(\mu_{11})^q)^{\varphi_0}\cdot(1-(s_{11})^q)^{\varphi_1}}, \nu_{11}^{\varphi_0}\cdot t_{11}^{\varphi_1}\right\rangle, \left\langle \sqrt[q]{1-(1-(\mu_{12})^q)^{\varphi_0}\cdot(1-(s_{12})^q)^{\varphi_1}}, \nu_{12}^{\varphi_0}\cdot t_{12}^{\varphi_1}\right\rangle\right)\\
&= \left\langle \sqrt[q]{\left(1-\left(1-\left(1-(1-(\mu_{11})^q)^{\varphi_0}\cdot(1-(s_{11})^q)^{\varphi_1}\right)\right)^{\varpi_1}\right)\cdot\left(1-\left(1-\left(1-(1-(\mu_{12})^q)^{\varphi_0}\cdot(1-(s_{12})^q)^{\varphi_1}\right)\right)^{\varpi_2}\right)},\right.\\
&\quad \left.\cdot\left(\nu_{11}^{\varphi_0}\cdot s_{11}^{\varphi_1}\right)^{\varpi_1}\cdot\left(\nu_{12}^{\varphi_0}\cdot s_{12}^{\varphi_1}\right)^{\varpi_2}\right\rangle\\
&= \left\langle \sqrt[q]{\left(1-\left(1-\left(1-\left(1-\left(\mu_{01}^1\right)^q\right)^{\varphi_0}\cdot\left(1-\left(\mu_{11}^1\right)^q\right)^{\varphi_1}\right)\right)^{\varpi_1}\right)\cdot\left(1-\left(1-\left(1-\left(1-\left(\mu_{02}^1\right)^q\right)^{\varphi_0}\cdot\left(1-\left(\mu_{12}^1\right)^q\right)^{\varphi_1}\right)\right)^{\varpi_2}\right)},\right.\\
&\quad \left.\cdot\left(\left(\nu_{01}^1\right)^{\varphi_0}\cdot\left(\nu_{11}^1\right)^{\varphi_1}\right)^{\varpi_1}\cdot\left(\left(\nu_{02}^1\right)^{\varphi_0}\cdot\left(\nu_{12}^1\right)^{\varphi_1}\right)^{\varpi_2}\right\rangle\\
&= \left\langle \sqrt[q]{\left(1-\left(1-\left(1-\prod_{i=0}^{1}\left(1-\left(\mu_{i1}^1\right)^q\right)^{\varphi_i}\right)\right)^{\varpi_1}\right)\cdot\left(1-\left(1-\left(1-\prod_{i=0}^{1}\left(1-\left(\mu_{i2}^1\right)^q\right)^{\varphi_i}\right)\right)^{\varpi_2}\right)},\right.\\
&\quad \left.\cdot\left(\prod_{i=0}^{1}\left(\nu_{i1}^1\right)^{\varphi_i}\right)^{\varpi_1}\cdot\left(\prod_{i=0}^{1}\left(\nu_{i2}^1\right)^{\varphi_i}\right)^{\varpi_2}\right\rangle\\
&= \left\langle \sqrt[q]{1-\prod_{k=1}^{2}\left(1-\left(1-\prod_{i=0}^{1}\left(1-\left(\mu_{ik}^1\right)^q\right)^{\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{2}\left(\prod_{i=0}^{1}\left(\left(\nu_{ik}^1\right)^q\right)^{\varphi_i}\right)^{\varpi_k}\right\rangle.
\end{aligned}\tag{28}$$

Hence, the theorem is valid for $n=2$. Considering that this result is fine for $n=n'$ that is

$$\begin{aligned}
&\mathrm{GWq-ROFA}\left(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{jn'}, \nu_{jn'}\rangle\right)\\
&\quad = \left\langle \sqrt[q]{1-\prod_{k=1}^{n'}\left(1-\left(1-\prod_{i=0}^{p}\left(1-\left(\mu_{ik}^j\right)^q\right)^{\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{n'}\left(\prod_{i=0}^{p}\left(\left(\nu_{ik}^j\right)^q\right)^{\varphi_i}\right)^{\varpi_k}\right\rangle.
\end{aligned}\tag{29}$$

Then, for $n=n'+1$, we have

$$\begin{aligned}
&\mathrm{GWq-ROFA}\left(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{j(n'+1)}, \nu_{j(n'+1)}\rangle\right)\\
&\quad = \left\langle \sqrt[q]{1-\prod_{k=1}^{n'+1}\left(1-\left(1-\prod_{i=0}^{p}\left(1-\left(\mu_{ik}^j\right)^q\right)^{\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{n'+1}\left(\prod_{i=0}^{p}\left(\left(\nu_{ik}^j\right)^q\right)^{\varphi_i}\right)^{\varpi_k}\right\rangle.
\end{aligned}\tag{30}$$

Hence, by mathematical induction, Theorem 1 satisfy for all positive integer n. □

Theorem 2. *(idempotency) If $\langle \mu_{jk}, \nu_{jk}\rangle = \langle \mu_j, \nu_j\rangle$ and $\langle \mu_{i'k}, \nu_{i'k}\rangle = \langle \mu, \nu\rangle = \langle \mu_j, \nu_j\rangle$ for all $k=1,2,\ldots,n$, then $GWq-ROFA(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{jn}, \nu_{jn}\rangle) = \langle \mu, \nu\rangle$.*

Proof. Given $\langle \mu_{jk}, \nu_{jk}\rangle = \langle \mu, \nu\rangle$ and $\langle \mu_{i'k}, \nu_{i'k}\rangle = \langle \mu, \nu\rangle$ for all $i'=1,2,\ldots,p$ and $k=1,2,\ldots,n$.

$$\Theta_j = \mathrm{GWq-ROFA}\left(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{jn}, \nu_{jn}\rangle\right)$$

$$=\left\langle \sqrt[q]{1-\prod_{k=1}^{n}\left(1-\left(1-\prod_{i=0}^{p}\left(1-\left(\mu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{n}\left(\prod_{i=0}^{p}\left(\left(\nu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)^{\varpi_k}\right\rangle$$

$$=\left\langle \sqrt[q]{1-\prod_{k=1}^{n}\left(1-\left(1-\left(1-(\mu)^{q}\right)^{\sum_{i=0}^{p}\varphi_i}\right)\right)^{\varpi_k}}, \prod_{k=1}^{n}\left(\left((\nu)^{q}\right)^{\sum_{i=0}^{p}\varphi_i}\right)^{\varpi_k}\right\rangle$$

$$=\left\langle \sqrt[q]{1-\prod_{k=1}^{n}\left(1-\left(1-\left(1-(\mu)^{q}\right)\right)\right)^{\varpi_k}}, \prod_{k=1}^{n}\left(\left((\nu)^{q}\right)\right)^{\varpi_k}\right\rangle$$

$$=\left\langle \sqrt[q]{1-\left(1-\left(1-\left(1-(\mu)^{q}\right)\right)\right)^{\sum_{k=1}^{n}\varpi_k}}, \left(\left((\nu)^{q}\right)\right)^{\sum_{k=1}^{n}\varpi_k}\right\rangle = \langle \mu,\nu\rangle,$$

$$\mathrm{GWq-ROFA}\left(\left\langle \mu_{j1},\nu_{j1}\right\rangle,\left\langle \mu_{j2},\nu_{j2}\right\rangle,\ldots,\left\langle \mu_{jn},\nu_{jn}\right\rangle\right)=\langle \mu,\nu\rangle. \tag{31}$$

□

Theorem 3. *(boundedness) If* $\langle \mu^{+},\nu^{-}\rangle = \left\langle (\mu_{ik}^{j})^{\max}, (\nu_{ik}^{j})^{\min}\right\rangle$ *and* $\langle \mu^{-},\nu^{+}\rangle = \left\langle (\mu_{ik}^{j})^{\min}, (\nu_{ik}^{j})^{\max}\right\rangle$ *for all* $j=1,2,\ldots,m$ *and* $i=0,1,2,\ldots,p$*, then* $\langle \mu^{-},\nu^{+}\rangle \le$ *GWq– ROFA* $(\langle \mu_{j1},\nu_{j1}\rangle, \langle \mu_{j2},\nu_{j2}\rangle,\ldots,\langle \mu_{jn},\nu_{jn}\rangle)\le \langle \mu^{+},\nu^{-}\rangle$.

Proof. Since $\mu^{-}\le \mu_{ik}^{j}\le \mu^{+} \Leftrightarrow (\mu^{-})^{q}\le (\mu_{ik}^{j})^{q}\le (\mu^{+})^{q}\Leftrightarrow 1-(\mu^{-})^{q} \ge 1-(\mu_{ik}^{j})^{q}\ge 1-(\mu^{+})^{q}\Leftrightarrow \prod_{i=0}^{p}(1-(\mu^{-})^{q})^{\varphi_i} \ge \prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i} \ge \prod_{i=0}^{p}(1-(\mu^{+})^{q})^{\varphi_i}\Leftrightarrow (1-(\mu^{-})^{q})^{\sum_{i=0}^{p}\varphi_i} \ge \prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}\ge (1-(\mu^{+})^{q})^{\sum_{i=0}^{p}\varphi_i} \Leftrightarrow (1-(\mu^{-})^{q}) \ge \prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}\ge (1-(\mu^{+})^{q})\Leftrightarrow 1-(1-(\mu^{-})^{q})\le 1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i} \le 1-(1-(\mu^{+})^{q})\Leftrightarrow (\mu^{-})^{q} \le 1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i} \le (\mu^{+})^{q}\Leftrightarrow 1-(\mu^{-})^{q}\ge 1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}) \ge 1-(\mu^{+})^{q}\Leftrightarrow \prod_{k=1}^{n}(1-(\mu^{-})^{q})^{\varpi_k} \ge \prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))^{\varpi_k}\ge \prod_{k=1}^{n}(1-(\mu^{+})^{q})^{\varpi_k}\Leftrightarrow (1-(\mu^{-})^{q})^{\sum_{k=1}^{n}\varpi_k}\ge \prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))\ge (1-(\mu^{+})^{q})^{\sum_{k=1}^{n}\varpi_k}\Leftrightarrow (1-(\mu^{-})^{q})\ge \prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))^{\varpi_k}\ge (1-(\mu^{+})^{q})\Leftrightarrow 1-(1-(\mu^{-})^{q})\le 1-\prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))^{\varpi_k}\le 1-(1-(\mu^{+})^{q})\Leftrightarrow (\mu^{-})^{q}\le 1-\prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))^{\varpi_k}\le (\mu^{+})^{q}\Leftrightarrow (\mu^{-})\le \sqrt[q]{1-\prod_{k=1}^{n}(1-(1-\prod_{i=0}^{p}(1-(\mu_{ik}^{j})^{q})^{\varphi_i}))^{\varpi_k}}\le (\mu^{+})$.

Similarly, nonmembership part is aggregated as $(\mu^{+})\le \prod_{k=1}^{n}(\prod_{i=0}^{p}((\nu_{ik}^{j})^{q})^{\varphi_i})^{\varpi_k}\le (\mu^{-})$. This concluded the proof of the theorem, $\langle \mu^{-},\nu^{+}\rangle \le \mathrm{GWq-ROFA}(\langle \mu_{j1},\nu_{j1}\rangle,\langle \mu_{j2},\nu_{j2}\rangle,\ldots,\langle \mu_{jn},\nu_{jn}\rangle)\le \langle \mu^{+},\nu^{-}\rangle$. □

Theorem 4. *(monotonicity) If* $\langle \mu_{ik}^{j},\nu_{ik}^{j}\rangle$ *and* $\langle \overline{\mu}_{ik}^{j},\overline{\nu}_{ik}^{j}\rangle$ *for all* $i=\{0,1,2,\ldots,p\}$, $j=\{1,2,\ldots,m\}$, $k=\{1,2,\ldots,n\}$ *are two IFVs such that* $\langle \mu_{ik}^{j},\nu_{ik}^{j}\rangle \le \langle \overline{\mu}_{ik}^{j},\overline{\nu}_{ik}^{j}\rangle$*, then* $GWq-ROFA$ $(\langle \mu_{j1},\nu_{j1}\rangle,\langle \mu_{j2},\nu_{j2}\rangle,\ldots,\langle \mu_{jn},\nu_{jn}\rangle)\le GWq-ROFA$ $(\langle \overline{\mu}_{j1},\overline{\nu}_{j1}\rangle,\langle \overline{\mu}_{j2},\overline{\nu}_{j2}\rangle,\ldots,\langle \overline{\mu}_{jn},\overline{\nu}_{jn}\rangle)$.

Proof. It can be concluded from Theorem 3. □

Proposition 2. *Let* $(\mathcal{Q},\mathcal{A},\eta)$ *be a GGq-ROFSS, given in Table 2. Then,*

(1) *If* $\langle s_{i'k},t_{i'k}\rangle=\langle 1,0\rangle$ *for all* i' *and* k*, then* $\Theta_j=\{\langle 1,0\rangle,\langle 1,0\rangle,\ldots,\langle 1,0\rangle\}$.

(2) *If* $\langle \mu_{0k}^{j},\nu_{0k}^{j}\rangle=\langle \mu_{jk},\nu_{jk}\rangle=\langle 1,0\rangle$ *for all* j *and* k*, then* $\Theta_j=\{\langle 1,0\rangle,\langle 1,0\rangle,\ldots,\langle 1,0\rangle\}$.

(3) *If* $\langle s_{i'k},t_{i'k}\rangle=\langle 1,0\rangle$ *for all* i',k *and* $\langle \mu_{0k}^{j},\nu_{0k}^{j}\rangle=\langle \mu_{jk},\nu_{jk}\rangle=\langle 1,0\rangle$ *for all* j,k*, then* $\Theta_j=\{\langle 1,0\rangle,\langle 1,0\rangle,\ldots,\langle 1,0\rangle\}$.

(4) *If* $\langle s_{i'k},t_{i'k}\rangle=\langle 0,1\rangle$ *for all* i',k *and* $\langle \mu_{0k}^{j},\nu_{0k}^{j}\rangle=\langle \mu_{jk},\nu_{jk}\rangle=\langle 0,1\rangle$ *for all* j,k *then* $\Theta_j=\{\langle 0,1\rangle,\langle 0,1\rangle,\ldots,\langle 0,1\rangle\}$.

Definition 23. Consider Definition 14, where GGq-ROFSS $(\mathcal{Q},\mathcal{A},\eta)$ is given in Table 2. Let $\Psi=[\varpi_1,\varpi_2,\ldots,\varpi_n]^{T}$ be the weighted vector over $\mathcal{A}$, such that $\sum_{k=1}^{n}\varpi_k=1$ and $\varpi_k>0$. Also take weighted vector $W=[\varphi_0,\varphi_1,\varphi_2,\ldots,\varphi_p]^{T}$ such that $\sum_{i=1}^{p}\varpi_i=1$ and $\varpi_i>0$, where $\varphi_1,\ldots,\varphi_p$ are weights for the judgements of the "p" number of senior moderators and φ_0 is the weight for each q-ROFV in the light gray part of Table 2. In other words, φ_0 is the weight of whole data in q-ROFSS (see the light gray part of Table 2). Assume $i=0,1,2,\ldots,p$. Then, the generalized weighted q-Rung orthopair fuzzy geometric operator (GWq-ROFG) undertook by GGq-ROFSS is contemplated as follows:

$$\Theta_j=\mathrm{GWq-ROFG}\left(\left\langle \mu_{j1},\nu_{j1}\right\rangle,\left\langle \mu_{j2},\nu_{j2}\right\rangle,\ldots,\left\langle \mu_{jn},\nu_{jn}\right\rangle\right)$$

$$= \mathrm{q-ROFWG}_k \begin{pmatrix} \mathrm{q-ROFWG}_i\left(\left\langle \mu_{01}^j, \nu_{01}^j\right\rangle, \left\langle \mu_{11}^j, \nu_{11}^j\right\rangle, \ldots, \left\langle \mu_{p1}^j, \nu_{p1}^j\right\rangle\right) \\ \mathrm{q-ROFWG}_i\left(\left\langle \mu_{02}^j, \nu_{02}^j\right\rangle, \left\langle \mu_{12}^j, \nu_{12}^j\right\rangle, \ldots, \left\langle \mu_{p2}^j, \nu_{p2}^j\right\rangle\right) \\ \cdots \\ \mathrm{q-ROFWG}_i\left(\left\langle \mu_{0n}^j, \nu_{0n}^j\right\rangle, \left\langle \mu_{1n}^j, \nu_{1n}^j\right\rangle, \ldots, \left\langle \mu_{pn}^j, \nu_{pn}^j\right\rangle\right) \end{pmatrix}$$

$$= \mathrm{q-ROFWG}_k \begin{pmatrix} \mathrm{q-ROFWG}_i\left(\left(\mu_{j1}, \nu_{j1}\right), \left\langle s_{11}, t_{11}\right\rangle, \left\langle s_{21}, t_{21}\right\rangle, \ldots, \left\langle s_{p1}^j, \nu_{p1}^j\right\rangle\right) \\ \mathrm{q-ROFWG}_i\left(\left\langle \mu_{j2}, \nu_{j2}\right\rangle, \left\langle s_{12}, t_{12}\right\rangle, \left\langle s_{22}, t_{22}\right\rangle, \ldots, \left\langle s_{p2}, t_{p2}\right\rangle\right) \\ \cdots \\ \mathrm{q-ROFWG}_i\left(\left\langle \mu_{jn}, \nu_{jn}\right\rangle, \left\langle s_{1n}, t_{1n}\right\rangle, \left\langle s_{2n}, t_{2n}\right\rangle, \ldots, \left\langle s_{pn}, t_{pn}\right\rangle\right) \end{pmatrix}, \tag{32}$$

where $\mathrm{q-ROFWG}_k$ is the q-rung orthopair fuzzy weighted geometric operator and it operates over the set of criteria, and $\mathrm{q-ROFWG}_i$ is the q-rung orthopair fuzzy weighted geometric operator and it operates mutually over q-ROFSS and on the set of senior moderators.

The set of all GWq-ROFG operators for m number of alternatives is indicated as $\Delta\prime = \{\Theta_1', \Theta_2', \ldots, \Theta_m'\}$. The above novel GWq-ROFG operators are realistic instruments for linear and entire aggregations of q-ROFVs in GGq-ROFSS.

The GWq-ROFG operators have a specific way of incorporating each component of GGq-ROFSS. They entirely compel q-ROFVs in a linear way towards attributes until the final q-ROFV appears.

Example 3. Consider GGq-ROFSS indicated in Table 4 in Example 3. Given $\mathrm{q-ROFV}(\rho_2) = (\langle 0.5, 0.3\rangle, \langle 0.5, 0.7\rangle, \langle 0.4, 0.3\rangle)$, we have to calculate the GWq-ROFG operator for ρ_1. The q-ROFVs are depicted as

$$\begin{aligned} &\mathrm{q-ROFWG}_{e_2}\left(\left\langle \mu_{21}, \nu_{21}\right\rangle, \left\langle s_{11}, t_{11}\right\rangle, \left\langle s_{21}, t_{21}\right\rangle\right) = (\langle 0.5, 0.3\rangle, \langle 0.4, 0.4\rangle, \langle 0.8, 0.2\rangle), \\ &\mathrm{q-ROFWG}_{e_4}\left(\left\langle \mu_{22}, \nu_{22}\right\rangle, \left\langle s_{12}, t_{12}\right\rangle, \left\langle s_{22}, t_{22}\right\rangle\right) = (\langle 0.5, 0.7\rangle, \langle 0.8, 0.7\rangle, \langle 0.7, 0.4\rangle), \\ &\mathrm{q-ROFWG}_{e_5}\left(\left\langle \mu_{23}, \nu_{23}\right\rangle, \left\langle s_{13}, t_{13}\right\rangle, \left\langle s_{23}, t_{23}\right\rangle\right) = (\langle 0.4, 0.3\rangle, \langle 0.7, 0.7\rangle, \langle 0.6, 0.6\rangle). \end{aligned} \tag{33}$$

Let $\Psi = [\varpi_1, \varpi_2, \varpi_3]^T = [(0.3/e_2), (0.4/e_4), (0.3/e_5)]^T$ be the weighted vector over $\mathscr{A}$. Also take weighted vector $W = [\varphi_0, \varphi_1, \varphi_2]^T = [0.40, 0.25, 0.35]^T$ for q-ROFSS and judgements of senior person/moderators. For $q = 4$, we have

$$\begin{aligned} &\mathrm{q-ROFWG}(\langle 0.5, 0.3\rangle, \langle 0.4, 0.4\rangle, \langle 0.8, 0.2\rangle) = \langle 0.6644, 0.2797\rangle, \\ &\mathrm{q-ROFWG}(\langle 0.5, 0.7\rangle, \langle 0.8, 0.7\rangle, \langle 0.7, 0.4\rangle) = \langle 0.6880, 0.5755\rangle, \\ &\mathrm{q-ROFWG}(\langle 0.4, 0.3\rangle, \langle 0.7, 0.7\rangle, \langle 0.6, 0.6\rangle) = \langle 0.5883, 0.4726\rangle. \end{aligned} \tag{34}$$

Now GWq-ROFG is given by

$$\begin{aligned} \Theta_2' &= \mathrm{GWq-ROFG}(\langle 0.5, 0.3\rangle, \langle 0.5, 0.7\rangle, \langle 0.4, 0.3\rangle) \\ &= \mathrm{q-ROFWG}(\langle 0.6644, 0.2797\rangle, \langle 0.6880, 0.5755\rangle, \langle 0.5883, 0.4726\rangle) \\ &= \langle 0.6560, 0.4369\rangle. \end{aligned} \tag{35}$$

Similarly, GWq-ROFG operators $\Theta_1', \Theta_3', \Theta_4'$, and Θ_5' can be obtained.

Theorem 5. *Let $a_{jk} = \left\langle \mu_j k, \nu_j k\right\rangle$ and $d_{i'k} = \left\langle s_{i'j}, t_{i'j}\right\rangle$ be the q-ROFVs in GGq-ROFSS in Table 2, where $k = 1, 2, \ldots, n$,*

$j = 1, 2, \ldots, m$, and $i' = 1, 2, \ldots, p$. If we consider $i = 0, 1, 2, \ldots, p$, then GWq-ROFG operator is given by

$$\begin{aligned}&\mathrm{GWq-ROFG}\left(\left\langle \mu_{j1}, \nu_{j1}\right\rangle, \left\langle \mu_{j2}, \nu_{j2}\right\rangle, \ldots, \left\langle \mu_{jn}, \nu_{jn}\right\rangle\right)\\&= \left\langle \prod_{k=1}^{n}\left(\prod_{i=0}^{p}\left(\left(\mu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)^{\varpi_k}, \sqrt[q]{1 - \prod_{k=1}^{n}\left(1 - \left(1 - \prod_{i=0}^{p}\left(1 - \left(\nu_{ik}^{j}\right)^{q}\right)^{\varphi_i}\right)\right)^{\varpi_k}} \right\rangle.\end{aligned} \tag{36}$$

Proof. Same as the proof of Theorem 1. □

Theorem 6. *(idempotency) If $\left\langle \mu_{jk}, \nu_{jk}\right\rangle = \left\langle \mu_j, \nu_j\right\rangle$ and $\langle \mu_{i'k}, \nu_{i'k}\rangle = \langle \mu, \nu\rangle = \left\langle \mu_j, \nu_j\right\rangle$ for all $k = 1, 2, \ldots, n$, then $GWq-ROFG(\left\langle \mu_{j1}, \nu_{j1}\right\rangle, \left\langle \mu_{j2}, \nu_{j2}\right\rangle, \ldots, \left\langle \mu_{jn}, \nu_{jn}\right\rangle) = \langle \mu, \nu\rangle$.*

Proof. Same as the proof of Theorem 2. □

Theorem 7. *(boundedness) If $\langle \mu^{+}, \nu^{-}\rangle = \left\langle (\mu_{ik}^{j})^{\max}, (\nu_{ik}^{j})^{\min}\right\rangle$ and $\langle \mu^{-}, \nu^{+}\rangle = \left\langle (\mu_{ik}^{j})^{\min}, (\nu_{ik}^{j})^{\max}\right\rangle$ for all $j = 1, 2, \ldots, m$ and $i = 0, 1, 2, \ldots, p$, then $\langle \mu^{-}, \nu^{+}\rangle \le GWq-ROFG(\langle \mu_{j1}, \nu_{j1}\rangle, \langle \mu_{j2}, \nu_{j2}\rangle, \ldots, \langle \mu_{jn}, \nu_{jn}\rangle) \le \langle \mu^{+}, \nu^{-}\rangle$.*

Proof. Same as the proof of Theorem 3. □

Theorem 8. *(monotonicity) If $\left\langle \mu_{ik}^{j}, \nu_{ik}^{j}\right\rangle$ and $\left\langle \overline{\mu}_{ik}^{j}, \overline{\nu}_{ik}^{j}\right\rangle$ for all $i = \{0, 1, 2, \ldots, p\}$, $j = \{1, 2, \ldots, m\}, k = \{1, 2, \ldots, n\}$ are two IFVs such that $\left\langle \mu_{ik}^{j}, \nu_{ik}^{j}\right\rangle \le \left\langle \overline{\mu}_{ik}^{j}, \overline{\nu}_{ik}^{j}\right\rangle$, then*

$$\begin{aligned}&\mathrm{GWq-ROFG}\left(\left\langle \mu_{j1}, \nu_{j1}\right\rangle, \left\langle \mu_{j2}, \nu_{j2}\right\rangle, \ldots, \left\langle \mu_{jn}, \nu_{jn}\right\rangle\right)\\&\le \mathrm{GWq-ROFG}\left(\left\langle \overline{\mu}_{j1}, \overline{\nu}_{j1}\right\rangle, \left\langle \overline{\mu}_{j2}, \overline{\nu}_{j2}\right\rangle, \ldots, \left\langle \overline{\mu}_{jn}, \overline{\nu}_{jn}\right\rangle\right).\end{aligned} \tag{37}$$

Proof. It can be concluded from Theorem 4. □

Proposition 3. *Let $(\mathcal{Q}, \mathcal{A}, \eta)$ be a GGq-ROFSS, given in Table 2. Then,*

(1) If $\langle s_{i'k}, t_{i'k}\rangle = \langle 0, 1\rangle$ for all i' and k, then $\Theta_j' = \{\langle 0, 1\rangle, \langle 0, 1\rangle, \ldots, \langle 0, 1\rangle\}$.

(2) If $\left\langle \mu_{0k}^{j}, \nu_{0k}^{j}\right\rangle = \left\langle \mu_{jk}, \nu_{jk}\right\rangle = \langle 0, 1\rangle$ for all j and k, then $\Theta_j' = \{\langle 0, 1\rangle, \langle 0, 1\rangle, \ldots, \langle 0, 1\rangle\}$.

(3) If $\langle s_{i'k}, t_{i'k}\rangle = \langle 0, 1\rangle$ for all i', k and $\left\langle \mu_{0k}^{j}, \nu_{0k}^{j}\right\rangle = \left\langle \mu_{jk}, \nu_{jk}\right\rangle = \langle 0, 1\rangle$ for all j, k then $\Theta_j' = \{\langle 0, 1\rangle, \langle 0, 1\rangle, \ldots, \langle 0, 1\rangle\}$.

(4) If $\langle s_{i'k}, t_{i'k}\rangle = \langle 1, 0\rangle$ for all i', k and $\left\langle \mu_{0k}^{j}, \nu_{0k}^{j}\right\rangle = \left\langle \mu_{jk}, \nu_{jk}\right\rangle = \langle 1, 0\rangle$ for all j, k then $\Theta_j' = \{\langle 1, 0\rangle, \langle 1, 0\rangle, \ldots, \langle 1, 0\rangle\}$.

6. Multicriteria Decision-Making Method

In this section, a methodology on proposed operators is introduced, and for application, a numerical application is investigated.

6.1. Methodology. Let $\mathcal{X} = \{\rho_1, \rho_2, \ldots, \rho_m\}$ be the set of alternatives for a problem in which we have to choose the best alternative. Every alternative possesses the specific q-ROFVs of a set of criteria/attributes E. Let $\mathcal{A} \subseteq E$. An expert's committee give the q-ROFVs on set $\mathcal{X}$ comprising attributes in the form of q-ROFSS. The final examination of the q-ROFSS is provided by some senior experts/moderators in the form of q-ROFs over $\mathcal{A} = \{e_1, e_2, \ldots, e_n\}$. This forms GGq-ROFSS $(\mathcal{Q}, \mathcal{A}, \eta)$ given in Table 2 in Definition 14. In real-life problem, usually, the importance of attributes is given in the form of a weighted vector. Let $\Psi = [\varphi_1, \varphi_2, \ldots, \varphi_n]^T$ be the weighted vector over $\mathcal{A}$, such that $\sum_{k=1}^{n} \varphi_k = 1$, $\varphi_k > 0$. Also take weighted vector $W = [\varpi_0, \varpi_1, \varpi_2, \ldots, \varpi_p]^T$ such that $\sum_{i'=0}^{p} \varpi_i = 1$, $\varpi_i > 0$ where $\varpi_1, \varpi_2, \ldots, \varpi_p$ are the weights for the judgements of the "p" number of senior moderators, and ϖ_0 is the weight for each q-ROFV in the light gray part of Table 2. In other words, ϖ_0 is the weight of whole data in q-ROFSS (see the light gray part of Table 2). Next, the normalization of the q-ROFVs in $(\mathcal{Q}, \mathcal{A}, \eta)\, \forall i, j, k$ is obtained by using the following equation:

$$\left\langle \mu_{ik}^{j}, \nu_{ik}^{j}\right\rangle = \begin{cases}\left\langle \mu_{ik}^{j}(e), \nu_{ik}^{j}(e)\right\rangle, & \text{if } e \text{ is benefit kind criteria,}\\ \left\langle \nu_{ik}^{j}(e), \mu_{ik}^{j}(e)\right\rangle, & \text{if } e \text{ is cost kind criteria.}\end{cases} \tag{38}$$

This designs a new GGq-ROFSS $(\mathcal{Q}\prime, \mathcal{A}, \eta\prime)$. Then, we compute GWq-ROFA or GWq-ROFG operators on $(\mathcal{Q}\prime, \mathcal{A}, \eta\prime)$ and calculate score function on each GWq-ROFA or GWq-ROFG. Finally, rank the alternatives on the real values of score functions.

In order to evaluate most suitable alternative, we numerate the steps of the MCDM method in an algorithm as follows.

This algorithm is shown in Figure 1.

6.2. Numerical Example. An automation company wants to choose a director for their Research and Development (R and D) department from four applicants ρ_1, ρ_2, ρ_3, and ρ_4. The crucial attributes for evaluation are

C_1: personality

C_2: self-confidence

C_3: score obtained in a college degree

C_4: administrative skills

C_5: experience

C_6: abilities in research areas

(1) Consider a set of alternatives say $\mathscr{X} = \{\rho_1, \rho_2, \ldots, \rho_s\}$ their attributes as a set E. $\mathscr{A} = \{e_1, e_2, \ldots, e_n\}$.
(2) Constitute r number of committees of experts $CM_1, CM_2, \ldots, CM_r$ comprising sets of attributes $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r$ respectively, where $\mathscr{A}_1 \subset E, \mathscr{A}_2 \subset E, \ldots, \mathscr{A}_r \subset E$.
(3) Obtain r number q-ROFSSs on the $CM_1, CM_2, \ldots, CM_r$ respectively.
(4) Obtain curial opinions of p number moderators on each q-ROFSSs. This formulate a GGq-ROFSSs $(\mathscr{Q}_1, \mathscr{A}_1, \eta_1), (\mathscr{Q}_2, \mathscr{A}_2, \eta_2), \ldots, (\mathscr{Q}_r, \mathscr{A}_r, \eta_r)$.
(5) Take union $(\mathscr{Q}_1, \mathscr{A}_1, \eta_1), (\mathscr{Q}_2, \mathscr{A}_2, \eta_2), \ldots, (\mathscr{Q}_r, \mathscr{A}_r, \eta_r)$ which is equal to $(\mathscr{Q}, \mathscr{A}, \eta)$, where $\mathscr{A} = \cup_{j'=1}^{r} \mathscr{A}_{j'} = \{e_1, e_2, \ldots, e_n\}$
(6) Normalize $(\mathscr{Q}, \mathscr{A}, \eta)$, which gives normalized GGq-ROFSS $(\mathscr{Q}\prime, \mathscr{A}, \eta\prime)$.
(7) Obtain the weighted vector $\Psi = [\varphi_1, \varphi_2, \ldots, \varphi_n]^T$ over set A of attributes. Obtain the weighted vector $W = [\varpi_0, \varpi_1, \varpi_2, \ldots, \varpi_p]^T$ comprising q-ROFSS and opinions of moderators.
(8) Compute GGq-ROFA (or GGq-ROFG) operators $\Theta_j\,(\text{or}\,\Theta'_j)\,(j = 1, 2, \ldots, m)$ on normalized GGq-ROFSS $(\mathscr{Q}\prime, \mathscr{A}, \eta\prime)$.
(9) Get score function S_{liu} or S_{px} on each GGq-ROFA (or GGq-ROFG) operator Θ_j or $\Theta'_j\,(j = 1, 2, \ldots, m)$ using Definition 4.
(10) Classify alternatives and foremost alternative is on the maximum score.

Algorithm 1: MCDM procedure.

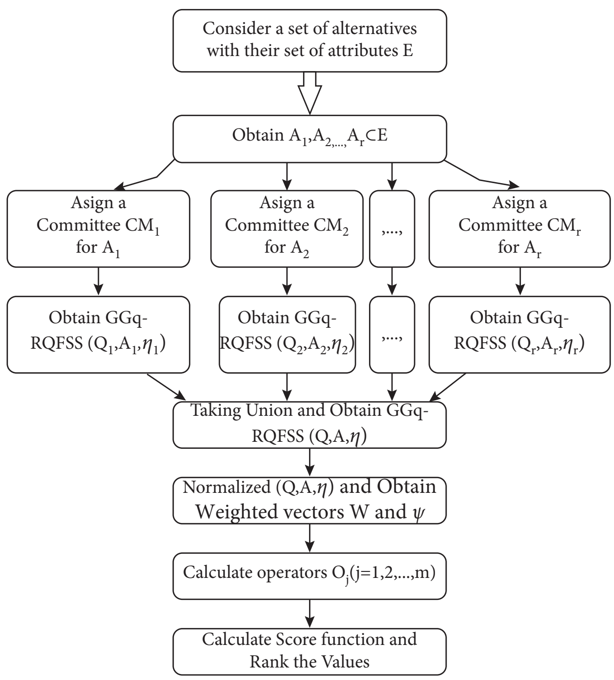

Figure 1: A flowchart of our algorithm.

Table 7: GGq-ROFSSs $(\mathscr{Q}, \mathscr{A}, \eta)$.

$\chi \vert \mathscr{A}$	C_1	C_2	C_5
ρ_1	$\langle 0.7, 0.7 \rangle$	$\langle 0.5, 0.7 \rangle$	$\langle 0.6, 0.8 \rangle$
ρ_2	$\langle 0.8, 0.5 \rangle$	$\langle 0.8, 0.6 \rangle$	$\langle 0.6, 0.7 \rangle$
ρ_3	$\langle 0.5, 0.4 \rangle$	$\langle 0.9, 0.3 \rangle$	$\langle 0.7, 0.4 \rangle$
ρ_4	$\langle 0.5, 0.7 \rangle$	$\langle 0.7, 0.7 \rangle$	$\langle 0.7, 0.8 \rangle$
α_1	$\langle 0.6, 0.7 \rangle$	$\langle 0.8, 0.7 \rangle$	$\langle 0.6, 0.5 \rangle$
α_2	$\langle 0.5, 0.6 \rangle$	$\langle 0.6, 0.5 \rangle$	$\langle 0.7, 0.7 \rangle$

Table 8: GGq-ROFSSs $(\mathscr{Q}\prime, \mathscr{B}, \eta\prime)$.

$\chi \vert \mathscr{B}$	C_3	C_4	C_6
ρ_1	$\langle 0.6, 0.3 \rangle$	$\langle 0.7, 0.7 \rangle$	$\langle 0.8, 0.4 \rangle$
ρ_2	$\langle 0.8, 0.4 \rangle$	$\langle 0.7, 0.6 \rangle$	$\langle 0.8, 0.7 \rangle$
ρ_3	$\langle 0.9, 0.3 \rangle$	$\langle 0.8, 0.3 \rangle$	$\langle 0.7, 0.5 \rangle$
ρ_4	$\langle 0.6, 0.7 \rangle$	$\langle 0.6, 0.7 \rangle$	$\langle 0.5, 0.6 \rangle$
α'_1	$\langle 0.4, 0.3 \rangle$	$\langle 0.6, 0.5 \rangle$	$\langle 0.7, 0.7 \rangle$
α'_2	$\langle 0.6, 0.6 \rangle$	$\langle 0.7, 0.5 \rangle$	$\langle 0.8, 0.3 \rangle$

In order to evaluate most suitable candidate, two different committees of experts, namely, CM_1 and CM_2, are constituted by the company. The attribute specifications for CM_1 and CM_2 are $\mathscr{A} = \{C_1, C_2, C_5\}$ and $\mathscr{B} = \{C_3, C_4, C_6\}$, respectively. The q-ROFV-based evaluations are q-ROFSSs $(\mathscr{Q}, \mathscr{A})$ (in the light gray part of Table 7) and $(\mathscr{Q}\prime, \mathscr{B})$ (in the light gray part of Table 8). To finalize q-ROFV-based data, two senior experts provided their opinion as q-ROFSs on the

Table 9: GGq-ROFSSs $(\mathscr{R}, E, \gamma)$.

$\chi\|E$	C_1	C_2	C_3	C_4	C_5	C_6
ρ_1	$\langle 0.7, 0.7\rangle$	$\langle 0.5, 0.7\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.7, 0.7\rangle$	$\langle 0.6, 0.8\rangle$	$\langle 0.8, 0.4\rangle$
ρ_2	$\langle 0.8, 0.5\rangle$	$\langle 0.8, 0.6\rangle$	$\langle 0.8, 0.4\rangle$	$\langle 0.7, 0.6\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.8, 0.7\rangle$
ρ_3	$\langle 0.5, 0.4\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.7, 0.4\rangle$	$\langle 0.7, 0.5\rangle$
ρ_4	$\langle 0.5, 0.7\rangle$	$\langle 0.7, 0.7\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.7, 0.8\rangle$	$\langle 0.5, 0.6\rangle$
β_1	$\langle 0.6, 0.7\rangle$	$\langle 0.8, 0.7\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.7, 0.7\rangle$
β_2	$\langle 0.5, 0.6\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.6, 0.6\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.7, 0.7\rangle$	$\langle 0.8, 0.3\rangle$

Table 10: GGq-ROFSS $(\mathscr{Q}, A, \eta)$.

$\chi\|\mathscr{A}$	e_1	e_2	e_3
ρ_1	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.5, 0.4\rangle$
ρ_2	$\langle 0.5, 0.4\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.5, 0.5\rangle$
ρ_3	$\langle 0.4, 0.4\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.4, 0.4\rangle$
α_1	$\langle 0.6, 0.4\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.3, 0.4\rangle$

Table 11: Comparison.

Method	Act	Ranking
Proposed method	$q=3$	$\rho_3 > \rho_1 > \rho_2$
Hayat et al. [55]	$q=1$	$\rho_3 > \rho_1 > \rho_2$
Hayat et al. [56]	$q=1$	$\rho_1 > \rho_3 > \rho_2$
Feng et al. [53]	$q=1$	$\rho_1 > \rho_3 > \rho_2$
Hayat et al. [59]	$q=1$	$\rho_3 > \rho_1 > \rho_2$

q-ROFSSs (see the brown part of Tables 7 and 8). The following steps are adopted to finalize decision-making process:

(1) Take union on GGq-ROFSSs $(\mathscr{Q}, \mathscr{A}, \eta)$ and $(\mathscr{Q}\prime, \mathscr{B}, \eta\prime)$, and obtain GGq-ROFSS $(\mathscr{R}, E, \gamma)$ in Table 9.

(2) Take $\Psi = [\varpi_1, \varpi_2, \ldots, \varpi_6]^T = [0.2, 0.3, 0.1, 0.2, 0.1, 0.1]^T$ be the weighted vector over E. Also take weighted vector $W = [\varphi_0, \varphi_1, \varphi_2]^T = [0.4, 0.25, 0.35]^T$ for q-ROFSS and judgements of senior experts.

(3) Calculate GWq-ROFA operators; we use $q = 4$ and obtain $\Theta_1 = \langle 0.6490, 0.5719\rangle$, $\Theta_2 = \langle 0.7065, 0.5555\rangle$, $\Theta_3 = \langle 0.7431, 0.4530\rangle$, and $\Theta_4 = \langle 0.6464, 0.6013\rangle$.

(4) Calculate score function; we obtain $S_{\text{liu}}(\Theta_1) = 0.0771$, $S_{\text{liu}}(\Theta_2) = 0.1510$, $S_{\text{liu}}(\Theta_3) = 0.2901$, and $S_{\text{liu}}(\Theta_4) = 0.0451$.

(5) Final ranking of candidates is given by $\rho_3 > \rho_2 > \rho_1 > \rho_4$. Thus, ρ_3 is the most appropriate candidate for the position of director in the company.

7. Comparative Study

In this part of the article, we compare our method with existing frameworks. The primely advantage of new aggregation operators is to characterize a weighted vector $W = [\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_p]^T$ on GGq-ROFSS, where $\varphi_1, \varphi_2, \ldots, \varphi_p$ are the weights on q-ROFSs of p number of moderators respectively, and φ_0 is a weight for q-ROFSS in GGq-ROFSS. Since assessments of moderators on q-ROFSS are the crucial component in GGq-ROFSS, in many real-life problems, their weights should be greater than φ_0. On the other hand, in some real-life situations, weight φ_0 on q-ROFSS is also important, the fact that it denotes a perception on fundamental data (or q-ROFSS) given by a committee of experts on crucial judgements on alternatives. Notably, we consider following example.

Table 12: Advantages.

Methods\ approches	Numbers of extra inputs	Weighted vectors	Operators type	Deficiency
Hayat et al. [55]	One	None	AND operation	Diminish the importance of reliable and unreliable grades on AND operation
Hayat et al. [56]	More than one	Over attributes	WA or WG	Not efficient on the same IF values of a moderator on each attribute
Hayat et al. [59]	More than one	Two weighted vectors over attributes and moderators respectively	WA or WG	Do not deal with larger space $q > 1$
Feng et al. [53]	One	Conversion of extra input into weighted vector	WA or WG	Diminish the importance of reliable and unreliable grades by the conversion of extra input into weighted vector
Proposed method	More than one	Two weighted vectors over attributes and moderators	WA or WG	Seem no deficiency

Example 4. Consider GGq-ROFSS as depicted in Table 6, where $\mathscr{A} = \{e_1, e_2, e_3\}$ is set of attributes with weighted vector $\Psi = \{(0.3/e_1), (0.4/e_2), (0.3/e_3)\}$ and $\mathscr{X} = \{\rho_1, \rho_2, \rho_3\}$ being the set of alternatives. α_1 is q-ROFS for moderator's assessments on q-ROFSS in Table 10.

Let $q = 3$ and a weighted vector $W = [(0.5/w_0), (0.5/w_1)]$, where w_0 is the weight for q-ROFSS and w_1 is the weight for q-ROFS α_1. By our new operators on GGq-ROFSSs, we obtain $S_{\text{liu}}(\Theta_1) = 0.0233$, $S_{\text{liu}}(\Theta_2) = 0.0225$, $S_{\text{liu}}(\Theta_3) = 0.0235$ and $\rho_3 > \rho_1 > \rho_2$.

By the operators in Hayat et al. [56], we obtain $S(\rho_1) = 0.7883$, $S(\rho_2) = 0.7646$, $S(\rho_3) = 0.7777$ and $\rho_1 > \rho_3 > \rho_2$.

By Feng et al. [54], $S(\rho_1) = 0.5931$, $S(\rho_2) = 0.5398$, $S(\rho_3) = 0.5509$, $\rho_1 > \rho_3 > \rho_2$.

By the operators in Hayat et al. [55, 59], we obtain $\rho_1 > \rho_3 > \rho_2$. A comparison for different methods on this example is shown in Table 11.

The main advantage our framework is that it aggregates information or data by concerning attributes until final ranking appears. It develop an internal mechanism that gently addresses the importance of parameters in aggregation operators for GGq-ROFSSs. The different components of the above methods are discussed in Table 12.

8. Conclusions

In this article, we have presented GGq-ROFSS and investigated some basic operations. Mainly, we have initiated new averaging and geometric aggregation operators on GGq-ROFSSs and investigated the underlying properties of these operators. We have given a MCDM framework and its validation through a numerical example. Finally, we have given comparison of our methodology with other existing methods. In GGq-ROFSSs, expert's judgements suggest the reliability of the evaluation of the alternatives on criteria. These judgements appear as generalization part in GGq-ROFSSs to fulfil and complete q-ROFSS-based data as some final examination. Therefore, for prospect MCDM, GGq-ROFSS, which indicates in $\mu^q + \nu^q \leq 1$ for some $q \geq 1$ suggests a better framework to deal with uncertainty.

The method of computation of q-ROFS-based data from newly introduced ideas can sight wide applications for available data in machine learning, applied intelligence, and supply chain management. Although there are some methods in machine learning, artificial intelligence, and supply chain management, proposed operators' beauty will be a notable sight in such domains. Our focus will be on these operators' new rudiments and their supply chain management applications in future work. Furthermore, we will define GGq-ROFSS-based order operators, Shapley Choquet operators, and distance aggregation operator.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare no conflicts of interest.

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