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Research Article

Bifurcation Analysis of a Discrete Food Chain Model with Harvesting

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We explore existence of fixed points, topological classifications around fixed points, existence of periodic points and prime period, and bifurcation analysis of a three-species discrete food chain model with harvesting. Finally, theoretical results are numerically verified.

1. Introduction

Many different types of interactions exist in nature between various species of organisms on this planet Earth and are studied under the discipline of ecology. Ecological interactions are most fundamental part in biology which determines community structure and development. Not all interactions are positive, some are negative also. One of the examples of negative correlation is ammensalism. Ammensalism is a type of ecological interactions between the members of two different species in which one is harmed, destroyed, or inhabited by the member of another species, while the other remains unaffected, neither harmed nor benefitted. It is a type of competitive behavior among different species and is frequently used to refer to asymmetrical competitive association. Research in the field of ecology draws the attention of several mathematicians such as Lotka [1] and Volterra [2]. Nowadays, ecologist and mathematician jointly contributed to the growth of this area of knowledge. Recently, many researchers have investigated the dynamical properties of discrete-time ecological models such as prey-predation, competitions, neutralism, and mutualism by studying fixed points, local and global attractivity, bounded, existence of bifurcation, and many more. For instance, Beddington et al. [3] have explored the behavior of following predator-prey model:

$$\begin{aligned} x_{n+1} &= x_n e^{r\left(1-\left(x_n/K\right)\right)-a y_n}, \\ y_{n+1} &= \alpha x_n\left(1-e^{-a y_n}\right). \end{aligned} \tag{1}$$

Chen [4] has explored global attractivity and permanence of the following discrete multispecies system:

$$\begin{aligned} x_{i_{n+1}} &= x_{i_n} e^{b_{i_n}-\sum_{l=1}^{k} a_{il_n} x_{l_n}-\sum_{l=1}^{k} c_{il_n} y_{l_n}}, \\ y_{j_{n+1}} &= y_{j_n} e^{-r_{j_n}+\sum_{l=1}^{k} d_{jl_n} x_{l_n}-\sum_{l=1}^{k} e_{jl_n} y_{l_n}}. \end{aligned} \tag{2}$$

Fang and Chen [5] have explored the permanence of multispecies Lotka–Volterra competition predator-prey system with delays. Furthermore, Fang et al. [6] have explored the dynamics of the following system:

$$\begin{aligned} x_{n+1} &= x_n e^{a_n-b_n x_n-\left(c_n y_n/\left(m_{1n}+m_{2n} x_n+m_{3n} y_n\right)\right)}, \\ y_{n+1} &= y_n e^{-d_n-e_n y_n+\left(f_n x_n/\left(m_{1n}+m_{2n} x_n+m_{3n} y_n\right)\right)}. \end{aligned} \tag{3}$$

Agiza et al. [7] have explored chaotic dynamics of the following discrete model with Holling type II:

$$\begin{aligned} x_{n+1} &= a x_n\left(1-x_n\right)-\frac{b x_n y_n}{1+\varepsilon x_n}, \\ y_{n+1} &= \frac{d x_n y_n}{1+\varepsilon x_n}. \end{aligned} \tag{4}$$

Huo and Li [8] have explored stable periodic solution of the following discrete model:

$$x_{n+1} = x_n e^{r_{1n} - b_{1n} x_n - a_{1n} y_n}, \quad y_{n+1} = y_n e^{r_{2n} - a_{2n} \left(y_n / x_n \right)}. \tag{5}$$

Lu and Zhang [9] have studied the permanence and global attractivity of the following discrete system:

$$x_{n+1} = x_n e^{a_n - b_n x_n - \left(m_n y_n / \left(A_n + x_n \right) \right)}, \quad y_{n+1} = y_n e^{d_n - e_n \left(y_n / x_n \right)}. \tag{6}$$

Zhao and Zhang [10] explored the chaos and permanence of the following discrete model:

$$x_{n+1} = (1-d) x_n e^{r\left(1 - \left(x_n / k\right) - a y_n\right)} + d\sigma\, x_{n-1} e^{r\left(1 - \left(x_{n-1}/k\right)\right)}, \quad y_{n+1} = (1-d) x_n \left(1 - e^{-a y_n}\right). \tag{7}$$

Zhao et al. [11] have investigated the dynamics of the following discrete model:

$$x_{n+1} = x_n e^{r\left(1 - \left(x_n / k\right)\right)} f\left(y_n, y_{n-1}\right), \quad y_{n+1} = x_n \left(1 - f\left(y_n, y_{n-1}\right)\right), \tag{8}$$

where $f\left(y_n, y_{n-1}\right) = e^{-a\left((1-d) y_n + d\sigma y_{n-1}\right)}$.

On the contrary, in recent years, many papers have been published to investigate the bifurcation analysis of certain discrete models by choosing step size as a bifurcation parameter. For example, Salman et al. [12] have explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + \frac{\delta}{2}\left(x_n\left(1 - x_n^2\right) - y_n\right), \quad y_{n+1} = y_n + \delta y_n \left(-s + c x_n\right), \tag{9}$$

by choosing step size δ as a bifurcation parameter. Liu and Xiao [13] have explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + \delta\left(r x_n \left(1 - x_n\right) - b x_n y_n\right), \quad y_{n+1} = y_n + \delta\left(-d + b x_n\right) y_n, \tag{10}$$

by choosing step size δ as a bifurcation parameter. Hasan and Hama [14] have explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + d x_n \left(1 - x_n - \frac{y_n}{1 + a x_n + b y_n}\right), \quad y_{n+1} = y_n + d y_n \left(\frac{c x_n}{1 + a x_n + b y_n} - e\right), \tag{11}$$

by choosing step size d as a bifurcation parameter. Wu and Zhang [15] have explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + \delta x_n \left(K_1 - \alpha_1 x_n - \beta_{12} y_n - \gamma_1 x_n y_n\right), \quad y_{n+1} = y_n + \delta y_n \left(K_2 - \alpha_2 y_n - \beta_{21} x_n - \gamma_2 x_n y_n\right), \tag{12}$$

by choosing step size δ as a bifurcation parameter. Rana [16] has explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + \delta\left(x_n\left(1 - x_n\right) - \frac{a x_n y_n}{x_n + y_n}\right), \quad y_{n+1} = x_n + \delta\left(-d y_n + \frac{b x_n y_n}{x_n + y_n}\right), \tag{13}$$

by choosing step size δ as a bifurcation parameter. Rana and Kulsum [17] have explored bifurcation analysis of the following discrete system:

$$x_{n+1} = x_n + \delta x_n \left(1 - x_n - \frac{y_n}{x_n^2 + a}\right), \quad y_{n+1} = y_n + \delta y_n \left(\alpha - \frac{\beta y_n}{x_n}\right), \tag{14}$$

by choosing step size δ as a bifurcation parameter. Motivated from the aforementioned studies, in this work, we explore existence of fixed points, topological classifications around fixed points, periodic points, and bifurcation analysis, by choosing step size h as a bifurcation parameter, of the following three species discrete food chain model with harvesting:

$$\begin{aligned} x_{n+1} &= x_n + h\left(a_1\left(1 - k_1\right) x_n - \alpha_{11} x_n^2\right), \\ y_{n+1} &= y_n + h\left(a_2\left(1 - k_2\right) y_n - \alpha_{22} y_n^2 - \alpha_{21} x_n y_n\right), \\ z_{n+1} &= z_n + h\left(a_3 z_n - \alpha_{33} z_n^2 - \alpha_{32} y_n z_n\right), \end{aligned} \tag{15}$$

which is a discrete form of the following model:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= a_1\left(1 - k_1\right) x - \alpha_{11} x^2, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= a_2\left(1 - k_2\right) y - \alpha_{22} y^2 - \alpha_{21} x y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= a_3 z - \alpha_{33} z^2 - \alpha_{32} y z, \end{aligned} \tag{16}$$

by Euler forward formula, where h is step size and t is customary denoted by n. It is noted that, in model (16), $x(t)$, $y(t)$, and $z(t)$, respectively, denote populations of first, second, and third species. Moreover $z(t)$ denotes growth rate of first, second, and third species; $\alpha_{ii}\,(i = 1, 2, 3)$ denotes the rate of decrease of first, second, and third species due to internal competitions; α_{21} denotes rate of decrease of second species due to attack of first species; α_{32} denotes the rate of decrease of third species due to attack of second species; k_1 and k_2, respectively, denote the harvesting rate of first and second species. It is also important to note that all parameters $h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}$, and α_{33} are positive. In addition, it is important here to mention that we will explore dynamical properties of the discrete-time model (15) instead of the continuous-time model, which is depicted in (16), because discrete-time models governed by difference equations are more realistic and appropriate than the continuous ones in the case where populations have non-overlapping generations, and moreover, discrete models can also provide efficient computational results for numerical simulations [12, 13].

This paper is structured as follows. In Section 2, we study the existence of fixed points of model (15) algebraically. The linearized form of model (15) is presented in Section 3. In Section 4, we explored topological classification around fixed points of the model. Existence of periodic points of model (15) is explored in Section 5. In Section 6, we explored detailed analysis of bifurcation around fixed points of model (15). Theoretical results are verified numerically in Section 7. Brief summary of the paper is presented in Section 8.

2. Study of Equilibrium Points

Here, we will study the boundary and interior equilibrium points of model (15) as follows.

Lemma 1. *Model (15) has atmost eight equilibrium points in $\mathbb{R}^3_+$. Precisely,*

(i) $\forall\, h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}, \alpha_{33} > 0$; *model (15) has a trivial equilibrium point:* $P_1 = (0,0,0)$.

(ii) $\forall\, a_3, \alpha_{33} > 0$; *model (15) has boundary equilibrium point:* $P_2 = (0,0,a_3/\alpha_{33})$.

(iii) $P_3 = (0, (1-k_2)a_2/\alpha_{22}, 0)$ *is a boundary equilibrium point of (15) if* $k_2 < 1$.

(iv) $P_4 = ((1-k_1)a_1/\alpha_{11}, 0, 0)$ *is a boundary equilibrium point of (15) if* $k_1 < 1$.

(v) $P_5 = (0, (1-k_2)a_2/\alpha_{22}, (a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33})$ *is a boundary equilibrium point of (15) if* $a_3 > \alpha_{32}(1-k_2)a_2/\alpha_{22}$ *with* $k_2 < 1$.

(vi) $P_6 = ((1-k_1)a_1/\alpha_{11}, 0, a_3/\alpha_{33})$ *is a boundary equilibrium point of (15) if* $k_1 < 1$.

(vii) $P_7 = ((1-k_1)a_1/\alpha_{11}, (a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}, 0)$ *is a boundary equilibrium point of (15) if* $a_2 > a_1(1-k_1)\alpha_{21}/(1-k_2)\alpha_{11}$ *with* $k_1, k_2 < 1$.

(viii) $P_8 = ((1-k_1)a_1/\alpha_{11}, (a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}, (a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})$ *is an interior equilibrium point of (15) if* $k_1 < 1$, $a_2 > a_1(1-k_1)\alpha_{21}/(1-k_2)\alpha_{11}$ *and* $a_3 > (a_2(1-k_2)\alpha_{11}\alpha_{32} - a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}$.

Proof. If model (15) has an equilibrium point, $P = (x, y, z)$, then

$$\begin{aligned} x &= x + h\left(a_1(1-k_1)x - \alpha_{11}x^2\right),\\ y &= y + h\left(a_2(1-k_2)y - \alpha_{22}y^2 - \alpha_{21}xy\right),\\ z &= z_n + h\left(a_3 z - \alpha_{33}z^2 - \alpha_{32}yz\right). \end{aligned} \tag{17}$$

The simple computation yields that, for the values of $P_i\,(i = 1,\dots,7)$, (17) satisfied identically. So, one can conclude that model (15) has seven boundary points: $P_i\,(i = 1,\dots,7)$. In order to find interior point, from (17), one obtains

$$\begin{aligned} a_1(1-k_1) - \alpha_{11}x &= 0,\\ a_2(1-k_2) - \alpha_{22}y - \alpha_{21}x &= 0,\\ a_3 - \alpha_{33}z - \alpha_{32}y &= 0. \end{aligned} \tag{18}$$

From 1st of (18), one obtains

$$x = \frac{(1-k_1)a_1}{\alpha_{11}}. \tag{19}$$

From 2nd equation of (18) and (19), one obtains

$$y = \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}. \tag{20}$$

From 3rd equation of (18) and (20), one obtains

$$z = \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}. \tag{21}$$

From (19)–(21), one can conclude that P_8 is an interior equilibrium point of (15) if $k_1 < 1$, $a_2 > (a_1(1-k_1)\alpha_{21}/(1-k_2)\alpha_{11})$, and $a_3 > (a_2(1-k_2)\alpha_{11}\alpha_{32} - a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}$. □

3. Linearized Form of Model (15)

The variational matrix $J|_P$ about P under the map:

$$(f_1, f_2, f_3) \mapsto (x_{n+1}, y_{n+1}, z_{n+1}), \tag{22}$$

where

$$\begin{aligned} f_1 &= x + h\left(a_1(1-k_1)x - \alpha_{11}x^2\right),\\ f_2 &= y + h\left(a_2(1-k_2)y - \alpha_{22}y^2 - \alpha_{21}xy\right),\\ f_3 &= z + h\left(a_3 z - \alpha_{33}z^2 - \alpha_{32}yz\right), \end{aligned} \tag{23}$$

is

$$J|_P = \begin{pmatrix} 1 + h(a_1(1-k_1) - 2\alpha_{11}x) & 0 & 0\\ -h\alpha_{21}y & 1 + h(a_2(1-k_2) - 2\alpha_{22}y - \alpha_{21}x) & 0\\ 0 & -h\alpha_{32}z & 1 + h(a_3 - 2\alpha_{33}z - \alpha_{32}y) \end{pmatrix}, \tag{24}$$

with

$$\begin{aligned}\lambda_1 &= 1 + h\left(a_1\left(1-k_1\right) - 2\alpha_{11}x\right),\\ \lambda_2 &= 1 + h\left(a_2\left(1-k_2\right) - 2\alpha_{22}y - \alpha_{21}x\right),\\ \lambda_3 &= 1 + h\left(a_3 - 2\alpha_{33}z - \alpha_{32}y\right).\end{aligned} \tag{25}$$

4. Dynamical Behavior: Topological Properties of Equilibrium Points

The dynamical behavior about fixed points $P_i\,(i = 1,\dots,8)$ of model (15) is explored in this section.

4.1. Dynamical Behavior about P_1. From (25), eigenvalues of $J|_{P_1}$ about P_1 are

$$\begin{aligned}\lambda_1 &= 1 + ha_1\left(1-k_1\right),\\ \lambda_2 &= 1 + ha_2\left(1-k_2\right),\\ \lambda_3 &= 1 + ha_3.\end{aligned} \tag{26}$$

The dynamical behavior about P_1 of model (15) is concluded as follows.

Lemma 2.

(i) *For all allowed parametric values, $h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}, \alpha_{33} > 0$, P_1 is not sink.*

(ii) *P_1 is a source if*

$$0 < h < \min\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2}{a_2\left(k_2-1\right)}\right\}. \tag{27}$$

(iii) *P_1 is a saddle if*

$$h > \max\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2}{a_2\left(k_2-1\right)}\right\}. \tag{28}$$

(iv) *P_1 is nonhyperbolic if*

$$h = \frac{2}{a_1\left(k_1-1\right)}, \tag{29}$$

or

$$h = \frac{2}{a_2\left(k_2-1\right)}. \tag{30}$$

4.2. Dynamical Behavior about P_2. From (25), eigenvalues of $J|_{P_2}$ about P_2 are

$$\begin{aligned}\lambda_1 &= 1 + ha_1\left(1-k_1\right),\\ \lambda_2 &= 1 + ha_2\left(1-k_2\right),\\ \lambda_3 &= 1 - ha_3.\end{aligned} \tag{31}$$

The dynamical behavior about P_2 is concluded as follows:

Lemma 3. (i) *P_2 is a sink if*

$$h > \max\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2}{a_2\left(k_2-1\right)}\right\}, \quad 0 < h < \frac{2}{a_3}. \tag{32}$$

(ii) *P_2 is a source if*

$$0 < h < \min\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2}{a_2\left(k_2-1\right)}\right\}, \quad h > \frac{2}{a_3}. \tag{33}$$

(iii) *P_2 is a saddle if*

$$h < \min\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2}{a_2\left(k_2-1\right)}\right\}, \quad 0 < h < \frac{2}{a_3}. \tag{34}$$

(iv) *P_2 is nonhyperbolic if*

$$h = \frac{2}{a_3}, \tag{35}$$

or

$$h = \frac{2}{a_1\left(k_1-1\right)}, \tag{36}$$

or

$$h = \frac{2}{a_2\left(k_2-1\right)}. \tag{37}$$

4.3. Dynamical Behavior about P_3. From (25), eigenvalues of $J|_{P_3}$ about P_3 are

$$\begin{aligned}\lambda_1 &= 1 + ha_1\left(1-k_1\right),\\ \lambda_2 &= 1 - ha_2\left(1-k_2\right),\\ \lambda_3 &= 1 + h\left(\frac{a_3\alpha_{22} - a_2\alpha_{32}\left(1-k_2\right)}{\alpha_{22}}\right).\end{aligned} \tag{38}$$

The dynamical behavior about P_3 is concluded as follows.

Lemma 4.

(i) *P_3 is a sink if*

$$\begin{aligned}&h > \max\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2\alpha_{22}}{a_2\alpha_{32}\left(1-k_2\right) - a_3\alpha_{22}}\right\},\\ &0 < h < \frac{2}{a_2\left(1-k_2\right)}.\end{aligned} \tag{39}$$

(ii) *P_3 is a source if*

$$\begin{aligned}&0 < h < \min\left\{\frac{2}{a_1\left(k_1-1\right)}, \frac{2\alpha_{22}}{a_2\alpha_{32}\left(1-k_2\right) - a_3\alpha_{22}}\right\},\\ &h > \frac{2}{a_2\left(1-k_2\right)}.\end{aligned} \tag{40}$$

(iii) P_3 is a saddle if

$$h > \max\left\{\frac{2}{a_1(k_1-1)}, \frac{2}{a_2(1-k_2)}\right\},$$
$$0 < h < \frac{2\alpha_{22}}{a_2\alpha_{32}(1-k_2) - a_3\alpha_{22}}. \quad (41)$$

(iv) P_3 is nonhyperbolic if

$$h = \frac{2}{a_2(1-k_2)}, \quad (42)$$

or

$$h = \frac{2}{a_1(k_1-1)}, \quad (43)$$

or

$$h = \frac{2\alpha_{22}}{a_2\alpha_{32}(1-k_2) - a_3\alpha_{22}}. \quad (44)$$

4.4. Dynamical Behavior about P_4. From (25), eigenvalues of $J|_{P_4}$ about P_4 are

$$\lambda_1 = 1 - ha_1(1-k_1),$$
$$\lambda_2 = 1 + ha_2(1-k_2), \quad (45)$$
$$\lambda_3 = 1 + ha_3.$$

The dynamical behavior about P_4 is concluded as follows.

Lemma 5.

(i) For all allowed parametric values, $h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}, \alpha_{33} > 0$, P_4 is not sink.

(ii) P_4 is a source if

$$h > \max\left\{\frac{2}{a_1(1-k_1)}, \frac{2}{a_2(k_2-1)}\right\}. \quad (46)$$

(iii) P_4 is a saddle if

$$0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2}{a_2(k_2-1)}\right\}. \quad (47)$$

(iv) P_4 is nonhyperbolic if

$$h = \frac{2}{a_1(1-k_1)}, \quad (48)$$

or

$$h = \frac{2}{a_2(k_2-1)}. \quad (49)$$

4.5. Dynamical Behavior about P_5. From (25), eigenvalues of $J|_{P_5}$ about P_5 are

$$\lambda_1 = 1 + ha_1(1-k_1),$$
$$\lambda_2 = 1 - ha_2(1-k_2), \quad (50)$$
$$\lambda_3 = 1 + h\left(-a_3 + \frac{\alpha_{32}(1-k_2)a_2}{\alpha_{22}}\right).$$

The dynamical behavior about P_5 is concluded as follows.

Lemma 6. (i) P_5 is a sink if

$$h > \max\left\{\frac{2}{a_1(k_1-1)}, \frac{2\alpha_{22}}{a_3\alpha_{22} - a_2\alpha_{32}(1-k_2)}\right\},$$
$$0 < h < \frac{2}{a_2(1-k_2)}. \quad (51)$$

(ii) P_5 is a source if

$$0 < h < \min\left\{\frac{2}{a_1(k_1-1)}, \frac{2\alpha_{22}}{a_3\alpha_{22} - a_2\alpha_{32}(1-k_2)}\right\},$$
$$h > \frac{2}{a_2(1-k_2)}. \quad (52)$$

(iii) P_5 is a saddle if

$$h > \max\left\{\frac{2}{a_1(k_1-1)}, \frac{2\alpha_{22}}{a_3\alpha_{22} - a_2\alpha_{32}(1-k_2)}\right\},$$
$$h > \frac{2}{a_2(1-k_2)}. \quad (53)$$

(iv) P_5 is nonhyperbolic if

$$h = \frac{2}{a_1(k_1-1)}, \quad (54)$$

or

$$h = \frac{a_3\alpha_{22} - 2\alpha_{22}}{a_2\alpha_{32}(1-k_2)}, \quad (55)$$

or

$$h = \frac{2}{a_2(1-k_2)}. \quad (56)$$

4.6. Dynamical Behavior about P_6. From (25), eigenvalues of $J|_{P_6}$ about P_6 are

$$\lambda_1 = 1 - ha_1(1-k_1),$$
$$\lambda_2 = 1 + h\left(a_2(1-k_2) - \frac{a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right), \quad (57)$$
$$\lambda_3 = 1 - ha_3.$$

The dynamical behavior about P_6 is concluded as follows.

Lemma 7.

(i) P_6 is a sink if

$$0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2}{a_3}\right\}$$
$$h > \frac{2\alpha_{11}}{a_1\alpha_{21}(1-k_1) - a_2\alpha_{11}(1-k_2)}. \tag{58}$$

(ii) P_6 is a source if

$$h > \max\left\{\frac{2}{a_1(1-k_1)}, \frac{2}{a_3}\right\},$$
$$0 < h < \frac{2\alpha_{11}}{a_1\alpha_{21}(1-k_1) - a_2\alpha_{11}(1-k_2)}. \tag{59}$$

(iii) P_6 is a saddle if

$$0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2}{a_3}\right\},$$
$$0 < h < \frac{2\alpha_{11}}{a_1\alpha_{21}(1-k_1) - a_2\alpha_{11}(1-k_2)}. \tag{60}$$

(iv) P_6 is nonhyperbolic if

$$h = \frac{2}{a_1(1-k_1)}, \tag{61}$$

or

$$h = \frac{2}{a_3}, \tag{62}$$

or

$$h = \frac{2\alpha_{11}}{a_1\alpha_{21}(1-k_1) - a_2\alpha_{11}(1-k_2)}. \tag{63}$$

4.7. Dynamical Behavior about P_7. From (25), eigenvalues of $J|_{P_7}$ about P_7 are

$$\lambda_1 = 1 - ha_1(1-k_1),$$
$$\lambda_2 = 1 - h\left(a_2(1-k_2) - \frac{2a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right),$$
$$\lambda_3 = 1 + h\left(a_3 - \frac{a_2(1-k_2)\alpha_{32}}{\alpha_{22}} + \frac{a_1\alpha_{21}\alpha_{32}(1-k_1)}{\alpha_{11}\alpha_{22}}\right). \tag{64}$$

The dynamical behavior about P_7 is concluded as follows:

Lemma 8.

(i) P_7 is a sink if

$$0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}\right\},$$
$$h > \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \tag{65}$$

(ii) P_7 is a source if

$$h > \max\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}\right\},$$
$$0 < h < \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \tag{66}$$

(iii) P_7 is a saddle if

$$0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}\right\},$$
$$0 < h < \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \tag{67}$$

(iv) P_7 is nonhyperbolic if

$$h = \frac{2}{a_1(1-k_1)}, \tag{68}$$

or

$$h = \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}, \tag{69}$$

or

$$h = \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \tag{70}$$

4.8. Dynamical Behavior about P_8. From (25), eigenvalues of $J|_{P_8}$ about P_8 are

$$\lambda_1 = 1 - ha_1(1-k_1),$$
$$\lambda_2 = 1 - h\left(a_2(1-k_2) - \frac{2a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right),$$
$$\lambda_3 = 1 - h\left(a_3 - \frac{a_2(1-k_2)\alpha_{32}}{\alpha_{22}} + \frac{a_1\alpha_{21}\alpha_{32}(1-k_1)}{\alpha_{11}\alpha_{22}}\right). \tag{71}$$

The dynamical behavior about P_8 is concluded as follows.

Lemma 9. *(i) P_8 is a sink if*

$$!0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}, \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}\right\}. \tag{72}$$

(ii) P_8 is a source if

$$!h > \max\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}, \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}\right\}. \tag{73}$$

(iii) P_8 is a saddle if

$$\begin{gathered} 0 < h < \min\left\{\frac{2}{a_1(1-k_1)}, \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}\right\}, \\ h > \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \end{gathered} \tag{74}$$

(iv) P_8 is nonhyperbolic if

$$h = \frac{2}{a_1(1-k_1)}, \tag{75}$$

or

$$h = \frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)}, \tag{76}$$

or

$$h = \frac{2\alpha_{11}\alpha_{22}}{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)}. \tag{77}$$

5. Periodic Points

We will prove that $P_i\,(i = 1, \ldots, 8)$ of model (15) are periodic points of period n.

Theorem 1. *Equilibrium points $P_i\,(i = 1, \ldots, 8)$ of model (15) are periodic points of prime period 1.*

Proof. From (15), define

$$F(x, y, z) := (f_1, f_2, f_3), \tag{78}$$

where f_1, f_2, and f_3 are represented in (23). From (78), the computation yields

$$\begin{aligned} F|_{P_1=(0,0,0)} &= P_1, \\ F|_{P_2=(0,0,(a_3/\alpha_{33}))} &= P_2, \\ F|_{P_3=(0,(1-k_2)a_2/\alpha_{22},0)} &= P_3, \\ F|_{P_4=((1-k_1)a_1/\alpha_{11},0,0)} &= P_4, \\ F|_{P_5=(0,(1-k_2)a_2/\alpha_{22},(a_3\alpha_{22}-\alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33})} &= P_5, \\ F|_{P_6=((1-k_1)a_1/\alpha_{11},0,a_3/\alpha_{33})} &= P_6, \\ F|_{P_7=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},0)} &= P_7, \\ F|_{P_8=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},(a_3\alpha_{11}\alpha_{22}-a_2(1-k_2)\alpha_{11}\alpha_{32}+a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})} &= P_8. \end{aligned} \tag{79}$$

Hence, from (79), we can say that equilibrium points $P_i\,(i = 1, \ldots, 8)$ of three species model (15) are periodic points of prime period 1. □

Now, it is proved that equilibrium points $P_i\,(i = 1, \ldots, 8)$ are period points of period n.

Theorem 2. *P_1 of model (15) is a periodic point of period n.*

Proof. From (78), the following computation yields the required statement:

$$\begin{aligned}
F^2 =& \left(f_1 + h\left[a_1(1-k_1)f_1 - \alpha_{11}(f_1)^2\right],\right.\\
& f_2 + h\left[a_2(1-k_2)f_2 - \alpha_{22}(f_2)^2 - \alpha_{21}f_1f_2\right],\\
& \left.f_3 + h\left[a_3f_3 - \alpha_{33}(f_2)^2 - \alpha_{32}f_2f_3\right]\right) \Rightarrow F^2|_{P_1} = P_1,\\
F^3 =& \left(f_1^2 + h\left[a_1(1-k_1)f_1^2 - \alpha_{11}\left(f_1^2\right)^2\right],\right.\\
& f_2^2 + h\left[a_2(1-k_2)f_2^2 - \alpha_{22}\left(f_2^2\right)^2 - \alpha_{21}f_1^2f_2^2(x,y,z)\right],\\
& \left.f_3^2 + h\left[a_3f_3^2 - \alpha_{33}\left(f_2^2\right)^2 - \alpha_{32}f_2^2f_3^2\right]\right) \Rightarrow F^3|_{P_1} = P_1,\\
&\vdots\\
F^n =& \left(f_1^n + h\left[a_1(1-k_1)f_1^n - \alpha_{11}(f_1^n)^2\right],\right.\\
& f_2^n + h\left[a_2(1-k_2)f_2^n - \alpha_{22}(f_2^n)^2 - \alpha_{21}f_1^nf_2^n\right],\\
& \left.f_3^n + h\left[a_3f_3^n - \alpha_{33}(f_2^n)^2 - \alpha_{32}f_2^nf_3^n\right]\right) \Rightarrow F^n|_{P_1} = P_1.
\end{aligned} \tag{80}$$

□

Theorem 3. *P_2 of model (15) is a periodic point of period n.*

Proof. Utilizing the computation as we have done in (80), one gets the following required statement:

$$\begin{aligned}
F^2|_{P_2=(0,0,a_3/\alpha_{33})} &= P_2,\\
F^3|_{P_2=(0,0,a_3/\alpha_{33})} &= P_2,\\
&\vdots\\
F^n|_{P_2=(0,0,a_3/\alpha_{33})} &= P_2.
\end{aligned} \tag{81}$$

□

Theorem 4. *P_3 of model (15) is a periodic point of period n.*

Proof. In view of (80), one gets the following required statement:

$$\begin{aligned}
F^2|_{P_3=(0,(1-k_2)a_2/\alpha_{22},0)} &= P_3,\\
F^3|_{P_3=(0,(1-k_2)a_2/\alpha_{22},0)} &= P_3,\\
&\vdots\\
F^n|_{P_3=(0,(1-k_2)a_2/\alpha_{22},0)} &= P_3.
\end{aligned} \tag{82}$$

□

Theorem 5. *P_4 of model (15) is a periodic point of period n.*

Proof. In view of (80), one gets the following required statement:

$$\begin{aligned}
F^2|_{P_4=((1-k_1)a_1/\alpha_{11},0,0)} &= P_4,\\
F^3|_{P_4=((1-k_1)a_1/\alpha_{11},0,0)} &= P_4,\\
&\vdots\\
F^n|_{P_4=((1-k_1)a_1/\alpha_{11},0,0)} &= P_4.
\end{aligned} \tag{83}$$

□

Theorem 6. *P_5 of model (15) is a periodic point of period n.*

Proof. From (80), one obtains

$$\begin{aligned}
F^2|_{P_5=(0,(1-k_2)a_2/\alpha_{22},(a_3\alpha_{22}-\alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33})} &= P_5,\\
F^3|_{P_5=(0,(1-k_2)a_2/\alpha_{22},(a_3\alpha_{22}-\alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33})} &= P_5,\\
&\vdots\\
F^n|_{P_5=(0,(1-k_2)a_2/\alpha_{22},(a_3\alpha_{22}-\alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33})} &= P_5.
\end{aligned} \tag{84}$$

□

Theorem 7. *P_6 of model (15) is a periodic point of period n.*

Proof. From (80), one obtains

$$\begin{aligned}
F^2|_{P_6=((1-k_1)a_1/\alpha_{11},0,a_3/\alpha_{33})} &= P_6,\\
F^3|_{P_6=((1-k_1)a_1/\alpha_{11},0,a_3/\alpha_{33})} &= P_6,\\
&\vdots\\
F^n|_{P_6=((1-k_1)a_1/\alpha_{11},0,a_3/\alpha_{33})} &= P_6.
\end{aligned} \tag{85}$$

□

Theorem 8. *P_7 of model (15) is a periodic point of period n.*

Proof. From (80), one obtains

$$\begin{aligned}
F^2|_{P_7=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},0)} &= P_7,\\
F^3|_{P_7=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},0)} &= P_7,\\
&\vdots\\
F^n|_{P_7=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},0)} &= P_7.
\end{aligned} \tag{86}$$

□

Theorem 9. *P_8 of model (15) is a periodic point of period n.*

Proof. From (80), one obtains

$$\begin{aligned}
&F^2|_{P_8=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},(a_3\alpha_{11}\alpha_{22}-a_2(1-k_2)\alpha_{11}\alpha_{32}+a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})} = P_8,\\
&F^3|_{P_8=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},(a_3\alpha_{11}\alpha_{22}-a_2(1-k_2)\alpha_{11}\alpha_{32}+a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})} = P_8,\\
&\vdots\\
&F^n|_{P_8=((1-k_1)a_1/\alpha_{11},(a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22},(a_3\alpha_{11}\alpha_{22}-a_2(1-k_2)\alpha_{11}\alpha_{32}+a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})} = P_8.
\end{aligned} \tag{87}$$

□

6. Analysis of Bifurcation

In this section, we give analysis of bifurcation about fixed points $P_i\,(i = 1, \ldots, 8)$ of model (15) by bifurcation theory [18, 19].

6.1. Analysis of Bifurcation at P_1. Here, we will study analysis of bifurcation at P_1 of model (15). From (26), the simple computation yields $\lambda_1|_{(29)} = -1$, but $\lambda_{2,3}|_{(29)} = 1 - (2a_2(1-k_2)/a_1(1-k_1)), 1 - (2a_3/\ a_1(1-k_1)) \neq 1$ or -1. This suggests that model (15) could undergo a flip bifurcation around P_1 if $\Omega = (h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}, \alpha_{33})$ passes the curve:

$$\mathcal{F}|_{P_1} = \left\{\Omega\colon h = \frac{2}{a_1(k_1-1)}\right\}. \tag{88}$$

However, flip bifurcation cannot occur by computation, so P_1 is degenerated with high co-dimension as $\Omega \in \mathcal{F}|_{P_1}$.

6.2. Analysis of Bifurcation at P_2. We will study analysis of bifurcation at P_2 of model (15). From (26), the simple computation yields $\lambda_3|_{(35)} = -1$, but $\lambda_{1,2}|_{(35)} = 1 + (2a_1(1-k_1)/a_3),\ 1 + (2a_2(1-k_2)/a_3) \neq 1$ or -1. This suggests that model (15) could undergo a flip bifurcation around P_2 if Ω passes the curve:

$$\mathcal{F}|_{P_2} = \left\{\Omega\colon h = \frac{2}{a_3}\right\}. \tag{89}$$

The proof of following theorem shows that model (15) undergoes flip bifurcation around P_2 if $\Omega \in \mathcal{F}|_{P_2}$.

Theorem 10. *Model (15) undergo flip bifurcation around P_2 if $\Omega \in \mathcal{F}|_{P_2}$.*

Proof. It is noticed that three-species model (15) is invariant with respect to $x = y = 0$. Thus, we restrict (15) on $x = y = 0$, to determine the bifurcation, where it takes the form

$$z_{n+1} = z_n + h\left(a_3 z_n - \alpha_{33} z_n^2\right). \tag{90}$$

From (90), define

$$f(z) \coloneqq z + h\left(a_3 z - \alpha_{33} z^2\right). \tag{91}$$

Now, one denotes $h = h^* = (2/a_3)$ and $z = z^* = (a_3/\alpha_{33})$. The computation yields

$$f_z|_{h=h^*=(2/a_3),z=z^*=(a_3/\alpha_{33})} = -1, \tag{92}$$

$$f_{zz}|_{h=h^*=(2/a_3),z=z^*=(a_3/\alpha_{33})} = -\frac{4\alpha_{33}}{a_3} \neq 0, \tag{93}$$

$$f_h|_{h=h^*=(2/a_3),z=z^*=(a_3/\alpha_{33})} = -\frac{a_3^2}{\alpha_{33}} \neq 0. \tag{94}$$

From (92)–(94), it can be concluded that the model undergoes flip bifurcation around P_2 if $\Omega \in \mathcal{F}|_{P_2}$. □

6.3. Analysis of Bifurcation at P_3. From (38), the computation yields $\lambda_2|_{(42)} = -1$, but $\lambda_{1,3}|_{(42)} = 1 + (2a_1(1-k_1)/a_2(1-k_2)), 1 + (2/a_2(1-k_2))[a_3\alpha_{22} - a_2\alpha_{32}(1-k_2)] \neq 1$ or -1. This suggests that model (15) could undergo a flip bifurcation around P_3 if Ω passes the curve:

$$\mathcal{F}|_{P_3} = \left\{\Omega\colon h = \frac{2}{a_2(1-k_2)}\right\}. \tag{95}$$

The proof of following theorem shows that model (15) undergoes flip bifurcation around P_3 if $\Omega \in \mathcal{F}|_{P_3}$.

Theorem 11. *Model (15) undergoes flip bifurcation around P_3 if $\Omega \in \mathcal{F}|_{P_3}$.*

Proof. It is noticed that, w.r.t $x = z = 0$, model (15) is invariant. So, one restricts model (15) on $x = z = 0$, where it becomes

$$y_{n+1} = y_n + h\left(a_2(1-k_2)y_n - \alpha_{22} y_n^2\right). \tag{96}$$

From (96), define

$$f(y) \coloneqq y + h a_2(1-k_2)y - h\alpha_{22} y^2. \tag{97}$$

Denote $h = h^* = (2/a_2(1-k_2))$ and $y = y^* = (a_2(1-k_2)/\alpha_{22})$. By computation, one obtains

$$f_y|_{h=h^*=(2/a_2(1-k_2)),y=y^*=(a_2(1-k_2)/\alpha_{22})} = -1, \tag{98}$$

$$f_{yy}|_{h=h^*=(2/a_2(1-k_2)),y=y^*=(a_2(1-k_2)/\alpha_{22})} = -\frac{4\alpha_{22}}{a_2(1-k_2)} \neq 0, \tag{99}$$

$$f_h|_{h=h^*=(2/a_2(1-k_2)),y=y^*=(a_2(1-k_2)/\alpha_{22})} = \frac{a_2(1-k_2)}{\alpha_{22}} \neq 0. \tag{100}$$

So, model (15) undergoes flip bifurcation by (98)–(100) if $\Omega \in \mathscr{F}|_{P_3}$. □

6.4. Analysis of Bifurcation at P_4. From (45), the computation yields $\lambda_1|_{(48)} = -1$, but $\lambda_{2,3}|_{(48)} = 1 + (2a_2(1-k_2)/a_1(1-k_1)), 1 + (2a_3/a_1(1-k_1)) \neq 1$ or -1. This suggests that model (15) could undergo a flip bifurcation around P_4 if Ω passes the curve:

$$\mathscr{F}|_{P_3} = \left\{\Omega\colon h = \frac{2}{a_1(1-k_1)}\right\}. \tag{101}$$

The proof of the following theorem shows that model (15) undergoes flip bifurcation around P_4 if $\Omega \in \mathscr{F}|_{P_4}$.

Theorem 12. *Model (15) undergoes flip bifurcation around P_4 if $\Omega \in \mathscr{F}|_{P_4}$.*

Proof. It is noticed that, w.r.t $y = z = 0$, model (15) is invariant. So, one restricts model (15) on $y = z = 0$, where it becomes

$$x_{n+1} = x_n + h\left(a_1(1-k_1)x_n - \alpha_{11}x_n^2\right). \tag{102}$$

From (102), define

$$f(x) \coloneqq x + ha_1(1-k_1)x - h\alpha_{11}x^2. \tag{103}$$

Denote $h = h^* = (2/a_1(1-k_1)), x = x^* = (a_1(1-k_1)/\alpha_{11})$. By computation, one obtains

$$f_x|_{h=h^*=(2/a_1(1-k_1)),x=x^*=(a_1(1-k_1)/\alpha_{11})} = -1, \tag{104}$$

$$f_{xx}|_{h=h^*=(2/a_1(1-k_1)),x=x^*=(a_1(1-k_1)/\alpha_{11})} = -\frac{4\alpha_{11}}{a_1(1-k_1)} \neq 0, \tag{105}$$

$$f_h|_{h=h^*=(2/a_1(1-k_1)),x=x^*=(a_1(1-k_1)/\alpha_{11})} = \frac{a_1(1-k_1)}{\alpha_{11}} \neq 0. \tag{106}$$

So, model (15) undergoes flip bifurcation by (104)–(106) if $\Omega \in \mathscr{F}|_{P_4}$. □

6.5. Analysis of Bifurcation at P_5. From (50), the computation yields $\lambda_1|_{(54)} = -1$, but $\lambda_{2,3}|_{(54)} = 1 + (2a_2(1-k_2)/a_1(1-k_1)), 1 - (2/a_1(1-k_1))[-a_3 + (\alpha_{32}(1-k_2)a_2/\alpha_{22})] \neq 1$ or -1. This suggests that model (15) could undergo flip bifurcation around P_5 if Ω passes the curve:

$$\mathscr{F}|_{P_5} = \left\{\Omega\colon h = \frac{2}{a_1(k_1-1)}\right\}. \tag{107}$$

The proof of the following theorem shows that model (15) undergoes flip bifurcation around P_5 if $\Omega \in \mathscr{F}|_{P_5}$.

Theorem 13. *Model (15) undergoes flip bifurcation around P_5 if $\Omega \in \mathscr{F}|_{P_5}$.*

Proof. Recall that if $\Omega \in \mathscr{F}|_{P_5}$, then $\lambda_1|_{(54)} = -1$, but $\lambda_{2,3}|_{(54)} = 1 + (2a_2(1-k_2)/a_1(1-k_1)), 1 - (2/a_1(1-k_1))[-a_3 + (\alpha_{32}(1-k_2)a_2/\alpha_{22})] \neq 1$ *or* -1. So, hereafter, detailed flip bifurcation is explored if Ω varies in the nbhd of h, i.e., $h = h + \varepsilon$, by assuming $h \neq (2\alpha_{22}/(a_3\alpha_{22} - a_2\alpha_{32}(1-k_2))), 2/a_2(1-k_2)$. Let

$$\begin{aligned} u_n &= x_n, \\ v_n &= y_n - \frac{(1-k_2)a_2}{\alpha_{22}}, \\ w_n &= z_n - \frac{a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}. \end{aligned} \tag{108}$$

Then, (15) gives

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + ha_1(1-k_1) & 0 & 0 \\ -h\alpha_{21}\dfrac{(1-k_2)}{\alpha_{22}}a_2 & 1 - ha_2(1-k_2) & 0 \\ 0 & -h\alpha_{32}\left(\dfrac{a_3\alpha_{32} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right) & 1 + h\left(-a_3 + \dfrac{\alpha_{32}(1-k_2)a_2}{\alpha_{22}}\right) \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$$

$$+ \begin{pmatrix} h\alpha_{11}u_n^2 + a_1(1-k_1)\varepsilon u_n - \alpha_{11}\varepsilon u_n^2 \\ h\left[\dfrac{(a_2(1-k_2))^2}{\alpha_{22}} - \alpha_{22}\left(v_n + \dfrac{(1-k_2)a_2}{\alpha_{22}}\right)^2 + \alpha_{21}u_n v_n\right] + \\ \varepsilon\left[a_2(1-k_2)\left(v_n + \dfrac{a_2(1-k_2)}{\alpha_{22}}\right) - \alpha_{22}\left(v_n + \dfrac{(1-k_2)a_2}{\alpha_{22}}\right)^2 - \alpha_{21}u_n\left(v_n + \dfrac{a_2(1-k_2)}{\alpha_{22}}\right)\right] \\ h\left[a_3\left(\dfrac{a_3\alpha_{32} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right) - \alpha_{33}\left(w_n + \dfrac{a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right)^2 - \right. \\ \left. \alpha_{32}v_n w_n - \dfrac{(1-k_2)a_2}{\alpha_{22}}\left(\dfrac{a_3\alpha_{32} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right)\right] + \varepsilon\left[a_3\left(w_n + \dfrac{a_3\alpha_{32} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right) - \right. \\ \left. \alpha_{33}\left(w_n + \dfrac{a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right)^2 - \alpha_{32}\left(v_n + \dfrac{(1-k_2)a_2}{\alpha_{22}}\right)\left(w_n + \dfrac{a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}}\right)\right] \end{pmatrix}. \tag{109}$$

By using transformation,

$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}. \tag{110}$$

(109) takes the form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} F(x_n, y_n, z_n, \varepsilon) \\ G(x_n, y_n, z_n, \varepsilon) \\ H(x_n, y_n, z_n, \varepsilon) \end{pmatrix}, \tag{111}$$

where

$$
\begin{aligned}
a_{11} &= \frac{(-a_1 - a_2 + a_1 k_1 + a_2 k_2)\alpha_{22}(a_1\alpha_{22} + a_3\alpha_{22} - a_1 k_1 \alpha_{22} - a_2\alpha_{32} + a_2 k_2 \alpha_{32})\alpha_{33}}{a_2(-1+k_2)(-a_2 + a_3 + a_2 k_2)\alpha_{21}\alpha_{32}^2}, \\
a_{21} &= -\frac{(a_1\alpha_{22} + a_3\alpha_{22} - a_1 k_1\alpha_{22} - a_2\alpha_{32} + a_2 k_2\alpha_{32})\alpha_{33}}{(-a_2 + a_3 + a_2 k_2)\alpha_{32}^2}, \\
a_{22} &= -\frac{(-a_2\alpha_{22} + a_3\alpha_{22} + a_2 k_2\alpha_{22} - a_2\alpha_{32} + a_2 k_2\alpha_{32})\alpha_{33}}{(-a_2 + a_3 + a_2 k_2)\alpha_{32}^2}, \\
F &= x_n a_1(1-k_1)\varepsilon - x_n^2 a_{11}\alpha_{11} + h x_n^2 a_{11}\alpha_{11} - x_n^2 a_{11}\alpha_{11}\varepsilon, \\
G &= -\frac{a_1 a_{11}(1-k_1)\varepsilon - x_n^2 a_{11}^2\alpha_{11} + h x_n^2 a_{11}^2\alpha_{11} - x_n^2\varepsilon a_{11}^2\alpha_{11}}{a_{11}} \\
&\quad + \frac{h\left(\left(a_2^2(1-k_2)^2/\alpha_{22}\right) + x_n a_{11}\alpha_{21}(x_n a_{21} + y_n\alpha_{22}) - \alpha_{22}(x_n a_{21} + y_n\alpha_{22} 1 - k_2/\alpha_{22})^2\right)}{a_{21}} \\
&\quad + \frac{\varepsilon a_2(1-k_2)(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{22})) - x_n\varepsilon a_{11}\alpha_{21}(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{22}))}{a_{21}} \\
&\quad - \frac{\varepsilon\alpha_{22}(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{11}))^2}{a_{21}}, \\
H &= h\left[-x_n(x_n + y_n + z_n)a_{11}\alpha_{32} - \frac{a_2(1-k_2)a_3\alpha_{32} - a_2\alpha_{32}(1-k_2)}{\alpha_{22}^2\alpha_{33}}\right. \\
&\quad \left. + \frac{a_3(a_3\alpha_{32} - a_2\alpha_{32}(1-k_2))}{\alpha_{22}\alpha_{33}} - \left(x_n + y_n + z_n + \frac{a_3\alpha_{32} - a_2\alpha_{32}(1-k_2)}{\alpha_{22}\alpha_{33}}\right)^2\alpha_{33}\right] \\
&\quad - \frac{h\left(\left(a_2^2(1-k_2)^2/\alpha_{22}\right) + x_n a_{11}\alpha_{21}(x_n a_{21} + y_n\alpha_{22}) - \alpha_{22}(x_n a_{21} + y_n\alpha_{22}((1-k_2)/\alpha_{22}))^2\right)}{a_{21}} \\
&\quad + \frac{\varepsilon a_2(1-k_2)(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{22})) - x_n\varepsilon a_{11}\alpha_{21}(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{22}))}{a_{21}} \\
&\quad - \frac{\varepsilon\alpha_{22}(x_n a_{21} + y_n\alpha_{22} + (a_2(1-k_2)/\alpha_{11}))^2}{a_{21}} \\
&\quad \varepsilon a_3\left(x_n + y_n + z_n + \frac{a_3\alpha_{32} - a_2\alpha_{32}(1-k_2)}{\alpha_{22}\alpha_{33}}\right) \\
&\quad - \varepsilon\left(x_n a_{21} + y_n\alpha_{22} + \frac{a_2(1-k_2)}{\alpha_{22}}\right)\alpha_{32}\left(x_n + y_n + z_n + \frac{a_3\alpha_{32} - a_2(1-k_2)\alpha_{32}}{\alpha_{22}\alpha_{33}}\right) \\
&\quad - \left(x_n + y_n + z_n + \frac{a_3\alpha_{32} - a_2\alpha_{32}(1-k_2)}{\alpha_{22}\alpha_{33}}\right)^2\alpha_{33}.
\end{aligned} \tag{112}
$$

Now, consider (111) on the center manifold, i.e.,

$$W^c(0) = \{(x_n, y_n, z_n) | (y_n, z_n) = (\chi_1(x_n), \chi_2(x_n)), \chi_i(0) = 0, D\chi_i(0) = 0, i = 1, 2\}, \tag{113}$$

where

$$\chi_i(x_n) = a_i x^2 + b_i x^3 + O(x)^4, \quad \text{for } i = 1, 2. \tag{114}$$

From (111) and (113), one obtains

$$\begin{aligned}\chi_1(-x_n + F(x_n, \chi_1, \chi_2)) &= \lambda_2 \chi_1(x_n) + G(x_n, \chi_1, \chi_2),\\ \chi_2(-x_n + F(x_n, \chi_1, \chi_2)) &= \lambda_3 \chi_2(x_n) + H(x_n, \chi_1, \chi_2).\end{aligned} \tag{115}$$

From (115), computation yields $a_1 = b_1 = a_2 = b_2 = 0$. Finally, map (111); restrict to $W^c(0)$ as

$$f(x_n) = -x_n + x_n a_1(1-k_1)\varepsilon - x_n^2 a_{11}\alpha_{11} + h x_n^2 a_{11}\alpha_{11} - x_n^2 a_{11}\alpha_{11}\varepsilon + O\left(\left(|x_n| + |\varepsilon|\right)^3\right). \tag{116}$$

For the model to undergo flip bifurcation, the following should be nonzero:

$$\begin{aligned}\Omega_1 &= \left(\frac{\partial^2 f}{\partial x_n \partial\varepsilon} + \frac{1}{2}\frac{\partial f}{\partial\varepsilon}\frac{\partial^2 f}{\partial x_n^2}\right)|_O = a_1(1-k_1) \neq 0,\\ \Omega_2 &= \left(\frac{1}{6}\frac{\partial^3 f}{\partial x_n^3} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_n^2}\right)^2\right)|_O = (a_{11}\alpha_{11} - h a_{11}\alpha_{11})^2 > 0.\end{aligned} \tag{117}$$

From (117), one can say that about P_5 model (15) undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_5}$. Moreover, period-2 points bifurcating from P_5 are stable since $\Omega_2 = (a_{11}\alpha_{11} - h a_{11}\alpha_{11})^2 > 0$. □

6.6. Analysis of Bifurcation at P_6. From (57), the computation yields $\lambda_1|_{(61)} = -1$, but $\lambda_{2,3}|_{(61)} = 1 + (2/a_1(1-k_1))[a_2(1-k_2) - (a_1\alpha_{21}(1-k_1)/\alpha_{11})], 1 - (2a_3/a_1(1-k_1)) \neq 1$ or -1. This suggests that model (15) could undergo flip bifurcation around P_6 if Ω passes the curve:

$$\mathscr{F}|_{P_6} = \left\{\Omega: h = \frac{2}{a_1(1-k_1)}\right\}. \tag{118}$$

The proof of the following theorem shows that model (15) undergoes flip bifurcation around P_6 if $\Omega \in \mathscr{F}|_{P_6}$.

Theorem 14. *Model (15) undergoes flip bifurcation around P_6 if $\Omega \in \mathscr{F}|_{P_6}$.*

Proof. Recall that if $\Omega \in \mathscr{F}|_{P_6}$, then $\lambda_1|_{(61)} = -1$, but $\lambda_{2,3}|_{(61)} = 1 + (2/a_1(1-k_1))[a_2(1-k_2) - (a_1\alpha_{21}(1-k_1)/\alpha_{11})], 1 - (2a_3/a_1(1-k_1)) \neq 1$ or -1. So, hereafter, detailed flip bifurcation is explored if Ω varies in the nbhd of h, i.e., $h = h + \varepsilon$, by assuming $h \neq (2/a_3), 2\alpha_{11}/(a_1\alpha_{21}(1-k_1) - a_2\alpha_{11}(1-k_2))$. Let

$$\begin{aligned}u_n &= x_n - \frac{(1-k_1)a_1}{\alpha_{11}},\\ v_n &= y_n,\\ w_n &= z_n - \frac{a_3}{\alpha_{33}}.\end{aligned} \tag{119}$$

Then, (15) gives

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - ha_1(1-k_1) & 0 & 0 \\ 0 & 1 + h\left(a_2(1-k_2) - \dfrac{a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right) & 0 \\ 0 & -h\alpha_{32}\dfrac{a_3}{\alpha_{32}} & 1 - ha_3 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$$
$$+\left(h\frac{(a_1(1-k_1))^2}{\alpha_{11}} - h\alpha_{11}\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2 + \varepsilon a_1(1-k_1)\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right) - \varepsilon\alpha_{11}\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\right.$$
$$- h\alpha_{22}v_n^2 - h\alpha_{22}u_nv_n + \varepsilon a_2(1-k_2)v_n - \varepsilon\alpha_{22}v_n^2 - \varepsilon\alpha_{21}v_n\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)$$
$$h\left(\frac{a_3^2}{\alpha_{33}} - \alpha_{33}\left(w_n + \frac{a_3}{\alpha_{33}}\right)^2 - \alpha_{32}v_nw_n\right)$$
$$\left.+\varepsilon a_3\left(w_n + \frac{a_3}{\alpha_{33}}\right) - \varepsilon\alpha_{33}\left(w_n + \frac{a_3}{\alpha_{33}}\right)^2 - \varepsilon\alpha_{32}v_n\left(w_n + \frac{a_3}{\alpha_{33}}\right)\right)\Bigg). \tag{120}$$

Using transformation,

$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & b_{23} \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}. \tag{121}$$

(120) becomes

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} F_1(x_n, y_n, z_n, \varepsilon) \\ G_1(x_n, y_n, z_n, \varepsilon) \\ H_1(x_n, y_n, z_n, \varepsilon) \end{pmatrix}, \tag{122}$$

where

$$\begin{aligned}
b_{23} &= \frac{\alpha_{33}(-a_2\alpha_{11} - a_3\alpha_{11} + a_2k_2\alpha_{11} + a_1\alpha_{21} - a_1k_1\alpha_{21})}{a_3\alpha_{11}\alpha_{32}}, \\
F_1 &= -z_na_2(1-k_2)\varepsilon + z_n\left(y_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)\alpha_{21}\varepsilon + hz_n\alpha_{22} + hy_nz_n\alpha_{22} + z_n^2b_{23}\alpha_{22}\varepsilon \\
&\quad + a_3\left(x_n + y_n + \frac{a_3}{\alpha_{33}}\right)\varepsilon - z_nb_{23}\alpha_{32}\left(x_n + y_n + \frac{a_3}{\alpha_{33}}\right)\varepsilon - \left(x_n + y_n + \frac{a_3}{\alpha_{33}}\right)^2\alpha_{33}\varepsilon \\
&\quad + h\left(-(x_n + y_n)z_nb_{23}\alpha_{32} + \frac{a_3^2}{\alpha_{33}} - \left(x_n + y_n + \frac{a_3}{\alpha_{33}}\right)^2\alpha_{33}\right), \\
G_1 &= a_1(1-k_1)\left(y_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)\varepsilon - h\left(y_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2 \\
&\quad + \frac{ha_1^2(1-k_1)^2}{\alpha_{11}} - \left(y_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\alpha_{11}\varepsilon, \\
H_1 &= -z_na_2(1-k_2)\varepsilon + z_n\left(y_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)\alpha_{21}\varepsilon + hz_n\alpha_{22} + hy_nz_n\alpha_{22} + z_n^2b_{23}\alpha_{22}\varepsilon.
\end{aligned} \tag{123}$$

Now, from model (122) on the center manifold,

$$W^c(0) = \{(x_n, y_n, z_n) \mid (y_n, z_n) = (\chi_3(x_n), \chi_4(x_n)), \chi_i(0) = 0, D\chi_i(0) = 0, i = 3, 4\}, \tag{124}$$

where

$$\chi_i(x_n) = a_i x^2 + b_i x^3 + O(x)^4, \quad \text{for } i = 3, 4. \tag{125}$$

From (122) and (124), one has

$$\begin{aligned} \chi_3(-x_n + F_1(x_n, \chi_3, \chi_4)) &= \lambda_2 \chi_3(x_n) + G_1(x_n, \chi_3, \chi_4), \\ \chi_4(-x_n + F_1(x_n, \chi_3, \chi_4)) &= \lambda_3 \chi_4(x_n) + H_1(x_n, \chi_3, \chi_4). \end{aligned} \tag{126}$$

From (126), the calculation yields: $a_3 = b_3 = a_4 = b_4 = 0$. Thus, map (122); restrict to $W^c(0)$ as

$$f(x_n) = -x_n + a_3\left(x_n + \frac{a_3}{\alpha_{33}}\right)\varepsilon - \left(x_n + \frac{a_3}{\alpha_{33}}\right)^2 \alpha_{33}\varepsilon + h\left(\frac{a_3^2}{\alpha_{33}} - \left(x_n + \frac{a_3}{\alpha_{33}}\right)^2 \alpha_{33}\right). \tag{127}$$

From (117) and (127), the computation yields: $\Omega_1 = 3a_3 - (ha_3^2/\alpha_{33}) \neq 0$ and $\Omega_2 = h^2\alpha_{33}^2 > 0$. This implies that about P_6 model (15) undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_6}$. Moreover, period-2 points bifurcating from P_6 are stable since $\Omega_2 = h^2\alpha_{33}^2 > 0$. □

6.7. Analysis of Bifurcation at P_7. From (64), the computation yields $\lambda_1|_{(68)} = -1$, but $\lambda_{2,3}|_{(68)} = 1 - (2/a_1(1-k_1))[a_2(1-k_2) - (2a_1\alpha_{21}(1-k_1)/\alpha_{11})]$, $1 + (2/a_1(1-k_1))[a_3 - (a_2(1-k_2)\alpha_{32}/\alpha_{22}) + (a_1\alpha_{21}\alpha_{32}(1-k_1)/\alpha_{11}\alpha_{22})] \neq 1$ or -1. This suggests that model (15) could undergo flip bifurcation around P_7 if Ω passes the curve:

$$\mathscr{F}|_{P_7} = \left\{\Omega\colon h = \frac{2}{a_1(1-k_1)}\right\}. \tag{128}$$

The proof of the following theorem shows that model (15) undergoes flip bifurcation around P_7 if $\Omega \in \mathscr{F}|_{P_7}$.

Theorem 15. *Model (15) undergoes flip bifurcation around P_7 if $\Omega \in \mathscr{F}|_{P_7}$.*

Proof. Recall that if $\Omega \in \mathscr{F}|_{P_7}$, then $\lambda_1|_{(68)} = -1$, but $\lambda_{2,3}|_{(68)} = 1 - (2/a_1(1-k_1))[a_2(1-k_2) - (2a_1\alpha_{21}(1-k_1)/\alpha_{11})]$, $1 + (2/a_1(1-k_1))[a_3 - (a_2(1-k_2)\alpha_{32}/\alpha_{22}) + (a_1\alpha_{21}\alpha_{32}(1-k_1)/\alpha_{11}\alpha_{22})] \neq 1$ or -1. So, in the following, flip bifurcation is explored by assuming $h \neq (2\alpha_{11}/(a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)))$, $(2\alpha_{11}\alpha_{22}/a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1))$. Let

$$\begin{aligned} u_n &= x_n - \frac{(1-k_1)a_1}{\alpha_{11}}, \\ v_n &= y_n - \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}, \\ w_n &= z_n. \end{aligned} \tag{129}$$

Then, (15) gives

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1-ha_1(1-k_1) & 0 & 0 \\ -h\alpha_{21}\left(\dfrac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right) & 1-h\left(a_2(1-k_2)-\dfrac{2a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right) & 0 \\ 0 & 0 & 1+h\left(a_3-\dfrac{a_2(1-k_2)\alpha_{32}}{\alpha_{22}}+\dfrac{a_1\alpha_{21}\alpha_{32}(1-k_1)}{\alpha_{11}\alpha_{22}}\right) \end{pmatrix}$$
$$\cdot \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$$
$$+\left(h\left(\frac{(a_1(1-k_1))^2}{\alpha_{11}}-\alpha_{11}\left(u_n+\frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\right)\right.$$
$$\left.+\varepsilon\left(a_1(1-k_1)\left(u_n+\frac{a_1(1-k_1)}{\alpha_{11}}\right)-\alpha_{11}\left(u_n+\frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\right)\right)$$
$$\cdot h\left[a_2(1-k_2)\left(\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)+\alpha_{22}\left(v_n+\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\right.$$
$$\left.-\alpha_{21}\frac{a_1(1-k_1)}{\alpha_{11}}\left(\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right]+\varepsilon\left[a_2(1-k_2)\left(v_n+\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right.$$
$$-\alpha_{22}\left(v_n+\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2$$
$$\left.-\alpha_{21}\left(u_n+\frac{a_1(1-k_1)}{\alpha_{11}}\right)\left(v_n+\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right]h\left(-\alpha_{33}w_n^2-\alpha_{32}v_nw_n\right)+\varepsilon\left(a_3w_n-\alpha_{33}w_n^2\right.$$
$$\left.\left.\left.-\alpha_{32}w_n\left(v_n+\frac{a_2(1-k_2)\alpha_{11}-a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right)\right)\right). \tag{130}$$

Using transformation,

$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}. \tag{131}$$

(130) gives

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} F_2(x_n, y_n, z_n, \varepsilon) \\ G_2(x_n, y_n, z_n, \varepsilon) \\ H_2(x_n, y_n, z_n, \varepsilon) \end{pmatrix}, \tag{132}$$

where

$$c_{11}=\frac{\alpha_{11}\alpha_{22}\left(ha_1\left(1-k_1\right)-ha_2\left(1-k_2\right)-\left(2ha_1\left(1-k_1\right)\alpha_{21}/\alpha_{11}\right)\right)}{h\alpha_{21}\left(a_2\left(1-k_2\right)\alpha_{11}-a_1\left(1-k_1\right)\alpha_{21}\right)},$$

$$F_2=\frac{h\left[\left(\left(a_1\left(1-k_1\right)\right)^2/\alpha_{11}\right)-\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)^2\alpha_{11}\right]+\varepsilon\left[a_1\left(1-k_1\right)\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)-\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)^2\alpha_{11}\right]}{c_{11}},$$

$$\begin{aligned}G_2={}&h\Bigg[\frac{a_2\left(1-k_2\right)\left(a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}\right)}{\alpha_{11}\alpha_{22}}-\frac{a_1\left(1-k_1\right)\alpha_{21}\left(a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}\right)}{\alpha_{11}^2\alpha_{22}}\\&+\left(x_n+y_n+\frac{a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\alpha_{22}\Bigg]\\&\cdot\frac{h\left[\left(\left(a_1\left(1-k_1\right)\right)^2/\alpha_{11}\right)-\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)^2\alpha_{11}\right]+\varepsilon\left[a_1\left(1-k_1\right)\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)-\left(x_nc_{11}+\left(a_1\left(1-k_1\right)/\alpha_{11}\right)\right)^2\alpha_{11}\right]}{c_{11}}\\&+\varepsilon\Bigg[a_2\left(1-k_2\right)\left(x_n+y_n+\frac{a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\\&+\frac{a_1\left(1-k_1\right)\alpha_{21}\left(x_n+y_n+\left(\left(a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}\right)/\alpha_{11}\alpha_{22}\right)\right)^3\left(x_nc_{11}\left(\left(a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}\right)/\alpha_{11}\alpha_{22}\right)\right)\alpha_{22}}{a_{11}}\Bigg],\end{aligned}$$

$$H_2=h\left(-\left(x_n+y_n\right)z_n\alpha_{32}-z^2\alpha_{33}\right)+\varepsilon\left[z_na_3-z_n\left(x_n+y_n+\frac{a_2a_{11}\left(1-k_2\right)-a_1\left(1-k_1\right)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\alpha_{32}-z_n^2\alpha_{33}\right]. \tag{133}$$

Now, using system (132) on the center manifold,

$$W^c(0)=\left\{\left(x_n,y_n,z_n\right)\mid\left(y_n,z_n\right)=\left(\chi_5\left(x_n\right),\chi_6\left(x_n\right)\right),\chi_i(0)=0,D\chi_i(0)=0,i=5,6\right\}, \tag{134}$$

where

$$\chi_i\left(x_n\right)=a_ix^2+b_ix^3+O(x)^4,\quad \text{for } i=5,6. \tag{135}$$

In view of (132) and (134), we obtain

$$\begin{aligned}\chi_5\left(-x_n+F_2\left(x_n,\chi_5,\chi_6\right)\right)&=\lambda_2\chi_5\left(x_n\right)+G_2\left(x_n,\chi_5,\chi_6\right),\\ \chi_6\left(-x_n+F_2\left(x_n,\chi_5,\chi_6\right)\right)&=\lambda_3\chi_6\left(x_n\right)+H_2\left(x_n,\chi_5,\chi_6\right).\end{aligned} \tag{136}$$

From (136), one gets: $a_5 = b_5 = a_6 = b_6 = 0$. Finally, map (132); restrict to $W^c(0)$ as

$$f(x_n) = -x_n + \frac{h\left[\left((a_1(1-k_1))^2/\alpha_{11}\right) - \left(x_n c_{11} + (a_1(1-k_1)/\alpha_{11})\right)^2\alpha_{11}\right] + \varepsilon\left[a_1(1-k_1)\left(x_n c_{11} + (a_1(1-k_1)/\alpha_{11})\right) - \left(x_n c_{11} + (a_1(1-k_1)/\alpha_{11})\right)^2\alpha_{11}\right]}{c_{11}}. \quad (137)$$

From (117) and (137), the computation yields: $\Omega_1 = a_1(1-k_1)(1-2c_{11}) \neq 0$. Moreover, $\Omega_2 = (h\alpha_{11}c_{11} + 2\alpha_{11}c_{11}\varepsilon)^2 > 0$. This implies that about P_7 model (15) undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_7}$. Moreover, period-2 points bifurcating from P_7 are stable since $\Omega_2 = (h\alpha_{11}c_{11} + 2\alpha_{11}c_{11}\varepsilon)^2 > 0$. □

6.8. Analysis of Bifurcation at P_8. From (71), the computation yields $\lambda_1|_{(75)} = -1$, but $\lambda_{2,3}|_{(75)} = 1 - (2/a_1(1-k_1))[a_2(1-k_2) - (2a_1\alpha_{21}(1-k_1)/\alpha_{11})], 1 - (2/a_1(1-k_1))[a_3 - (a_2(1-k_2)\alpha_{32}/\alpha_{22}) + (a_1\alpha_{21}\alpha_{32}(1-k_1)/\alpha_{11}\alpha_{22})] \neq 1$ *or* -1. This suggests that model (15) could undergo flip bifurcation around P_8 if Ω passes the curve:

$$\mathscr{F}|_{P_8} = \left\{\Omega\colon h = \frac{2}{a_1(1-k_1)}\right\}. \quad (138)$$

The proof of the following theorem shows that model (15) undergoes flip bifurcation around P_8 if $\Omega \in \mathscr{F}|_{P_8}$.

Theorem 16. *Model (15) undergoes flip bifurcation around P_8 if $\Omega \in \mathscr{F}|_{P_8}$.*

Proof. Recall that if $\Omega \in \mathscr{F}|_{P_8}$, then $\lambda_1|_{(75)} = -1$, but $\lambda_{2,3}|_{(75)} = 1 - (2/a_1(1-k_1))[a_2(1-k_2) - (2a_1\alpha_{21}(1-k_1)/\alpha_{11})]$, $1 - (2/a_1(1-k_1))[a_3 - (a_2(1-k_2)\alpha_{32}/\alpha_{22}) + (a_1\alpha_{21}\alpha_{32}(1-k_1)/\alpha_{11}\alpha_{22})] \neq 1$ or -1. So, in the following, flip bifurcation is explored by assuming $h \neq (2\alpha_{11}/(a_2\alpha_{11}(1-k_2) - 2a_1\alpha_{21}(1-k_1)))$, $(2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32}(1-k_2) + a_1\alpha_{21}\alpha_{32}(1-k_1)))$. Let

$$u_n = x_n - \frac{(1-k_1)a_1}{\alpha_{11}},$$
$$v_n = y_n - \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}},$$
$$w_n = z_n - \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}. \quad (139)$$

Then, (15) becomes

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - ha_1(1-k_1) & 0 & 0 \\ -h\alpha_{21}\left(\frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right) & 1 - h\left(a_2(1-k_2) - \frac{2a_1\alpha_{21}(1-k_1)}{\alpha_{11}}\right) & 0 \\ 0 & -h\alpha_{32}\left(\frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right) & 1 - h\left(a_3 - \frac{a_2(1-k_2)\alpha_{32}}{\alpha_{22}} + \frac{a_1\alpha_{21}\alpha_{32}(1-k_1)}{\alpha_{11}\alpha_{22}}\right) \end{pmatrix}$$
$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} + \begin{pmatrix} F_3(x_n, y_n, z_n, \varepsilon) \\ G_3(x_n, y_n, z_n, \varepsilon) \\ H_3(x_n, y_n, z_n, \varepsilon) \end{pmatrix}, \quad (140)$$

where

$$\begin{aligned}
F_3 &= h\left(\frac{(a_1(1-k_1))^2}{\alpha_{11}} - \alpha_{11}\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\right) + \varepsilon\left(a_1(1-k_1)\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right) - \alpha_{11}\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)^2\right), \\
G_3 &= h\left(a_2(1-k_2)\left(\frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right. \\
&\quad - \alpha_{21}\left(v_n + \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2 - \alpha_{21}u_n v_n \\
&\quad \left. - \alpha_{21}\frac{a_1(1-k_1)}{\alpha_{11}}\left(\frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right) \\
&\quad + \varepsilon\left(a_2(1-k_2)\left(v_n + \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right) - \alpha_{22}\left(v_n + \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\right. \\
&\quad \left. - \alpha_{21}\left(u_n + \frac{a_1(1-k_1)}{\alpha_{11}}\right)\left(v_n + \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right), \\
H_3 &= h\left(a_3\left(\frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right) - \alpha_{32}v_n w_n\right. \\
&\quad - \alpha_{33}\left(w_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)^2 \\
&\quad \left. - \alpha_{32}\frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\left(\frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)\right) \\
&\quad + \varepsilon\left(a_3\left(w_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)\right. \\
&\quad \left. - \alpha_{33}\left(w_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)^2\right) - \alpha_{32}\left(v_n + \frac{a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right) \\
&\quad \varepsilon \times \left(w_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right).
\end{aligned} \tag{141}$$

Now, by utilizing transformation,

$$\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, \tag{142}$$

gives

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} F_2^*(x_n, y_n, z_n, \varepsilon) \\ G_2^*(x_n, y_n, z_n, \varepsilon) \\ H_2^*(x_n, y_n, z_n, \varepsilon) \end{pmatrix}, \tag{143}$$

where

$$
\begin{aligned}
d_{11} &= \frac{\alpha_{11}^2 (ha_1 - ha_1k_1 - ha_2 (1-k_2) - (2ha_1 (1-k_1)\alpha_{21}/\alpha_{11}))\alpha_{22}^2\alpha_{33} (ha_1 - ha_1k_1 - h(a_3 - (a_2 (1-k_2)\alpha_{32}/\alpha_{22}) + (a_1 (1-k_1)\alpha_{21}\alpha_{32}/\alpha_{11}\alpha_{22})))}{h^2\alpha_{21} (a_2 (1-k_2)\alpha_{11} - a_1 (1-k_1)\alpha_{21})\alpha_{32} (a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32})}, \\
d_{21} &= \frac{\alpha_{11}\alpha_{22} (ha_1 - ha_1k_1 - h(a_3 - (a_2 (1-k_2)\alpha_{32}/\alpha_{22}) + (a_1 (1-k_1)\alpha_{21}\alpha_{32}/\alpha_{11}\alpha_{22})))\alpha_{33}}{h\alpha_{32} (a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32})}, \\
d_{22} &= -\frac{\alpha_{33} (-a_2\alpha_{11}\alpha_{22} + a_3\alpha_{11}\alpha_{22} + a_2k_2\alpha_{11}\alpha_{22} - 2a_1\alpha_{21}\alpha_{22} + 2a_1k_1\alpha_{21}\alpha_{22} - a_2\alpha_{11}\alpha_{32} + a_2k_2\alpha_{11}\alpha_{32} + a_1\alpha_{21}\alpha_{32} - a_1k_1\alpha_{21}\alpha_{32})}{\alpha_{32} (a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32} + a_2k_2\alpha_{11}\alpha_{32} + a_1\alpha_{21}\alpha_{32} - a_1k_1\alpha_{21}\alpha_{32})}, \\
F_2^* &= \frac{h\big(\big((a_1 (1-k_1))^2/\alpha_{11}\big) - (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big) + \varepsilon\big(a_1 (1-k_1) (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11})) - (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big)}{d_{11}}, \\
G_2^* &= -\frac{d_{21}\big(h\big(\big((a_1 (1-k_1))^2/\alpha_{11}\big) - (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big) + \varepsilon\big(a_1 (1-k_1) (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11})) - (x_nd_{11} + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big)\big)}{d_{11}d_{22}} \\
&\quad + \frac{h}{d_{22}}\left(-x_nd_{11} (d_{21}x_n + d_{22}y_n)\alpha_{21} - \alpha_{21}\left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\right. \\
&\quad \left. + \frac{a_2 (1-k_2) (a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})}{\alpha_{11}\alpha_{22}} - \frac{a_1 (1-k_1)\alpha_{21} (a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})}{\alpha_{11}^2\alpha_{22}}\right) \\
&\quad + \frac{\varepsilon}{d_{22}}\left(a_2 (1-k_2)\left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right. \\
&\quad - \frac{a_1 (1-k_1)\alpha_{21} (d_{11}x + (a_1 (1-k_1)/\alpha_{11})) (d_{21}x_n + d_{22}y_n + ((a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}))}{\alpha_{11}} \\
&\quad \left. - \left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\alpha_{22}\right), \\
H_2^* &= h\left(-(y_n + 2z_n) (d_{21}x_n + d_{22}y_n)\alpha_{32} - \alpha_{32}\left(x_n + y_n + z_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)^2\right. \\
&\quad + \frac{a_3 (a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32})}{\alpha_{11}\alpha_{22}\alpha_{33}} \\
&\quad \left. - \frac{(a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})\alpha_{32} (a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32})}{\alpha_{11}^2\alpha_{22}^2\alpha_{33}}\right) \\
&\quad + \frac{(d_{21} - d_{22})\big(h\big(\big((a_1 (1-k_1))^2/\alpha_{11}\big) - (d_{11}x_n + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big) + \epsilon\big(a_1 (1-k_1) (d_{11}x_n + (a_1 (1-k_1)/\alpha_{11})) - (d_{11}x_n + (a_1 (1-k_1)/\alpha_{11}))^2\alpha_{11}\big)\big)}{d_{11}d_{22}} \\
&\quad - \frac{h}{d_{22}}\left(-d_{11}x_n (d_{21}x_n + d_{22}y_n)\alpha_{21} - \alpha_{21}\left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\right. \\
&\quad \left. + \frac{a_2 (1-k_2) (a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})}{\alpha_{11}\alpha_{22}} - \frac{a_1 (1-k_1)\alpha_{21} (a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})}{\alpha_{11}^2\alpha_{22}}\right) \\
&\quad + \frac{\varepsilon}{d_{22}}\left(a_2 (1-k_2)\left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)\right. \\
&\quad - \frac{a_1 (1-k_1)\alpha_{21} (d_{11}x_n + (a_1 (1-k_1)/\alpha_{11})) (d_{21}x_n + d_{22}y_n + ((a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}))}{\alpha_{11}} \\
&\quad \left. - \left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right)^2\alpha_{22}\right) \\
&\quad + \varepsilon\left(a_3\left(x_n + y_n + z_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)\right. \\
&\quad - \alpha_{32}\left(d_{21}x_n + d_{22}y_n + \frac{a_2a_{11} (1-k_2) - a_1 (1-k_1)\alpha_{21}}{\alpha_{11}\alpha_{22}}\right) \\
&\quad \times\left(x_n + y_n + z_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right) \\
&\quad \left. - \left(x_n + y_n + z_n + \frac{a_3\alpha_{11}\alpha_{22} - a_2 (1-k_2)\alpha_{11}\alpha_{32} + a_1 (1-k_1)\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\right)^2\alpha_{33}\right).
\end{aligned}
\tag{144}
$$

Now, from system (143) on the center manifold

$$W^c(0) = \{(x_n, y_n, z_n) | (y_n, z_n) = (\chi_7(x_n), \chi_8(x_n)), \chi_i(0) = 0, D\chi_i(0) = 0, i = 7, 8\}, \tag{145}$$

where

$$\chi_i(x_n) = a_i x^2 + b_i x^3 + O(x)^4, \quad \text{for } i = 7, 8. \tag{146}$$

From (143) and (145), we obtain

$$\begin{aligned} \chi_7(-x_n + F_2^*(x_n, \chi_7, \chi_8)) &= \lambda_2 \chi_7(x_n) + G_2^*(x_n, \chi_7, \chi_8), \\ \chi_8(-x_n + F_2^*(x_n, \chi_7, \chi_8)) &= \lambda_3 \chi_8(x_n) + H_2^*(x_n, \chi_7, \chi_8). \end{aligned} \tag{147}$$

From (147), computation yields: $a_7 = b_7 = a_8 = b_8 = 0$. Finally, map (143); restrict to $W^c(0)$ as

$$f_1^*(x_n) = -x_n + \frac{h\left[\left(\left(a_1(1-k_1)\right)^2/\alpha_{11}\right) - \left(x_n d_{11} + \left(a_1(1-k_1)/\alpha_{11}\right)\right)^2 \alpha_{11}\right] + \varepsilon\left[a_1(1-k_1)\left(x_n d_{11} + \left(a_1(1-k_1)/\alpha_{11}\right)\right) - \left(x_n d_{11} + \left(a_1(1-k_1)/\alpha_{11}\right)\right)^2 \alpha_{11}\right]}{d_{11}}. \tag{148}$$

From (117) and (148), the computation yields: $\Omega_1 = a_1(1-k_1)(1-2d_{11}) \neq 0$. Moreover, $\Omega_2 = (h\alpha_{11}d_{11} + 2\alpha_{11}d_{11}\varepsilon)^2 > 0$. This implies that about P_8 model (15) undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_8}$. Moreover, period-2 points bifurcating from P_8 are stable since $\Omega_2 = (h\alpha_{11}d_{11} + 2\alpha_{11}d_{11})^2 > 0$. □

7. Numerical Simulations

Numerical simulations of three-species model (15) are performed in this section to check previous theoretical findings and to show rich dynamical behaviors. In this regard, following eight cases are presented to address the accuracy of theoretical results obtained about fixed points for model (15):

Case I: if $a_1 = 4.1, a_2 = 4.2, a_3 = 0.08, k_1 = 2.2, k_2 = 0.6, \alpha_{11} = 0.04, \alpha_{22} = 0.4, \alpha_{21} = 0.09, \alpha_{32} = 0.09$, and $\alpha_{33} = 0.4$, then, from (29), one gets $h = 0.4065040650406504$. From (27), if $h = 0.01 < \min\{0.4065040650406504, 0.20703933747412007\}$ and starting from $(0.9, 0.1, 0.2)$, then Figure 1(a) indicates that P_1 of (15) is a source. However, if $h = 0.5 > \max\{0.406504065040, 0.2070393374741\}$, then Figure 1(b) indicates that P_1 of (15) is a saddle. Hence, theoretical results obtained in Lemma 2 coincide with numerical simulations.

Case II: if $a_1 = 4.1, a_2 = 4.2, a_3 = 2.7, k_1 = 2.2, k_2 = 0.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 1.9, \alpha_{32} = 0.9$, and $\alpha_{33} = 0.4$, then, from (35), one gets $h = 0.7407407407407407$. Figure 2(a) indicates if $h = 0.01 < 0.7407407407407407$, then P_2 of (15) is a sink. However, if $h = 0.95 > 0.7407407407407407$, then Figure 2(b) indicates that P_2 is unstable. Moreover, if $h = 0.7407407407407407$, then P_2 exchanges the stability, and in fact, flip bifurcation takes place by Theorem 10. Therefore, the flip bifurcation diagrams are presented in Figure 3. Finally, maximum Lyapunov exponents corresponding to Figure 3 are drawn in Figure 4.

Case III: if $a_1 = 4.1, a_2 = 4.2, a_3 = 2.7, k_1 = 2.2, k_2 = 1.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 1.9, \alpha_{32} = 0.9$, and $\alpha_{33} = 0.4$, then, from (42), one gets $h = 0.7936507936507935$. Hence, P_3 is stable if $h < 0.7936507936507935$, and exchange stability is $h = 0.7407407407407407$, and in fact, flip bifurcation takes place by Theorem 11. Therefore, the flip bifurcation diagrams with initial value $(0, 0.3, 0.1)$ are presented in Figure 5. Finally, maximum Lyapunov exponents corresponding to Figure 5 are drawn in Figure 6.

Case IV: if $a_1 = 5.1, a_2 = 4.2, a_3 = 2.7, k_1 = 0.09, k_2 = 1.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 1.9, \alpha_{32} = 0.9$, and $\alpha_{33} = 0.4$, then, from (48), one gets $h = 0.43094160741219567$. Hence, P_4 is stable if $h < 0.43094160741219567$, and exchange stability is $h = 0.43094160741219567$, and in fact, flip bifurcation takes place by Theorem 12. Therefore, the flip bifurcation diagrams with initial value $(0.1, 0.2, 0.1)$ are presented in Figure 7. Finally, maximum Lyapunov exponents corresponding to Figure 7 are drawn in Figure 8.

Case V: if $a_1 = 7.1, a_2 = 4.2, a_3 = 2.7, k_1 = 1.9, k_2 = 1.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 1.9, \alpha_{32} = 0.9$, and $\alpha_{33} = 0.4$, then, from (54), one gets $h = 0.31298904538341166$. Hence, P_5 is stable if $h < 0.31298904538341166$, and exchange stability is $h = 0.31298904538341166$, and in fact, flip bifurcation takes place by Theorem 13. Therefore, the flip bifurcation diagrams with initial value $(0, 0.1, 0.1)$ are presented in Figure 9. Finally, maximum Lyapunov exponents corresponding to Figure 9 are drawn in Figure 10.

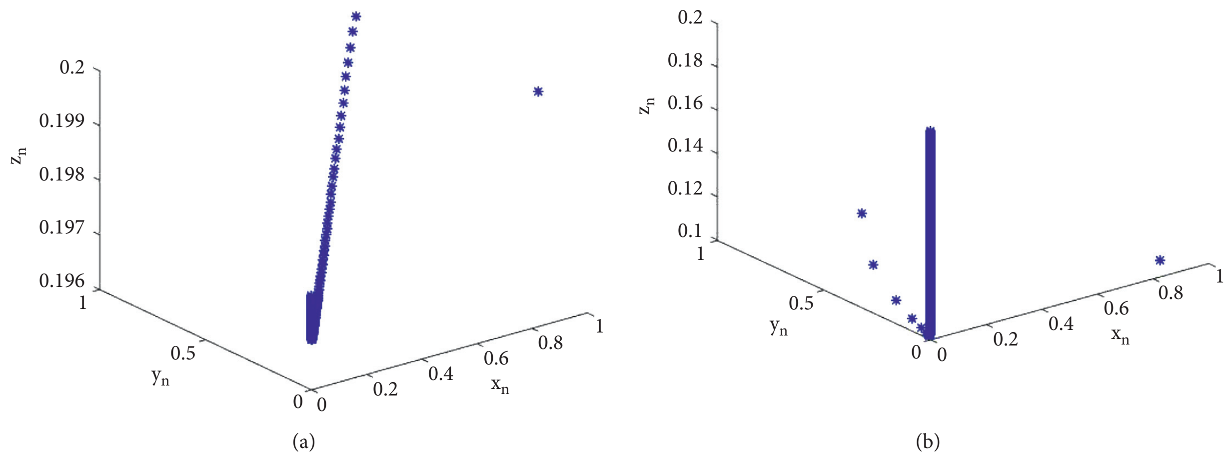


FIGURE 1: Phase portrait about P_1.

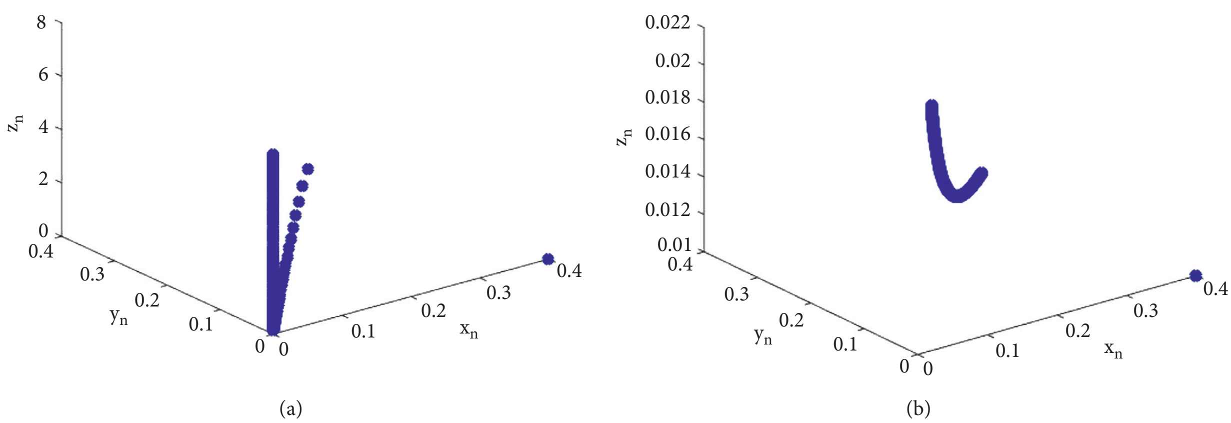


FIGURE 2: Phase portrait about P_2.

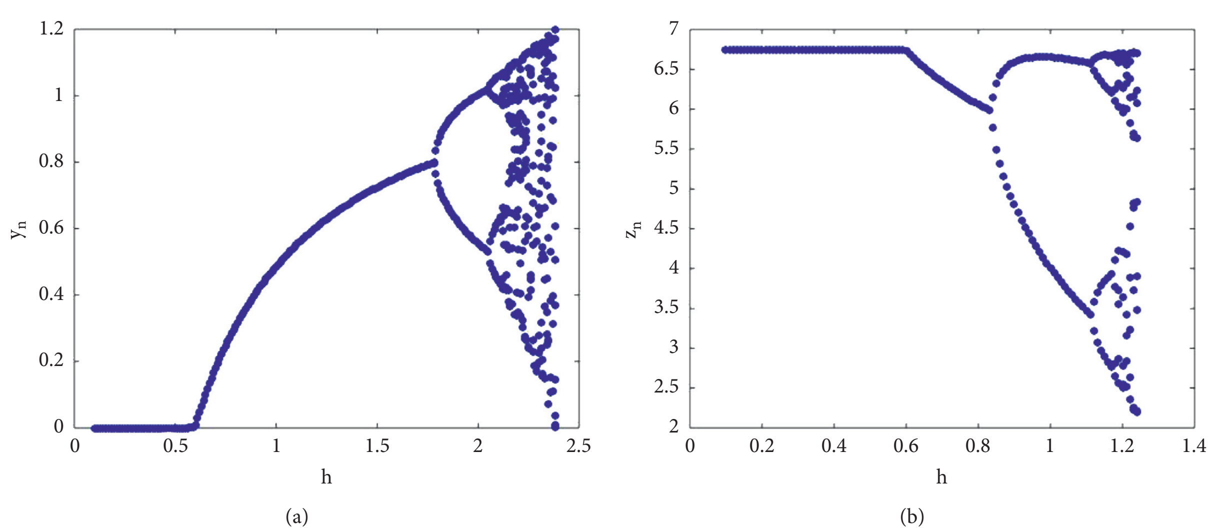


FIGURE 3: Continued.

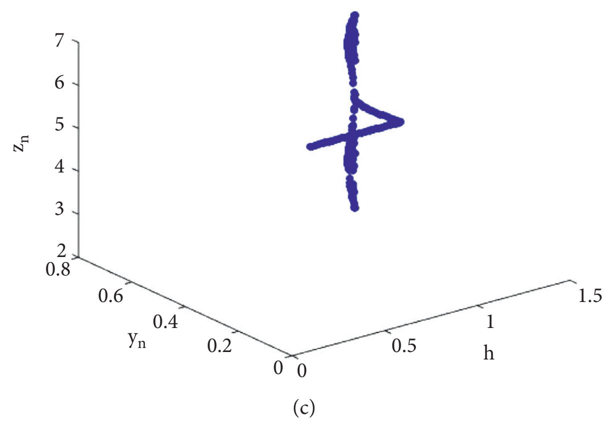


Figure 3: Flip bifurcation diagrams at P_2, where $h \in [0.1, 6.4]$.

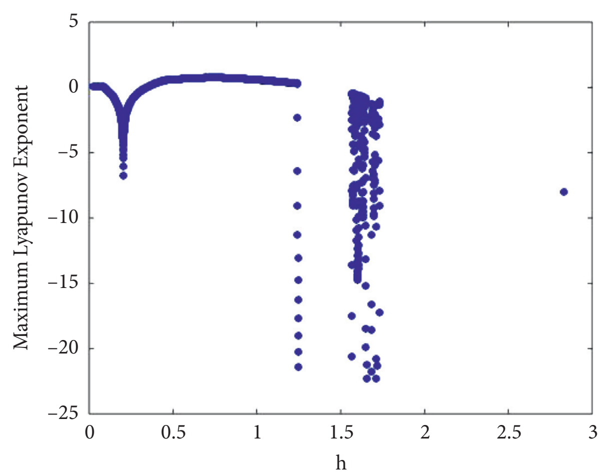


Figure 4: Maximum Lyapunov exponents corresponding to Figure 3.

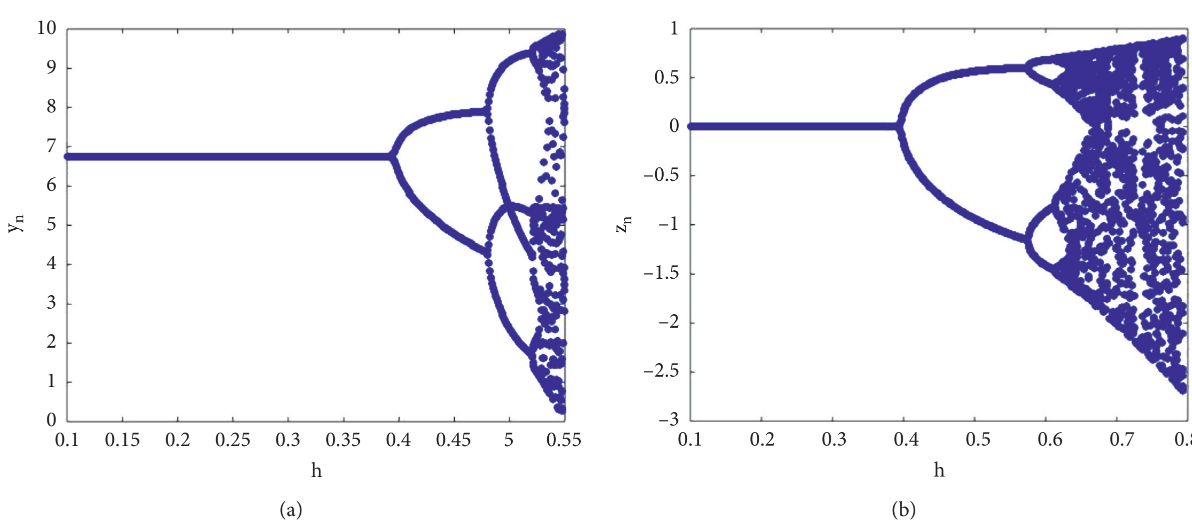
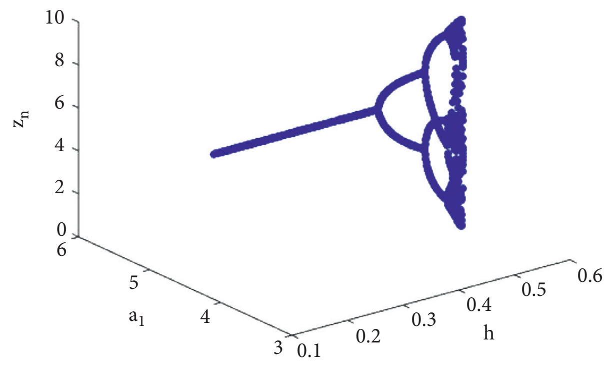


Figure 5: Continued.



(c)

Figure 5: Flip bifurcation diagrams at P_3, where $h \in [0.1, 1.4]$.

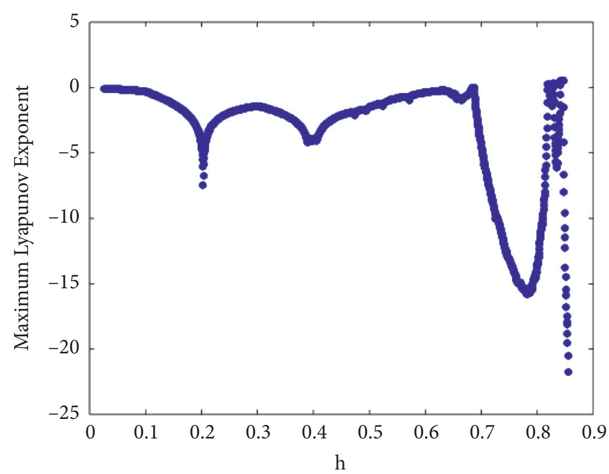
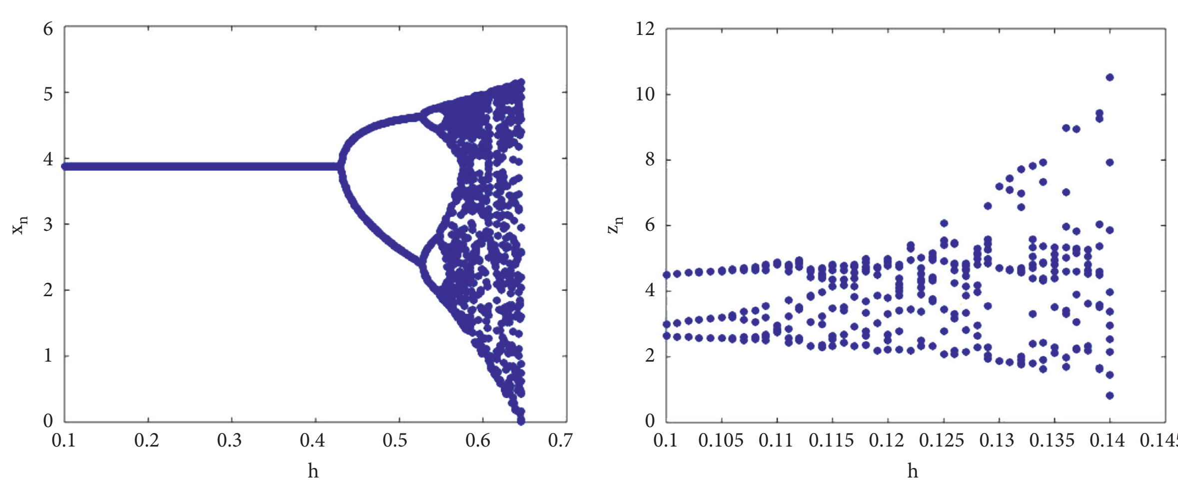
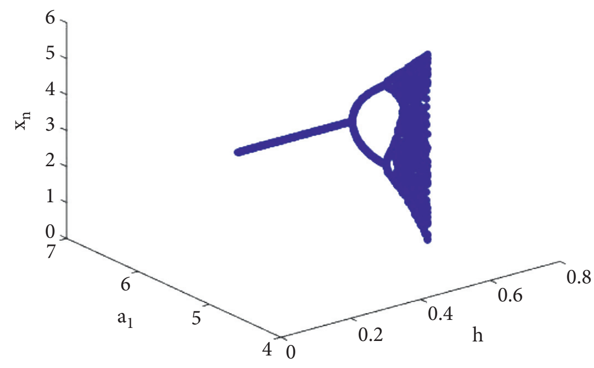


Figure 6: Maximum Lyapunov exponents corresponding to Figure 5.



(a) (b)

Figure 7: Continued.



(c)

Figure 7: Flip bifurcation diagrams at P_4, where $h \in [0:1; 1:4]$.

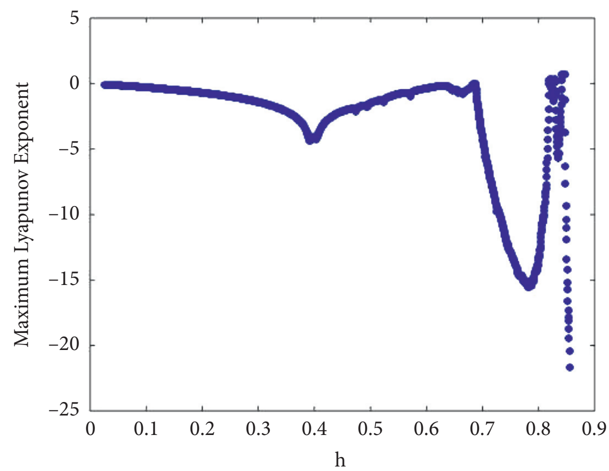
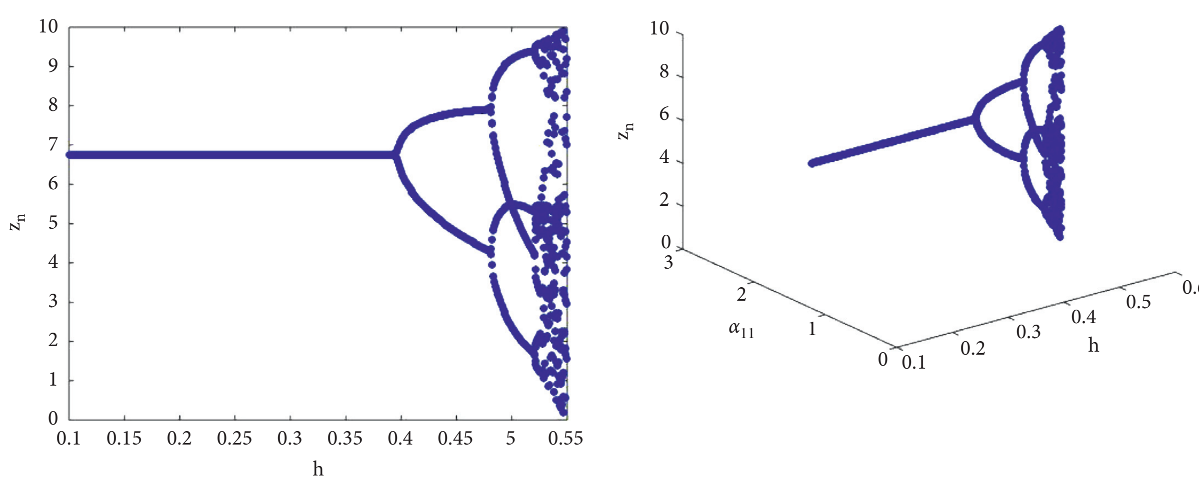


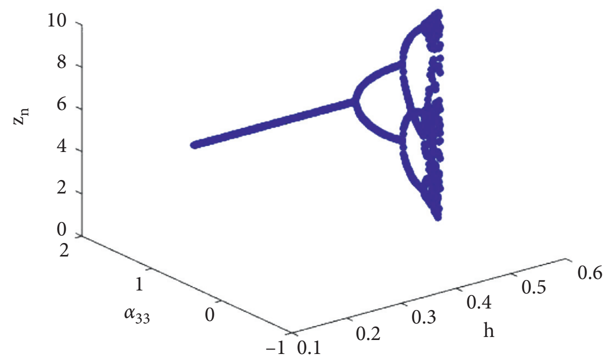
Figure 8: Maximum Lyapunov exponents corresponding to Figure 7.



(a)

(b)

Figure 9: Continued.



(c)

Figure 9: Flip bifurcation diagrams at P_5, where $h \in [0.1, 1.4]$.

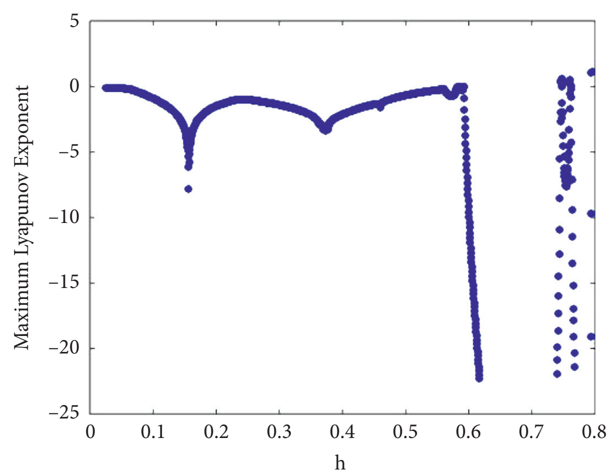
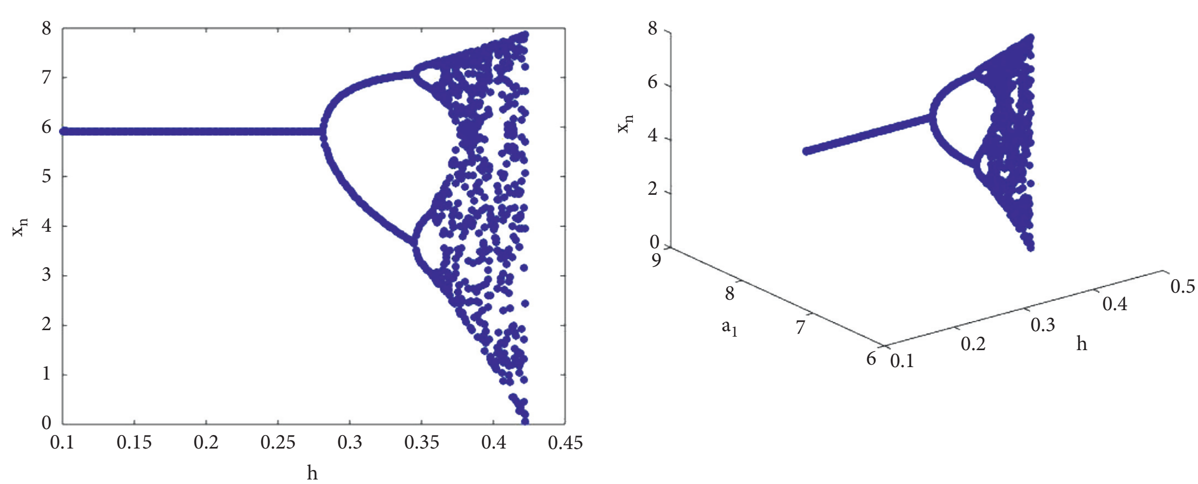


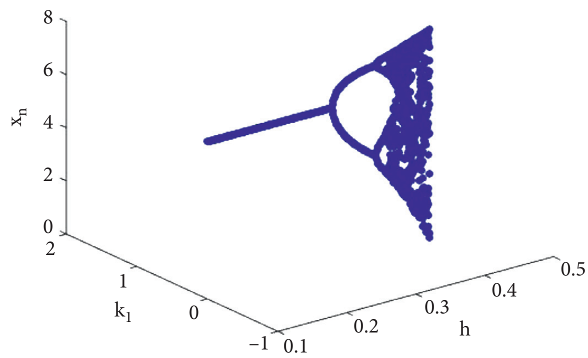
Figure 10: Maximum Lyapunov exponents corresponding to Figure 9.



(a)

(b)

Figure 11: Continued.



(c)

FIGURE 11: Flip bifurcation diagrams at P_6, where $h \in [0.1, 1.4]$.

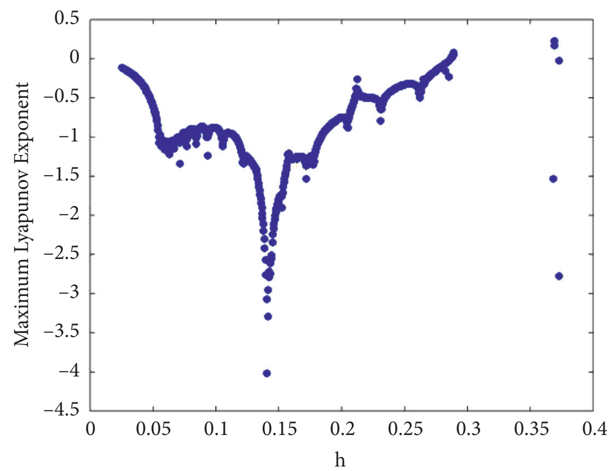
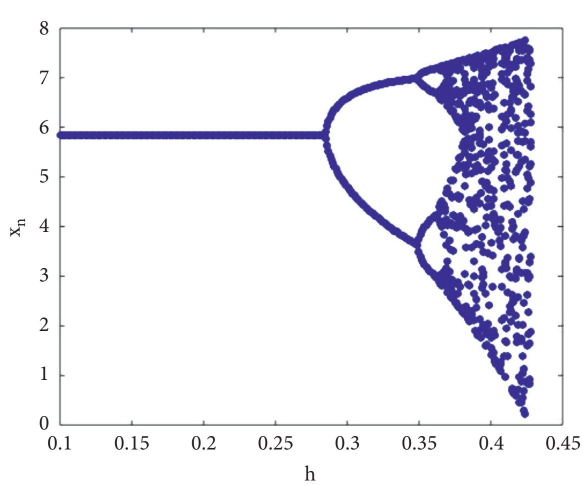
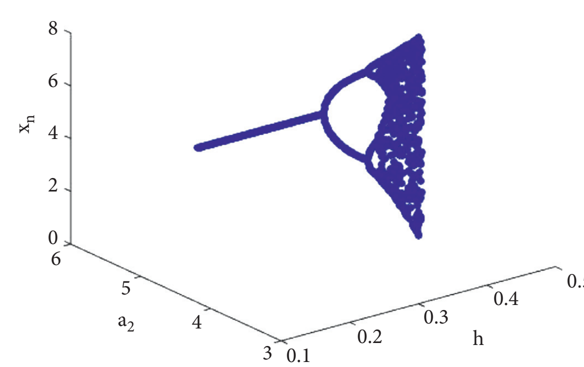


FIGURE 12: Maximum Lyapunov exponents corresponding to Figure 11.

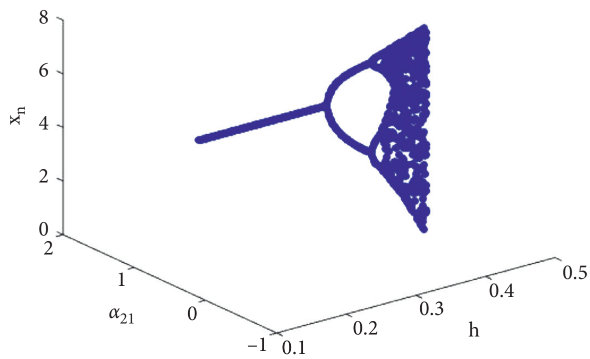


(a)



(b)

FIGURE 13: Continued.



(c)

Figure 13: Flip bifurcation diagrams at P_7, where $h \in [0.1, 1.0]$.

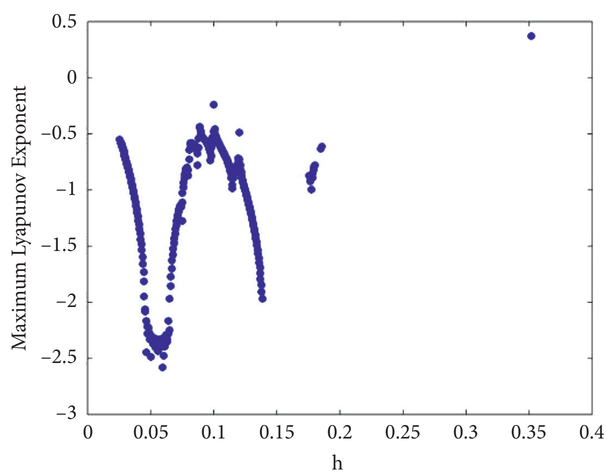
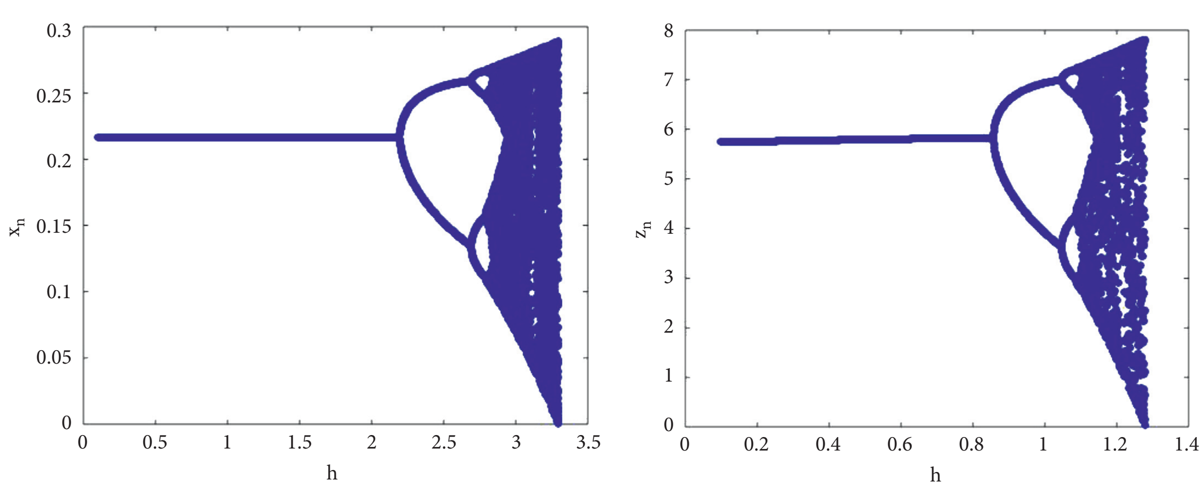


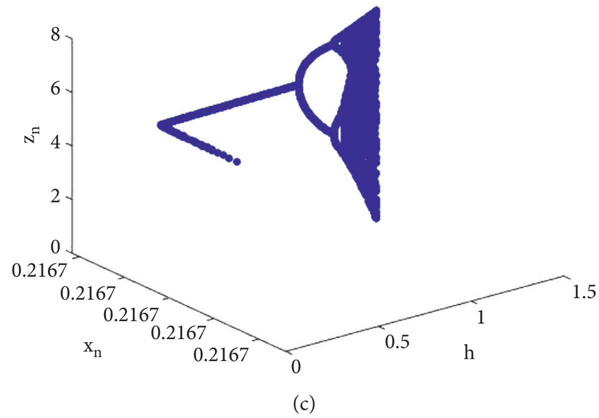
Figure 14: Maximum Lyapunov exponents corresponding to Figure 13.



(a)

(b)

Figure 15: Continued.



(c)

Figure 15: Flip bifurcation diagrams at P_8, where $h \in [0.1, 4.2]$.

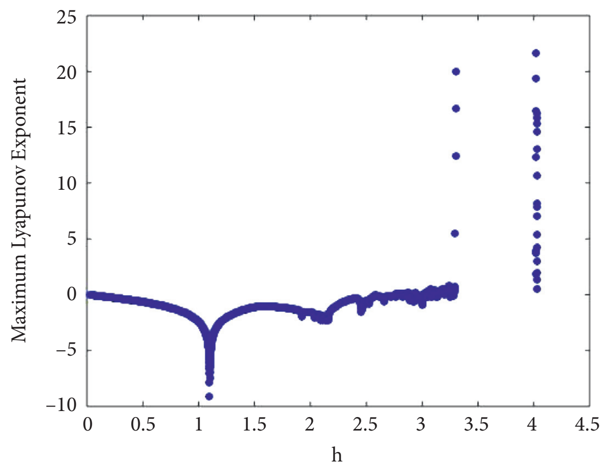


Figure 16: Maximum Lyapunov exponents corresponding to Figure 15.

Table 1: Dynamical classifications around fixed points of model (15).

Fixed points	Corresponding behavior
P_1	Not sink; source if $0 < h < \min\{2/a_1(k_1-1), 2/a_2(k_2-1)\}$; saddle if $h > \max\{2/a_1(k_1-1), 2/a_2(k_2-1)\}$; nonhyperbolic if $h = 2/a_1(k_1-1)$ or $h = 2/a_2(k_2-1)$
P_2	Sink if $h > \max\{2/a_1(k_1-1), 2/a_2(k_2-1)\}$ and $0 < h < (2/a_3)$ Source if $0 < h < \min\{2/a_1(k_1-1), 2/a_2(k_2-1)\}$ and $h > (2/a_3)$ Saddle if $0 < h < \min\{2/a_1(k_1-1), 2/a_2(k_2-1)\}$ and $0 < h < (2/a_3)$ Nonhyperbolic if $h = (2/a_3)$ or $h = (2/a_1(k_1-1))$ or $h = (2/a_2(k_2-1))$
P_3	Sink if $h > \max\{2/a_1(k_1-1), 2\alpha_{22}/(a_2\alpha_{32}(1-k_2) - a_3\alpha_{22})\}$ and $0 < h < 2/a_2(1-k_2)$ Source if $0 < h < \min\{2/a_1(k_1-1), 2\alpha_{22}/(a_2\alpha_{32}(1-k_2) - a_3\alpha_{22})\}$ and $h > 2/a_2(1-k_2)$ Saddle if $h > \max\{2/a_1(k_1-1), 2/a_2(1-k_2)\}$ and $0 < h < (2\alpha_{22}/(a_2\alpha_{32}(1-k_2) - a_3\alpha_{22}))$ Nonhyperbolic if $h = 2/a_2(1-k_2)$ or $h = 2/a_1(k_1-1)$ or $h = 2\alpha_{22}/(a_2\alpha_{32}(1-k_2) - a_3\alpha_{22})$
P_4	Not sink; source if $h > \max\{2/a_1(1-k_1), 2/a_2(k_2-1)\}$ Saddle if $0 < h < \min\{2/a_1(1-k_1), 2/a_2(k_2-1)\}$ Nonhyperbolic if $h = (2/a_1(1-k_1))$ or $h = (2/a_2(k_2-1))$
P_5	Sink if $h > \max\{2/a_1(k_1-1), 2\alpha_{22}/(a_3\alpha_{22} - a_2\alpha_{32}(1-k_2))\}$ and $0 < h < 2/a_2(1-k_2)$ Source if $0 < h < \min\{2/a_1(k_1-1), 2\alpha_{22}/(a_3\alpha_{22} - a_2\alpha_{32}(1-k_2))\}$ and $h > 2/a_2(1-k_2)$ Saddle if $h > \max\{2/a_1(k_1-1), 2\alpha_{22}/(a_3\alpha_{22} - a_2\alpha_{32}(1-k_2))\}$ and $h > 2/a_2(1-k_2)$ Nonhyperbolic if $h = 2/a_1(k_1-1)$ or $h = (a_3\alpha_{22} - 2\alpha_{22})/a_2\alpha_{32}(1-k_2)$ or $h = 2/a_2(1-k_2)$

Table 1: Continued.

Fixed points	Corresponding behavior
P_6	Sink if $0<h<\min\{2/a_1(1-k_1), 2/a_3\}$ and $h>2\alpha_{11}/(a_1\alpha_{21}(1-k_1)-a_2\alpha_{11}(1-k_2))$; source if $h>\max\{2/a_1(1-k_1), 2/a_3\}$ and $0<h<2\alpha_{11}/(a_1\alpha_{21}(1-k_1)-a_2\alpha_{11}(1-k_2))$; saddle if $0<h<\min\{2/a_1(1-k_1), 2/a_3\}$ and $0<h<2\alpha_{11}/(a_1\alpha_{21}(1-k_1)-a_2\alpha_{11}(1-k_2))$; nonhyperbolic if $h=2/a_1(1-k_1)$ or $h=(2/a_3)$ or $h=2\alpha_{11}/(a_1\alpha_{21}(1-k_1)-a_2\alpha_{11}(1-k_2))$.
P_7	Sink if $0<h<\min\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1))\}$ and $h>2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$ Source if $h>\max\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1))\}$ and $0<h<2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$ Saddle if $0<h<\min\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1))\}$ and $h<2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$ Nonhyperbolic if $h=2/a_1(1-k_1)$ or $h=2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1))$ or $h=2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$
P_8	Sink if $0<h<\min\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1)), 2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))\}$ Source if $h>\max\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1)), 2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))\}$ Saddle if $0<h<\min\{2/a_1(1-k_1), 2\alpha_{11}/(a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1))\}$ and $h>2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$ Nonhyperbolic if $h=2/a_1(1-k_1)$ or $h=\frac{2\alpha_{11}}{a_2\alpha_{11}(1-k_2)-2a_1\alpha_{21}(1-k_1)}$ or $h=2\alpha_{11}\alpha_{22}/(a_3\alpha_{11}\alpha_{22}-a_2\alpha_{11}\alpha_{32}(1-k_2)+a_1\alpha_{21}\alpha_{32}(1-k_1))$

Case VI: if $a_1 = 7.1, a_2 = 4.2, a_3 = 2.7, k_1 = 0.0009, k_2 = 1.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 1.9, \alpha_{32} = 0.9, \alpha_{33} = 0.4$ then from (61) one gets: $h = 0.2819438903463822$. Hence P_6 is stable if $h < 0.2819438903463822$, and exchange stability if $h = 0.2819438903463822$ and infact flip bifurcation takes place by Theorem 14. Therefore the flip bifurcation diagrams with initial value $(0.1, 0, 0.1)$ are presented in Figure 11. Finally maximum lypunov exponents corresponding to Figure 11 are drawn in Figure 12.

Case VII: If $a_1 = 9.1, a_2 = 4.2, a_3 = 2.7, k_1 = 0.23, k_2 = 4.6, \alpha_{11} = 1.2, \alpha_{22} = 1.4, \alpha_{21} = 0.099, \alpha_{32} = 5.9$, and $\alpha_{33} = 2.4$, then, from (68), one gets $h = 0.285428 8568574283$. Hence, P_7 is stable if $h < 0.2854 288568574283$, and exchange stability is $h = 0.2854 288568574283$, and in fact, flip bifurcation takes place by Theorem 15. Therefore, the flip bifurcation diagrams with initial value $(0.2, 0.1, 0)$ are presented in Figure 13. Finally, maximum Lyapunov exponents corresponding to Figure 13 are drawn in Figure 14.

Case VIII: if $a_1 = 9.1, a_2 = 1.2, a_3 = 2.7, k_1 = 0.9, k_2 = 0.006, \alpha_{11} = 4.2, \alpha_{22} = 1.4, \alpha_{21} = 4.9, \alpha_{32} = 1.9$, and $\alpha_{33} = 0.4$, then, from (75), one gets $h = 2.19780 21978021984$. Hence, $P_7 = (0.21666666666666, 0.09366666666666, 6.305083333333)$ is stable if $h < 2.1978021978021984$, and exchange stability is $h = 2.1978021978021984$, and in fact, flip bifurcation takes place by Theorem 16. Therefore, flip bifurcation diagrams with $(0.2, 0.2, 0.4)$ are presented in Figure 15 which indicates that period-2 points bifurcate from P_8 are stable, since $\Omega_2 = (h\alpha_{11}d_{11} + 2\alpha_{11}d_{11})^2 = 310.844023668 > 0$. Finally, maximum Lyapunov exponents corresponding to Figure 15 are drawn in Figure 16.

8. Conclusion

The work is about the existence of fixed points, topological classifications around fixed points, periodic points, and bifurcations of a three-species discrete food chain model with harvesting in the region: $\mathbb{R}^3_+ = \{(x, y, z)\colon x, y, z \geq 0\}$. We proved that, for all parametric values $h, a_1, a_2, a_3, k_1, k_2, \alpha_{11}, \alpha_{22}, \alpha_{21}, \alpha_{32}$, and α_{33}, model (15) has trivial fixed point: $P_1 = (0, 0, 0)$; boundary fixed points: $P_2 = (0, 0, a_3/\alpha_{33})$ $\forall a_3, \alpha_{33} > 0$; $P_3 = (0, (1-k_2)a_2/\alpha_{22}, 0)$if $k_2 < 1$; $P_4 = ((1-k_1)a_1/\alpha_{11}, 0, 0)$if $k_1 < 1$; $P_5 = (0, (1-k_2)a_2/\alpha_{22}, ((a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2)/\alpha_{22}\alpha_{33}))$if $a_3 > (\alpha_{32}(1-k_2)a_2/\alpha_{22})$ with $k_2 < 1$; $P_6 = ((1-k_1)a_1/\alpha_{11}, 0, a_3/\alpha_{33})$if $k_1 < 1$; $P_7 = ((1-k_1)a_1/\alpha_{11}, (a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}, 0)$if $a_2 > (a_1(1-k_1)\alpha_{21}/(1-k_2)\alpha_{11})$ with $k_1, k_2 < 1$. We also proved that if $k_1 < 1$, $a_2 > (a_1(1-k_1)\alpha_{21}/(1-k_2)\alpha_{11})$ and $a_3 > (a_2(1-k_2)\alpha_{11}\alpha_{32} - a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}$; then, $P_8 = (((1-k_1)a_1/\alpha_{11}), (a_2(1-k_2)\alpha_{11} - a_1(1-k_1)\alpha_{21})/\alpha_{11}\alpha_{22}, (a_3\alpha_{11}\alpha_{22} - a_2(1-k_2)\alpha_{11}\alpha_{32} + a_1(1-k_1)\alpha_{21}\alpha_{32})/\alpha_{11}\alpha_{22}\alpha_{33})$ is an interior equilibrium point of (15). Furthermore, we studied the local stability with different topological classifications around each fixed points whose main findings are presented in Table 1. Next, for under consideration model (15), we also studied existence of periodic points by existing theory. Furthermore, we explored the existence of possible bifurcations about each fixed points in order to understand dynamics of model (15) deeply. It is proved that (i) around P_1 model undergoes no flip bifurcation if $\Omega \in \mathscr{F}|_{P_1} = \{\Omega\colon h = 2/a_1(k_1 - 1)\}$, (ii) around P_2 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_2} = \{\Omega\colon h = 2/a_3\}$, (iii) around P_3 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_3} = \{\Omega\colon h = 2/a_2(1-k_2)\}$, (iv) around P_4 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_4} = \{\Omega\colon h = 2/a_1(1-k_1)\}$, (v) around P_5 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_5} = \{\Omega\colon h = 2/a_1(k_1 - 1)\}$, (vi) around P_6 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_6} = \{\Omega\colon h = 2/a_1(1-k_1)\}$, (vii) around P_7 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_7} = \{\Omega\colon h = 2/a_1(1-k_1)\}$, and (viii) around P_8 model undergoes flip bifurcation if $\Omega \in \mathscr{F}|_{P_8} = \{\Omega\colon h = 2/a_1(1-k_1)\}$. Finally, obtained results are verified numerically. This research can provide a framework for theoretical basis and help for the research in different aspects of biology specifically in the field of ecology.

Data Availability

All the data used in this study are included within the article and the sources from where they were adopted are cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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