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Research Article

On a Unified Mittag-Leffler Function and Associated Fractional Integral Operator

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The aim of this paper is to unify the extended Mittag-Leffler function and generalized Q function and define a unified Mittag-Leffler function. Both the extended Mittag-Leffler function and generalized Q function can be obtained from the unified Mittag-Leffler function. The Laplace, Euler beta, and Whittaker transformations are applied for this function, and generalized formulas are obtained. These formulas reproduce integral transformations of various deduced Mittag-Leffler functions and Q function. Also, the convergence of this unified Mittag-Leffler function is proved, and an associated fractional integral operator is constructed.

1. Introduction

The exponential function naturally exists in the solution of differential equations and plays a very vital role in solving real-world problems modeled in the form of differential mathematical systems. At the same time, the Mittag-Leffler function provides assistance in the formulation of solutions of complicated fractional dynamical systems. The aim of this paper is to unify two types of functions, namely, an extended generalized Mittag-Leffler function given in (7) and the Q function given in (8). We study Laplace, Euler beta, and Whittaker transformations of extended generalized Mittag-Leffler function given in (7) and the Q function given in (8) in the compact formulas. Also we will define a compact form of fractional integral operator.

First, we give some basic definitions and notations which will be helpful to understand later definitions. These include the Laplace transform, Euler beta transform, Whittaker transform, gamma function (Γ), beta function (B), p-beta function (B_p), Mittag-Leffler function ($E_{\alpha,\beta}$), extended Mittag-Leffler function ($E^{\lambda,r,k,\theta}_{\alpha,\beta,\gamma}$), the fractional integral operator associated with extended Mittag-Leffler function and generalized Q function ($Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}$).

Definition 1 (see [1]). Laplace transform of an integrable function f on $[0,\infty)$ is defined as follows:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\mathrm{d}t, \tag{1}$$

where $s \in \mathbb{C}$ is the variable of the transform.

Definition 2 (see [2]). The Euler beta transform of a function f is defined by the following definite integral:

$$B[f(t); a, b] = \int_0^1 t^{a-1}(1-t)^{b-1} f(t)\mathrm{d}t, \tag{2}$$

where a and b are any complex number with $\Re(a) > 0$ and $\Re(b) > 0$.

Definition 3 (see [2]). The Whittaker transform is defined by the following improper integral:

$$\int_0^\infty e^{-(t/2)} t^{\nu-1} \omega_{\lambda,\mu}(t) \mathrm{d}t = \frac{\Gamma((1/2)+\mu+\nu)\Gamma((1/2)-\mu+\nu)}{\Gamma(1-\lambda+\nu)}, \tag{3}$$

where $\Re(\mu \pm \nu) > -(1/2)$ and $\omega_{\lambda,\mu}$ is the Whittaker confluent hypergeometric function.

Definition 4 (see [3]). The gamma function is defined by the following improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t, \tag{4}$$

where $\Re(z) > 0$.

Definition 5 (see [2]). The beta function is defined by a definite integral and is given by

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1} \mathrm{d}t, \tag{5}$$

where $\Re(m)$, $\Re(n) > 0$.

Definition 6 (see [4]). The Mittag-Leffler function with two parameters is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{l=0}^\infty \frac{z^l}{\Gamma(\alpha l+\beta)}, \tag{6}$$

where $\Re(\alpha) > 0$.

Definition 7 (see [5]). An extended and generalized Mittag-Leffler function ($E_{\alpha,\beta,\gamma}^{\delta,\mu,k,\nu}$) is defined by the following series:

$$E_{\alpha,\beta,\gamma}^{\lambda,r,k,\theta}(z;p) = \sum_{l=0}^\infty \frac{B_p(\lambda+lk,\theta-\lambda)(\theta)_{lk} z^l}{B(\lambda,\theta-\lambda)(\gamma)_{lr}\Gamma(\alpha l+\beta)}, \tag{7}$$

where $z,\alpha,\beta,\gamma,\theta,\lambda \in \mathbb{C}, \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\theta), \Re(\lambda) > 0$ with $p \geq 0$, $r > 0$, $0 < k \leq r + \Re(\alpha)$, and $(\theta)_{lk} = (\Gamma(\theta+lk)/\Gamma(\theta))$.

Definition 8 (see [2]). A generalized Q function ($Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}$) is defined by the following series:

$$Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(z;\underline{a},\underline{b}) = \sum_{l=0}^\infty \frac{\prod_{i=1}^n B(b_i,l)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^n B(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}, \tag{8}$$

where $\underline{a} = (a_1,a_2,\ldots,a_n), \underline{b} = (b_1,b_2,\ldots, b_n), \alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho,\theta,a_i,b_i \in \mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\theta),\Re(\lambda),\Re(\delta),\Re(\rho)\} > 0, k \in (0,1)\cup\mathbb{N}$.

The Mittag-Leffler function takes place naturally similar to that of the exponential function in the solutions of fractional integro-differential equations having the arbitrary order. The Mittag-Leffler functions have to gain more recognition due to their wide applications in diverse fields [5–9]. They are used to define new fractional integral operators, and the fractional integral operators are used to generalize mathematical inequalities, see [5, 8, 10–14].

Our motivation is to introduce the unified Mittag-Leffler function. In this paper, we unify the extended Mittag-Leffler function (7) and generalized Q function (8) in a single function named unified Mittag-Leffler function. We studied the Laplace, Euler beta, and Whittaker transformation of the unified Mittag-Leffler function and obtained the compact formulas which reproduce integral transformations of Mittag-Leffler function and generalized Q function. Furthermore, the convergence of unified Mittag-Leffler function is proved, associated fractional integral operator is defined, and its boundedness is provided.

In the next section, we give the definition of unified Mittag-Leffler function and deduce extended generalized Mittag-Leffler function and generalized Q functions.

2. Unified Mittag-Leffler Function

We define a Mittag-Leffler function (will be called the unified Mittag-Leffler function) which unifies the functions given in (7) and (8) as follows:

$$M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(z;\underline{a},\underline{b},\underline{c},p) = \sum_{l=0}^\infty \frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{z^l}{\Gamma(\alpha l+\beta)}, \tag{9}$$

where $\underline{a} = (a_1,a_2,\ldots,a_n)$, $\underline{b} = (b_1,b_2,\ldots,b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, $a_i,b_i,c_i \in \mathbb{C}; i=1,\ldots,n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0, \forall i$. Also let $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho, \theta, z \in \mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\theta)\} > 0$, and $k \in (0,1)\cup\mathbb{N}$ with $k + \Re(\rho) < \Re(\delta+\nu+\alpha)$, $\operatorname{Im}(\rho) = \operatorname{Im}(\delta+\nu+\alpha)$.

For $n=1$, (9) will obtain the following form:

$$M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,1}(z;\underline{a},\underline{b},\underline{c},p) = \sum_{l=0}^\infty \frac{B_p(b_1,a_1)(\lambda)_{\rho l}(\theta)_{kl}}{B(c_1,a_1)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{z^l}{\Gamma(\alpha l+\beta)}. \tag{10}$$

By setting $b_1 = c_1 + lk, a_1 = \theta - \lambda, c_1 = \lambda, \rho = \nu = 0, \delta > 0$ in (10), we will get (7). Also, by substituting $a_i = l$, $p = 0$ and $\Re(\rho) > 0$ in (9), we will obtain (8). Hence the newly defined Mittag-Leffler function provides different kinds of related functions by setting the specific values of the parameters. The functions defined in [2, 4–9, 15] are particular cases of this newly defined function.

2.1. Integral Transforms of Unified Mittag-Leffler Function. Now we give the integral transforms of the unified Mittag-Leffler function. These transformations include the Laplace transform, Euler beta transform, and Whittaker transform.

2.1.1. Laplace Transform. First we give the Laplace transform of the unified Mittag-Leffler function.

Theorem 1. *For* $\underline{a} = (a_1,a_2,\ldots,a_n), \underline{b} = (b_1,b_2, \ldots,b_n)$, $\underline{c} = (c_1,c_2,\ldots,c_n), a_i,b_i,c_i \in \mathbb{C}; i=1,\ldots,n$ *such that* $\Re(a_i),\Re(b_i),\Re(c_i) > 0, \forall i.$ *Also let* $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho, \theta,t \in \mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\theta)\} > 0$, *and* $k \in (0,1)\cup\mathbb{N}$ *with* $k+\Re(\rho) < \Re(\delta+\nu+\alpha)$, $\operatorname{Im}(\rho) = \operatorname{Im}(\delta+\nu+\alpha)$, *and the Laplace transform of unified Mittag-Leffler function is given as follows:*

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=s^{-1}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,1,n}(s^{-1};\underline{a},\underline{b},\underline{c},p). \tag{11}$$

Proof. By the definition of the Laplace transform of a function, we have

$$\begin{aligned}L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]&=\int_0^\infty e^{-st}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{t^l}{\Gamma(\alpha l+\beta)}\mathrm{d}t\\&=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{1}{\Gamma(\alpha l+\beta)}\frac{l!}{s^{l+1}}\\&=\frac{1}{s}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{1}{\Gamma(\alpha l+\beta)}s^{-l}(1)_l.\end{aligned} \tag{12}$$

Hence the Laplace transform of the unified Mittag-Leffler function can be given as follows:

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=s^{-1}M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,1,n}(s^{-1};\underline{a},\underline{b},\underline{c},p). \tag{13}$$

□

Corollary 1. *For $a_i=l$, $p=0$, and $\Re(\rho)>0$, the Laplace transform of the unified Mittag-Leffler function will become*

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=L\left[Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{b},\underline{c})\right]. \tag{14}$$

Proof. From Theorem 1, we have

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=s^{-1}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{(1)_l s^{-l}}{\Gamma(\alpha l+\beta)}. \tag{15}$$

For $p=0$, $B_p(x,y)=B(x,y)$.

Therefore the above expression becomes

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=s^{-1}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{(1)_l s^{-l}}{\Gamma(\alpha l+\beta)}. \tag{16}$$

Putting $a_i=l$ and $\Re(\rho)>0$, we get the following:

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=s^{-1}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^n B(b_i,l)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{l!s^{-l}}{\Gamma(\alpha l+\beta)}. \tag{17}$$

Hence we get

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=L\left[Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{b},\underline{c})\right]. \tag{18}$$

Similarly for $n=1$, $b1=c1+lk$, $a1=\theta-\lambda$, $c1=\lambda$, $\rho=\nu=0$ and $\delta>0$ one can have

$$L\left[M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\right]=L\left[E_{\alpha,\beta,\gamma}^{\lambda,\delta,k,\theta}(t;p)\right]. \tag{19}$$

□

2.1.2. Euler Beta Transform. The Euler beta transformation of the unified Mittag-Leffler function is given in the next theorem.

Theorem 2. *For $\underline{a}=(a_1,a_2,\ldots,a_n)$, $\underline{b}=(b_1,b_2,\ldots,b_n)$, $\underline{c}=(c_1,c_2,\ldots,c_n)$, $a_i,b_i,c_i\in\mathbb{C}$; $i=1,\ldots,n$ such that $\Re(a_i),\Re(b_i),\Re(c_i)>0,\forall i$. Also let $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho,\theta,t\in\mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\theta)\}>0$, and $k\in(0,1)\cup\mathbb{N}$ with $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$, $\mathrm{Im}(\rho)=\mathrm{Im}(\delta+\nu+\alpha)$, and the Euler beta transform of the unified Mittag-Leffler function is given as follows:*

$$B\left[M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p);m,n\right]=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{B(l+m,n)}{\Gamma(\alpha l+\beta)}. \tag{20}$$

Proof. By the definition of beta transform of an integrable function, we have the following:

$$\begin{aligned}B\left[M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p);m,n\right]&=\int_0^1 t^{m-1}(1-t)^{n-1}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{t^l}{\Gamma(\alpha l+\beta)}\mathrm{d}t\\&=\sum_{l=0}^{\infty}\frac{\Pi_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\Pi_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{1}{\Gamma(\alpha l+\beta)}\int_0^1 t^{(l+m)-1}(1-t)^{n-1}\mathrm{d}t\\&=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{B(l+m,n)}{\Gamma(\alpha l+\beta)}.\end{aligned} \tag{21}$$

□

Corollary 2. *For $a_i=l$, $p=0$, and $\Re(\rho)>0$, the Euler beta transform of unified Mittag-Leffler function will become*

$$B\left[M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)\right]=B\left[Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{b},\underline{c})\right]. \tag{22}$$

Similarly for $n=1$, $b1=c1+lk$, $a1=\theta-\lambda$, $c1=\lambda$, $\rho=\nu=0$ and $\delta>0$ one can have

$$B\left[M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)\right]=B\left[E^{\lambda,\delta,k,\theta}_{\alpha,\beta,\gamma}(t;p)\right]. \tag{23}$$

2.1.3. Whittaker Transform. The Whittaker transformation of the unified Mittag-Leffler function is given in the next theorem.

Theorem 3. *For $\underline{a}=(a_1,a_2,\dots,a_n)$, $\underline{b}=(b_1,b_2,\dots,b_n)$, $\underline{c}=(c_1,c_2,\dots,c_n)$, $a_i,b_i,c_i\in\mathbb{C}$; $i=1,\dots,n$ such that $\Re(a_i),\Re(b_i),\Re(c_i)>0,\forall i$. Also let $\alpha,\beta,\gamma,\delta,\mu,\nu,\zeta,\rho,\theta,t\in\mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\zeta),\Re(\theta)\}>0$, and $k\in(0,1)\cup\mathbb{N}$ with $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$, $\mathrm{Im}(\rho)=\mathrm{Im}(\delta+\nu+\alpha)$, and we have*

$$\begin{aligned}&\int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)M^{\zeta,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega t^{\eta};\underline{a},\underline{b},\underline{c},p)\mathrm{d}t\\&\quad=\frac{\phi^{-\xi}(f-1)!}{(g-1)!}M^{\zeta,\rho,\theta,k,f,\eta,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu,g,\eta,1}(\omega\phi^{-\eta};\underline{a},\underline{b},\underline{c},p),\end{aligned} \tag{24}$$

where $f=(1/2)\pm\psi+\xi$ and $g=1-\lambda+\xi$.

Proof. Consider the following improper integral:

$$\int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)M^{\zeta,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega t^{\eta};\underline{a},\underline{b},\underline{c},p)\mathrm{d}t. \tag{25}$$

By substituting $\phi t=q$ in the above integral, we obtain the following:

$$\begin{aligned}&\int_0^{\infty}e^{-(q/2)}\left(\frac{q}{\phi}\right)^{\xi-1}\omega_{\lambda,\psi}(q)M^{\zeta,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega\left(\frac{q}{\phi}\right)^{\eta};\underline{a},\underline{b},\underline{c},p\right)\frac{\mathrm{d}q}{\phi}\\&\quad=\phi^{-\xi}\int_0^{\infty}e^{-(q/2)}q^{\xi-1}\omega_{\lambda,\psi}(q)\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(q/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\mathrm{d}q\\&\quad=\phi^{-\xi}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(1/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\int_0^{\infty}e^{-(q/2)}q^{(\xi+\eta l)-1}\omega_{\lambda,\psi}(q)\mathrm{d}q.\end{aligned} \tag{26}$$

By using the definition of Whittaker transformation, we get

$$\begin{aligned}
&= \phi^{-\xi}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(1/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\frac{\Gamma((1/2)+\psi+\xi+\eta l)\Gamma((1/2)-\psi+\xi+\eta l)}{\Gamma(1-\lambda+\xi+\eta l)}\\
&= \phi^{-\xi}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(1/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\frac{\Gamma((1/2)\pm\psi+\xi+\eta l)}{\Gamma(1-\lambda+\xi+\eta l)\Gamma(1+l)}\\
&= \phi^{-\xi}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(1/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\frac{\Gamma(g)\Gamma(f+\eta l)\Gamma(f)}{\Gamma(g)\Gamma(g+\eta l)\Gamma(1+l)\Gamma(f)}\\
&= \phi^{-\xi}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\zeta)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l(1/\phi)^{\eta l}}{\Gamma(\alpha l+\beta)}\frac{(f-1)!(f)_{\eta l}}{(g-1)!(g)_{\eta l}(1)_l}\\
&= \phi^{-\xi}\frac{(f-1)!}{(g-1)!}M_{\alpha,\beta,\gamma,\delta,\mu,\nu,g,\eta,1}^{\zeta,\rho,\theta,k,f,\eta,n}(\omega\phi^{-\eta};\underline{a},\underline{b},\underline{c},p),
\end{aligned} \tag{27}$$

where $f: = (1/2)\pm\psi+\xi$ and $g: = 1-\lambda+\xi$, and the required result is obtained. □

Corollary 3. *For $a_i = l$, $p = 0$ and $\Re(\rho) > 0$, we have*

$$\begin{aligned}
&\int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\zeta,\rho,\theta,k,n}(\omega t^{\eta};\underline{a},\underline{b},\underline{c},p)dt\\
&\quad = \int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\zeta,\rho,\theta,k,n}(\omega t^{\eta};\underline{b},\underline{c})dt.
\end{aligned} \tag{28}$$

Similarly for $n=1$, $b1=c1+lk$, $a1=\theta-\lambda$, $c1=\lambda$, $\rho=\nu=0$ and $\delta>0$ one can have

$$\begin{aligned}
&\int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\zeta,\rho,\theta,k,n}(\omega t^{\eta};\underline{a},\underline{b},\underline{c},p)dt\\
&\qquad \int_0^{\infty}e^{-(\phi t/2)}t^{\xi-1}\omega_{\lambda,\psi}(\phi t)E_{\alpha,\beta,\gamma}^{\zeta,\delta,k,n}(\omega t^{\eta}p)dt.
\end{aligned} \tag{29}$$

3. Convergence of Unified Mittag-Leffler Function

Before stating the theorem for the convergence of the unified Mittag-Leffler function, we give an important formula that will be used in the proof of our theorem.

Definition 9. The asymptotic formula for the gamma function is given in [5] the following:

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} = z^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2z}+O\left(\frac{1}{z^2}\right)\right], \quad |z|\longrightarrow\infty, |\arg z|<\pi. \tag{30}$$

Theorem 4. *The unified Mittag-Leffler function ($M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}$) converges absolutely for all values of $t\in\mathbb{C}$ if $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$ with $\operatorname{Im}(\rho)=\operatorname{Im}(\delta+\nu+\alpha)$.*

Proof. By the definition of the unified Mittag-Leffler function, we have the following series:

$$\begin{aligned}
&M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(t;\underline{a},\underline{b},\underline{c},p)\\
&\quad = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{t^l}{\Gamma(\alpha l+\beta)}\\
&\quad = \sum_{l=0}^{\infty}a_l t^l,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
a_l &= \frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{1}{\Gamma(\alpha l+\beta)}\\
\left|\frac{a_l}{a_{l+1}}\right| &= \left|\frac{(\lambda)_{\rho l}}{(\lambda)_{\rho l+\rho}}\cdot\frac{(\theta)_{kl}}{(\theta)_{kl+k}}\cdot\frac{(\gamma)_{\delta l+\delta}}{(\gamma)_{\delta l}}\cdot\frac{(\mu)_{\nu l+\nu}}{(\mu)_{\nu l}}\cdot\frac{\Gamma(\alpha l+\alpha+\beta)}{\Gamma(\alpha l+\beta)}\right|.
\end{aligned} \tag{32}$$

Applying limit on both sides, we get the following:

$$\lim_{l\longrightarrow\infty}\left|\frac{a_l}{a_{l+1}}\right| = \lim_{l\longrightarrow\infty}\left|\frac{(\lambda)_{\rho l}}{(\lambda)_{\rho l+\rho}}\cdot\frac{(\theta)_{kl}}{(\theta)_{kl+k}}\cdot\frac{(\gamma)_{\delta l+\delta}}{(\gamma)_{\delta l}}\cdot\frac{(\mu)_{\nu l+\nu}}{(\mu)_{\nu l}}\cdot\frac{\Gamma(\alpha l+\alpha+\beta)}{\Gamma(\alpha l+\beta)}\right|. \tag{33}$$

Using (30), the fractions involving Pochhammer symbols and gamma function in (33) become the following:

$$\frac{(\lambda)_{\rho l}}{(\lambda)_{\rho l+\rho}} = (\rho l)^{-\rho}\left[1-\frac{2\lambda+\rho-1}{2l}+O\left(\frac{1}{(\rho l)^2}\right)\right], \quad (34)$$

$$\frac{(\theta)_{kl}}{(\theta)_{kl+k}} = (kl)^{-k}\left[1-\frac{2\theta+k-1}{2l}+O\left(\frac{1}{(kl)^2}\right)\right], \quad (35)$$

$$\frac{(\gamma)_{\delta l+\delta}}{(\gamma)_{\delta l}} = (\delta l)^{\delta}\left[1+\frac{2\gamma+\delta-1}{2l}+O\left(\frac{1}{(\delta l)^2}\right)\right], \quad (36)$$

$$\frac{(\mu)_{\nu l+\nu}}{(\mu)_{\nu l}} = (\nu l)^{\nu}\left[1+\frac{2\mu+\nu-1}{2l}+O\left(\frac{1}{(\nu l)^2}\right)\right], \quad (37)$$

$$\frac{\Gamma(\alpha l+\alpha+\beta)}{\Gamma(\alpha l+\beta)} = (\alpha l)^{\alpha}\left[1+\frac{2\beta+\alpha-1}{2l}+O\left(\frac{1}{(\alpha l)^2}\right)\right]. \quad (38)$$

Using (34)–(38) in (33), we get the following:

$$\begin{aligned}
\lim_{l\longrightarrow\infty}\left|\frac{a_l}{a_{l+1}}\right| &\approx \lim_{l\longrightarrow\infty}|(\rho l)^{-\rho}\left[1-\frac{2\lambda+\rho-1}{2l}+O\left(\frac{1}{(\rho l)^2}\right)\right]\\
&\times (kl)^{-k}\left[1-\frac{2\theta+k-1}{2l}+O\left(\frac{1}{(kl)^2}\right)\right](\delta l)^{\delta}\left[1+\frac{2\gamma+\delta-1}{2l}+O\left(\frac{1}{(\delta l)^2}\right)\right]\\
&\times (\nu l)^{\nu}\left[1+\frac{2\mu+\nu-1}{2l}+O\left(\frac{1}{(\nu l)^2}\right)\right](\alpha l)^{\alpha}\left[1+\frac{2\beta+\alpha-1}{2l}+O\left(\frac{1}{(\alpha l)^2}\right)\right]|,\\
\lim_{l\longrightarrow\infty}\left|\frac{a_l}{a_{l+1}}\right| &\approx \lim_{l\longrightarrow\infty}\frac{\delta^{\delta}\nu^{\nu}\alpha^{\alpha}}{\rho^{\rho}k^{k}}\cdot l^{(\delta+\nu+\alpha)-(\rho+k)}.
\end{aligned} \quad (39)$$

The formula for the radius of convergence of a series is

$$\lim_{l\longrightarrow\infty}\left|\frac{a_l}{a_{l+1}}\right| = R. \quad (40)$$

Therefore, the function $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}$ converges absolutely for all values of t if $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$ with $\mathrm{Im}(\rho)=\mathrm{Im}(\delta+\nu+\alpha)$. □ □

Next, we give recurrence relations of unified Mittag-Leffler function.

Theorem 5. *Let* $\underline{a}=(a_1,a_2,\ldots,a_n)$, $\underline{b}=(b_1,b_2,\ldots,b_n)$, $\underline{c}=(c_1,c_2,\ldots,c_n)$, $a_i,b_i,c_i\in\mathbb{C}; i=1,\ldots,n$ *such that* $\Re(a_i),\Re(b_i),\Re(c_i)>0,\forall i$. *Also let* $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho,\theta,t\in\mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\theta)\}>0$, *and* $k\in(0,1)\cup\mathbb{N}$ *with* $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$, $\mathrm{Im}(\rho)=\mathrm{Im}(\delta+\nu+\alpha)$, *and then the difference of two consecutive unified Mittag-Leffler functions is given as follows:*

$$\begin{aligned}
&M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)-M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma-1,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)\\
&=\frac{t\delta}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}t}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p),
\end{aligned} \quad (41)$$

with $\Re(\gamma)>1$.

Proof. By the definition of the unified Mittag-Leffler function, we have

$$\begin{aligned}
&M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)-M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma-1,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p)\\
&=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\mu)_{\nu l}}\frac{t^l}{\Gamma(\alpha l+\beta)}\left[\frac{1}{(\gamma)_{\delta l}}-\frac{1}{(\gamma-1)_{\delta l}}\right]\\
&=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\mu)_{\nu l}}\frac{t^l}{\Gamma(\alpha l+\beta)}\frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta l)}\frac{\delta l}{1-\gamma}\\
&=\frac{\delta t}{1-\gamma}\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n}B(c_i,a_i)(\mu)_{\nu l}(\gamma)_{\delta l}}\frac{lt^{l-1}}{\Gamma(\alpha l+\beta)}\\
&=\frac{t\delta}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}t}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b},\underline{c},p).
\end{aligned} \quad (42)$$

□

Theorem 6. *For* $m \in \mathbb{Z}^{+}$, $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, *where* $a_i, b_i, c_i \in \mathbb{C}$; $i = 1, \ldots, n$ *such that* $\Re(a_i), \Re(b_i), \Re(c_i) > 0, \forall i$. *Also let* $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda), \Re(\theta)\} > 0$ *and* $k \in (0,1) \cup \mathbb{N}$ *with* $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$, $\mathrm{Im}(\rho) = \mathrm{Im}(\delta + \nu + \alpha)$, *and mth derivative of unified Mittag-Leffler function is given by*

$$\left(\frac{d}{dt}\right)^{m} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t; \underline{a}, \underline{b}, \underline{c}, p) = \frac{(\lambda)_{m\rho}(\theta)_{mk}}{(\gamma)_{m\delta}(\mu)_{m\nu}} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_p(b_i, a_i)(\lambda + \rho m)_{\rho l}(\theta + km)_{kl}}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma + \delta m)_{\delta l}(\mu + \nu m)_{\nu l}} \cdot \frac{(1+l)_m t^l}{\Gamma(\alpha(l+m)+\beta)}. \tag{43}$$

Proof. Differentiating the unified Mittag-Leffler function m times, we get

$$\begin{aligned}\left(\frac{d}{dt}\right)^{m} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t; \underline{a}, \underline{b}, \underline{c}, p) &= \sum_{l=m}^{\infty} \frac{\prod_{i=1}^{n} B_p(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{[l(l-1), \ldots, (l-(m-1))]t^{l-m}}{\Gamma(\alpha l + \beta)} \\ &= \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_p(b_i, a_i)(\lambda)_{\rho(l+m)}(\theta)_{k(l+m)}}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma)_{\delta(l+m)}(\mu)_{\nu(l+m)}} \frac{[(l+m)(l+m-1), \ldots, (l+1)]t^{l}}{\Gamma(\alpha l + \beta)}.\end{aligned} \tag{44}$$

We know that $(\theta)_{a+b} = (\theta + a)_b (\theta)_a$. Therefore, we obtain

$$\left(\frac{d}{dt}\right)^{m} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t; \underline{a}, \underline{b}, \underline{c}, p) = \frac{(\lambda)_{m\rho}(\theta)_{mk}}{(\gamma)_{m\delta}(\mu)_{m\nu}} \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_p(b_i, a_i)(\lambda + \rho m)_{\rho l}(\theta + km)_{kl}}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma + \delta m)_{\delta l}(\mu + \nu m)_{\nu l}} \cdot \frac{(1+l)_m t^l}{\Gamma(\alpha(l+m)+\beta)}. \tag{45}$$

□

Next, we give the definition of fractional integral operator with unified Mittag-Leffler (Mfunction) as the kernel.

Definition 10. Let $f \in L_1[a,b]$. Then $\forall\, \xi \in [a,b]$, the fractional integral operator with Mfunction as its kernel is defined as follows:

$$\begin{aligned} I^{\omega,\lambda,\rho,\theta,k,n}_{a^{+},\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi; \underline{a}, \underline{b}, \underline{c}, p) &= \int_{a}^{\xi} (\xi - t)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi - t)^{\alpha}; \underline{a}, \underline{b}, \underline{c}, p) f(t)dt, \\ I^{\omega,\lambda,\rho,\theta,k,n}_{b^{-},\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi; \underline{a}, \underline{b}, \underline{c}, p) &= \int_{\xi}^{b} (t - \xi)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(t - \xi)^{\alpha}; \underline{a}, \underline{b}, \underline{c}, p) f(t)dt, \end{aligned} \tag{46}$$

with $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, where $a_i, b_i, c_i, \omega \in \mathbb{C}$; $i = 1, \ldots, n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0, \forall i$. Also let $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda), \Re(\theta)\} > 0$, and $k \in (0,1) \cup \mathbb{N}$ with $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$, $\mathrm{Im}(\rho) = \mathrm{Im}(\delta + \nu + \alpha)$.

Remark 1. For $n=1$, $b_1=c_1+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, $\delta>0$, we obtain the fractional integral operator containing extended generalized Mittag-Leffler function in its kernel and is given by [5];

$$\begin{aligned}\varepsilon^{\omega,\lambda,\delta,k,\theta}_{a^+,\alpha,\beta,\gamma} f(\xi,p) &= \int_a^\xi (\xi-t)^{\beta-1} E^{\lambda,\delta,k,\theta}_{\alpha,\beta,\gamma}\left(\omega(\xi-t)^\alpha, p\right) f(t)\mathrm{d}t,\\ \varepsilon^{\omega,\lambda,\delta,k,\theta}_{b^-,\alpha,\beta,\gamma} f(\xi,p) &= \int_\xi^b (t-\xi)^{\beta-1} E^{\lambda,\delta,k,\theta}_{\alpha,\beta,\gamma}\left(\omega(t-\xi)^\alpha, p\right) f(t)\mathrm{d}t.\end{aligned} \tag{47}$$

Now we give the proof of boundedness of the fractional integral operator defined above.

Theorem 7. *Let $f\in L_1[a,b]$. If $\underline{a}=(a_1,a_2,\ldots,a_n)$, $\underline{b}=(b_1,b_2,\ldots,b_n)$, $\underline{c}=(c_1,c_2,\ldots,c_n)$, where $a_i,b_i,c_i,\omega\in\mathbb{C}; i=1,\ldots,n$ such that $\Re(a_i),\Re(b_i),\Re(c_i)>0,\forall i$. Also let $\alpha,\beta,\gamma,\delta,\mu,\nu,\lambda,\rho,\quad\theta,t\in\mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda),\Re(\theta)\}>0$, and $k\in(0,1)\cup\mathbb{N}$ with $k+\Re(\rho)<\Re(\delta+\nu+\alpha)$, $\mathrm{Im}(\rho)=\mathrm{Im}(\delta+\nu+\alpha)$, and then the fractional integral operator $I^{\omega,\lambda,\rho,\theta,k,n}_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}f$ is bounded on $L_1[a,b]$.*

Proof. Applying 1-norm to the fractional integral operator $I^{\omega,\lambda,\rho,\theta,k,n}_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}f$, we get the following:

$$\begin{aligned}\left\|I^{\omega,\lambda,\rho,\theta,k,n}_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}f\right\|_1 &= \int_a^b\left|\int_a^\xi (\xi-t)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\xi-t)^\alpha;\underline{a},\underline{b},\underline{c},p\right)f(t)\mathrm{d}t\right|\mathrm{d}\xi\\ &\le \int_a^b |f(t)|\left[\int_t^b (\xi-t)^{\Re(\beta)-1}\left|M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\xi-t)^\alpha;\underline{a},\underline{b},\underline{c},p\right)\right|\mathrm{d}\xi\right]\mathrm{d}t.\end{aligned} \tag{48}$$

By substituting $\xi-t=s$, we obtain the following inequality:

$$\begin{aligned}\left\|I^{\omega,\lambda,\rho,\theta,k,n}_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}f\right\|_1 &\le \int_a^b |f(t)|\left[\int_0^{b-t} s^{\Re(\beta)-1}\left|M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega s^\alpha;\underline{a},\underline{b},\underline{c},p\right)\right|\mathrm{d}s\right]\mathrm{d}t\\ &\le \int_a^b |f(t)|\left[\int_0^{b-a} s^{\Re(\beta)-1}\left|M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega s^\alpha;\underline{a},\underline{b},\underline{c},p\right)\right|\mathrm{d}s\right]\mathrm{d}t\\ &\le \left|\sum_{l=0}^\infty \frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l}{\Gamma(\alpha l+\beta)}\right|\\ &\quad\times\int_0^{b-a} s^{\Re(\alpha)l+\Re(\beta)-1}\mathrm{d}s\|f\|_1\\ &\le \left|\sum_{l=0}^\infty \frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l}{\Gamma(\alpha l+\beta)}\frac{(b-a)^{\Re(\alpha)l}}{\Re(\alpha)l+\Re(\beta)}\right|\times(b-a)^{\Re(\beta)}\|f\|_1,\\ \left\|I^{\omega,\lambda,\rho,\theta,k,n}_{a^+,\alpha,\beta,\gamma,\delta,\mu,\nu}f\right\|_1 &\le K\|f\|_1,\end{aligned} \tag{49}$$

where

$$K=\left|\frac{\prod_{i=1}^n B_p(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\prod_{i=1}^n B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}}\frac{\omega^l}{\Gamma(\alpha l+\beta)}\frac{(b-a)^{\Re(\alpha)l}}{\Re(\alpha)l+\Re(\beta)}\right|(b-a)^{\Re(\beta)}. \tag{50}$$

□

4. Conclusions

In this paper, we extended the Mittag-Leffler function and generalized Q function simultaneously. By applying the Laplace, Euler beta, and Whittaker transformations on the unified Mittag-Leffler function, compact formulas are established from which formulas for generalized Q function and extended generalized Mittag-Leffler function are deduced. These formulas also reproduce integral transformations of various deduced Mittag-Leffler functions. Moreover, we proved the convergence of this unified Mittag-Leffler function and constructed the associated fractional integral operator. Our proposed unified Mittag-Leffler function and constructed fractional integral operator will give new directions to the researcher working in this field.

Data Availability

There are no additional data required for the finding of results of this paper.

Conflicts of Interest

It is declared that the authors have no conflicts of interests.

Authors' Contributions

All authors have equal contribution in this article.

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