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Research Article

Qualitative Behavior of a Nonlinear Generalized Recursive Sequence with Delay

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Difference equations are of growing importance in engineering in view of their applications in discrete time-systems used in association with microprocessors. We will check out the global stability and boundedness for a nonlinear generalized high-order difference equation with delay.

1. Introduction

Recently, there is a tremendous rate of interest in examining difference formulas. Among the factors, this is a necessity for techniques that we can use in checking out equations emerging in mathematical models.

Difference formulas have been investigated in different mathematical branches for an extended period.

Camouzis et al. [1] studied

$$\Upsilon_{n+1} = \frac{\beta\Upsilon_n^2}{1+\Upsilon_{n-1}^2}. \tag{1}$$

Elabbasy et al. [2] dealt with

$$\Upsilon_{n+1} = \frac{\alpha\Upsilon_{n-k}}{\beta+\gamma\prod_{i=0}^{k}\Upsilon_{n-i}}. \tag{2}$$

Grove et al. [3] presented a summary of

$$\Upsilon_{n+1} = \frac{A_1+B_1\Upsilon_n+C_1\Upsilon_{n-1}+D_1\Upsilon_{n-2}}{A_2+B_2\Upsilon_n+C_2\Upsilon_{n-1}+D_2\Upsilon_{n-2}}. \tag{3}$$

Kulenovic et al. [4] studied

$$\Upsilon_{n+1} = \frac{A_1\Upsilon_n+B_1\Upsilon_{n-1}}{C_1\Upsilon_n+D_1\Upsilon_{n-1}}. \tag{4}$$

Kulenovic and Ladas [5] studied

$$\Upsilon_{n+1} = \frac{A_1+B_1\Upsilon_n+C_1\Upsilon_{n-1}}{A_2+B_2\Upsilon_n+C_2\Upsilon_{n-1}}. \tag{5}$$

Stevic in [6] studied the positive solution of

$$\Upsilon_{n+1} = \frac{\Upsilon_{n-k}}{\Upsilon_n+\cdots+\Upsilon_{n-k+1}+1}. \tag{6}$$

Agarwal and Elsayed [7] studied

$$\Upsilon_{n+1} = a\Upsilon_n+\frac{b\Upsilon_n\Upsilon_{n-3}}{c\Upsilon_{n-2}+d\Upsilon_{n-3}}. \tag{7}$$

For other works, we refer to [6, 8–27].

Our objective is to check out global stability and boundedness of solutions for

$$\Upsilon_{n+1} = \alpha+\sum_{i=0}^{k}a_i\Upsilon_{n-i}+\frac{\Upsilon_n\Upsilon_{n-k}}{\beta+\sum_{j=0}^{k}b_j\Upsilon_{n-j}}, \tag{8}$$

where a_i and $b_i \in (0,\infty)$ and $\alpha, \beta \geq 1$ with the initials $\Upsilon_{-1}, \Upsilon_{-p+1}, \dots, \Upsilon_{-p}$ and $\Upsilon_0 \in (0,\infty)$.

2. Local Stability of Equilibrium

Theorem 1. *Equation (8) has the following equilibriums:*

$$\overline{\Upsilon}_1 = -\frac{1}{2}D_1,$$
$$\overline{\Upsilon}_2 = -\frac{1}{2}D_2,$$
$$D_1 = \frac{-\beta + \alpha B + A\beta - C}{-B + AB + 1},$$
$$D_2 = \frac{-\beta + \alpha B + A\beta + C}{-B + AB + 1}, \tag{9}$$

where

$$C = \sqrt{\beta^2 + 2\beta\alpha B - 2A\beta^2 + \alpha^2\beta^2 - 2\alpha\beta AB + A^2\beta^2 - 4\alpha\beta},$$
$$A = \sum_{i=0}^{k} a_i,$$
$$B = \sum_{j=0}^{k} b_j. \tag{10}$$

Let f be defined by

$$f(v_0, v_1, v_2, \dots, v_k) = \alpha + \sum_{i=0}^{k} a_i v_i + \frac{v_0 v_k}{\beta + \sum_{i=0}^{k} b_i v_i}. \tag{11}$$

Therefore,

$$\frac{\partial f(v_0, v_1, v_2, \dots, v_k)}{\partial v_0} = a_0 + \frac{\left(\beta + \sum_{i=0}^{k} b_i v_i\right) v_k - (v_0 v_k) b_0}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2},$$
$$\frac{\partial f(v_0, v_1, v_2, \dots, v_p)}{\partial v_1} = a_1 + \frac{\left(\beta + \sum_{i=0}^{k} b_i v_i\right) 0 - (v_0 v_k) b_1}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2} = a_1 - \frac{(v_0 v_k) b_1}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2},$$
$$\frac{\partial f(v_0, v_1, v_2, \dots, v_p)}{\partial v_2} = a_2 + \frac{\left(\beta + \sum_{i=0}^{k} b_i v_i\right) 0 - (v_0 v_k) b_2}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2} = a_2 - \frac{(v_0 v_k) b_2}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2},$$
$$\vdots$$
$$\frac{\partial f(v_0, v_1, v_2, \dots, v_p)}{\partial v_{k-1}} = a_{k-1} - \frac{(v_0 v_k) b_{k-1}}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2},$$
$$\frac{\partial f(v_0, v_1, v_2, \dots, v_p)}{\partial v_k} = a_k + \frac{\left(\beta + \sum_{i=0}^{k} b_i v_i\right) v_0 - (v_0 v_k) b_k}{\left(\beta + \sum_{i=0}^{k} b_i v_i\right)^2}. \tag{12}$$

At $\overline{\Upsilon}_1 = -(1/2)D_1$, we obtain

$$\begin{aligned}
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})}{\partial v_0} &= a_0 + \frac{(\beta + B\overline{\Upsilon})\overline{\Upsilon} - \left(\overline{\Upsilon}^2\right)b_0}{(\beta+\overline{x}B)^2} = a_0 + \frac{-(1/2)\beta D_1 + (1/4)BD_1^2 - (1/4)D_1^2 b_0}{(\beta - (1/2)BD_1)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})}{\partial v_1} &= a_1 - (1/4)\frac{D_1^2 b_1}{(\beta - (1/2)BD_1)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})}{\partial v_2} &= a_2 - (1/4)\frac{D_1^2 b_2}{(\beta - (1/2)BD_1)^2},\\
&\vdots\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})}{\partial v_{k-1}} &= a_{k-1} - (1/4)\frac{D_1^2 b_{k-1}}{(\beta - (1/2)BD_1)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})}{\partial v_k} &= a_k + \frac{-(1/2)\beta D_1 + (1/4)BD_1^2 - (1/4)D_1^2 b_k}{(\beta - (1/2)BD_1)^2},
\end{aligned} \tag{13}$$

and we have

$$y_{n+1} + \sum_{i=0}^{k} d_i y_{n-i} = 0, \tag{14}$$

where $d_i = -f_{v_i}(\overline{\Upsilon},\overline{\Upsilon},\ldots,\overline{\Upsilon})$, for $i = 0,1,\ldots,k$, where

$$\lambda^{k+1} + \sum_{i=0}^{k} d_i \lambda^i = 0. \tag{15}$$

Theorem 2. *(i) $\overline{\Upsilon}_1 = -(1/2)D_1$ of (8) is locally asymptotically stable if*

$$A + \frac{D_1[-\beta + (1/4)BD_1]}{(\beta - (1/2)BD_1)^2} < 1; \tag{16}$$

(ii) $\overline{\Upsilon}_1 = -(1/2)D_2$ of (8) is locally asymptotically stable if

$$A + \frac{D_2[-\beta + (1/4)BD_2]}{(\beta - (1/2)BD_2)^2} < 1. \tag{17}$$

Proof

$$\begin{aligned}
&\left|a_0 + \frac{-(1/2)\beta D_1 + (1/4)BD_1^2 - (1/4)D_1^2 b_0}{(\beta - (1/2)BD_1)^2}\right| + \left|a_1 - \frac{1}{4}\frac{D_1^2 b_1}{(\beta - (1/2)BD_1)^2}\right| + \left|a_2 - \frac{1}{4}\frac{D_1^2 b_2}{(\beta - (1/2)BD_1)^2}\right|\\
&\quad + \cdots + \left|a_{k-1} - \frac{1}{4}\frac{D_1^2 b_{k-1}}{(\beta - (1/2)BD_1)^2}\right| + \left|a_k + \frac{-(1/2)\beta D_1 + (1/4)BD_1^2 - (1/4)D_1^2 b_k}{(\beta - (1/2)BD_1)^2}\right| < 1,\\
&\sum_{i=0}^{k} a_i + \frac{1}{(\beta-(1/2)BD_1)^2}\left\{\left(-\frac{1}{2}\beta D_1 + \frac{1}{4}BD_1^2 - \frac{1}{4}D_1^2 b_0\right) + \left(-\frac{1}{2}\beta D_1 + \frac{1}{4}BD_1^2 - \frac{1}{4}D_1^2 b_k\right)\right\}\\
&\quad - \frac{D_1^2}{4(\beta - (1/2)BD_1)^2}\left[\sum_{i=1}^{k-1} b_i\right] < 1,\\
&\sum_{i=0}^{k} a_i + \frac{1}{(\beta-(1/2)BD_1)^2}\left\{-\beta D_1 + \frac{1}{2}BD_1^2 - \frac{1}{4}D_1^2(b_0 + b_k)\right\} - \frac{D_1^2}{4(\beta - (1/2)BD_1)^2}\left[\sum_{i=1}^{k-1} b_i\right] < 1,\\
&A + \frac{1}{(\beta-(1/2)BD_1)^2}\left\{-\beta D_1 + \frac{1}{2}BD_1^2 - \frac{1}{4}D_1^2(b_0 + b_k)\right\} - \frac{D_1^2}{4(\beta - (1/2)BD_1)^2}\left[B - b_0 - b_k\right] < 1,\\
&A + \frac{1}{(\beta-(1/2)BD_1)^2}\left[-\beta D_1 + \frac{1}{2}BD_1^2 - \frac{D_1^2}{4}b_0 - \frac{D_1^2}{4}b_k - \frac{D_1^2}{4}B + \frac{D_1^2}{4}b_0 + \frac{D_1^2}{4}b_k\right] < 1,\\
&A + \frac{D_1[-\beta + (1/4)BD_1]}{(\beta - (1/2)BD_1)^2} < 1,
\end{aligned} \tag{18}$$

where

$$D_1 = \frac{-\beta + \alpha B + A\beta - C}{-B + AB + 1}, \tag{19}$$

i.e.,

$$A + \frac{((-\beta + \alpha B + A\beta - C)/(-B + AB + 1))[-\beta + (1/4)B((-\beta + \alpha B + A\beta - C)/(-B + AB + 1))]}{(\beta - (1/2)B((-\beta + \alpha B + A\beta - C)/(-B + AB + 1)))^2} < 1. \tag{20}$$

Proof of (ii) is the same as the proof of (i). □

3. Solutions Boundedness for (8)

In Theorem 3, every solution of (8) is bounded.

Proof

$$\Upsilon_{n+1} = \alpha + \sum_{i=0}^{k} a_i \Upsilon_{n-i} + \frac{\Upsilon_n \Upsilon_{n-k}}{\beta + \sum_{j=0}^{k} b_j \Upsilon_{n-j}} \le \alpha + \sum_{i=0}^{k} a_i \Upsilon_{n-i} + \frac{\Upsilon_n \Upsilon_{n-k}}{b_k \Upsilon_{n-k}} \le \alpha + \sum_{i=0}^{k} a_i \Upsilon_{n-i} + \frac{\Upsilon_n}{b_k}. \tag{21}$$

□

4. Applications

4.1. Case 1: $\alpha = 0$ and $\beta = 0$. We have

$$\Upsilon_{n+1} = \sum_{i=0}^{k} a_i \Upsilon_{n-i} + \frac{\Upsilon_n \Upsilon_{n-k}}{\sum_{j=0}^{k} b_j \Upsilon_{n-j}}. \tag{22}$$

Equation (22) has equilibrium $\overline{\Upsilon} = 0$.

Theorem 3. *Suppose that*

$$A + \frac{1}{B} < 1, \tag{23}$$

where $A = \sum_{i=0}^{k} a_i$ and $B = \sum_{j=0}^{k} b_j$. Then, the equilibrium of (22) is locally asymptotically stable.

Proof

$$\begin{aligned}
&\left|a_0 + \frac{(1/4)B - (1/4)b_0}{(-(1/2)B)^2}\right| + \left|a_1 - \frac{1}{4}\frac{b_1}{(-(1/2)B)^2}\right| + \left|a_2 - \frac{1}{4}\frac{b_2}{(-(1/2)B)^2}\right| \\
&\quad + \cdots + \left|a_{k-1} - \frac{1}{4}\frac{b_{k-1}}{(-(1/2)B)^2}\right| + \left|a_k + \frac{(1/4)B - (1/4)b_k}{(-(1/2)B)^2}\right| < 1, \\
&\sum_{i=0}^{k} a_i + \frac{1}{(-(1/2)B)^2}\left\{\left(\frac{1}{4}B - \frac{1}{4}b_0\right) + \left(\frac{1}{4}B - \frac{1}{4}b_k\right)\right\} - \frac{1}{4(-(1/2)B)^2}\left[\sum_{i=1}^{k} b_i\right] < 1, \\
&\sum_{i=0}^{k} a_i + \frac{1}{(-(1/2)B)^2}\left\{\frac{1}{2}B - \frac{1}{4}b_0 - \frac{1}{4}b_k\right\} - \frac{1}{4(-(1/2)B)^2}\left[\sum_{i=1}^{k-1} b_i\right] < 1, \\
&\sum_{i=0}^{k} a_1 + \frac{1}{(-(1/2)B)^2}\left\{\frac{1}{2}B - \frac{1}{4}b_0 - \frac{1}{4}b_k\right\} - \frac{1}{4(-(1/2)B)^2}\left[B - b_0 - b_k\right] < 1, \\
&\sum_{i=0}^{k} a_i + \frac{1}{(-(1/2)B)^2}\left[\frac{1}{2}B - \frac{1}{4}b_0 - \frac{1}{4}b_k\right] - \frac{1}{4}\left[B - b_0 - b_k\right] < 1, \\
&\sum_{i=0}^{k} a_i + \frac{1}{(-(1/2)B)^2}\left[\frac{1}{2}B - \frac{1}{4}B\right] < 1, \\
&A + \frac{1}{B} < 1.
\end{aligned} \tag{24}$$

□

4.2. Case 2 [7]. $\alpha = 0$ and $\beta = 0$ with $k = 3$, $\{a_i\}_{i=0}^{k}$ & $\{b_i\}_{i=0}^{k} = 0$ except $a_0, b_2, b_3 \neq 0$

We have

$$\Upsilon_{n+1} = a_0 \Upsilon_n + \frac{\Upsilon_n \Upsilon_{n-3}}{b_2 \Upsilon_{n-2} + b_3 \Upsilon_{n-3}}. \quad (25)$$

Equation (25) has equilibriums $\overline{\Upsilon} = 0$.

Let f be defined by

$$f(v_0, v_2, v_3) = a_0 v_0 + \frac{v_0 v_3}{b_2 v_2 + b_3 v_3},$$

$$\left| a_0 + \frac{(1/4)(b_2 + b_3)}{(-(1/2)(b_2 + b_3))^2} \right| + \left| -\frac{1}{4} \frac{b_2}{(-(1/2)(b_2 + b_3))^2} \right| + \left| \frac{(1/4)(b_2 + b_3) - (1/4) b_3}{(-(1/2)(b_2 + b_3))^2} \right| < 1,$$

$$\left| a_0 + \frac{(b_2 + b_3)}{(b_2 + b_3)^2} \right| + \left| -\frac{b_2}{(b_2 + b_3)^2} \right| + \left| \frac{(b_2 + b_3) - b_3}{(b_2 + b_3)^2} \right| < 1,$$

$$\left| a_0 + \frac{b_2 + b_3}{(b_2 + b_3)^2} \right| + \left| -\frac{b_2}{(b_2 + b_3)^2} \right| + \left| \frac{b_2 + b_3 - b_3}{(b_2 + b_3)^2} \right| < 1, \quad (26)$$

$$\left| a_0 + \frac{1}{(b_2 + b_3)} \right| + \left| -\frac{b_2}{(b_2 + b_3)^2} \right| + \left| \frac{b_2}{(b_2 + b_3)^2} \right| < 1,$$

$$a_0 + \frac{1}{(b_2 + b_3)} + \frac{2 b_2}{(b_2 + b_3)^2} < 1,$$

$$\frac{(3 b_2 + b_3)}{(b_2 + b_3)^2} < 1 - a_0.$$

Hence, the equilibrium $\overline{\Upsilon} = 0$ of (25) is locally asymptotically stable if $((3b_2 + b_3)/(b_2 + b_3)^2) < 1 - a_0$.

4.3. Case 3. $\alpha = 0$ and $\beta = 1$ with $k = 2$, $\{a_i\}_{i=0}^{k}$ & $\{b_i\}_{i=0}^{k} = 0$ except $a_0, a_1, b_1, b_2 \neq 0$

Here, we have

$$\Upsilon_{n+1} = a_0 \Upsilon_n + a_1 \Upsilon_{n-1} + \frac{\Upsilon_n \Upsilon_{n-2}}{1 + b_1 \Upsilon_{n-1} + b_2 \Upsilon_{n-2}}. \quad (27)$$

Equation (27) has the following equilibriums:

$$\overline{\Upsilon}_1 = 0,$$

$$\overline{\Upsilon}_2 = -\frac{a_0 + a_1 - 1}{1 + (a_0 + a_1 - 1)(b_1 + b_2)}. \quad (28)$$

Let f defined by

$$f(v_0, v_1, v_2) = a_0 v_0 + a_1 v_1 + \frac{v_0 v_2}{1 + b_1 v_1 + b_2 v_2},$$

$$\frac{\partial f(v_0, v_1, v_2)}{\partial v_0} = a_0 + \frac{(b_1 v_1 + 1 + b_2 v_2) v_2}{(b_1 v_1 + 1 + b_2 v_2)^2} = a_0 + \frac{v_2}{(b_1 v_1 + 1 + b_2 v_2)},$$

$$\frac{\partial f(v_0, v_1, v_2)}{\partial v_1} = a_1 + \frac{-v_0 v_2 (b_1)}{(b_1 v_1 + 1 + b_2 v_2)^2},$$

$$\frac{\partial f(v_0, v_1, v_2)}{\partial v_2} = \frac{(1 + b_1 v_1 + b_2 v_2) v_0 - v_0 v_2 b_2}{(1 + b_1 v_1 + b_2 v_2)^2} = \frac{(b_1 v_1 + 1) v_0}{(b_1 v_1 + 1 + b_2 v_2)^2},$$

$$\left| a_0 + \frac{v_2}{(b_1 v_1 + 1 + b_2 v_2)} \right| + \left| a_1 + \frac{-v_0 v_2 (b_1)}{(b_1 v_1 + 1 + b_2 v_2)^2} \right| + \left| \frac{(b_1 v_1 + 1) v_0}{(b_1 v_1 + 1 + b_2 v_2)^2} \right| < 1, \tag{29}$$

$$a_0 + \frac{v_2}{(b_1 v_1 + 1 + b_2 v_2)} + a_1 - \frac{v_0 v_2 (b_1)}{(b_1 v_1 + 1 + b_2 v_2)^2} + \frac{(b_1 v_1 + 1) v_0}{(b_1 v_1 + 1 + b_2 v_2)^2} < 1,$$

$$a_0 + \frac{(b_1 v_1 + 1 + b_2 v_2) v_2}{(b_1 v_1 + 1 + b_2 v_2)^2} + a_1 - \frac{v_0 v_2 (b_1)}{(b_1 v_1 + 1 + b_2 v_2)^2} + \frac{(b_1 v_1 + 1) v_0}{(b_1 v_1 + 1 + b_2 v_2)^2} < 1,$$

$$\frac{(b_1 v_1 + 1 + b_2 v_2) v_2 - v_0 v_2 (b_1) + (1 + b_1 v_1) v_0}{(b_1 v_1 + 1 + b_2 v_2)^2} < 1 - a_0 - a_1,$$

$$\frac{v_2 + b_1 v_1 v_2 + b_2 v_2^2 - v_0 v_2 b_1 + v_0 + b_1 v_0 v_1}{(b_1 v_1 + 1 + b_2 v_2)^2} < 1 - a_0 - a_1.$$

So, we have

Theorem 4

(i) *The equilibrium* $\overline{\Upsilon}_1 = 0$ *of equation (27) is locally asymptotically stable if*

$$0 < 1 - a_0 - a_1; \tag{30}$$

(ii) *The equilibrium* $\overline{\Upsilon}_1 = -((a_0 + a_1 - 1)/(1 + (a_0 + a_1 - 1)(b_1 + b_2)))$ *of equation (27) is locally asymptotically stable if*

$$\frac{(-1 + a_0 + a_1)(2a_0(b_1 + b_2) + 2a_1(b_1 + b_2) - b_1 + b_1 a_0 + b_1 a_1 - b_2 + b_2 a_0 + b_2 a_1)}{(a_0(b_1 + b_2) + a_1(b_1 + b_2) - b_1 + b_1 a_0 + b_1 a_1 - b_2 + b_2 a_0 + b_2 a_1)^2} < 1 - a_0 - a_1. \tag{31}$$

4.4. Case 4. We will consider the difference equation as a particular case of (8):

$$\Upsilon_{n+1} = a_0 \Upsilon_n + a_2 \Upsilon_{n-2} + a_4 \Upsilon_{n-4} + \frac{\Upsilon_n \Upsilon_{n-4}}{b_0 \Upsilon_n + b_2 \Upsilon_{n-2} + b_4 \Upsilon_{n-4}}. \tag{32}$$

Theorem 5. *Suppose*

$$1 < (1 - a_0 - a_2 - a_4)(b_0 + b_4 + b_2). \tag{33}$$

Then, the equilibrium $\overline{\Upsilon} = 0$ of (32) is locally asymptotically stable.

Proof. Let f be defined by

$$f(v_0, v_2, v_4) = a_0 v_0 + a_2 v_2 + a_4 v_4 + \frac{v_0 v_4}{b_0 v_0 + b_2 v_2 + b_4 v_4}. \tag{34}$$

Therefore,

$$\begin{aligned}
\frac{\partial f(v_0,v_2,v_4)}{\partial v_0} &= a_0+\frac{(b_0v_0+b_2v_2+b_4v_4)v_4-v_0v_4(b_0)}{(b_0v_0+b_2v_2+b_4v_4)^2}=a_0+\frac{b_2v_2v_4+b_4v_4^2}{(b_0v_0+b_2v_2+b_4v_4)^2},\\
\frac{\partial f(v_0,v_2,v_4)}{\partial v_2} &= a_2+\frac{(b_0v_0+b_2v_2+b_4v_4)0-v_0v_4(b_2)}{(b_0v_0+b_2v_2+b_4v_4)^2}=a_2+\frac{-v_0v_4b_2}{(b_0v_0+b_2v_2+b_4v_4)^2},\\
\frac{\partial f(v_0,v_2,v_4)}{\partial v_4} &= a_4+\frac{(b_0v_0+b_2v_2+b_4v_4)v_0-v_0v_4(b_4)}{(b_0v_0+b_2v_2+b_4v_4)^2}=a_4+\frac{b_0v_0^2+b_2v_2v_0}{(b_0v_0+b_2v_2+b_4v_4)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon})}{\partial v_0} &= a_0+\frac{b_2+b_4}{(b_0+b_4+b_2)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon})}{\partial v_2} &= a_2+\frac{-b_2}{(b_0+b_4+b_2)^2},\\
\frac{\partial f(\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon})}{\partial v_4} &= a_4+\frac{b_0+b_2}{(b_0+b_4+b_2)^2},\\
&\left|a_0+\frac{b_2+b_4}{(b_0+b_4+b_2)^2}\right|+\left|a_2+\frac{-b_2}{(b_0+b_4+b_2)^2}\right|+\left|a_4+\frac{b_0+b_2}{(b_0+b_4+b_2)^2}\right|<1,\\
&\frac{b_4+b_0+b_2}{(b_0+b_4+b_2)^2}<1-a_0-a_2-a_4,\\
&1<(1-a_0-a_2-a_4)(b_0+b_4+b_2).
\end{aligned}\tag{35}$$

□

Theorem 6. *Every solution of (32) is bounded.*

Proof

$$\Upsilon_{n+1}=a_0\Upsilon_n+a_2\Upsilon_{n-2}+a_4\Upsilon_{n-4}+\frac{\Upsilon_n\Upsilon_{n-4}}{b_0\Upsilon_n+b_2\Upsilon_{n-2}+b_4\Upsilon_{n-4}}<\left(a_0+\frac{1}{b_4}\right)\Upsilon_n+a_2\Upsilon_{n-2}+a_4\Upsilon_{n-4}.\tag{36}$$

There are many cases in which the solution of (32) is bounded:

(1) If $(a_0+(1/b_4))<1, a_2=0$ and $a_4=0$.

(2) If $(a_0+a_2+(1/b_4))<1, \Upsilon_{n-2}<\Upsilon_n$ and $a_4=0$.

(3) If $(a_0+a_4+(1/b_4))<1, \Upsilon_{n-4}<\Upsilon_n$ and $a_2=0$.

(4) If $(a_0+a_2+a_4+(1/b_4))<1, \Upsilon_{n-4}<\Upsilon_n$ and $\Upsilon_{n-2}<\Upsilon_n$. □

Data Availability

The data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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