Research Article

A Descent Four-Term Conjugate Gradient Method with Global Convergence Properties for Large-Scale Unconstrained Optimisation Problems

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The conjugate gradient method is a useful method to solve large-scale unconstrained optimisation problems and to be used in some applications in several fields such as engineering, medical science, image restorations, neural network, and many others. The main benefit of the conjugate gradient method is not using the second derivative or its approximation, such as Newton's method or its approximation. Moreover, the algorithm of the conjugate gradient method is simple and easy to apply. This study proposes a new modified conjugate gradient method that contains four terms depending on popular two- and three-term conjugate gradient methods. The new algorithm satisfies the descent condition. In addition, the new CG algorithm possesses the convergence property. In the numerical results part, we compare the new algorithm with famous methods such as CG-Descent. We conclude from numerical results that the new algorithm is more efficient than other popular CG methods such as CG-Descent 6.8 in terms of number of function evaluations, number of gradient evaluations, number of iterations, and CPU time.

1. Introduction

To solve large-scale unconstrained optimisation problems, we prefer to use conjugate gradient (CG) since it is efficient and robust and does not use the second derivative. Consider the following problem:

$$\min f(x) \quad \mathbb{R}^n, \quad (1)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous and differentiable function, and the gradient is available. The CG method generates a sequence of x_k given by the equation

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 1, 2, \dots \quad (2)$$

The step length α_k was obtained using line search. d_k is a search direction of the CG method defined as follows:

$$d_k = \begin{cases} -g_k, & \text{if } k = 1, \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 2, \end{cases} \quad (3)$$

where $g_k = g(x_k) = \nabla f$ and β_k is known as the CG formula.

To obtain the step length, we use the strong Wolfe-Powell (SWP) [1, 2] line search given by the following equations:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \quad (4)$$

$$|g(x_k + \alpha_k d_k)^T d_k| \leq \sigma |g_k^T d_k|, \quad (5)$$

where $0 < \delta < \sigma < 1$.

The weak Wolfe-Powell (WWP) line search is given by equation (4) and

$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k. \quad (6)$$

The popular classical formulas of CG methods Hestenes–Stiefel (HS) [3], Polak–Ribiere–Polyak (PRP) [4], Liu–Storey (LS) [5], Fletcher–Reeves (FR) [6], Fletcher (CD) [7], and Dai–Yuan (DY) [8] are as follows:

$$\begin{aligned} \beta_k^{\text{HS}} &= \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \\ \beta_k^{\text{PRP}} &= \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \\ \beta_k^{\text{LS}} &= \frac{g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}}, \\ \beta_k^{\text{FR}} &= \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \\ \beta_k^{\text{CD}} &= -\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}, \\ \beta_k^{\text{DY}} &= \frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}, \end{aligned} \quad (7)$$

where $y_{k-1} = g_k - g_{k-1}$.

Powell [9] proposed an example to show that a function does not satisfy the convergence even if the exact line search is employed with the PRP formula. Gilbert and Nocedal [10] used Powell [9] suggestion for the PRP formula. They proved that if $\beta_k^{\text{PRP}+} = \max\{0, \beta_k^{\text{PRP}}\}$, the WWP line search is employed and the sufficient descent Wolfe–Powell line search condition is satisfied; then $\beta_k^{\text{PRP}+}$ is globally convergent. Moreover, Zoutendijk [11] showed that the FR formula with CG method and the exact line search has convergence analysis. Finally, Al-Baali [12] proved that the CG method with the FR coefficient is globally convergent when $\sigma \leq 1/2$ SWP is employed.

In 2006, Wei et al. [13] presented a nonnegative CG formula that can satisfy the descent property and global convergence property as follows:

$$\beta_k^{\text{WYL}} = \frac{g_k^T (g_k - (\|g_k\|/\|g_{k-1}\|)g_{k-1})}{\|g_{k-1}\|^2}, \quad (8)$$

where $y_{k-1} = g_k - g_{k-1}$. In 2016, Alhawarat et al. [14] presented the following nonnegative CG formula with new restart property:

$$\beta_k^{\text{AZPRP}} = \begin{cases} \frac{\|g_k\|^2 - \mu_k |g_k^T g_{k-1}|}{\|g_{k-1}\|^2}, & \text{if } \|g_k\|^2 > \mu_k |g_k^T g_{k-1}|, \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

where $\|\cdot\|$ represents the Euclidean norm and μ_k is defined as follows:

$$\mu_k = \frac{\|x_k - x_{k-1}\|}{\|y_{k-1}\|}. \quad (10)$$

In 2020, Kaelo et al. [15] proposed a nonnegative CG formula and compared the new formula with β_k^{AZPRP} . The proposed equation is as follows:

$$\beta_k^{\text{PKT}} = \begin{cases} \frac{\|g_k\|^2 - g_k^T g_{k-1}}{\max\{d_{k-1}^T y_{k-1}, -g_{k-1}^T d_{k-1}\}}, & \text{if } 0 < g_k^T g_{k-1} < \|g_k\|^2, \\ \frac{\|g_k\|^2}{\max\{d_{k-1}^T y_{k-1}, -g_{k-1}^T d_{k-1}\}}, & \text{otherwise.} \end{cases} \quad (11)$$

In 2001, Dai and Liao [16] proposed the following conjugacy condition:

$$d_k^T y_{k-1} = -t g_k^T s_{k-1}, \quad (12)$$

where $s_{k-1} = x_k - x_{k-1}$ and $t \geq 0$. In the case of $t = 0$, equation (13) becomes the classical conjugacy condition. By using equations (2) and (12), Dai and Liao [16] proposed the following CG formula:

$$\beta_k^{\text{DL}} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} = \beta_k^{\text{HS}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}. \quad (13)$$

However, β_k^{DL} inherits the same problem as β_k^{PRP} and β_k^{HS} ; that is, β_k^{DL} is not nonnegative in general. Thus, Gilbert and Nocedal [10] replaced equation (13) by

$$\beta_k^{\text{DL}+} = \max\{\beta_k^{\text{HS}}, 0\} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}. \quad (14)$$

In 2006, Hager and Zhang [17, 18] presented the following CG formula based on (13):

$$\beta_k^{\text{HZ}} = \max\{\beta_k^{\text{N}}, \eta_k\}, \quad (15)$$

where $\beta_k^{\text{N}} = (1/d_k^T y_k)(y_k - 2d_k(\|y_k\|^2/d_k^T y_k))^T g_k$, $\eta_k = -(1/\|d_k\| \min\{\eta, \|g_k\|\})$, and $\eta > 0$ is a constant.

Note that if $t = 2(\|y_k\|^2/s_k^T y_k)$, then $\beta_k^{\text{N}} = \beta_k^{\text{DY}}$.

Based on equation (13), Andrei [19, 20] proposed two three-term CG (TTTCG) methods in 2013 as follows:

$$\begin{aligned} d_k &= -g_k + \left(\frac{g_k^T y_{k-1} - t g_k^T s_{k-1}}{y_{k-1}^T d_{k-1}} \right) d_{k-1} - \frac{g_k^T d_{k-1}}{y_{k-1}^T d_{k-1}} y_{k-1}, \\ t &= \left(1 + \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \right), \\ d_k &= -g_k + \left(\frac{g_k^T y_{k-1} - t g_k^T s_{k-1}}{y_{k-1}^T d_{k-1}} \right) d_{k-1} - \frac{g_k^T d_{k-1}}{y_{k-1}^T d_{k-1}} y_{k-1}, \\ t &= \left(1 + 2 \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \right). \end{aligned} \quad (16)$$

Also, based on equation (13), Babaie-Kafaki and Ghanbari [21] proposed the following CG method:

$$d_k = -g_k + \left(\frac{g_k^T y_{k-1} - t g_k^T s_{k-1}}{y_{k-1}^T d_{k-1}} \right) d_{k-1} - \frac{g_k^T d_{k-1}}{y_{k-1}^T d_{k-1}} y_{k-1},$$

$$t = \left(\max \left(\varsigma, 1 - \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \right) \right), \quad (17)$$

where $\varsigma > 0$.

Similar to [19–21], Deng and Wan [22] in 2015 presented the following CG method:

$$d_k = -g_k + \left(\frac{g_k^T y_{k-1} - t g_k^T s_{k-1}}{y_{k-1}^T d_{k-1}} \right) d_{k-1} - \frac{g_k^T d_{k-1}}{y_{k-1}^T d_{k-1}} y_{k-1},$$

$$t = \left(\max \left(0, 1 - \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}} \right) \right). \quad (18)$$

In 2019, Liu et al. [23] proposed

$$d_k = -g_k + \left(\beta_k^{LS} - \frac{\|g_{k-1}\|^2 g_k^T d_{k-1}}{(d_{k-1}^T g_{k-1})^2} \right) d_{k-1} + \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_{k-1}, \quad (19)$$

with the following assumption:

$$\left(\frac{g_k^T d_{k-1}}{d_{k-1}^T g_{k-1}} \right) > \nu \in (0, 1). \quad (20)$$

Moreover, Liu et al. [23] proved the sufficient descent condition for no convex functions to solve nonlinear monotone equations. To avoid using the condition $(g_k^T d_{k-1}/d_{k-1}^T g_{k-1}) > \nu \in (0, 1)$, Liu et al. [24] constructed the following three-term CG method:

$$d_k = -g_k + \left(\beta_k^{LS} - \frac{\|g_{k-1}\|^2 g_k^T s_{k-1}}{(d_{k-1}^T g_{k-1})^2} \right) d_{k-1} - \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_{k-1}, \quad (21)$$

and they used it to solve equation (1). Liu et al. [24] proposed wide choices using equation (21) as follows:

$$d_k^{\text{MTTPRP}} = -g_k + \left(\beta_k^{\text{PRP}} - \frac{g_k^T s_{k-1}}{\|g_{k-1}\|^2} \right) d_{k-1} + \left(\frac{g_k^T d_{k-1}}{\|g_{k-1}\|^2} \right) g_{k-1},$$

$$d_k^{\text{MTTHS}} = -g_k + \left(\beta_k^{\text{HS}} - \frac{\|g_{k-1}\|^2 g_k^T s_{k-1}}{(d_{k-1}^T y_{k-1})^2} \right) d_{k-1} + \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_{k-1}, \quad (22)$$

$$d_k^{\text{ADAP}} = -g_k + \left(\frac{g_k^T y_{k-1}}{t_{k-1}} - \frac{\|g_{k-1}\|^2 g_k^T s_{k-1}}{t_{k-1}^2} \right) d_{k-1} + \left(\frac{g_k^T d_{k-1}}{t_{k-1}} \right) g_{k-1},$$

$$t_{k-1} = \max \left\{ -d_{k-1}^T g_{k-1}, \|g_{k-1}\|^2, d_{k-1}^T y_{k-1} \right\}.$$

In 2020, Yao et al. [25] proposed three terms of CG with a new choice of t as follows:

$$d_{k+1} = -g_{k+1} + \left(\frac{g_k^T y_k - t_k g_{k+1}^T s_k}{y_k^T d_k} \right) d_k + \frac{g_{k+1}^T d_k}{y_k^T d_k} y_k. \quad (23)$$

Based on the SWP line search, Yao et al. [25] selected t_k to satisfy the descent condition as follows:

$$t_k > \frac{\|y_k\|^2}{y_k^T s_k}. \quad (24)$$

Yao et al. [25] also proposed a theorem stating that if t_k is close to $(\|y_k\|^2/y_k^T s_k)$, then the search direction results in a zigzag search path. Therefore, they selected the following choice for t_k :

$$t_k = 1 + 2 \frac{\|y_k\|^2}{y_k^T s_k}. \quad (25)$$

As an application of the CG method, we can restore actual images from damaged images by minimising the z algorithm with an efficient number of iterations and CPU time. Moreover, we can test the quality using the root mean square error (RMSE) between the actual image (original) and the restored image as follows:

$$\text{rmse} = \frac{\|\zeta - \zeta_k\|_2}{\|\zeta\|}. \quad (26)$$

Here, ζ and ζ_k are the actual image and restored image, accordingly. Moreover, the CG method can be applied in several fields: neural network, image restoration, medical science, machine learning, finance and economics, and many other fields. The reader can refer to [26–36] for more about the CG method and its applications.

2. The New Search Direction

In the literature review, we note that the general form of the three-term CG method can be written as follows:

$$d_k = -g_k + \eta_k d_{k-1} - \theta_k y_{k-1}, \quad (27)$$

where θ_k and η_k are CG parameters. Based on equation (13), we extend equation (27) to the following form:

$$d_k^{\text{FTCGHS}} = -g_k + \left(\beta_k^{\text{HS}} - t_k \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right) d_{k-1} - \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) (y_{k-1} + s_{k-1}), \quad (28)$$

where $t_k = (\|s_k\|/\|y_{k-1}\|)$.

In this paper, we assume the following for equation (28):

$$\begin{aligned} \eta_k &= \beta_k^{\text{HS}} - t_k \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \\ \theta_k &= \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right). \end{aligned} \quad (29)$$

Note that equation (28) can be reduced to the HSCG method if we use an exact line search.

By using exact line search, we obtain

$$g_k^T d_{k-1} = 0, \quad (30)$$

and since $g_k^T s_{k-1} = \alpha_k g_k^T d_{k-1}$, we obtain

$$d_k^{\text{FTCGHS}} = -g_k + (\beta_k^{\text{HS}}) d_{k-1}. \quad (31)$$

Note that η_k is similar to β_k^{DL} with different values of t_k , where β_k^{DL} inherits the conjugacy condition with inexact line search. For θ_k , it is worth noting that it contains the term $g_k^T d_{k-1}$ and is divided by the denominator of the HS method. Meanwhile, for $(y_{k-1} + s_{k-1})$, the direction y_{k-1} is an essential term in obtaining the descent condition when multiplied by negative gradient. It is also useful in terms of efficiency, i.e., if this term goes to zero, the search direction will restart using the steepest descent method, avoiding equation (28) to cycle without reaching the solution.

Figure 1 describes the steps of the CG method to obtain the stationary point using SWP line search and equation (28) with stopping criteria of $\|g_k\| \leq 10^{-6}$.

3. Global Convergence Analysis of the CG Method with Figure 1

The following assumption is crucial to satisfy the convergence analysis of CG methods.

Assumption 1

(A). The level set $\Omega = \{x | f(x) \leq f(x_1)\}$ is bounded; that is, a positive constant ρ exists such that

$$\|x\| \leq \rho, \quad \forall x \in \Omega. \quad (32)$$

(B). In some neighbourhood Q of Ω , f is continuously differentiable, and its gradient is Lipschitz continuous; that is, for all, $x, y \in Q$, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|. \quad (33)$$

In addition to this assumption, we can conclude that there exists a positive constant B such that

$$\|g(u)\| \leq B, \quad \forall u \in N. \quad (34)$$

The descent condition (downhill condition),

$$g_k^T d_k < 0, \quad \forall k \geq 1, \quad (35)$$

is useful in studying the CG method and plays an important role in the proof of global convergence analysis. Abaali [12] modified (35) to the following form and used it to prove the FR method:

$$g_k^T d_k \leq -c\|g_k\|^2, \quad \forall k \geq 1, \quad (36)$$

where $c \in (0, 1)$. Equation (36) is the sufficient descent condition. Moreover, using equation (36) is better than equation (35) since we can control the quantity of $g_k^T d_k$ by using $\|g_k\|^2$.

3.1. The Descent Property of the New Search Direction. The following theorem shows that the search direction in equation (28) ensures that the sufficient descent condition (35) is satisfied with the SWP line search.

Theorem 1. *Let the sequences generate (1) and (28), computed by SWP line search. Then, the sufficient descent condition holds.*

Proof. Multiplying (28) by g_k^T yields

$$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + \left(\frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} - \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T (y_{k-1} + s_{k-1}) \\ &= -\|g_k\|^2 + \left(\frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} - t \left(\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} - \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T y_{k-1} - \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T s_{k-1} \end{aligned}$$

FIGURE 1: The steps of the CG method to obtain the optimum point.

$$\begin{aligned}
 &= -\|g_k\|^2 - t\alpha_k \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} - \alpha_k \left(\frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} \\
 &= -\|g_k\|^2 - \alpha_k \left(\frac{\|g_k^T d_{k-1}\|^2}{d_{k-1}^T y_{k-1}} \right) (t + 1).
 \end{aligned} \tag{37}$$

From Wolfe–Powell search, we obtain

$$d_{k-1}^T y_{k-1} > 0, \tag{38}$$

and thus

$$g_k^T d_k \leq -\|g_k\|^2. \tag{39}$$

Zoutendijk [11] presented a useful lemma for analysing the global convergence property of the CG method. The lemma is given as follows. $\square$

Lemma 1. *Let Assumption 1 hold. Consider any CG method in the form of (1) and (2), and α_k satisfies the WWP line search of (5) and (6), in which the search direction is descent. Then, the following condition holds:*

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty. \quad (40)$$

3.2. Global Convergence of Figure 1 with Convex Functions. Dai and Liao [16] presented a useful theorem to obtain the global convergence theorem of the CG method as follows.

Theorem 2. Suppose that Assumption 1 holds. Consider any conjugate gradient method in the form of equations (1) and (2), where d_k is a descent direction and α_k is obtained from the strong Wolfe line search. If

$$\sum_{k \geq 1} \frac{1}{\|d_k\|^2} = \infty, \quad (41)$$

then

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (42)$$

The following theorem shows that the new search direction strongly satisfies the convergences analysis with convex functions.

Theorem 3. Suppose that Assumption 1 holds. Consider the CG method in the forms of equations (2) and (28) and d_k as a descent direction, where α_k is obtained using equations (3) and (4). If $f(x)$ is a uniformly convex function, then $\liminf_{k \rightarrow \infty} \|g_k\| = 0$.

Proof. Because function $f(x)$ is uniformly convex, there exists a positive constant $\bar{\omega}$ such that

$$\bar{\omega} \|x - y\|^2 \leq (\nabla f(x) - \nabla f(y))^T (x - y), \quad (43)$$

for all $x, y \in P$. Thus,

$$d_{k-1}^T y_{k-1} \geq \bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2. \quad (44)$$

By using equation (28) and triangular inequality, we obtain

$$\begin{aligned} \|d_k\| &\leq \|g_k\| + \left(\frac{|g_k^T y_{k-1}|}{d_{k-1}^T y_{k-1}} + t \frac{|g_k^T s_{k-1}|}{d_{k-1}^T y_{k-1}} \right) \|d_{k-1}\| \\ &\quad + \left(\frac{|g_k^T d_{k-1}|}{d_{k-1}^T y_{k-1}} \right) (\|y_{k-1}\| + \|s_{k-1}\|). \end{aligned} \quad (45)$$

By using equation (44), we have

$$\begin{aligned} \|d_k\| &\leq \|g_k\| + \left(\frac{|g_k^T y_{k-1}|}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} + t \frac{|g_k^T s_{k-1}|}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} \right) \|d_{k-1}\| \\ &\quad + \left(\frac{|g_k^T d_{k-1}|}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} \right) (\|y_{k-1}\| + \|s_{k-1}\|). \end{aligned} \quad (46)$$

Then, using triangular inequality and Assumption 1 yields

$$\begin{aligned} \|d_k\| &\leq \|g_k\| + \left(\frac{L \alpha_{k-1} \|g_k\| \|d_{k-1}\|^2}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} + t \frac{\alpha_{k-1} \|g_k\| \|d_{k-1}\|^2}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} \right) + \left(\frac{\|g_k\| \|d_{k-1}\|^2}{\bar{\omega} \alpha_{k-1} \|d_{k-1}\|^2} \right) (L \alpha_{k-1} + \alpha_{k-1}) \\ &\leq \|g_k\| + \left(\frac{L \|g_k\|}{\bar{\omega}} + t \frac{\|g_k\|}{\bar{\omega}} \right) + \left(\frac{\|g_k\|}{\bar{\omega}} \right) (L + 1) \\ &\leq \|g_k\| + \frac{\|g_k\|}{\bar{\omega}} (2L + t + 1). \end{aligned} \quad (47)$$

By using Assumption 1, we now have

$$\|d_k\| \leq B + \frac{B}{\bar{\omega}} (2L + t + 1). \quad (48)$$

Let $B + (B/\bar{\omega})(2L + t + 1) = M$, where M is constant. Thus, we obtain

$$\|d_k\| \leq M. \quad (49)$$

By using Theorem 2, we now have

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (50)$$

□

3.3. Global Convergence of Figure 1 with General Nonlinear Functions. The following restriction for η_k is very important to establish the convergence analysis for our new search direction. The main importance of this restriction is to avoid the CG method multiplayer, which will be nonnegative.

$$\eta_k^+ = \max \left\{ 0, \beta_k^{\text{HS}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right\}. \quad (51)$$

Thus, equation (28) becomes

$$d_k = -g_k + \eta_k^+ d_{k-1} - \theta_k (y_{k-1} + s_{k-1}). \quad (52)$$

Lemma 2. Assume that Assumption 1 holds and the sequences $\{g_k\}$ and $\{d_k\}$ are generated using Figure 1, where the step size α_k is computed via the SWP line search of (4) and (5) or weak Wolfe–Powell line search of (4) and (6), such that the sufficient descent condition holds. If $\beta_k \geq 0$, there exists a constant $\gamma > 0$, such that $\|g_k\| > \gamma$ for all $k \geq 1$. Then, $d_k \neq 0$ and

$$\sum_{k=0}^{\infty} \|u_{k+1} - u_k\|^2 < \infty, \quad (53)$$

where $u_k = (d_k / \|d_k\|)$.

Proof. First, if $d_k = 0$, then, from the sufficient descent condition, we obtain $g_k = 0$. Thus, we suppose that $d_k \neq 0$ and

$$\bar{\gamma} \geq \|g_k\| \geq \gamma > 0, \quad \forall k \geq 1. \quad (54)$$

We define

$$u_k = w_k + \delta_k u_{k-1}, \quad (55)$$

where

$$w_k = \frac{-g_k - \theta_k (y_{k-1} + s_{k-1})}{\|d_k\|}, \quad (56)$$

$$\delta_k = \eta_k^+ \frac{\|d_{k-1}\|}{\|d_k\|}.$$

Since u_k is a unit vector, then

$$\|w_k\| = \|u_k - \delta_k u_{k-1}\| = \|\delta_k u_k - u_{k-1}\|. \quad (57)$$

Using the triangular inequality and $\delta_k \geq 0$, we now have

$$\begin{aligned} \|u_k - u_{k-1}\| &\leq (1 + \delta_k) \|u_k - u_{k-1}\| = \|u_k - \delta_k u_{k-1} - (u_{k-1} - \delta_k u_k)\| \\ &\leq \|u_k - \delta_k u_{k-1}\| + \|u_{k-1} - \delta_k u_k\| = 2\|w_k\|. \end{aligned} \quad (58)$$

We define

$$\nu = -g_k - \theta_k (y_{k-1} + s_{k-1}). \quad (59)$$

Using the triangular inequality, we obtain

$$\|\nu\| \leq \|g_k\| + |\theta_k| \|y_{k-1} + s_{k-1}\|. \quad (60)$$

Now, utilising the equations of SWP of (4) and (5), we can conclude the two following inequalities given by

$$\begin{aligned} d_{k-1}^T y_{k-1} &\geq (\sigma - 1) g_{k-1}^T d_{k-1}, \\ |\theta_k| &= \left| \frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right| \leq \left(\frac{\sigma}{1 - \sigma} \right) = \xi. \end{aligned} \quad (61)$$

Also, by using weak Wolfe–Powell line search of (4) and (6), we obtain

$$d_{k-1}^T y_{k-1} \geq (\sigma - 1) g_{k-1}^T d_{k-1}. \quad (62)$$

Suppose that $\alpha_k \geq 1$; thus

$$|\theta_k| = \left| \frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} \right| \leq \frac{\|g_k\| \alpha_{k-1} \|d_{k-1}\|}{(1 - \sigma) \|g_{k-1}\|^2} = \frac{\|g_k\| \|s_{k-1}\|}{(1 - \sigma) \|g_{k-1}\|^2} \leq \frac{\bar{\gamma} \rho}{(1 - \sigma) \gamma^2} = \zeta, \quad (63)$$

where ζ is some positive constant. Since $|\theta_k| \leq \zeta$ with $\alpha_k \geq 1$, then $|\theta_k| \leq \zeta$ with $\alpha_k < 1$. Thus $|\theta_k| \leq \zeta$ for all α_k .

By using the triangular inequality and Assumption 1, we obtain

$$\|y_{k-1} + s_{k-1}\| \leq \|y_{k-1}\| + \|s_{k-1}\| \leq 2B + 2\rho. \quad (64)$$

Let $M = \max\{\xi, \zeta\}$. Thus, the inequality in (60) can be written as follows:

$$\|\nu\| \leq B + M(2B + 2\rho). \quad (65)$$

Let

$$T = B + M(2B + 2\rho), \quad (66)$$

then

$$\|\nu\| \leq T. \quad (67)$$

From equation (54), we now have

TABLE 1: The set of test functions with the numerical results.

Function	Dim	FTCGHS				CG-Descent				DL+			
		No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time
AKIVA	2	8	20	15	0.02	10	21	11	0.02	8	20	15	0.02
ALLINITU	4	9	25	18	0.02	12	29	18	0.02	9	25	18	0.03
ARGLINC	200	2	98	98	0.08	11	106	110	0.02	5	79	78	0.06
ARWHEAD	5000	7	26	22	0.03	7	15	8	0.02	6	16	12	0.02
BARD	3	12	32	22	0.02	16	33	17	0.02	12	32	22	0.02
BDEXP	5000	2	7	7	0.02	5	11	6	0.02	2	7	7	0.02
BEALE	2	11	33	26	0.02	15	31	16	0.02	11	33	26	0.02
BIGGS3	6	79	207	144	0.02	110	231	125	0.02	79	207	144	0.02
BIGGS5	6	79	207	144	0.02	110	231	125	0.02	79	207	144	0.02
BIGGS6	6	24	64	44	0.02	27	57	31	0.02	24	64	44	0.02
BIGGSB1	5000	2500	2507	4995	2.41	2500	2507	4995	2.47	8328	8335	16651	10.86
BOX2	3	10	23	14	0.02	11	24	13	0.02	10	23	14	0.02
BOX3	3	10	23	14	0.02	11	24	13	0.02	10	23	14	0.02
BOX	10000	7	24	20	0.08	8	25	19	0.08	7	25	21	0.09
BRKMCC	2	5	11	6	0.02	5	11	6	0.02	5	11	6	0.02
BROWNAL	200	8	19	12	0.02	9	25	18	0.02	10	29	21	0.02
BROWNDEN	4	16	38	31	0.02	16	31	19	0.02	16	38	31	0.02
BROYDN7D	5000	54	100	76	0.26	1411	2810	1429	5.22	75	138	112	0.36
BRYBND	5000	32	86	62	0.15	85	174	90	0.28	149	317	174	0.55
CAMEL6	2	6	22	18	0.02	13	34	22	0.02	6	22	18	0.02
CHNROSNB	50	299	590	343	0.02	287	564	299	0.02	1009	1998	1180	0.01
CLIFF	2	10	46	39	0.02	18	70	54	0.02	10	46	39	0.01
COSINE	10000	12	56	50	0.19	11	39	32	0.19	12	52	43	0.2
CUBE	2	17	48	34	0.02	32	77	47	0.02	17	48	34	0.02
CURLY10	10000	49915	70576	79203	175.06	47808	67294	76156	171.25	68087	88635	105434	240.64
CURLY20	10000	67275	90273	11160	369.84	66587	89245	110540	377.8	88068	112535	142271	540.5
CURLY30	10000	73708	98014	123274	514	79030	102516	134682	635.36	93324	121387	183418	712.27
DENSCHNA	2	6	16	12	0.02	9	19	10	0.02	6	16	12	0.02
DENSCHNB	2	6	18	15	0.02	7	15	8	0.02	6	18	15	0.02
DENSCHNC	2	11	36	31	0.02	12	26	14	0.02	11	36	31	0.02
DENSCHND	3	14	46	40	0.02	47	98	51	0.02	14	46	40	0.02
DENSCHNE	3	12	43	38	0.02	18	49	32	0.02	12	43	38	0.02
DIXMAANA	3000	8	19	13	0.02	7	15	8	0.02	6	15	11	0.02
DIXMAANB	3000	19	111	110	0.05	6	13	7	0.02	6	15	11	0.02
DIXMAAND	3000	26	153	148	0.08	7	15	8	0.02	7	17	12	0.02
DIXMAANE	3000	254	286	481	0.28	222	239	429	0.23	394	428	764	0.5
DIXMAANH	3000	162	334	177	0.14	173	347	174	0.2	332	671	343	0.45
DIXMAANI	3000	2811	2893	5548	3.63	3856	3926	7644	4.09	3522	3623	6953	4.66
DIXMAANJ	3000	315	634	321	0.34	327	655	328	0.25	476	957	486	0.56
DIXMAANK	3000	275	555	282	0.28	283	567	284	0.22	425	854	432	0.49
DIXON3DQ	10000	10000	10007	19995	20.3	10000	10007	19995	19.48	15258	15265	30511	37.63
DJTL	2	75	1163	1148	0.02	82	917	880	0.02	75	1163	1148	0.02
DQDRTIC	5000	5	11	6	0.02	5	11	6	0.02	15	32	18	0.02
ECKERLE4LS.SIF	3	2	6	4	0.02	3	7	4	0.02	2	6	4	0.02
EGGCRATE	2	6	15	10	0.02	6	15	10	0.02	6	15	10	0.02
EIGENALS	2550	10197	18439	12170	172.55	10083	18020	12244	172.67	9534	18450	18540	185.64
EIGENBLS	2550	19030	38071	19044	311.06	15301	30603	15302	225.23	22540	45340	24700	350.43
EIGENCLS	2652	9958	18370	11529	179.77	10136	19292	11118	167.52	13450	26740	18450	203.45
ELATVIDU	2	8	32	29	0.02	11	25	15	0.02	8	32	29	0.02
ENGVAL1	5000	24	53	46	0.08	27	50	36	0.06	21	48	37	0.06
ENGVAL2	3	26	73	55	0.02	26	61	37	0.02	26	73	55	0.02
ENSOLS	9	22	47	27	0.02	23	45	26	0.02	22	47	27	0.02
EXTROSNB	1000	227	490	280	0.02	3808	7759	3982	1.05	7182	12662	10680	2.39
exp2	2	7	16	9	0.02	8	17	9	0.02	7	16	9	0.02
FBRAINLS	2	9	27	21	0.03	10	23	14	0.02	9	27	21	0.02
FBRAIN2LS	4	79	259	204	0.47	118	339	248	0.66	79	259	204	0.47
FLETCHBV2	5000	1	1	1	0.02	1	1	1	0.02	1	1	1	0.02
FLETCHCR	1000	227	446	254	0.02	152	290	176	0.05	199	412	217	0.08

TABLE 1: Continued.

Function	Dim	FTCGHS				CG-Descent				DL+			
		No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time
GENHUMPS	5000	8	107	106	0.37	6475	12964	6493	19.89	4938	14763	10180	25.27
GROWTHLS	3	109	431	369	0.02	156	456	319	0.02	109	431	369	0.02
GULF	3	33	95	72	0.02	37	84	48	0.02	33	95	72	0.02
HAHN1LS	7	5	56	53	0.02	37	121	86	0.02	5	56	53	0.02
HAIRY	2	17	82	68	0.02	36	99	65	0.02	17	82	68	0.02
HATFLDD	3	17	49	37	0.02	20	43	24	0.02	17	49	37	0.02
HATFLDE	3	13	37	30	0.02	30	72	45	0.02	13	37	30	0.02
HATFLDFL	3	21	68	54	0.02	39	92	55	0.02	21	68	54	0.02
HEART6LS	6	375	1137	876	0.02	684	1576	941	0.02	375	1137	876	0.02
HIELOW	3	13	30	21	0.03	14	30	16	0.02	13	30	21	0.05
HILBERTA	2	2	5	3	0.02	2	5	3	0.02	2	5	3	0.02
HILBERTB	10	4	9	5	0.02	4	9	5	0.02	4	9	5	0.02
HIMMELBB	2	4	18	18	0.02	10	28	21	0.02	4	18	18	0.02
HIMMELBG	2	7	22	17	0.02	8	20	13	0.02	7	22	17	0.02
HIMMELBH	2	5	13	9	0.02	7	16	9	0.02	5	13	9	0.02
HYDCAR6LS.SIF	29	120	242	123	0.02	14401	29028	14875	0.45	1001	2027	1174	0.03
INDEF	5000	1	46	147	0.39	1	46	147	0.44	1	46	147	0.42
INTEQNELS.SIF	12	7	15	8	0.02	6	13	7	0.02	6	13	7	0.02
JIMACK	3549	8332	16666	8334	1212	8314	16629	8315	1169.4	11978	23971	12235	1732.94
JUDGE	2	9	24	18	0.02	10	23	13	0.02	9	24	18	0.02
KOWOSB	4	16	46	32	0.02	17	39	23	0.02	16	46	32	0.02
KSSLS	1000	2	15	15	0.56	10	25	16	0.58	6	19	16	0.55
LANCZOS1LS	6	61	177	135	0.02	148	325	181	0.02	61	177	135	0.02
LANCZOS2LS	6	60	169	125	0.02	169	379	215	0.02	60	169	125	0.02
LANCZOS3LS	6	61	164	118	0.02	179	392	219	0.02	61	164	118	0.02
LIARWHD	5000	11	36	31	0.05	21	45	25	0.06	15	41	31	0.03
LSC1LS	3	31	108	89	0.02	36	101	71	0.02	31	108	89	0.02
LSC2LS	3	37	106	86	0.02	54	119	67	0.02	37	106	86	0.02
LUKSAN11LS	100	957	1957	1007	0.03	955	1912	957	0.03	2434	5355	3048	0.13
LUKSAN12LS	98	44	130	98	0.02	160	302	233	0.02	252	529	407	0.01
LUKSAN13LS	98	90	182	168	0.02	84	158	121	0.02	142	279	243	0.02
MANCINO	100	12	29	18	0.09	11	23	12	0.06	11	23	12	0.06
MARATOSB	2	589	2885	2585	0.02	1145	3657	2779	0.02	589	2885	2585	0.02
MEXHAT	2	14	59	55	0.02	20	56	39	0.02	14	59	55	0.02
MEYER3	3	19	76	63	0.02	19	67	52	0.02	19	76	63	0.02
MGH09LS	4	25	82	72	0.02	57	137	86	0.02	25	82	72	0.02
MGH10LS	3	1082	4052	4968	0.03	1134	4464	5357	0.03	1082	4052	4968	0.02
MGH10SLS	3	19	112	102	0.03	146	505	401	0.03	19	112	102	0.02
MGH17LS	5	84	323	265	0.03	228	564	363	0.02	84	323	265	0.02
MISRA1BLS.SIF	2	26	113	101	0.03	35	139	117	0.02	26	113	101	0.02
MODBEALE.SIF	20000	233	481	254	4.3	517	1157	1010	13.17	224	473	304	4.89
MOREBV	5000	161	168	317	0.31	161	168	317	0.42	117	124	229	0.23
MSQRTALS	1024	2795	5599	2806	8.2	2905	5815	2911	8.89	8953	17316	9581	28.81
NCB20	5010	588	1314	801	7.02	879	1511	1463	11.38	11026	20505	15341	129.2
NONDIA	5000	7	17	11	0.01	7	25	20	0.03	7	25	19	0.03
NONDQUAR	5000	69	261	227	0.16	1942	3888	1947	2.3	2349	4787	2488	2.83
PALMER1C	8	12	27	28	0.02	11	26	26	0.02	12	27	28	0.02
PALMER1D	7	10	24	23	0.02	11	25	25	0.02	10	24	23	0.02
PALMER2C	8	11	21	22	0.02	11	21	21	0.02	11	21	22	0.02
PALMER3C	8	11	21	21	0.02	11	20	20	0.02	11	21	21	0.02
PALMER4C	8	11	21	21	0.02	11	20	20	0.02	11	21	21	0.02
PALMER5C	6	6	13	7	0.02	6	13	7	0.02	6	13	7	0.02
PALMER6C	8	11	24	24	0.02	11	24	24	0.02	11	24	24	0.02
PALMER7C	8	11	20	20	0.02	11	20	20	0.02	11	20	20	0.02
PALMER8C	8	11	19	17	0.02	11	18	17	0.02	11	19	19	0.02
PARKCH	15	672	1385	1128	0.02	672	1385	1128	0.02	412	982	611	16.02
PENALTY1	1000	2	28	28	0.02	28	69	44	0.02	14	51	43	0.02
PENALTY2	200	191	225	369	0.02	191	221	354	0.03	337	480	758	0.06

TABLE 1: Continued.

Function	Dim	FTCGHS				CG-Descent				DL+			
		No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter	No. Fun.Ev	No. Grad.Ev	CPU time	No. Iter.	No. Fun.Ev	No. Grad.Ev	CPU time
PENALTY3	200	51	148	110	0.84	99	285	219	1.74	102	346	290	2.19
POWELLBSLS	2	50	211	234	0.02	61	247	246	0.02	50	211	234	0.02
POWER	10000	355	724	376	0.59	372	754	384	0.58	356	733	391	0.58
POWERSUM	4	4	10	6	0.59	5	11	6	0.02	4	10	6	0.02
QING	100	67	134	85	0.03	68	135	84	0.02	85	179	96	0.02
RAT43LS.SIF	4	44	156	122	0.02	54	145	97	0.02	44	156	122	0.02
RECIPELS.SIF	3	16	49	38	0.02	27	58	32	0.02	16	49	38	0.02
ROSENBR	2	28	84	65	0.02	34	77	44	0.02	28	84	65	0.02
S308	2	7	21	17	0.02	8	19	12	0.02	7	21	17	0.02
SCHMVETT	5000	38	69	52	0.17	43	73	60	0.02	59	103	88	0.28
SENSORS	100	25	68	47	0.38	21	50	34	0.02	24	71	53	0.47
SINQUAD	5000	14	43	34	0.08	14	40	33	0.08	13	46	38	0.09
SISSER	2	5	19	19	0.08	6	18	14	0.02	5	19	19	0.02
SROSENBR	5000	9	25	19	1.14	11	23	12	0.02	9	23	15	0.02
STRATEC	10	170	419	283	6.33	462	1043	796	19	170	419	283	6.3
STRTCHDV.SIF	10	12	38	32	0.02	16	35	20	0.02	12	38	32	0.02
TESTQUAD	5000	1580	1587	3155	1.52	1577	1584	3149	1.28	20325	20361	40674	21.61
TOINTGOR	50	118	216	154	0.02	135	233	174	0.02	192	348	270	0.02
TOINTGSS	5000	4	10	7	0.02	4	9	5	0.02	4	9	5	0.02
TOINTQOR	50	29	36	53	0.02	29	36	53	0.02	49	56	93	0.02
TRIDIA	5000	780	787	1555	1.03	782	7889	1559	0.89	4699	4721	9408	6.38
TRIGON1.SIF	10	19	41	22	0.02	22	45	23	0.02	19	41	22	0.02
VANDANMSLS.SIF	22	8	25	18	0.02	5	12	7	0.02	5	11	6	0.02
VARDIM	200	1	4	4	0.02	10	21	11	0.02	9	20	15	0.02
VAREIGVL	50	27	60	37	0.02	23	47	21	0.02	28	71	51	0.02
VESUVIALS	8	1262	1954	3155	0.02	1519	2317	4111	1.56	1262	1954	3155	1.22
VESUVIOULS	8	79	211	173	0.02	80	180	131	0.06	79	211	173	0.09
VIBRBEAM	8	98	255	174	0.02	138	323	199	0.02	98	255	174	0.02
YATP1CLS	1E+05	14	41	31	5.76	23	53	31	6.13	17	48	36	7.12
YATP2CLS	1E+05	7	29	24	2.78	8	23	17	2.09	7	27	22	2.67
YFITU	3	68	208	167	0.02	84	197	118	0.02	68	208	167	0.03
ZANGWIL2	2	1	3	2	0.02	1	3	2	0.02	1	3	2	0.02

$$\|u_k - u_{k-1}\| \leq 2w. \quad (68)$$

By using equations (53) and (54), we obtain the following:

$$\sum_{k=0}^{\infty} \|u_{k+1} - u_k\|^2 \leq 4 \sum_{k=0}^{\infty} \|w\|^2 \leq 4T^2 \sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} < \infty, \quad (69)$$

which completes the proof.

The following property, which is referred to as Property 1, was presented by Gilbert and Nocedal in [10]. $\square$

Property 1. Consider a method of the form of (2) and (3). Assume that (54) is satisfied for all $k \geq 1$. Then, the CG method has Property 1 if there exist constants $b > 1$ and $\lambda > 0$

such that, for all $k \geq 1$, $|\eta_k| \leq b$ and if $\|s_k\| \leq \lambda$, we obtain $|\eta_k| \leq (1/2b)$.

Lemma 3. Consider the CG method of the form of (1) and (2) with η_k^+ , where the step size satisfies SWP line search of (4) and (5). If equation (54) holds, then η_k^+ possesses Property*. Namely, suppose that (54) holds; then there exist $b > 1$ and $\lambda > 0$ such that, for all $k \geq 1$, we have $|\eta_k^+| \leq b$ and if $\|s_k\| \leq \lambda$, we obtain $|\eta_k^+| \leq (1/2b)$.

Proof. As a result, set $b = (2(L+t)\bar{\gamma}B)/(1-\sigma)\gamma^2 \geq 1$ and $\lambda = ((1-\sigma)c\gamma^2/2b(L+t)\bar{\gamma})$.

Using SWP of (4) and (5) with equation (54), we obtain

$$|\eta_k^+| \leq \left| \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} \right| + t \left| \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right| \leq \frac{L\|g_k\|\|s_{k-1}\| + t\|g_k\|\|s_{k-1}\|}{(1-\sigma)\gamma^2} \leq \frac{2(L+t)\bar{\gamma}B}{(1-\sigma)\gamma^2} = b > 1. \quad (70)$$

FIGURE 2: Performance measure based on the number of iterations.

FIGURE 3: Performance measure based on the function evaluation.

Also, if $\|s_k\| \leq \lambda$, we obtain

$$\begin{aligned}
 |\eta_k^+| &\leq \left| \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} \right| + t \left| \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right| \leq \frac{L \|g_k\| \|s_{k-1}\| + t \|g_k\| \|s_{k-1}\|}{(1-\sigma)\gamma^2} \\
 &\leq \frac{(L+t)\bar{\gamma}\lambda}{(1-\sigma)\gamma^2},
 \end{aligned} \tag{71}$$

where

$$|\eta_k^+| \leq \frac{1}{2b}. \tag{72}$$

By using Lemma 4.1 and Lemma 4.2 in [10], we obtain the following result. $\square$

Theorem 4. Let the sequences $\{x_k\}$ and $\{d_k\}$ be generated by (2) and (3) with the CG method in equation (28), where the step size satisfies (4) and (5). Then, we obtain the results by using Lemmas 2 and 3, as well as Lemmas 4.1 and 4.2 in [10], where $\liminf_{k \rightarrow \infty} \|g_k\| = 0$.

4. Numerical Results and Discussion

To test the new search direction (28), we selected some test functions in Table 1 from CUTer [37]. A comparison with other popular and strong CG coefficients is employed. The comparison includes the CG-Descent 6.8 and DL + CG formula based on the CPU time, number of iterations, number of function evaluations, and t number of gradient evaluations. We use the SWP line search to obtain the step

FIGURE 4: Performance measure based on the CPU time.

length with $\delta = 0.01$ and $\sigma = 0.1$ for all algorithms except CG-Descent 6.8. We use approximate WWP line search as mentioned by the authors. The results of FTCGHS and DL + CG methods are obtained by running the modified code of CG-Descent; the code can be obtained from the Hager web page: <https://people.clas.ufl.edu/hager/software/>.

The norm of the gradient was employed as the stopping criterion, specifically $\|g_k\| \leq 10^{-6}$ for all algorithms. The host computer is AMD A4-7210 APU Radeon R3 Graphics, where the installed memory is 4 GB with operating system Ubuntu 20.04.2.0 LTS. The results are shown in Figures 2–5, in which a performance measure introduced by Dolan and More [38] was employed. In Table 1, the following notations are used:

No. Iter indicates number of iterations

No. Fun.Ev indicates number of function evaluations

No. Grad.Ev indicates number of gradient evaluations

Figure 2 shows that FTCGHS strongly outperformed all methods in the number of iterations. Meanwhile, Figure 3 presents the number of function evaluations, and we can note that the FTCGHS method also strongly outperforms CG-Descent 6.8 and DL+ methods. From Figures 4 and 5, we can note that FTCGHS method slightly outperforms CG-Descent and DL+, since we use SWP line search. However, let the SWP line search in equation (5) be extended as follows:

$$\begin{aligned} \sigma_1 g_k^T d_k &\leq g(x_k + \alpha_k d_k)^T d_k \leq \sigma_2 g_k^T d_k, \\ 0 < \delta < \sigma_1 &\leq \sigma_2 < 1. \end{aligned} \quad (73)$$

Then, the results in Figures 4 and 5 will be more efficient. For more about equation (73), the reader can refer to [39].

In addition, we present an example for two-dimensional three-hump camel back function, which is given by the following formula:

FIGURE 5: Performance measure based on the gradient evaluation.

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} - x_1x_2 + x_2^2. \quad (74)$$

Here, we have the following:

Number of variables (n): 2

Initial points: (1, 1), (5, 5), (10, 10), and (15, 15)

This function has three local minima, one of which is global. Therefore, this function is a multimodal function usually used to test global minima. The global minimum is $x^* = (0, 0)$ and the function value is $f(x^*) = 0$. This function looks like the back of an upside-down camel. See Figure 6 for the three-dimensional graph.

FIGURE 6: Three-hump camel back function in 3D.

5. Conclusions

In this paper, we proposed a new four-term CG method as presented in equation (28) by using the following direction:

$$g_k, d_{k-1}, y_{k-1}, s_{k-1}. \quad (75)$$

The search direction inherits the following properties:

- (1) It presents four-term CG method
- (2) It satisfies the sufficient descent condition
- (3) The convergence analyses for convex and nonconvex functions are obtained
- (4) The numerical results show that the new modification outperforms popular methods such as CG-Descent and DL+ methods
- (5) The new search direction can be developed by using different values of t

In addition, we will attempt to extend the SWP line search in the future by different values of sigma to reduce the number of gradient evolutions and CPU time.

Data Availability

The data used to support the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

- [1] P. Wolfe, "Convergence conditions for ascent methods," *SIAM Review*, vol. 11, no. 2, pp. 226–235, 1969.
- [2] P. Wolfe, "Convergence conditions for ascent methods. II: some corrections," *SIAM Review*, vol. 13, no. 2, pp. 185–188, 1971.
- [3] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *Journal of Research of the National Bureau of Standards*, vol. 49, no. 6, pp. 409–436, 1952.
- [4] E. Polak and G. Ribiere, "Note sur la convergence de méthodes de directions conjuguées," *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, vol. 3, pp. 35–43, 1969.
- [5] Y. Liu and C. Storey, "Efficient generalized conjugate gradient algorithms, part 1: theory," *Journal of Optimization Theory and Applications*, vol. 69, no. 1, pp. 129–137, 1991.
- [6] R. Fletcher and C. M. Reeves, "Function minimization by conjugate gradients," *The Computer Journal*, vol. 7, no. 2, pp. 149–154, 1964.
- [7] R. Fletcher, *Practical Method of Optimisation. Unconstrained Optimisation*, Wiley, Hoboken, NJ, USA, 1997.
- [8] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM Journal on Optimization*, vol. 10, no. 1, pp. 177–182, 1999.
- [9] M. J. D. Powell, "Nonconvex minimization calculations and the conjugate gradient method," *Lecture Notes in Mathematics*, vol. 1066, pp. 122–141, 1984.
- [10] J. C. Gilbert and J. Nocedal, "Global convergence properties of conjugate gradient methods for optimization," *SIAM Journal on Optimization*, vol. 2, no. 1, pp. 21–42, 1992.
- [11] G. Zoutendijk, "Nonlinear programming," *Computational Methods. Integer Nonlinear Program*, vol. 143, pp. 37–86, 1970.
- [12] M. Al-Baali, "Descent property and global convergence of the Fletcher-Reeves method with inexact line search," *IMA Journal of Numerical Analysis*, vol. 5, no. 1, pp. 121–124, 1985.
- [13] Z. Wei, S. Yao, and L. Liu, "The convergence properties of some new conjugate gradient methods," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1341–1350, 2006.
- [14] A. Alhawarat, Z. Salleh, M. Mamat, and M. Rivaie, "An efficient modified Polak-Ribière-Polyak conjugate gradient method with global convergence properties," *Optimization Methods and Software*, vol. 32, no. 6, pp. 1299–1312, 2017.
- [15] P. Kaelo, P. Mtshali, and M. V. Thuto, "A globally convergent hybrid conjugate gradient method with strong Wolfe conditions for unconstrained optimization," *Mathematical Sciences*, vol. 14, no. 1, pp. 1–9, 2020.
- [16] Y.-H. Dai and L. Z. Liao, "New conjugacy conditions and related nonlinear conjugate gradient methods," *Applied Mathematics and Optimization*, vol. 43, no. 1, pp. 87–101, 2001.
- [17] W. W. Hager and H. Zhang, "A new conjugate gradient method with guaranteed descent and an efficient line search," *SIAM Journal on Optimization*, vol. 16, no. 1, pp. 170–192, 2005.
- [18] W. W. Hager and H. Zhang, "The limited memory conjugate gradient method," *SIAM Journal on Optimization*, vol. 23, no. 4, pp. 2150–2168, 2013.

- [19] N. Andrei, "A simple three-term conjugate gradient algorithm for unconstrained optimization," *Journal of Computational and Applied Mathematics*, vol. 241, pp. 19–29, 2013.
- [20] N. Andrei, "On three-term conjugate gradient algorithms for unconstrained optimization," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 6316–6327, 2013.
- [21] S. Babaie-Kafaki and R. Ghanbari, "Two modified three-term conjugate gradient methods with sufficient descent property," *Optimization Letters*, vol. 8, no. 8, pp. 2285–2297, 2014.
- [22] S. Deng and Z. Wan, "A three-term conjugate gradient algorithm for large-scale unconstrained optimization problems," *Applied Numerical Mathematics*, vol. 92, pp. 70–81, 2015.
- [23] J. K. Liu, J. L. Xu, and L. Q. Zhang, "Partially symmetrical derivative-free Liu-Storey projection method for convex constrained equations," *International Journal of Computer Mathematics*, vol. 96, no. 9, pp. 1787–1798, 2019.
- [24] J. K. Liu, Y. X. Zhao, and X. L. Wu, "Some three-term conjugate gradient methods with the new direction structure," *Applied Numerical Mathematics*, vol. 150, pp. 433–443, 2020.
- [25] S. Yao, L. Ning, H. Tu, and J. Xu, "A one-parameter class of three-term conjugate gradient methods with an adaptive parameter choice," *Optimization Methods and Software*, vol. 35, no. 6, pp. 1051–1064, 2020.
- [26] G. Yuan, J. Lu, and Z. Wang, "The PRP conjugate gradient algorithm with a modified WWP line search and its application in the image restoration problems," *Applied Numerical Mathematics*, vol. 152, pp. 1–11, 2020.
- [27] S. Yao, Y. Wu, J. Yang, and J. Xu, "A three-term gradient descent method with subspace techniques," *Mathematical Problems in Engineering*, vol. 2021, Article ID 8867309, 7 pages, 2021.
- [28] A. Alhawarat, N.-T. Trung, and Z. Salleh, "Conjugate gradient Method: a developed version to resolve unconstrained optimization problems," *Journal of Computer Science*, vol. 16, no. 9, pp. 1220–1228, 2020.
- [29] Z. Dai and J. Kang, "Some new efficient mean-variance portfolio selection models," *International Journal of Finance & Economics*, 2021.
- [30] Z. Dai, J. Kang, and F. Wen, "Predicting stock returns: a risk measurement perspective," *International Review of Financial Analysis*, vol. 74, Article ID 101676, 2021.
- [31] Z. Dai and F. Wen, "Some improved sparse and stable portfolio optimization problems," *Finance Research Letters*, vol. 27, pp. 46–52, 2018.
- [32] Z. Dai, X. Chen, and F. Wen, "A modified Perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations," *Applied Mathematics and Computation*, vol. 270, pp. 378–386, 2015.
- [33] A. Alhawarat, M. Mamat, M. Rivaie, and Z. Salleh, "An efficient hybrid conjugate gradient method with the strong Wolfe-Powell line search," *Mathematical Problems in Engineering*, vol. 2015, Article ID 103517, 7 pages, 2015.
- [34] M. Al-Baali, A. Caliciotti, G. Fasano, and M. Roma, "A class of approximate inverse preconditioners based on krylov-subspace methods for large-scale nonconvex optimization," *SIAM Journal on Optimization*, vol. 30, no. 3, pp. 1954–1979, 2020.
- [35] A. Alhawarat, T. T. Nguyen, R. Sabra, and Z. Salleh, "An efficient modified AZPRP conjugate gradient method for large-scale unconstrained optimisation problem," *Journal of Mathematics*, vol. 2021, Article ID 6692024, 9 pages, 2021.
- [36] I. A. Masmali, Z. Salleh, Z. Salleh, and A. Alhawarat, "A decent three term conjugate gradient method with global convergence properties for large scale unconstrained optimization problems," *AIMS Mathematics*, vol. 6, no. 10, pp. 10742–10764, 2021.
- [37] I. Bongartz, A. R. Conn, N. Gould, and P. L. Toint, "CUTE: constrained and unconstrained testing environment," *ACM Transactions on Mathematical Software*, vol. 21, no. 1, pp. 123–160, 1995.
- [38] E. D. Dolan and J. J. Moré, "Benchmarking optimization software with performance profiles," *Mathematical Programming*, vol. 91, no. 2, pp. 201–213, 2002.
- [39] M. Al-Baali, Y. Narushima, and H. Yabe, "A family of three-term conjugate gradient methods with sufficient descent property for unconstrained optimization," *Computational Optimization and Applications*, vol. 60, no. 1, pp. 89–110, 2015.