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Research Article

An Efficient Model for the Approximation of Intuitionistic Fuzzy Sets in terms of Soft Relations with Applications in Decision Making

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The basic notions in rough set theory are lower and upper approximation operators defined by a fixed binary relation. This paper proposes an intuitionistic fuzzy rough set (IFRS) model which is a combination of intuitionistic fuzzy set (IFS) and rough set. We approximate an IFS by using soft binary relations instead of fixed binary relations. By using this technique, we get two pairs of intuitionistic fuzzy (IF) soft sets, called the upper approximation and lower approximation with respect to foresets and aftersets. Properties of newly defined rough set model (IFRS) are studied. Similarity relations between IFS with respect to this rough set model (IFRS) are also studied. Finally, an algorithm is constructed depending on these approximations of IFSs and score function for decision-making problems, although a method of decision-making algorithm has been introduced for fuzzy sets already. But, this new IFRS model is more accurate to solve the problem because IFS has degree of nonmembership and degree of hesitant.

1. Introduction

To control uncertainty usually, probability theory is deliberated as an applicable tool, but for its practical work, a randomly stable system must be a very basic requirement. To establish such kind of system, a lot of time is needed. In today's speedy life, as everyone has shortage of time, in an unreliable environment, researchers have introduced many updated methods and techniques to solve uncertainties. Rough set (RS), fuzzy set (FS), and soft set (SS) are expressive methods to control uncertainty, vagueness, and incompleteness in the information systems. The abovementioned sets have their own operations and properties. These sets are much applicable in real life, computer science, and artificial intelligence. We are encountered by different types of real-world problems everyday which have uncertainty and vagueness rather than preciseness. Precise and complete reasoning would not be possible if our information data are inexact, vague, and incomplete. Recently, the gap between traditional mathematics with precise concepts and the world full of uncertainty become much smaller than earlier. In different fields, nature of vagueness can be different. Researchers are very active and interested to study many newly defined theories to solve this problem [1].

Fuzzy set (FS) theory was introduced by Zadeh in 1965 [2] which is a very revolutionary attempt to deal with uncertainty. The FS theory is a generalization of classical set theory. It has greater richness in application than the classical set theory. It has ability to translate human linguistic terms mathematically. Although FS has the membership degree, but often the nonmembership degree is required also to handle critical situations in real-world problems. Atanassov presented IFS in 1986 [3, 4] which is the generalization of FS. The IFS describes the fuzzy characteristic of things more comprehensively than FS and thus is a powerful and successful tool to express fuzzy information of real-world problems. Elements of IFS are written in the form of ordered pairs and these ordered pairs are said to be intuitionistic fuzzy numbers or intuitionistic fuzzy values. Each intuitionistic fuzzy value is characterized by a membership degree, a nonmembership degree, and a hesitant degree. The sum of these three degrees is equal to 1. The IFSs are important in fuzzy mathematics due to its wide applications in real life, such as in pattern recognition, career determination, medical diagnosis, electoral system, and machine learning [5, 6].

In 1982, rough set (RS) theory introduced by Pawlak [7] is one of the untraditional methods to control uncertainty. A subset distinguished by lower approximation and upper approximation is known as RS. Pawlak used equivalence relations to prepare approximations in a set [7, 8]. However, the equivalence relations in RS seem to be very restrictive that may limit the scope of RS model. For instance, a frequent and significant problem in the medical field is the stomach pain in the children, which can be expected to some reasons and it is a demanding job to diagnose the reason correctly. The RS theory can help the doctors to diagnose the correct reason by discharge comments. Also, it has a broad scale of applications in image processing, knowledge finding, recognition of optical characters, and pattern recognition and to recognize various facial expressions in artificial intelligence, in data clustering, in decision-making problems with precised accuracy, and in business and finance because of their capacity to find the rule induction and knowledge. Figure 1 shows the graphical representation of RS with lower approximation and upper approximation. Upper approximation is a set which has elements having possible belonging with the target set and lower approximation set has objects having positive belonging with the target set [9–11].

Figure 1 shows that the target set is in red line circle, yellow box is the lower approximated set, blue box is the upper approximated set, and green box is the universal set.

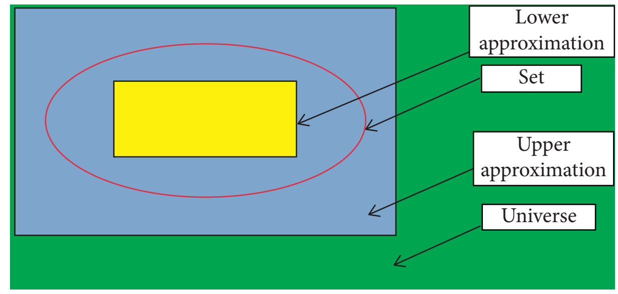

Figure 1: Graphical representation of rough set.

In 1999, Molodtsov [12, 13] introduced the key notion of SS to deal with uncertainty. This new technique is free from the problems related with existing techniques of uncertainty. An appropriate number of parameters is available in this theory which makes it possible. The SS theory has a wide variety of applications in many fields, such as operational research, the smoothness of functions, Riemann integration, and game theory. Moreover, the SSs have a rich number of operations which are very helpful to deal with uncertainty in different types of situations. The concept of parametric reduction in SSs has been studied by many authors [14, 15]. Ali et al. [16] initiated some new operations in SS theory. Abbas et al. [17] initiated various generalized operations in SS theory by applying many relaxed conditions on parameters. Applications of SSs can be found in [14, 18–26].

Many extensions of SSs have been presented such as probabilistic SS theory, bijective SS theory, fuzzy bipolar SS theory, and intuitionistic fuzzy soft set (IFSS) theory. Akram et al. [27] introduced three hybrid models, namely, N-soft rough IF sets, IF N-soft sets, and IF N-soft RSs with real-life applications of decision-making algorithm. Alcantud et al. [28] presented covering-based fuzzy RS model by t-norm and fuzzy logical implicator. This fuzzy RS model is useful to characterize the covering-based optimistic and pessimistic multigranulation. They also presented two kinds of decision-making methods to analyze this model theoretically. Alcantud et al. [29] presented a tool which aggregates infinite chains of IF sets over time. They presented IF sets along an indefinitely long number of periods by using score and accuracy degrees of temporal IF elements.

A parameterized collection of ordinary binary relations is called a soft binary relation on a universe and this is generalization of binary relations. In RS theory, rough approximations just address single binary relations but rough approximations by using soft binary relations can deal with different binary relations. The idea of soft relation over U is given by Feng et al. [30] in 2013. Babitha and Sunil [31] presented some results on soft set relations. Many authors have generalized the notion of Pawlak RS model by using dominance relations, covering relations, similarity relations, tolerance relations, fuzzy relations, neighbourhood relations, and other indiscernibility relations, see [28, 32–42].

1.1. Related Works. Feng et al. [1] presented a hybrid model of SSs which is rough approximation of SS. They used an SS instead of an equivalence relation to granulate the universe. In the result, soft approximation space and soft RSs have been introduced as a deviation of rough approximation space. Furthermore, they also extended Dubois and Prade's RSs by approximating a FS in a soft approximation space and called soft rough FSs. Feng et al. have done a lot of work by combining SSs, RSs, and FSs and defined new models [1, 43, 44]. Ali and Shabir [45] studied fuzzy SSs. Roy and Maji [46] initiated the study of fuzzy SSs. In 2020, Bashir et al. introduced a model of rough fuzzy ternary semigroups based on three-dimensional congruence relation [47]. Many authors introduced RS approximations in IFSs [48, 49]. Kanwal and Shabir approximated the ideals and fuzzy sets in semigroups based on soft relations [50, 51]. Later in 2020, Shabir and Kanwal used soft relations to define lower and upper approximations of a set in [51]. We have generalized the concept in [51] in terms of IFS by introducing nonmembership degree. Our model IFRS gives approximations corresponding to every attribute or parameter. In this way, we get more accurate results by reducing errors than all previous ones.

1.2. Connection of IFRS Model with Rough Sets. The IF set theory deals mainly with vagueness, while the RS theory deals with incompleteness. The study of the combination of these two theories is useful to deal with impreciseness. It means that the rough IF sets are useful to deal with both vagueness and incompleteness. Recently, RS approximations have been discussed in IF environment. In the result, IF rough sets, rough IF sets, and generalized IF rough sets have been presented. Zhou et al. studied different relation-based IF approximation operators in the axiomatic and constructive approaches [52].

1.3. Innovative Contribution. Some researchers have words that one theory is better than other theory to deal with inexact data. Majority of researchers admitted that RSs and FSs are very closely related with each other, but distinction is there that they model different types of vagueness. The RS is a coarsely described crisp set, whereas the FS is viewed as a class with blunt boundaries. Since the FS has only membership degree but the IF set has also degree of nonmembership which is more useful in medical science. To diagnose a decease, IF environment is better than fuzzy environment due to the presence of nonmembership degree. In 2020, Shabir et al. [51] presented a model of RS which is combination of SS and FS. They approximated FS in terms of soft binary relations. In our paper, we considered an IF set instead of a FS and an IF set has more accurate results than a FS in medical science. The IFS with other algebraic structures generalizes hybrid models which are very useful in medical science, computer science, and other fields. Samanta and Mondal [53] presented the IF rough set (A, B) which is generalized IFS in terms of fuzzy rough sets A and B. On the other hand, an IF rough set (A, B) presented by Chakrabarty et al. [48] is the generalization of fuzzy rough set in terms of IFSs A and B. Zhou [54] proposed IF rough sets induced by IF approximation spaces and discussed their properties. Recently, IF set has combined with rough set approximations and resulting sets are called IF rough sets and rough IF and generalized IF rough sets. In axiomatic and constructive approach, Zhou et al. presented a useful framework and studied different IF rough approximation operators by using a special type of IF triangular norm min. Zhou (2014) presented IF soft rough sets and soft rough If sets. These newly presented models were very useful as new approaches for decision-making problems. By integrating IF sets with SSs, Maji et al. presented IF soft sets. Jiang et al. discussed an approach of IF soft sets-based decision making and they also presented interval-valued IF soft set model. IF soft set is an important combination of IF sets and SSs. It makes more accurate and realistic descriptions of materialistic world. Zhang presented a useful model combining IF soft sets with RSs [1, 52, 55, 56].

1.4. Motivation. In the present paper, we extend the concept given by Shabir et al. [51] in terms of FSs. We use IFSs instead of FSs which is more valuable to manage uncertainty in many scientific fields, such as medical diagnosis and pattern recognition. Our proposed model based on soft relations is very useful due to importance of IFS in real-life situations [5, 6]. In our research, we propose a decision-making algorithm by using our model based on soft relations. Then, we present an example to illustrate the validity of our proposed decision-making algorithm. Our IFRS model is the combination of RS, IFS, and SS which is helpful to control impreciseness and uncertainty.

1.5. Organization of the Paper. The pattern of this paper is as follows. In Section 2, some foundational concepts are identified with FSs, IFS, RSs, and SSs and soft binary relations are described. In Section 3, we presented IFRS model based on soft relations and discussed some properties. Soft similarity relations have been examined in Section 4. In Section 5, we gave an approach to a decision-making problem based on an IFS. Moreover, an example is presented to illustrate this decision-making algorithm in Section 6.

2. Preliminaries and Basic Concepts

In this section, some basic notions about binary relations, IFS, soft sets, and intuitionistic fuzzy soft sets are given. Throughout this paper, U_1 and U_2 represent two nonempty finite sets unless stated otherwise.

A binary relation $\mathscr{R}$ from U_1 to U_2 is a subset of $U_1 \times U_2$ and a subset of $U \times U$ is called a binary relation on U. If $\mathscr{R}$ is a binary relation on U, then $\mathscr{R}$ is said to be reflexive if $(u, u) \in \mathscr{R}$ for all $u \in U$, symmetric if $(u, v) \in \mathscr{R}$ implies $(v, u) \in \mathscr{R}$ for all $u, v \in U$, and transitive if $(u, v) \in \mathscr{R}$ and $(v, w) \in \mathscr{R}$ imply $(u, w) \in \mathscr{R}$ for all $u, v, w \in U$. If a binary relation $\mathscr{R}$ is reflexive, symmetric, and transitive, then it is called an equivalence relation. An equivalence relation partitions the set into disjoint classes.

Let U be a nonempty universe. An IF set M in the universe U is an object having the form $M = \{\langle x, \mu_M(x), \gamma_M(x)\rangle \colon x \in U\}$, where $\mu_M \colon U \longrightarrow [0,1]$ and $\gamma_M \colon U \longrightarrow [0,1]$, satisfying $0 \leq \mu_M(x) + \gamma_M(x) \leq 1$ for all $x \in U$. The values $\mu_M(x)$ and $\gamma_M(x)$ are called degree of membership and degree of nonmembership of $x \in U$ to M, respectively. The number $\pi_M(x) = 1 - \mu_M(x) - \gamma_M(x)$ is called the degree of hesitancy of $x \in U$ to M. The collection of all IFSs in U is denoted by $\mathrm{IF}(U)$. In the remaining paper, we shall write an IFS by $M = \langle \mu_M, \gamma_M \rangle$ instead of $M = \{\langle x, \mu_M(x), \gamma_M(x)\rangle \colon x \in U\}$. Let $M = \langle \mu_M, \gamma_M \rangle$ and $N = \langle \mu_N, \gamma_N \rangle$ be two IFSs in U. Then, $M \subseteq N$ if and only if $\mu_M(x) \leq \mu_N(x)$ and $\gamma_N(x) \leq \gamma_M(x)$ for all $x \in U$. Two IFSs M and N are said to be equal if and only if $M \subseteq N$ and $N \subseteq M$. The union and intersection of two IFSs M and N in U are denoted and defined by $M \cap N = \langle \mu_M \cap \mu_N, \gamma_M \cup \gamma_N \rangle$ and $M \cup N = \langle \mu_M \cup \mu_N, \gamma_M \cap \gamma_N \rangle$, where $(\mu_M \cap \mu_N)(x) = \inf\{\mu_M(x), \mu_N(x)\}$, $(\gamma_M \cup \gamma_N)(x) = \sup\{\gamma_M(x), \gamma_N(x)\}$, $(\mu_M \cup \mu_N)(x) = \sup\{\mu_M(x), \mu_N(x)\}$, $(\gamma_M \cap \gamma_N)(x) = \inf\{\gamma_M(x), \gamma_N(x)\}$.

Next, we define two special types of (IFSs) as follows:

The IF universe set $U = 1_U = \langle 1, 0 \rangle$ and IF empty set $\varnothing = 0_U = \langle 0, 1 \rangle$, where $1(x) = 1$ and $0(x) = 0$ for all $x \in U$. The complement of an IFS $M = \langle \mu, \gamma \rangle$ is denoted and defined as $M^c = \langle \gamma, \mu \rangle$ [3].

For a fixed $x \in U$, the pair $(\mu_M(x), \gamma_M(x))$ is called IF value or IF number. In order to define the order between two IFNs, Chen and Tan [57] presented the score function as $S(x) = \mu_M(x) - \gamma_M(x)$ and Hong and Choi [58] defined the accuracy function as $H(x) = \mu_M(x) + \gamma_M(x)$, where x is an IFV. Xu [59,60] combined the accuracy and score functions and formed the order relations between any pair of IFVs (x, y) as follows:

(i) If $S(x) > S(y)$, then $x > y$;

(ii) If $S(x) = S(y)$, then

(a) If $H(x) = H(y)$, then $x = y$;

(b) If $H(x) < H(y)$, then $x < y$.

A pair (F, A) is called a soft set over U if F is a mapping given by $F\colon A \longrightarrow P(U)$, where A is a subset of E (the set of parameters) and $P(U)$ is the power set of U. Thus, $F(e)$ is a subset of U for all $e \in A$. Hence, a soft set over U is a parametrized collection of subsets of U. A pair (F, A) is called an intuitionistic fuzzy soft set over U if F is a mapping given by $F\colon A \longrightarrow \mathrm{IF}(U)$ and A is a subset of E (the set of parameters). Thus, $F(e)$ is an IF set in U for all $e \in A$. Hence, an IF soft set over U is a parametrized collection of IF sets in U. For two IF soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is an IF soft subset of (G, B) if (1) $A \subseteq B$ and (2) $F(e)$ is an IF subset of $G(e)$ for all $e \in A$. Two IF soft sets (F, A) and (G, B) over a common universe U are said to be IF soft equal if (F, A) is an IF soft subset of (G, B) and (G, B) is an IF soft subset of (F, A). The union of two IF soft sets (F, A) and (G, A) over the common universe U is the IF soft set (H, A), where $H(e) = F(e) \cup G(e)$ for all $e \in A$. The intersection of two IF soft sets (F, A) and (G, A) over the common universe U is the IF soft set (K, A), where $K(e) = F(e) \cap G(e)$ for all $e \in A$ [12, 61, 62].

An IF soft set can be represented by a table, which is shown in the following example.

Example 1. Let $U = \{x, y, z, s, t\}$, $A = \{e_1, e_2\}$. Consider an IF soft set (F, A) over U defined by $F(e_1)(x) = (0.3, 0.4)$, $F(e_1)(y) = (0.4, 0.3)$, $F(e_1)(z) = (0.4, 0.2)$, $F(e_1)(s) = (0.8, 0.1)$, $F(e_1)(t) = (0.2, 0.6)$ and $(Fe_2)(x) = (0.9, 0)$, $F(e_2)(y) = (0.5, 0.5)$, $F(e_2)(z) = (0.4, 0.5)$, $F(e_2)(s) = (0.3, 0.7)$, $F(e_2)(t) = (0.6, 0.3)$.

The above intuitionistic fuzzy soft set can be represented as in Table 1.

3. Approximations of an IFS by Soft Binary Relation

In this section, we consider soft binary relation from U_1 to U_2 and approximate an IFS of U_2 by using aftersets and get two IF soft sets of U_1. Similarly, we approximate an IFS of U_1 by using foresets and get two IF soft sets of U_2. We also study some properties of these approximations.

Definition 1 (see [30]). A soft binary relation (σ, A) from U_1 to U_2 is a soft set over $U_1 \times U_2$, that is, $\sigma\colon A \longrightarrow P(U_1 \times U_2)$, where A is a subset of the set of parameters E.

Table 1: Representation of IF soft set.

U	e_1	e_2
x	(0.3, 0.4)	(0.9, 0)
y	(0.4, 0.3)	(0.5, 0.5)
z	(0.4, 0.2)	(0.4, 0.5)
s	(0.8, 0.1)	(0.3, 0.7)
t	(0.2, 0.6)	(0.6, 0.3)

Of course, (σ, A) is a parameterized collection of binary relations from U_1 to U_2. That is, for each $e \in A$, we have a binary relation $\sigma(e)$ from U_1 to U_2.

Definition 2. Let (σ, A) be a soft binary relation from U_1 to U_2 and $M = \langle \mu_M, \gamma_M \rangle$ be an IFS in U_2. Then, we define lower approximation $\underline{\sigma}^M = (\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M})$ and upper approximation $\overline{\sigma}^M = (\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M})$ of $M = \langle \mu_M, \gamma_M \rangle$ with respect to aftersets as follows:

$$
\begin{aligned}
\underline{\sigma}^{\mu_M}(e)(u_1) &= \begin{cases} \wedge_{a \in u_1\sigma(e)} \mu_M(a), & \text{if } u_1\sigma(e) \neq \varnothing, \\ 1, & \text{if } u_1\sigma(e) = \varnothing, \end{cases} \\
\underline{\sigma}^{\gamma_M}(e)(u_1) &= \begin{cases} \vee_{a \in u_1\sigma(e)} \gamma_M(a), & \text{if } u_1\sigma(e) \neq \varnothing, \\ 0, & \text{if } u_1\sigma(e) = \varnothing, \end{cases} \\
\overline{\sigma}^{\mu_M}(e)(u_1) &= \begin{cases} \vee_{a \in u_1\sigma(e)} \mu_M(a), & \text{if } u_1\sigma(e) \neq \varnothing, \\ 0, & u_1\sigma(e) = \varnothing, \end{cases} \\
\overline{\sigma}^{\gamma_M}(e)(u_1) &= \begin{cases} \wedge_{a \in u_1\sigma(e)} \gamma_M(a), & \text{if } u_1\sigma(e) \neq \varnothing, \\ 1, & \text{if } u_1\sigma(e) = \varnothing, \end{cases}
\end{aligned}
\tag{1}
$$

where $u_1\sigma(e) = \{a \in U_2\colon (u_1, a) \in \sigma(e)\}$ and is called the afterset of u_1 for $u_1 \in U_1$ and $e \in A$.

(i) $\underline{\sigma}^{\mu_M}(e)(u_1)$ indicates the degree to which u_1 definitely have the property e.

(ii) $\underline{\sigma}^{\gamma_M}(e)(u_1)$ indicates the degree to which u_1 probably do not have the property e.

(iii) $\overline{\sigma}^{\mu_M}(e)(u_1)$ indicates the degree to which u_1 probably have the property e.

(iv) $\overline{\sigma}^{\gamma_M}(e)(u_1)$ indicates the degree to which u_1 definitely do not have the property e.

In Definition 2, soft binary relation from U_1 to U_2 is given and IFS in U_2 can be approximated as lower and upper approximations with respect to the aftersets. The resulting sets are two pairs of IF soft sets.

Definition 3. Let (σ, A) be a soft binary relation from U_1 to U_2 and $M = \langle \mu_M, \gamma_M \rangle$ be an IFS in U_1. Then, we define lower approximation ${}^M\underline{\sigma} = ({}^{\mu_M}\underline{\sigma}, {}^{\gamma_M}\underline{\sigma})$ and upper approximation ${}^M\overline{\sigma} = ({}^{\mu_M}\overline{\sigma}, {}^{\gamma_M}\overline{\sigma})$ of $M = \langle \mu_M, \gamma_M \rangle$ with respect to foresets as follows:

$$\begin{aligned}
{}^{\mu_M}\underline{\sigma}(e)(u_2) &= \begin{cases} \wedge_{a\in\sigma(e)u_2}\mu_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 1, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
{}^{\gamma_M}\underline{\sigma}(e)(u_2) &= \begin{cases} \vee_{a\in\sigma(e)u_2}\gamma_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 0, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
{}^{\mu_M}\overline{\sigma}(e)(u_2) &= \begin{cases} \vee_{a\in\sigma(e)u_2}\mu_M(a) & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 0 & \text{if } \sigma(e)u_2 = \emptyset, \end{cases} \\
{}^{\gamma_M}\overline{\sigma}(e)(u_2) &= \begin{cases} \wedge_{a\in\sigma(e)u_2}\gamma_M(a), & \text{if } \sigma(e)u_2 \neq \emptyset, \\ 1, & \text{if } \sigma(e)u_2 = \emptyset, \end{cases}
\end{aligned} \tag{2}$$

where $\sigma(e)u_2 = \{a \in U_1\colon (a, u_2) \in \sigma(e)\}$ and is called the foreset of u_2 for $u_2 \in U_2$ and $e \in A$. Of course, $\underline{\sigma}^M\colon A \longrightarrow \mathrm{IF}(U_1)$, $\overline{\sigma}^M\colon A \longrightarrow \mathrm{IF}(U_1)$ and ${}^M\underline{\sigma}\colon A \longrightarrow \mathrm{IF}(U_2)$, ${}^M\overline{\sigma}\colon A \longrightarrow \mathrm{IF}(U_2)$. The following example explains these concepts.

In Definition 3, soft binary relation from U_1 to U_2 is given and IFS in U_1 can be approximated as lower and upper approximations with respect to the foresets. The resulting sets are two pairs of IF soft sets.

Example 2. Suppose that $Mr.\ X$ wants to buy a shirt for his own use. Let $U_1 = \{\text{the set of all shirts designs}\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $U_2 = \{\text{the colors of all designs}\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{\text{the set of stores near his house}\}$.

Define $\sigma\colon A \longrightarrow P(U_1 \times U_2)$ by

$$\begin{aligned}
\sigma(e_1) &= \left\{ \begin{array}{l} (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \\ (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \end{array} \right\}, \\
\sigma(e_2) &= \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\}, \\
\sigma(e_3) &= \{(d_2, c_4), (d_3, c_1), (d_3, c_3), (d_5, c_3), (d_5, c_4)\},
\end{aligned} \tag{3}$$

which represents the relation between designs and colors available on store e_i for $1 \leq i \leq 3$. Then,

$$\begin{aligned}
d_1\sigma(e_1) &= \{c_1, c_2, c_3\}, \\
d_2\sigma(e_1) &= \{c_2, c_4\}, \\
d_3\sigma(e_1) &= \emptyset, \\
d_4\sigma(e_1) &= \{c_2, c_3\}, \\
d_5\sigma(e_1) &= \{c_3, c_4\}, \\
d_6\sigma(e_1) &= \{c_1\} \\
d_1\sigma(e_2) &= \{c_3\}, \\
d_2\sigma(e_2) &= \{c_3\}, \\
d_3\sigma(e_2) &= \emptyset, \\
d_4\sigma(e_2) &= \{c_1\}, \\
d_5\sigma(e_2) &= \{c_1\}, \\
d_6\sigma(e_2) &= \{c_2, c_3\} \\
d_1\sigma(e_3) &= \emptyset, \\
d_2\sigma(e_3) &= \{c_4\}, \\
d_3\sigma(e_3) &= \{c_1, c_3\}, \\
d_4\sigma(e_3) &= \emptyset, \\
d_5\sigma(e_3) &= \{c_3, c_4\}, \\
d_6\sigma(e_3) &= \emptyset,
\end{aligned} \tag{4}$$

where $d_i\sigma(e_j)$ represents the color of the design d_i available on the store e_j.

Also,

$$\begin{aligned}
\sigma(e_1)c_1 &= \{d_1, d_6\}, \\
\sigma(e_1)c_2 &= \{d_1, d_2, d_4\}, \\
\sigma(e_1)c_3 &= \{d_1, d_4, d_5\}, \\
\sigma(e_1)c_4 &= \{d_2, d_5\}, \\
\sigma(e_2)c_1 &= \{d_4, d_5\}, \\
\sigma(e_2)c_2 &= \{d_6\}, \\
\sigma(e_2)c_3 &= \{d_1, d_2, d_6\}, \\
\sigma(e_2)c_4 &= \emptyset, \\
\sigma(e_3)c_1 &= \{d_3\}, \\
\sigma(e_3)c_2 &= \emptyset, \\
\sigma(e_3)c_3 &\{d_3, d_5\}, \\
\sigma(e_3)c_4 &= \{d_2, d_5\},
\end{aligned} \tag{5}$$

where $\sigma(e_j)c_i$ represents the design of the color c_i available on the store e_j.

$$\begin{aligned}
&\text{Define } M = \langle \mu_M, \gamma_M \rangle\colon U_2 \longrightarrow [0,1] \text{ which represents the preference of the} \\
&\qquad \text{colors given by Mr.} X \text{ such that} \\
&\mu_M(c_1) = 0.9, \mu_M(c_2) = 0.8, \mu_M(c_3) = 0.4, \mu_M(c_4) = 0 \\
&\gamma_M(c_1) = 0.0, \gamma_M(c_2) = 0.2, \gamma_M(c_3) = 0.5, \gamma_M(c_4) \\
&\text{Define } N = \langle \mu_N, \gamma_N \rangle\colon U_1 \longrightarrow [0,1] \text{ which represents the preference of the} \\
&\qquad \text{designs given by Mr.} X \text{ such that} \\
&\mu_N(d_1) = 1, \mu_N(d_2) = 0.7, \mu_N(d_3) = 0.5, \mu_N(d_4) = 0.1, \\
&\mu_N(d_5) = 0, \mu_N(d_6) = 0.4 \\
&\gamma_N(d_1) = 0, \gamma_N(d_2) = 0.2, \gamma_N(d_3) = 0.5, \gamma_N(d_4) = 0.7, \\
&\gamma_N(d_5) = 1, \gamma_N(d_6) = 0.5.
\end{aligned} \tag{6}$$

Therefore, the lower and upper approximations (with respect to the aftersets as well as with respect to the foresets) are

$$\begin{aligned}
\underline{\sigma}^M &= \left(\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M}\right) \text{(given in Table 2)}, \\
\overline{\sigma}^M &= \left(\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M}\right) \text{(given in Table 3)},
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
{}^N\underline{\sigma} &= \left({}^{\mu_N}\underline{\sigma}, {}^{\gamma_N}\underline{\sigma}\right) \text{(given in Table 4)}\ {}^N\overline{\sigma} \\
&= \left({}^{\mu_N}\overline{\sigma}, {}^{\gamma_N}\overline{\sigma}\right) \text{(given in Table 5)}.
\end{aligned} \tag{8}$$

Table 2 shows the lower approximation of IFS M with respect to the aftersets by using Definition 2. Table 3 shows the upper approximation of IFS M with respect to the aftersets by using Definition 2. Table 4 shows the lower approximation of IFS N with respect to the foresets by using Definition 3. Table 5 shows the upper approximation of IFS N with respect to the foresets by using Definition 3.

Theorem 1. *Let (σ, A) be a soft binary relation from U_1 to U_2, that is, $\sigma\colon A \longrightarrow P(U_1 \times U_2)$. For any IFSs, $M = \langle \mu_M, \gamma_M \rangle$, $N = \langle \mu_N, \gamma_N \rangle$, and $P = \langle \mu_P, \gamma_P \rangle$ of U_2, the following are true:*

(1) If $N \subseteq P$, then $\underline{\sigma}^N \subseteq \underline{\sigma}^P$;
(2) If $N \subseteq P$, then $\overline{\sigma}^N \subseteq \overline{\sigma}^P$;
(3) $\underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{N \cap P}$;
(4) $\overline{\sigma}^N \cap \overline{\sigma}^P \supseteq \overline{\sigma}^{N \cap P}$;
(5) $\underline{\sigma}^N \cup \underline{\sigma}^P \subseteq \underline{\sigma}^{N \cup P}$;
(6) $\overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^{N \cup P}$;
(7) $\underline{\sigma}^{1_{U_2}} = 1_{U_1}$ if $u_1\sigma(e) \neq \varnothing$;
(8) $\overline{\sigma}^{1_{U_2}} = 1_{U_1}$ if $u_1\sigma(e) \neq \varnothing$;
(9) $\underline{\sigma}^M = (\overline{\sigma}^{M^c})^c$ if $u_1\sigma(e) \neq \varnothing$;
(10) $\overline{\sigma}^M = (\underline{\sigma}^{M^c})^c$ if $u_1\sigma(e) \neq \varnothing$;
(11) $\underline{\sigma}^{0_{U_2}} = 0_{U_1} = \overline{\sigma}^{0_{U_2}}$ if $u_1\sigma(e) \neq \varnothing$.

Proof

(1) Let $u_1 \in U_1$. If $u_1\sigma(e) = \varnothing$, then $\underline{\sigma}^{\mu_N}(e)(u_1) = 1 = \underline{\sigma}^{\mu_P}(e)(u_1)$ and $\underline{\sigma}^{\gamma_N}(e)(u_1) = 0 = \underline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1\sigma(e) \neq \varnothing$, then $\underline{\sigma}^{\mu_N}(e)(u_1) = \wedge_{a \in u_1\sigma(e)}\mu_N(a) \leq \wedge_{a \in u_1\sigma(e)}\mu_P(a)$ because $\mu_N(a) \leq \mu_P(a) = \underline{\sigma}^{\mu_P}(e)(u_1)$. Thus, $\underline{\sigma}^{\mu_N}(e)(u_1) \leq \underline{\sigma}^{\mu_P}(e)(u_1)$.
Also,
$\underline{\sigma}^{\gamma_N}(e)(u_1) = \vee_{a \in u_1\sigma(e)}\gamma_N(a) \geq \vee_{a \in u_1\sigma(e)}\gamma_P(a)$ because $\gamma_N(a) \geq \gamma_P(a)$
$= \underline{\sigma}^{\gamma_P}(e)(u_1)$.
Thus, $\underline{\sigma}^{\gamma_N}(e)(u_1) \geq \underline{\sigma}^{\gamma_P}(e)(u_1)$. Hence, $\underline{\sigma}^N \subseteq \underline{\sigma}^P$.

(2) Let $u_1 \in U_1$. If $u_1\sigma(e) = \varnothing$, then $\overline{\sigma}^{\mu_N}(e)(u_1) = 0 = \overline{\sigma}^{\mu_P}(e)(u_1)$ and $\overline{\sigma}^{\gamma_N}(e)(u_1) = 1 = \overline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1\sigma(e) \neq \varnothing$, then
$\overline{\sigma}^{\mu_N}(e)(u_1) = \vee_{a \in u_1\sigma(e)}\mu_N(a) \leq \vee_{a \in u_1\sigma(e)}\mu_P(a)$ because $\mu_N(a) \leq \mu_P(a) = \overline{\sigma}^{\mu_P}(e)(u_1)$.
Thus, $\overline{\sigma}^{\mu_N}(e)(u_1) \leq \overline{\sigma}^{\mu_P}(e)(u_1)$.
Also,
$\overline{\sigma}^{\gamma_N}(e)(u_1) = \wedge_{a \in u_1\sigma(e)}\gamma_N(a) \geq \wedge_{a \in u_1\sigma(e)}\gamma_P(a)$ because $\gamma_N(a) \geq \gamma_P(a)$
$= \overline{\sigma}^{\gamma_P}(e)(u_1)$.
Thus, $\overline{\sigma}^{\gamma_N}(e)(u_1) \geq \overline{\sigma}^{\gamma_P}(e)(u_1)$. Hence, $\overline{\sigma}^N \subseteq \overline{\sigma}^P$.

(3) Let $u_1 \in U_1$. If $u_1\sigma(e) = \varnothing$, then $\underline{\sigma}^{\mu_{N \cap P}}(e)(u_1) = 1 = \underline{\sigma}^{\mu_N}(e)(u_1) \cap \underline{\sigma}^{\mu_P}(e)(u_1)$ and $\underline{\sigma}^{\gamma_{N \cap P}}(e)(u_1) = 0 = \underline{\sigma}^{\gamma_N}(e)(u_1) \cup \underline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1\sigma(e) \neq \varnothing$, then
$(\underline{\sigma}^{\mu_N} \cap \underline{\sigma}^{\mu_P})(e)(u_1) = \underline{\sigma}^{\mu_N}(e)(u_1) \wedge \underline{\sigma}^{\mu_P}(e)(u_1) = (\wedge_{a \in u_1\sigma(e)}\mu_N(a)) \wedge (\wedge_{a \in u_1\sigma(e)}\mu_P(a)) = \wedge_{a \in u_1\sigma(e)}(\mu_N(a) \wedge \mu_P(a)) = \wedge_{a \in u_1\sigma(e)}(\mu_N \wedge \mu_P)(a) = \wedge_{a \in u_1\sigma(e)}(\mu_{N \cap P})(a) = \underline{\sigma}^{\mu_{N \cap P}}(e)(u_1)$.

Table 2: Lower approximation of intuitionistic fuzzy set M.

$\underline{\sigma}^M$	e_1	e_2	e_3
d_1	$(0.4, 0.5)$	$(0.4, 0.5)$	$(1, 0)$
d_2	$(0, 0.8)$	$(0.4, 0.5)$	$(0, 0.8)$
d_3	$(1, 0)$	$(1, 0)$	$(0.4, 0.5)$
d_4	$(0.4, 0.5)$	$(0.9, 0)$	$(1, 0)$
d_5	$(0, 0.8)$	$(0.9, 0)$	$(0, 0.8)$
d_6	$(0.9, 0)$	$(0.4, 0.5)$	$(1, 0)$

Table 3: Upper approximation of intuitionistic fuzzy set M.

$\overline{\sigma}^M$	e_1	e_2	e_3
d_1	$(0.9, 0)$	$(0.4, 0.5)$	$(0, 1)$
d_2	$(0.8, 0.2)$	$(0.4, 0.5)$	$(0, 0.8)$
d_3	$(0, 1)$	$(0, 1)$	$(0.9, 0)$
d_4	$(0.8, 0.2)$	$(0.9, 0)$	$(0, 1)$
d_5	$(0.4, 0.5)$	$(0.9, 0)$	$(0.4, 0.5)$
d_6	$(0.9, 0)$	$(0.8, 0.2)$	$(0, 1)$

Table 4: Lower approximation of intuitionistic fuzzy set N.

${}^N\underline{\sigma}$	e_1	e_2	e_3
c_1	$(0.4, 0.5)$	$(0, 1)$	$(0.5, 0.5)$
c_2	$(0.1, 0.7)$	$(0.4, 0.5)$	$(1, 0)$
c_3	$(0, 1)$	$(0.4, 0.5)$	$(0, 1)$
c_4	$(0, 1)$	$(1, 0)$	$(0, 1)$

Table 5: Upper approximation of intuitionistic fuzzy set N.

${}^N\overline{\sigma}$	e_1	e_2	e_3
c_1	$(1, 0)$	$(0.1, 0.7)$	$(0.5, 0.5)$
c_2	$(1, 0)$	$(0.4, 0.5)$	$(0, 1)$
c_3	$(1, 0)$	$(1, 0)$	$(0.5, 0.5)$
c_4	$(0.7, 0.2)$	$(0, 1)$	$(0.7, 0.2)$

Also,
$(\underline{\sigma}^{\gamma_N} \cup \underline{\sigma}^{\gamma_P})(e)(u_1) = \underline{\sigma}^{\gamma_N}(e)(u_1) \vee \underline{\sigma}^{\gamma_P}(e)(u_1) = (\vee_{a\in u_1\sigma(e)}\gamma_N(a)) \vee (\vee_{a\in u_1\sigma(e)}\gamma_P(a)) = \vee_{a\in u_1\sigma(e)}(\gamma_N(a) \vee \gamma_P(a)) = \vee_{a\in u_1\sigma(e)}(\gamma_N \vee \gamma_P)(a) = \vee_{a\in u_1\sigma(e)}(\gamma_{N\cap P})(a) = \underline{\sigma}^{\gamma_{N\cap P}}(e)(u_1)$.

This shows that $\underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{N\cap P}$.

(4) Since $N\cap P \subseteq N$ and $N \cap P \subseteq P$, we have from part (2) $\overline{\sigma}^{N\cap P} \subseteq \overline{\sigma}^N$ and $\overline{\sigma}^{N\cap P} \subseteq \overline{\sigma}^P$. Thus, $\overline{\sigma}^{N\cap P} \subseteq \overline{\sigma}^N \cap \overline{\sigma}^P$.

(5) Since $N\cup P \supseteq N$ and $N\cup P \supseteq P$, we have from part (1) $\underline{\sigma}^{N\cup P} \supseteq \underline{\sigma}^N$ and $\underline{\sigma}^{N\cup P} \supseteq \underline{\sigma}^P$. Thus, $\underline{\sigma}^{N\cup P} \supseteq \underline{\sigma}^N \cup \underline{\sigma}^P$.

(6) Let $u_1 \in U_1$. If $u_1\sigma(e) = \varnothing$, then $\overline{\sigma}^{\mu_{N\cup P}}(e)(u_1) = 0 = \overline{\sigma}^{\mu_N}(e)(u_1) \cup \overline{\sigma}^{\mu_P}(e)(u_1)$ and $\overline{\sigma}^{\gamma_{N\cup P}}(e)(u_1) = 1 = \overline{\sigma}^{\gamma_N}(e)(u_1) \cap \overline{\sigma}^{\gamma_P}(e)(u_1)$. If $u_1\sigma(e) \neq \varnothing$, then $(\overline{\sigma}^{\mu_N} \cup \overline{\sigma}^{\mu_P})(e)(u_1) = \overline{\sigma}^{\mu_N}(e)(u_1) \vee \overline{\sigma}^{\mu_P}(e)(u_1) = (\vee_{a\in u_1\sigma(e)}\mu_N(a)) \vee (\vee_{a\in u_1\sigma(e)}\mu_P(a)) = \vee_{a\in u_1\sigma(e)}(\mu_N(a) \vee \mu_P(a)) = \vee_{a\in u_1\sigma(e)}(\mu_N \vee \mu_P)(a) = \vee_{a\in u_1\sigma(e)}(\mu_{N\cup P})(a) = \overline{\sigma}^{\mu_{N\cup P}}(e)(u_1)$.

Also,
$(\overline{\sigma}^{\gamma_N} \cap \overline{\sigma}^{\gamma_P})(e)(u_1) = \overline{\sigma}^{\gamma_N}(e)(u_1) \wedge \overline{\sigma}^{\gamma_P}(e)(u_1) = (\wedge_{a\in u_1\sigma(e)}\gamma_N(a)) \wedge (\wedge_{a\in u_1\sigma(e)}\gamma_P(a)) = \wedge_{a\in u_1\sigma(e)}(\gamma_N(a) \wedge \gamma_P(a)) = \wedge_{a\in u_1\sigma(e)}(\gamma_N \wedge \gamma_P)(a) = \wedge_{a\in u_1\sigma(e)}(\gamma_{N\cup P})(a) = \overline{\sigma}^{\gamma_{N\cup P}}(e)(u_1)$.

This shows that $\overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^{N\cup P}$.

(7) Consider $\underline{\sigma}^1(e)(u_1) = \wedge_{a\in u_1\sigma(e)}1(a) = \wedge_{a\in u_1\sigma(e)}(1) = 1$, because $u_1\sigma(e) \neq \varnothing$

and $\underline{\sigma}^0(e)(u_1) = \vee_{a\in u_1\sigma(e)}0(a) = \vee_{a\in u_1\sigma(e)}(0) = 0$, because $u_1\sigma(e) \neq \varnothing$.

Thus, $\underline{\sigma}^{1_{U_2}} = 1_{U_1}$.

(8) The proof is similar to the proof of part (7).

(9) Let $M = <\mu_M, \gamma_M>$ be an IFS on U_2. Then, $M^c = <\mu_{M^c}, \gamma_{M^c}> = <\gamma_M, \mu_M>$, that is, $\mu_{M^c} = \gamma_M$ and $\gamma_{M^c} = \mu_M$. Now, $\overline{\sigma}^{M^c} = (\overline{\sigma}^{\mu_{M^c}}, \overline{\sigma}^{\gamma_{M^c}}) = (\overline{\sigma}^{\gamma_M}, \overline{\sigma}^{\mu_M})$. Thus, $\overline{\sigma}^{\mu_{M^c}}(e)(u_1) = \vee_{a\in u_1\sigma(e)}\mu_{M^c}(a) = \vee_{a\in u_1\sigma(e)}\gamma_M(a) = \underline{\sigma}^{\gamma_M}(e)(u_1)$ and $\overline{\sigma}^{\gamma_{M^c}}(e)(u_1) = \wedge_{a\in u_1\sigma}(e)\gamma_{M^c}(a) = \wedge_{a\in u_1\sigma(e)}\mu_M(a) = \underline{\sigma}^{\mu_M}(e)(u_1)$. Hence, $\overline{\sigma}^{M^c} = (\overline{\sigma}^{\mu_{M^c}}, \overline{\sigma}^{\gamma_{M^c}}) = (\underline{\sigma}^{\gamma_M}, \underline{\sigma}^{\mu_M}) = (\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M})^c$, that is, $(\overline{\sigma}^{M^c})^c = \overline{\sigma}^M$.

(10) follows from part (9).

(11) Straightforward.

Theorem 1 describes the properties of newly defined IFRS model based on soft relations. It shows that if an IFS N is the subset of IFS P, then the lower approximation of N is also a subset of the lower approximation of P, and if an IFS N is subset of IFS P, then the upper approximation of N is also a subset of the upper approximation of P. Similarly, the empirical relations among the operations union, intersection, and complement have been described. □

Theorem 2. *Let (σ, A) be a soft binary relation from U_1 to U_2, that is, $\sigma: A \longrightarrow P(U_1 \times U_2)$. For any IFSs, $M = \langle\mu_M, \gamma_M\rangle$, $N = \langle\mu_N, \gamma_N\rangle$, and $P = \langle\mu_P, \gamma_P\rangle$ of U_1, the following are true:*

(1) If $N \subseteq P$, then ${}^N\underline{\sigma} \subseteq {}^P\underline{\sigma}$;

(2) If $N \subseteq P$, then ${}^N\overline{\sigma} \subseteq {}^P\overline{\sigma}$;

(3) ${}^N\underline{\sigma} \cap {}^P\underline{\sigma} = {}^{N\cap P}\underline{\sigma}$;

(4) ${}^N\overline{\sigma} \cap {}^P\overline{\sigma} \supseteq {}^{N\cap P}\overline{\sigma}$;

(5) ${}^N\underline{\sigma} \cup {}^P\underline{\sigma} \subseteq {}^{N\cup P}\underline{\sigma}$;

(6) ${}^N\overline{\sigma} \cup {}^P\overline{\sigma} = {}^{N\cup P}\overline{\sigma}$;

(7) $\underline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_1\sigma(e) \neq \varnothing$;

(8) $\overline{\sigma}^{1_{U_1}} = 1_{U_2}$ if $u_1\sigma(e) \neq \varnothing$;

(9) ${}^M\underline{\sigma} = (M^c\overline{\sigma})^c$ if $u_1\sigma(e) \neq \varnothing$;

(10) ${}^M\overline{\sigma} = (M^c\underline{\sigma})^c$ if $u_1\sigma(e) \neq \varnothing$;

(11) ${}^{0_{U_1}}\underline{\sigma} = 0_{U_2} = {}^{0_{U_1}}\overline{\sigma}$.

Proof. The proof is similar to the proof of Theorem 1.

The following example shows that equality does not hold in (4) and (5) assertions of above theorems in general. □

Table 6: Intuitionistic fuzzy set P.

	c_1	c_2	c_3	c_4
μ_P	0.1	0.6	0.5	1
γ_P	0.9	0.1	0.5	0

Example 3. Consider Example 2. *Let* $P = \langle \mu_P, \gamma_P \rangle$: $U_2 \longrightarrow [0, 1]$ (given in Table 6).

Table 6 simply shows the degree of membership and degree of nonmembership of IFS P.

Then, $M \cup P = \langle \mu_{M\cup P}, \gamma_{M\cup P} \rangle = \langle \mu_M \cup \mu_P, \gamma_M \cap \gamma_P \rangle$ (given in Table 7) and $M \cap P = \langle \mu_{M\cap P}, \gamma_{M\cap P} \rangle = \langle \mu_M \cap \mu_P, \gamma_M \cup \gamma_P \rangle$ (given in Table 7).

Table 7 shows the calculations of union and intersection of two IFSs M and P, respectively.

Now, $\underline{\sigma}^P = (\underline{\sigma}^{\mu_P}, \underline{\sigma}^{\gamma_P})$ (given in Table 8) and $\underline{\sigma}^{M\cup P} = (\underline{\sigma}^{\mu_{M\cup P}}, \underline{\sigma}^{\gamma_{M\cup P}})$ (given in Table 9).

Now, $\underline{\sigma}^M \cup \underline{\sigma}^P$ (given in Table 10).

In Table 10, we calculated $\underline{\sigma}^M \cup \underline{\sigma}^P$. Tables 9 and 10 show that $\underline{\sigma}^M \cup \underline{\sigma}^P \neq \underline{\sigma}^{M\cup P}$.

Now, $\overline{\sigma}^P = (\overline{\sigma}^{\mu_P}, \overline{\sigma}^{\gamma_P})$ (given in Table 11).

In Table 11, we calculated upper approximation of P.

Now, $\overline{\sigma}^{M\cap P} = (\overline{\sigma}^{\mu_{M\cap P}}, \overline{\sigma}^{\gamma_{M\cap P}})$ (given in Table 12).

Now, $\overline{\sigma}^M \cap \overline{\sigma}^P$ (given in Table 13).

In Table 12, we calculated upper approximation of $M \cap P$. In Table 13, we calculated the intersection of upper approximations of M and P. Tables 12 and 13 show that $\overline{\sigma}^M \cap \overline{\sigma}^P \neq \overline{\sigma}^{M\cap P}$.

Theorem 3. *Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2, such that $(\sigma_1, A) \subseteq (\sigma_2, A)$, that is, $\sigma_1(e) \subseteq \sigma_2(e)$ for all $e \in A$. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_2, the following are true:*

(i) $\underline{\sigma_2}^M \subseteq \underline{\sigma_1}^M$;

(ii) $\overline{\sigma}_1^M \subseteq \overline{\sigma}_2^M$.

Proof

(1) Let $u_1 \in U_1$. If $u_1\sigma_1(e) = \varnothing$, then $\underline{\sigma_1}^{\mu_M}(e)(u_1) = 1 \geq \underline{\sigma_2}^{\mu_M}(e)(u_1)$ and $\underline{\sigma_1}^{\gamma_M}(e)(u_1) = 0 \leq \underline{\sigma_2}^{\gamma_M}(e)(u_1)$. If $u_1\sigma_1(e) \neq \varnothing$, then $u_1\sigma_2(e) \neq \varnothing$, we have

$\underline{\sigma_1}^{\mu_M}(e)(u_1) = \wedge_{a\in u_1\sigma_1(e)}\mu_M(a) \geq \wedge_{a\in u_1\sigma_2(e)}\mu_M(a)$ because $u_1\sigma_1(e) \subseteq u_1\sigma_2(e) = \underline{\sigma_2}^{\mu_M}(e)(u_1)$.

Also,

$\underline{\sigma_1}^{\gamma_M}(e)(u_1) = \vee_{a\in u_1\sigma_1(e)}\gamma_M(a) \leq \vee_{a\in u_1\sigma_2(e)}\gamma_M(a)$ because $u_1\sigma_1(e) \subseteq u_1\sigma_2(e)$

$= \underline{\sigma_2}^{\gamma_M}(e)(u_1)$.

Hence, $\underline{\sigma_2}^M \subseteq \underline{\sigma_1}^M$.

(2) Let $u_1 \in U_1$. If $u_1\sigma_1(e) = \varnothing$, then $\overline{\sigma_1}^{\mu_M}(e)(u_1) = 0 \leq \overline{\sigma_2}^{\mu_M}(e)(u_1)$ and $\overline{\sigma}_1^{\gamma_M}(e)(u_1) = 1 \geq \overline{\sigma_2}^{\gamma_M}(e)(u_1)$. If $u_1\sigma_1(e) \neq \varnothing$, then $u_1\sigma_2(e) \neq \varnothing$, and we have

$\overline{\sigma_1}^{\mu_M}(e)(u_1) = \vee_{a\in u_1\sigma_1(e)}\mu_M(a) \leq \vee_{a\in u_1\sigma_2(e)}\mu_M(a)$ because $u_1\sigma_1(e) \subseteq u_1\sigma_2(e) = \overline{\sigma_2}^{\mu_M}(e)(u_1)$.

Table 7: Intuitionistic fuzzy sets $M \cup P$, $M \cap P$.

	c_1	c_2	c_3	c_4
$M \cup P$	(0.9, 0)	(0.8, 0.1)	(0.5, 0.5)	(1, 0)
$M \cap P$	(0.1, 0.9)	(0.6, 0.2)	(0.4, 0.5)	(0, 0.8)

Table 8: Lower approximation of P.

$\underline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.1, 0.9)	(0.5, 0.5)	(1, 0)
d_2	(0.6, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.1, 0.9)
d_4	(0.5, 0.5)	(0.1, 0.9)	(1, 0)
d_5	(0.5, 0.5)	(0.1, 0.9)	(0.5, 0.5)
d_6	(0.1, 0.9)	(0.5, 0.5)	(1, 0)

Table 9: Lower approximation of $M \cup P$.

$\underline{\sigma}^{M\cup P}$	e_1	e_2	e_3
d_1	(0.5, 0.5)	(0.5, 0.5)	(1, 0)
d_2	(0.8, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.5, 0.5)
d_4	(0.5, 0.5)	(0.9, 0)	(1, 0)
d_5	(0.5, 0.5)	(0.9, 0)	(0.5, 0.5)
d_6	(0.9, 0)	(0.5, 0.5)	(1, 0)

Table 10: Union of lower approximations of M and P.

$\underline{\sigma}^M \cup \underline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.4, 0.5)	(0.5, 0.5)	(1, 0)
d_2	(0.6, 0.1)	(0.5, 0.5)	(1, 0)
d_3	(1, 0)	(1, 0)	(0.4, 0.5)
d_4	(0.5, 0.5)	(0.9, 0)	(1, 0)
d_5	(0.5, 0.5)	(0.9, 0)	(0.5, 0.5)
d_6	(0.9, 0)	(0.5, 0.5)	(1, 0)

Table 11: Upper approximation of P.

$\overline{\sigma}^P$	e_1	e_2	e_3
d_1	(0.6, 0.1)	(0.5, 0.5)	(0, 1)
d_2	(1, 0)	(0.5, 0.5)	(1, 0)
d_3	(0, 1)	(0, 1)	(0.5, 0.5)
d_4	(0.6, 0.1)	(0.1, 0.9)	(0, 1)
d_5	(1, 0)	(0.1, 0.9)	(1, 0)
d_6	(0.1, 0.9)	(0.6, 0.1)	(0, 1)

Also,

$\overline{\sigma_1}^{\gamma_M}(e)(u_1) = \wedge_{a\in u_1\sigma_1(e)}\gamma_M(a) \geq \wedge_{a\in u_1\sigma_2(e)}\gamma_M(a)$ because $u_1\sigma_1(e) \subseteq u_1\sigma_2(e)$

$= \overline{\sigma_2}^{\gamma_M}(e)(u_1)$.

Hence, $\overline{\sigma_1}^M \subseteq \overline{\sigma_2}^M$.

Theorem 3 shows that if any soft relation $(\sigma_1, A) \subseteq (\sigma_2, A)$, then for any IFS M in U_2, the lower approximation associated with (σ_2, A) is a subset of (σ_1, A). Similarly, if any soft relation $(\sigma_1, A) \subseteq (\sigma_2, A)$, then for any

Table 12: Upper approximation of $M \cap P$.

$\overline{\sigma}^{M\cap P}$	e_1	e_2	e_3
d_1	$(0.6, 0.2)$	$(0.4, 0.5)$	$(0, 1)$
d_2	$(0.6, 0.2)$	$(0.4, 0.5)$	$(0, 0.8)$
d_3	$(0, 1)$	$(0, 1)$	$(0.4, 0.5)$
d_4	$(0.6, 0.2)$	$(0.1, 0.9)$	$(0, 1)$
d_5	$(0.4, 0.5)$	$(0.1, 0.9)$	$(0.4, 0.5)$
d_6	$(0.1, 0.9)$	$(0.6, 0.2)$	$(0, 1)$

Table 13: Intersection of upper approximations of M and P.

$\overline{\sigma}^{M} \cap \overline{\sigma}^{P}$	e_1	e_2	e_3
d_1	$(0.6, 0.1)$	$(0.4, 0.5)$	$(0, 1)$
d_2	$(0.8, 0.2)$	$(0.4, 0.5)$	$(0, 0.8)$
d_3	$(0, 1)$	$(0, 1)$	$(0.5, 0.5)$
d_4	$(0.6, 0.2)$	$(0.1, 0.9)$	$(0, 1)$
d_5	$(0.4, 0.5)$	$(0.1, 0.9)$	$(0.4, 0.5)$
d_6	$(0.1, 0.9)$	$(0.6, 0.2)$	$(0, 1)$

IFS M in U_2, the upper approximation associated with (σ_1, A) is a subset of (σ_2, A). □

Theorem 4. *Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2, such that $(\sigma_1, A) \subseteq (\sigma_2, A)$, that is, $\sigma_1(e) \subseteq \sigma_2(e)$ for all $e \in A$. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_1, the following are true:*

(1) $^{M}\underline{\sigma_2} \subseteq {}^{M}\underline{\sigma_1}$

(2) $^{M}\overline{\sigma}_1 \subseteq {}^{M}\overline{\sigma}_2$

Proof

(1) Let $u_2 \in U_2$. If $\sigma_1(e)u_2 = \varnothing$, then ${}^{\mu_M}\underline{\sigma_1}(e)(u_2) = 1 \geq {}^{\mu_M}\underline{\sigma_2}(e)(u_2)$ and ${}^{\gamma_M}\underline{\sigma_1}(e)(u_2) = 0 \leq {}^{\gamma_M}\underline{\sigma_2}(e)(u_2)$. If $\sigma_1(e)u_2 \neq \varnothing$, then $\sigma_2(e)u_2 \neq \varnothing$, we have

${}^{\mu_M}\underline{\sigma_1}(e)(u_2) = \wedge_{a\in\sigma_1(e)u_2}\mu_M(a) \geq \wedge_{a\in\sigma_2(e)u_2}\mu_M(a)$ because $\sigma_1(e)u_2 \subseteq \sigma_2(e)u_2 = {}^{\mu_M}\underline{\sigma_2}(e)(u_2)$.

Also,

${}^{\gamma_M}\underline{\sigma_1}(e)(u_2) = \vee_{a\in\sigma_1(e)u_2}\gamma_M(a) \leq \vee_{a\in\sigma_2(e)u_2}\gamma_M(a)$ because $\sigma_1(e)u_2 \subseteq \sigma_2(e)u_2$

$= {}^{\gamma_M}\underline{\sigma_2}(e)(u_2)$.

Hence, $^{M}\underline{\sigma_2} \subseteq {}^{M}\underline{\sigma_1}$.

(2) Let $u_2 \in U_2$. If $\sigma_1(e)u_2 = \varnothing$, then ${}^{\mu_M}\overline{\sigma_1}(e)(u_2) = 0 \leq {}^{\mu_M}\overline{\sigma_2}(e)(u_2)$ and ${}^{\gamma_M}\overline{\sigma_1}(e)(u_2) = 1 \geq {}^{\gamma_M}\overline{\sigma_2}(e)(u_2)$. If $\sigma_1(e)u_2 \neq \varnothing$, then $\sigma_2(e)u_2 \neq \varnothing$, and we have

${}^{\mu_M}\overline{\sigma_1}(e)(u_2) = \vee_{a\in\sigma_1(e)u_2}\mu_M(a) \leq \vee_{a\in\sigma_2(e)u_2}\mu_M(a)$ because $\sigma_1(e)u_2 \subseteq \sigma_2(e)u_2 = {}^{\mu_M}\overline{\sigma_2}(e)(u_2)$.

Also,

${}^{\gamma_M}\overline{\sigma_1}(e)(u_2) = \wedge_{a\in\sigma_1(e)u_2}\gamma_M(a) \geq \wedge_{a\in\sigma_2(e)u_2}\gamma_M(a)$ because $\sigma_1(e)u_2 \subseteq \sigma_2(e)u_2$

$= {}^{\gamma_M}\overline{\sigma_2}(e)(u_2)$.

Hence, $^{M}\overline{\sigma_1} \subseteq {}^{M}\overline{\sigma_2}$. □

Theorem 5. *Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_2, the following are true:*

(1) $\underline{\sigma_1}^{M} \subseteq \underline{(\sigma_1 \cap \sigma_2)}^{M}$;

(2) $\underline{\sigma_2}^{M} \subseteq \underline{(\sigma_1 \cap \sigma_2)}^{M}$;

(3) $\overline{(\sigma_1 \cap \sigma_2)}^{M} \subseteq \overline{\sigma}_1^{M}$;

(4) $\overline{(\sigma_1 \cap \sigma_2)}^{M} \subseteq \overline{\sigma}_2^{M}$.

Proof

(1) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 3 part (1), $\underline{\sigma_1}^{M} \subseteq \underline{(\sigma_1 \cap \sigma_2)}^{M}$.

(2) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 3 part (1), $\underline{\sigma_2}^{M} \subseteq \underline{(\sigma_1 \cap \sigma_2)}^{M}$.

(3) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 3 part (2), $\overline{(\sigma_1 \cap \sigma_2)}^{M} \subseteq \overline{\sigma}_1^{M}$.

(4) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 3 part (2), $\overline{(\sigma_1 \cap \sigma_2)}^{M} \subseteq \overline{\sigma}_2^{M}$. □

Theorem 6. *Let (σ_1, A) and (σ_2, A) be two soft binary relations from U_1 to U_2. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U_1, the following are true:*

(1) $^{M}\underline{\sigma_1} \subseteq {}^{M}\underline{(\sigma_1 \cap \sigma_2)}$;

(2) $^{M}\underline{\sigma_2} \subseteq {}^{M}\underline{(\sigma_1 \cap \sigma_2)}$;

(3) $^{M}\overline{(\sigma_1 \cap \sigma_2)} \subseteq {}^{M}\overline{\sigma}_1$;

(4) $^{M}\overline{(\sigma_1 \cap \sigma_2)} \subseteq {}^{M}\overline{\sigma}_2$.

Proof

(1) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 4 part (1), $^{M}\underline{\sigma_1} \subseteq {}^{M}\underline{(\sigma_1 \cap \sigma_2)}$.

(2) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 4 part (1), $^{M}\underline{\sigma_2} \subseteq {}^{M}\underline{(\sigma_1 \cap \sigma_2)}$.

(3) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$, therefore from Theorem 4 part (2), $^{M}\overline{(\sigma_1 \cap \sigma_2)} \subseteq {}^{M}\overline{\sigma}_1$.

(4) As $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, therefore from Theorem 4 part (2), $^{M}\overline{(\sigma_1 \cap \sigma_2)} \subseteq {}^{M}\overline{\sigma}_2$.

In Theorems 5 and 6, some empirical relations have been discussed about union and intersection of two soft relations (σ_1, A) and (σ_2, A) with respect to the aftersets and with respect to the foresets, respectively. □

Definition 4. If (σ, A) is a soft set over $U \times U$, then (σ, A) is called a soft binary relation on U.

In fact, (σ, A) is a parameterized collection of binary relations on U. That is, for each parameter $e \in A$, we have a binary relation $\sigma(e)$ on U. A soft binary relation (σ, A) on U is said to be soft reflexive relation on U if $\sigma(e)$ is a reflexive relation on U for all $e \in A$. If (σ, A) is a soft reflexive binary relation on U, then $u\sigma(e)(\text{resp.}\sigma(e)u)$ is nonempty and $u \in u\sigma(e)(\text{resp.}u \in \sigma(e)u)$. It is not necessary that $u\sigma(e) = \sigma(e)u$. A soft binary relation (σ, A) on U is said to be soft symmetric relation on U if $\sigma(e)$ is a symmetric relation on U for all $e \in A$. A soft binary relation (σ, A) on U is said to be soft transitive relation on U if $\sigma(e)$ is a transitive relation on U for all $e \in A$.

A soft binary relation (σ, A) over U is soft equivalence relation over U if it is soft reflexive, soft symmetric, and soft transitive relation over U. A soft binary relation (σ, A) over U is a soft equivalence relation over U if $\sigma(e)$ for all $e \in A$ is an equivalence relation over U. In this case, $u\sigma(e) = \sigma(e)u$ and $\{u\sigma(e)\colon u \in U\}$ is a partition of U. Also, in this case, ${}^{M}\overline{\sigma}(e) = \overline{\sigma}^{M}(e)$ and ${}^{M}\overline{\sigma}(e) = \overline{\sigma}^{M}(e)$, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U.

The approximation operators have additional properties with respect to soft reflexive binary relation as follows.

Theorem 7. *Let (σ, A) be a soft reflexive binary relation on U. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U, the following are true:*

(1) $\underline{\sigma}^{\mu_M}(e) \leq \mu_M$ for all $e \in A$;
(2) $\mu_M \leq \overline{\sigma}^{\mu_M}(e)$ for all $e \in A$;
(3) $\underline{\sigma}^{\mu_M}(e) \leq \overline{\sigma}^{\mu_M}(e)$ for all $e \in A$;
(4) $\underline{\sigma}^{\gamma_M}(e) \geq \gamma_M$ for all $e \in A$;
(5) $\gamma_M \geq \overline{\sigma}^{\gamma_M}(e)$ for all $e \in A$;
(6) $\underline{\sigma}^{\gamma_M}(e) \geq \overline{\sigma}^{\gamma_M}(e)$ for all $e \in A$.

Proof

(1) Let $u \in U$. Then,
$\underline{\sigma}^{\mu_M}(e)(u) = \wedge_{a \in u\sigma_1(e)}\mu_M(a) \leq \mu_M(u)$ because $u \in u\sigma(e)$.
(2) Hence, $\underline{\sigma}^{\mu_M}(e) \leq \mu_M$.
Let $u \in U$. Then,
$\overline{\sigma}^{\mu_M}(e)(u) = \vee_{a \in u\sigma_1(e)}\mu_M(a) \geq \mu_M(u)$ because $u \in u\sigma(e)$
Hence, $\mu_M \leq \overline{\sigma}^{\mu_M}(e)$.
(3) It follows from part (1) and part (2).
(4) Let $u \in U$. Then,
$\underline{\sigma}^{\gamma_M}(e)(u) = \vee_{a \in u\sigma_1(e)}\gamma_M(a) \geq \gamma_M(u)$ because $u \in u\sigma(e)$
Hence, $\underline{\sigma}^{\gamma_M}(e) \geq \gamma_M$.
(5) Let $u \in U$. Then,
$\overline{\sigma}^{\gamma_M}(e)(u) = \wedge_{a \in u\sigma_1(e)}\gamma_M(a) \leq \gamma_M(u)$ because $u \in u\sigma(e)$
Hence, $\gamma_M \geq \overline{\sigma}^{\gamma_M}(e)$.
It follows from part (4) and part (5).

Theorem 7 shows the empirical relations between IFS M and a soft reflexive relation (σ, A). □

Theorem 8. *Let (σ, A) be a soft reflexive binary relation on U. Then, for any IFS $M = \langle \mu_M, \gamma_M \rangle$ of U, the following are true:*

(1) ${}^{\mu_M}\underline{\sigma}(e) \leq \mu_M$ for all $e \in A$;
(2) $\mu_M \leq {}^{\mu_M}\overline{\sigma}(e)$ for all $e \in A$;
(3) ${}^{\mu_M}\underline{\sigma}(e) \leq {}^{\mu_M}\overline{\sigma}(e)$ for all $e \in A$;
(4) ${}^{\gamma_M}\underline{\sigma}(e) \geq \gamma_M$ for all $e \in A$;
(5) $\gamma_M \geq {}^{\gamma_M}\overline{\sigma}(e)$ for all $e \in A$;
(6) ${}^{\gamma_M}\underline{\sigma}(e) \geq {}^{\gamma_M}\overline{\sigma}(e)$ for all $e \in A$.

Proof

(1) Let $u \in U$. Then,
${}^{\mu_M}\underline{\sigma}(e)(u) = \wedge_{a \in \sigma_1(e)u}\mu_M(a) \leq \mu_M(u)$ because $u \in \sigma(e)u$.
Hence, ${}^{\mu_M}\underline{\sigma}(e) \leq \mu_M$.
(2) Let $u \in U$. Then,
${}^{\mu_M}\overline{\sigma}(e)(u) = \vee_{a \in \sigma_1(e)u}\mu_M(a) \geq \mu_M(u)$ because $u \in \sigma(e)u$
Hence, $\mu_M \leq {}^{\mu_M}\overline{\sigma}(e)$.
(3) It follows from part (1) and part (2).
(4) Let $u \in U$. Then,
${}^{\gamma_M}\underline{\sigma}(e)(u) = \vee_{a \in \sigma_1(e)u}\gamma_M(a) \geq \gamma_M(u)$ because $u \in \sigma(e)u$
Hence, ${}^{\gamma_M}\underline{\sigma}(e) \geq \gamma_M$.
(5) Let $u \in U$. Then,
${}^{\gamma_M}\overline{\sigma}(e)(u) = \wedge_{a \in \sigma_1(e)u}\gamma_M(a) \leq \gamma_M(u)$ because $u \in \sigma(e)u$
(i) Hence, $\gamma_M \geq {}^{\gamma_M}\overline{\sigma}(e)$.
(6) It follows from part (4) and part (5). □

4. Similarity Relations

In this section, we define some relations between IFS of U_2 with the help of a soft relation from U_1 to U_2. We say that two intuitionistic fuzzy sets in U_2 are related if the lower (upper) approximations in U_1 are equal. Similarly, we define relations between intuitionistic fuzzy sets of U_1.

Definition 5. Let (σ, A) be a soft binary relation from U_1 to U_2. Then, for any IFS $N = \langle \mu_N, \gamma_N \rangle$ and $P = \langle \mu_P, \gamma_P \rangle$ of U_2, we define

$N \simeq_A P$ if and only if $\underline{\sigma}^N = \underline{\sigma}^P$ $N \eqsim_A P$ if and only if $\overline{\sigma}^N = \overline{\sigma}^P$ $N \approx_A P$ if and only if $\underline{\sigma}^N = \underline{\sigma}^P$ and $\overline{\sigma}^N = \overline{\sigma}^P$.

Definition 6. Let (σ, A) be a soft binary relation from U_1 to U_2. Then, for any IFS $N = \langle \mu_N, \gamma_N \rangle$ and $P = \langle \mu_P, \gamma_P \rangle$ of U_1, we define

$N \eqsim_{\sigma} P$ if and only if ${}^{N}\underline{\sigma} = {}^{P}\underline{\sigma} N \eqsim_{\sigma} P$ if and only if ${}^{N}\overline{\sigma} = {}^{P}\overline{\sigma} N \approx_{\sigma} P$ if and only if ${}^{N}\underline{\sigma} = {}^{P}\underline{\sigma}$ and ${}^{N}\overline{\sigma} = {}^{P}\overline{\sigma}$.

These binary relations may be called the lower similarity relation, upper similarity relation, and similarity relation, respectively.

Definitions 5 and 6 show that if an IFS N has upper(lower) similarity relation with an IFS P, then its associated lower(upper) approximation has also upper(-lower) similarity relation.

Proposition 1. *The relations $\simeq_A$, $\eqsim_A$, and $\approx_A$ are equivalence relations on $IF(U_2)$.*

Proof. $\simeq_A$ is reflexive: let N be an IFS of U_2. Since $\underline{\sigma}^N = \underline{\sigma}^N$, so we have $N \simeq N$. $\simeq_A$ is symmetric: let N and P be IFSs of U_2 such that $N \simeq P$; this implies $\underline{\sigma}^N = \underline{\sigma}^P$, so $\underline{\sigma}^P = \underline{\sigma}^N$; this implies $P \simeq N$.

$\simeq_A$ is transitive: let N, P and Q be IFSs of U_2 such that $N \simeq P$ and $P \simeq Q$; this implies $\underline{\sigma}^N = \underline{\sigma}^P$ and $\underline{\sigma}^P = \underline{\sigma}^Q$, so $\underline{\sigma}^N = \underline{\sigma}^Q$; this implies $N \simeq Q$.

Thus, $\simeq_A$ is an equivalence relation on $\mathrm{IF}(U_2)$.

Similarly, $\eqsim_A$ and $\approx_A$ are equivalence relations on $\mathrm{IF}(U_2)$. □

Proposition 2. *The relations $\simeq_{\sigma}$, $\eqsim_{\sigma}$, and $\approx_{\sigma}$ are equivalence relations on $IF(U_1)$.*

Proof. $\simeq_{\sigma}$ is reflexive: let N be an IFS of U_1. Since ${}^{N}\underline{\sigma} = {}^{N}\underline{\sigma}$, so we have $N \simeq N$.

$\simeq_{\sigma}$ is symmetric: let N and P be IFSs of U_1 such that $N \simeq P$; this implies ${}^{N}\underline{\sigma} = {}^{P}\underline{\sigma}$, so ${}^{P}\underline{\sigma} = {}^{N}\underline{\sigma}$; this implies $P \simeq N$.

$\simeq_{\sigma}$ is transitive: let N, P, and Q be IFSs of U_1 such that $N \simeq P$ and $P \simeq Q$; this implies ${}^{N}\underline{\sigma} = {}^{P}\underline{\sigma}$ and ${}^{P}\underline{\sigma} = {}^{Q}\underline{\sigma}$, so ${}^{N}\underline{\sigma} = {}^{Q}\underline{\sigma}$; this implies $N \simeq Q$.

Thus, $\simeq_{\sigma}$ is an equivalence relation on $\mathrm{IF}(U_1)$.

Similarly, $\eqsim_{\sigma}$ and $\approx_{\sigma}$ are equivalence relations on $\mathrm{IF}(U_1)$. □

Theorem 9. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_2. Then, the following are true:*

(1) $N \eqsim_A P$ if and only if $N \eqsim_A (N \cup P) \eqsim_A P$;

(2) $N \eqsim_A P$ and $Q \eqsim_A T$ imply that $(N \cup Q) \eqsim_A (P \cup T)$;

(3) $N \subseteq P$ and $P \eqsim_A 0_{U_2}$ imply that $N \eqsim_A 0$;

(4) $(N \cup P) \eqsim_A 0_{U_2}$ if and only if $N \eqsim_A 0_{U_2}$ and $P \eqsim_A 0_{U_2}$;

(5) $N \subseteq P$ and $N \eqsim_A 1_{U_2}$ imply that $P \eqsim_A 1$;

(6) If $(N \cap P) \eqsim_A 1_{U_2}$, then $N \eqsim_A 1_{U_2}$ and $P \eqsim_A 1_{U_2}$.

Proof

(1) Let $N \eqsim_A P$. Then, $\overline{\sigma}^N = \overline{\sigma}^P$. By Theorem 1, we get $\overline{\sigma}^{N \cup P} = \overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^N = \overline{\sigma}^P$. This implies that $N \eqsim_A (N \cup P) \eqsim_A P$. Conversely, it holds due to transitive property of relation $\eqsim_A$.

(2) Let $N \eqsim_A P$ and $Q \eqsim_A T$. Then, $\overline{\sigma}^N = \overline{\sigma}^P$ and $\overline{\sigma}^Q = \overline{\sigma}^T$.

By Theorem 1, we get $\overline{\sigma}^{N \cup Q} = \overline{\sigma}^N \cup \overline{\sigma}^Q = \overline{\sigma}^P \cup \overline{\sigma}^T = \overline{\sigma}^{P \cup T}$. This implies that $(N \cup Q) \eqsim_A (P \cup T)$.

(3) Let $N \subseteq P$ and $P \eqsim_A 0_{U_2}$. Then, $\overline{\sigma}^P = \overline{\sigma}^{0_{U_2}}$.

Also, by Theorem 1, $N \subseteq P$ implies that $\overline{\sigma}^N \subseteq \overline{\sigma}^P = \overline{\sigma}^{0_{U_2}}$. But $\overline{\sigma}^{0_{U_2}} \subseteq \overline{\sigma}^N$. Thus, $\overline{\sigma}^N = \overline{\sigma}^{0_{U_2}}$. This implies that $N \eqsim_A 0$.

(4) If $N \eqsim_A 0_{U_2}$ and $P \eqsim_A 0_{U_2}$, then $\overline{\sigma}^N = \overline{\sigma}^{0_{U_2}}$ and $\overline{\sigma}^P = \overline{\sigma}^{0_{U_2}}$. Now, by Theorem 1, we have $\overline{\sigma}^{N \cup P} = \overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^{0_{U_2}} \cup \overline{\sigma}^{0_{U_2}} = \overline{\sigma}^{0_{U_2}}$. This implies that $(N \cup P) \eqsim_A 0_{U_2}$. Conversely, if $(N \cup P) \eqsim_A 0_{U_2}$, then by part (3), we have $N \eqsim_A 0_{U_2}$ and $P \eqsim_A 0_{U_2}$.

(5) Suppose $N \eqsim_A 1_{U_2}$. Then, $\overline{\sigma}^N = \overline{\sigma}^{1_{U_2}}$. As

$N \subseteq P$, we have $\overline{\sigma}^P \supseteq \overline{\sigma}^N = \overline{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$, so we have $\overline{\sigma}^P \subseteq \overline{\sigma}^{1_{U_2}}$. This implies that $\overline{\sigma}^P = \overline{\sigma}^{1_{U_2}}$, that is, $P \eqsim_A 1_{U_2}$.

(6) It follows from (5).

Theorem 9 shows some lower similarity relations of union and intersection of IFSs N, P, Q, and T in U_2 with respect to the aftersets. □

Theorem 10. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_1. Then, the following are true:*

(1) $N \eqsim_{\sigma} P$ if and only if $N \eqsim_{\sigma} (N \cup P) \eqsim_{\sigma} P$;

(2) $N \eqsim_{\sigma} P$ and $Q \eqsim_{\sigma} T$ imply that $(N \cup Q) \eqsim_{\sigma} (P \cup T)$;

(3) $N \subseteq P$ and $P \eqsim_{\sigma} 0_{U_1}$ imply that $N \eqsim_{\sigma} 0_{U_1}$;

(4) $(N \cup P) \eqsim_{\sigma} 0_{U_1}$ if and only if $N \eqsim_{\sigma} 0_{U_1}$ and $P \eqsim_{\sigma} 0_{U_1}$;

(5) $N \subseteq P$ and $N \eqsim_{\sigma} 1_{U_1}$ imply that $P \eqsim_{\sigma} 1_{U_1}$;

(6) If $(N \cap P) \eqsim_{\sigma} 1_{U_1}$, then $N \eqsim_{\sigma} 1_{U_1}$ and $P \eqsim_{\sigma} 1_{U_1}$.

Proof

(1) Let $N \eqsim_{\sigma} P$. Then, ${}^{N}\overline{\sigma} = {}^{P}\overline{\sigma}$. By Theorem 2, we get ${}^{N \cup P}\overline{\sigma} = {}^{N}\overline{\sigma} \cup {}^{P}\overline{\sigma} = {}^{N}\overline{\sigma} = {}^{P}\overline{\sigma}$. This implies that $N \eqsim_{\sigma} (N \cup P) \eqsim_{\sigma} P$. Converse holds by the transitivity of the relation $\eqsim_{\sigma}$.

(2) Let $N \eqsim_{\sigma} P$ and $Q \eqsim_{\sigma} T$. Then, ${}^{N}\overline{\sigma} = {}^{P}\overline{\sigma}$ and ${}^{Q}\overline{\sigma} = {}^{T}\overline{\sigma}$.

By Theorem 2, we get ${}^{N \cup Q}\overline{\sigma} = {}^{N}\overline{\sigma} \cup {}^{Q}\overline{\sigma} = {}^{P}\overline{\sigma} \cup {}^{T}\overline{\sigma} = {}^{P \cup T}\overline{\sigma}$. This implies that $(N \cup Q) \eqsim_{\sigma} (P \cup T)$.

(3) Let $N \subseteq P$ and $P \eqsim_{\sigma} 0_{U_1}$. Then, ${}^{P}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$.

Also, by Theorem 2, $N \subseteq P$ implies that ${}^{N}\overline{\sigma} \subseteq {}^{P}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. But, ${}^{0_{U_1}}\overline{\sigma} \subseteq {}^{N}\overline{\sigma}$. Thus, ${}^{N}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. This implies that $N \eqsim_{\sigma} 0$.

(4) If $N \eqsim_{\sigma} 0_{U_1}$ and $P \eqsim_{\sigma} 0_{U_1}$, then ${}^{N}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$ and ${}^{P}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. Now, by Theorem 2, we have ${}^{N \cup P}\overline{\sigma} = {}^{N}\overline{\sigma} \cup {}^{P}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma} \cup {}^{0_{U_1}}\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. This implies that $(N \cup P) \eqsim_{\sigma} 0_{U_1}$. Conversely, if $(N \cup P) \eqsim_{\sigma} 0_{U_1}$, then by part (3), we have $N \eqsim_{\sigma} 0_{U_1}$ and $P \eqsim_{\sigma} 0_{U_1}$.

(5) Suppose $N \eqsim_{\sigma} 1_{U_1}$. Then, ${}^{N}\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$. As

$N \subseteq P$, we have ${}^{P}\overline{\sigma} \supseteq {}^{N}\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$. On the other hand, $P \subseteq 1_{U_1}$, so we have ${}^{P}\overline{\sigma} \subseteq {}^{1_{U_1}}\overline{\sigma}$. This implies that ${}^{P}\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$, that is, $P \eqsim_{\sigma} 1_{U_1}$.

(6) It follows from (5).

Theorem 10 shows some lower similarity relations of union and intersection of IFSs N, P, Q, and T in U_1 with respect to the foresets. □

Theorem 11. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_2. Then, the following are true:*

(1) $N \simeq_A P$ if and only if $N \simeq_A (N \cap P) \simeq_A P$;

(2) $N \simeq_A P$ and $Q \simeq_A T$ imply that $(N \cap Q) \simeq_A (P \cap T)$;

(3) $N \subseteq P$ and $P \simeq_A 0_{U_2}$ imply that $N \simeq_A 0_{U_2}$;

(4) $(N \cap P) \simeq_A 0_{U_2}$ if and only if $N \simeq_A 0_{U_2}$ and $P \simeq_A 0_{U_2}$;

(5) $N \subseteq P$ and $N \simeq_A 1_{U_2}$ imply that $P \simeq_A 1_{U_2}$;

(6) If $(N \cap P) \simeq_A 1_{U_2}$, then $N \simeq_A 1_{U_2}$ and $P \simeq_A 1_{U_2}$.

Proof

(1) Let $N \simeq_A P$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$. By Theorem 1, we get $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^N = \underline{\sigma}^P$. This implies that $N \simeq_A (N \cap P) \simeq_A P$. Converse holds by the transitivity of the relation $\simeq_A$.

(2) Let $N \simeq_A P$ and $Q \simeq_A T$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$ and $\underline{\sigma}^Q = \underline{\sigma}^T$. By Theorem 1, we get $\underline{\sigma}^{N \cap Q} = \underline{\sigma}^N \cap \underline{\sigma}^Q = \underline{\sigma}^P \cap \underline{\sigma}^T = \underline{\sigma}^{P \cap T}$. This implies that $(N \cap Q) \simeq_A (P \cap T)$.

(3) Let $N \subseteq P$ and $P \simeq_A 0_{U_2}$. Then, $\underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$.
Also, by Theorem 1, $N \subseteq P$ implies that $\underline{\sigma}^N \subseteq \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$. But, $\underline{\sigma}^{0_{U_2}} \subseteq \underline{\sigma}^N$. Thus, $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$. This implies that $N \simeq_A 0_{U_2}$.

(4) If $N \simeq_A 0_{U_2}$ and $P \simeq_A 0_{U_2}$, then $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$. Now, by Theorem 1, we have $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}} \cap \underline{\sigma}^{0_{U_2}} = \underline{\sigma}^{0_{U_2}}$, so $\underline{\sigma}^{N \cap P} = \underline{\sigma}^{0_{U_2}}$ This implies that $(N \cap P) \simeq_A 0_{U_2}$. Conversely, if $(N \cap P) \simeq_A 0_{U_2}$, then by part (3), we have $N \simeq_A 0_{U_2}$ and $P \simeq_A 0_{U_2}$.

(5) Suppose $N \simeq_A 1_{U_2}$. Then, $\underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^P \supseteq \underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$, so we have $\underline{\sigma}^P \subseteq \underline{\sigma}^{1_{U_2}}$. This implies that $\underline{\sigma}^P = \underline{\sigma}^{1_{U_2}}$, that is, $P \simeq_A 1_{U_2}$.

(6) It follows from (5).

Theorem 11 shows some upper similarity relations of union and intersection of IFSs N, P, Q, and T in U_2 with respect to the aftersets. □

Theorem 12. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_1. Then, the following are true:*

(1) $N \simeq_\sigma P$ if and only if $N \simeq_\sigma (N \cap P) \simeq_\sigma P$;

(2) $N \simeq_\sigma P$ and $Q \simeq_\sigma T$ imply that $(N \cap Q) \simeq_\sigma (P \cap T)$;

(3) $N \subseteq P$ and $P \simeq_\sigma 0_{U_1}$ imply that $N \simeq_\sigma 0_{U_1}$;

(4) $(N \cap P) \simeq_\sigma 0_{U_1}$ if and only if $N \simeq_\sigma 0_{U_1}$ and $P \simeq_\sigma 0_{U_1}$;

(5) $N \subseteq P$ and $N \simeq_\sigma 1_{U_1}$ imply that $P \simeq_\sigma 1_{U_1}$;

(6) If $(N \cap P) \simeq_\sigma 1_{U_1}$, then $N \simeq_\sigma 1_{U_1}$ and $P \simeq_\sigma 1_{U_1}$.

Proof

(1) Let $N \simeq_\sigma P$. Then, ${}^N\underline{\sigma} = {}^P\underline{\sigma}$. By Theorem 2, we get ${}^{N \cap P}\underline{\sigma} = {}^N\underline{\sigma} \cap {}^P\underline{\sigma} = {}^N\underline{\sigma} = {}^P\underline{\sigma}$. This implies that $N \simeq_\sigma (N \cap P) \simeq_\sigma P$. Converse holds by the transitivity of the relation $\simeq_\sigma$.

(2) Let $N \simeq_\sigma P$ and $Q \simeq_\sigma T$. Then, ${}^N\underline{\sigma} = {}^P\underline{\sigma}$ and ${}^Q\underline{\sigma} = {}^T\underline{\sigma}$.
By Theorem 2, we get ${}^{N \cap Q}\underline{\sigma} = {}^N\underline{\sigma} \cap {}^Q\underline{\sigma} = {}^P\underline{\sigma} \cap {}^T\underline{\sigma} = {}^{P \cap T}\underline{\sigma}$. This implies that $(N \cap Q) \simeq_\sigma (P \cap T)$.

(3) Let $N \subseteq P$ and $P \simeq_\sigma 0_{U_1}$. Then, ${}^P\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$.
Also, by Theorem 2, $N \subseteq P$ implies that ${}^N\underline{\sigma} \subseteq {}^P\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$. But, ${}^{0_{U_1}}\underline{\sigma} \subseteq {}^N\underline{\sigma}$. Thus, ${}^N\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$. This implies that $N \simeq_\sigma 0_{U_1}$.

(4) If $N \simeq_\sigma 0_{U_1}$ and $P \simeq_\sigma 0_{U_1}$, then $\underline{\sigma}^N = {}^{0_{U_1}}\underline{\sigma}$ and ${}^P\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$. Now, by Theorem 2, we have ${}^{N \cap P}\underline{\sigma} = {}^N\underline{\sigma} \cap {}^P\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}^{0_{U_1}} \cap {}^{0_{U_1}}\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$, so ${}^{N \cap P}\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$. This implies that $(N \cap P) \simeq_\sigma 0_{U_1}$. Conversely, if $(N \cap P) \simeq_\sigma 0_{U_1}$, then by part (3), we have $N \simeq_\sigma 0_{U_1}$ and $P \simeq_\sigma 0_{U_1}$.

(5) Suppose $N \simeq_\sigma 1_{U_1}$. Then, ${}^N\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$. As $N \subseteq P$, we have ${}^P\underline{\sigma} \supseteq {}^N\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$. On the other hand, $P \subseteq 1_{U_1}$, so we have ${}^P\underline{\sigma} \subseteq {}^{1_{U_1}}\underline{\sigma}$. This implies that ${}^P\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$, that is, $P \simeq_\sigma 1_{U_1}$.

(6) It follows from (5). □

Theorem 13. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_2. Then, the following are true:*

(1) $N \subseteq P$ and $P \approx {}_A 0_{U_2}$ imply that $N \approx {}_A 0_{U_2}$;

(2) $N \subseteq P$ and $N \approx {}_A 1_{U_2}$ imply that $P \approx {}_A 1_{U_2}$;

(3) $(N \cup P) \approx {}_A 0_{U_2}$, then $N \approx {}_A 0_{U_2}$ and $P \approx {}_A 0_{U_2}$;

(4) $(N \cap P) \approx {}_A 1_{U_2}$, then $N \approx {}_A 1_{U_2}$ and $P \approx {}_A 1_{U_2}$;

(5) $N \approx {}_A P$ if and only if $N \simeq_A (N \cup P) \simeq_A P$ and $N \simeq_A (N \cap P) \simeq_A P$.

Proof

(1) Suppose $P \approx {}_A 0_{U_2}$, this implies that $\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\overline{\sigma}^P = \overline{\sigma}^{0_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^N \subseteq \underline{\sigma}^P = \underline{\sigma}^{0_{U_2}}$ and $\overline{\sigma}^N \subseteq \overline{\sigma}^P = \overline{\sigma}^{0_{U_2}}$. On the other hand, $N \supseteq 0_{U_2}$; this implies that $\underline{\sigma}^N \supseteq \underline{\sigma}^{0_{U_2}}$ and $\overline{\sigma}^N \supseteq \overline{\sigma}^{0_{U_2}}$. So,
$\underline{\sigma}^N = \underline{\sigma}^{0_{U_2}}$ and $\overline{\sigma}^N = \overline{\sigma}^{0_{U_2}}$, thus $N \approx {}_A 0_{U_2}$.

(2) Suppose $N \approx {}_A 1_{U_2}$, this implies that $\underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$ and $\overline{\sigma}^P = \overline{\sigma}^{1_{U_2}}$. As $N \subseteq P$, we have $\underline{\sigma}^P \supseteq \underline{\sigma}^N = \underline{\sigma}^{1_{U_2}}$ and $\overline{\sigma}^P \supseteq \overline{\sigma}^N = \overline{\sigma}^{1_{U_2}}$. On the other hand, $P \subseteq 1_{U_2}$; this implies that $\underline{\sigma}^P \subseteq \underline{\sigma}^{1_{U_2}}$ and $\overline{\sigma}^P \subseteq \overline{\sigma}^{1_{U_2}}$. So,
$\underline{\sigma}^P = \underline{\sigma}^{1_{U_2}}$ and $\overline{\sigma}^P = \overline{\sigma}^{1_{U_2}}$, thus $P \approx {}_A 1_{U_2}$.

(3) It follows from part (1).

(4) It follows from part (2).

(5) Let $N \approx {}_A P$. Then, $\underline{\sigma}^N = \underline{\sigma}^P$ and $\overline{\sigma}^P = \overline{\sigma}^N$. By Theorem 1, we get $\underline{\sigma}^{N \cap P} = \underline{\sigma}^N \cap \underline{\sigma}^P = \underline{\sigma}^N = \underline{\sigma}^P$ and $\overline{\sigma}^{N \cup P} = \overline{\sigma}^N \cup \overline{\sigma}^P = \overline{\sigma}^N = \overline{\sigma}^P$. This implies that $N \simeq_A (N \cap P) \simeq_A P$ and $N \simeq_A (N \cup P) \simeq_A P$.

Converse holds by the transitivity of the relation $\approx_A$.

Theorem 13 shows some similarity relations of union and intersection of IFSs N, P, Q, and T in U_2 with respect to the aftersets. □

Theorem 14. *Let (σ, A) be a soft binary relation from U_1 to U_2. Let N, P, Q, and T be IFSs of U_1. Then, the following are true:*

(1) $N \subseteq P$ and $P \approx_\sigma 0_{U_1}$ imply that $N \approx_\sigma 0_{U_1}$;

(2) $N \subseteq P$ and $N \approx_\sigma 1_{U_1}$ imply that $P \approx_\sigma 1_{U_1}$;

(3) $(N \cup P) \approx_\sigma 0_{U_1}$, then $N \approx_\sigma 0_{U_1}$ and $P \approx_\sigma 0_{U_1}$;

(4) $(N \cap P) \approx_\sigma 1_{U_1}$, then $N \approx_\sigma 1_{U_1}$ and $P \approx_\sigma 1_{U_1}$;

(5) $N \approx_\sigma P$ if and only if $N \eqsim_\sigma (N \cup P) \eqsim_\sigma P$ and $N \simeq_\sigma (N \cap P) \simeq_\sigma P$.

Proof

(1) Suppose $P \approx_\sigma 0U_1$, this implies that ${}^N\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$ and ${}^P\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. As $N \subseteq P$, we have ${}^N\underline{\sigma} \subseteq {}^P\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$ and ${}^N\overline{\sigma} \subseteq {}^P\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$. On the other hand, $N \supseteq 0_{U_1}$; this implies that $\underline{\sigma} \supseteq {}^{0_{U_1}}\underline{\sigma}$ and ${}^N\overline{\sigma} \supseteq {}^{0_{U_1}}\overline{\sigma}$. So,
${}^N\underline{\sigma} = {}^{0_{U_1}}\underline{\sigma}$ and ${}^N\overline{\sigma} = {}^{0_{U_1}}\overline{\sigma}$, thus $N \approx_\sigma 0_{U_1}$.

(2) Suppose $N \approx_\sigma 1_{U_1}$, this implies that ${}^N\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$ and ${}^P\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$. As $N \subseteq P$, we have ${}^P\underline{\sigma} \supseteq {}^N\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$ and ${}^P\overline{\sigma} \supseteq {}^N\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$. On the other hand, $P \subseteq 1_{U_1}$; this implies that ${}^P\underline{\sigma} \subseteq {}^{1_{U_1}}\underline{\sigma}$ and ${}^P\overline{\sigma} \subseteq {}^{1_{U_1}}\overline{\sigma}$. So,
${}^P\underline{\sigma} = {}^{1_{U_1}}\underline{\sigma}$ and ${}^P\overline{\sigma} = {}^{1_{U_1}}\overline{\sigma}$, thus $P \approx_\sigma 1_{U_1}$.

(3) It follows from part (1).

(4) It follows from part (2).

(5) Let $N \approx_\sigma P$. Then, ${}^N\underline{\sigma} = {}^P\underline{\sigma}$ and ${}^P\overline{\sigma} = {}^N\overline{\sigma}$. By Theorem 2, we get ${}^{N \cap P}\underline{\sigma} = {}^N\underline{\sigma} \cap {}^P\underline{\sigma} = {}^N\underline{\sigma} = {}^P\underline{\sigma}$ and ${}^{N \cup P}\overline{\sigma} = {}^N\overline{\sigma} \cup {}^P\overline{\sigma} = {}^N\overline{\sigma} = {}^P\overline{\sigma}$. This implies that $N \simeq_\sigma (N \cap P) \simeq_\sigma P$ and $N \eqsim_\sigma (N \cup P) \eqsim_\sigma P$.

Converse holds by the transitivity of the relation $\approx_\sigma$. □

5. Application in Decision-Making Problem

A major area of study in all kinds of data analysis is decision making. Many experts and researchers introduced many methods to find a wise decision. RS theory [7], SS theory [12], and IFS theory [3] are the theories which are mostly used in the decision-making problems. In the above sections, we develop a rough set model using soft binary relations. We used soft binary relation to approximate an IFS. We used score function defined by Chen and Tan [57] and accuracy function defined by Hong and Choi [58] to define order between objects. Now, we present an algorithm for the approach to a decision-making problem and this problem is depended on IF soft rough set theory based on soft binary relations. This algorithm extends the already existing approach which is described by Kanwal and Shabir [51]. For our new approach, data information is only needed which is provided by the decision-making problem and no need of any additional information by any supplementary ways. So, the decision results can be avoided by the effect of subjective information. Therefore, the outcomes could avoid the inconsistent results for the same problem and could be better objective. The decision Algorithm 1 is as follows:

Now, we show this approach step by step to decision making which is proposed by using the following example. The following example discusses algorithm to make wise decision for the selection of a car.

Example 4. Suppose a person Mr. X wants to select a car out of available models. Let $U_1 = \{$the set of all models available in range$\} = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $U_2 = \{$the colors of all models$\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{$the set of brands$\} = \{e_1 = \text{Suzuki}, e_2 = \text{Toyota}, e_3 = \text{Honda}\}$.

Define σ: $A \longrightarrow P(U_1 \times U_2)$ by

$$\begin{aligned}
\sigma(e_1) &= \left\{ \begin{array}{l} (m_1, c_1), (m_1, c_2), (m_1, c_3), (m_2, c_2), (m_2, c_4), \\ (m_4, c_2), (m_4, c_3), (m_5, c_3), (m_5, c_4), (m_6, c_1) \end{array} \right\}, \\
\sigma(e_2) &= \{(m_1, c_3), (m_2, c_3), (m_4, c_1), (m_5, c_1), (m_6, c_2), (m_6, c_3)\}, \\
\sigma(e_3) &= \{(m_3, c_3), (m_3, c_1), (m_2, c_4), (m_5, c_3), (m_5, c_4)\},
\end{aligned} \tag{9}$$

which represents the relation between models and colors available in brand e_i for $1 \le i \le 3$. Then,

$$\begin{aligned}
&m_1\sigma(e_1) = \{c_1, c_2, c_3\}, m_2\sigma(e_1) = \{c_2, c_4\}, m_3\sigma(e_1) = \varnothing, \\
&m_4\sigma(e_1) = \{c_2, c_3\}, m_5\sigma(e_1) = \{c_3, c_4\}, m_6\sigma(e_1) = \{c_1\}, \\
&m_1\sigma(e_2) = \{c_3\}, m_2\sigma(e_2) = \{c_3\}, m_3\sigma(e_2) = \varnothing, \\
&m_4\sigma(e_2) = \{c_1\}, m_5\sigma(e_2) = \{c_1\}, m_6\sigma(e_2) = \{c_2, c_3\}, \\
&m_1\sigma(e_3) = \varnothing, m_2\sigma(e_3) = \{c_4\}, m_3\sigma(e_3) = \{c_1, c_3\}, \\
&m_4\sigma(e_3) = \varnothing, m_5\sigma(e_3) = \{c_3, c_4\}, m_6\sigma(e_3) = \varnothing,
\end{aligned} \tag{10}$$

where $m_i\sigma(e_j)$ represents the color of the model m_i available in the brand e_j.

Also,

$$\begin{aligned}
&\sigma(e_1)c_1 = \{m_1, m_6\}, \sigma(e_1)c_2 = \{m_1, m_2, m_4\}, \\
&\sigma(e_1)c_3 = \{m_1, m_4, m_5\}, \sigma(e_1)c_4 = \{m_2, m_5\}, \\
&\sigma(e_2)c_1 = \{m_4, m_5\}, \sigma(e_2)c_2 = \{m_6\}, \\
&\sigma(e_2)c_3 = \{m_1, m_2\}, \sigma(e_2)c_4 = \varnothing, \\
&\sigma(e_3)c_1 = \{m_3\}, \sigma(e_3)c_2 = \varnothing, \\
&\sigma(e_3)c_3 = \{m_3, m_5\}, \sigma(e_3)c_4 = \{m_2, m_5\},
\end{aligned} \tag{11}$$

(1) Compute the upper IF soft set approximation $\overline{\sigma}^M$ and lower IF soft set approximation $\underline{\sigma}^M$ of an IF set $M = \langle \mu_M, \gamma_M \rangle$ with respect to the aftersets;
(2) Compute the score values for each of the entries of the $\underline{\sigma}^M$ and $\overline{\sigma}^M$ and denote them by $\underline{S}_{ij}(x_i, e_j)$ and $\overline{S}_{ij}(x_i, e_j)$ for all i, j;
(3) Compute the aggregated score $\underline{S}(x_i) = \sum_{j=1}^{n} \underline{S}_{ij}(x_i, e_j)$ and $\overline{S}(x_i) = \sum_{j=1}^{n} \overline{S}_{ij}(x_i, e_j)$;
(4) Compute $S(x_i) = \underline{S}(x_i) + \overline{S}(x_i)$;
(5) The best decision is $x_k = \max_i S(x_i)$;
(6) If k has more than one value, say k_1, k_2, then we calculate the accuracy values $\underline{H}_{ij}(x_i, e_j)$ and $\overline{H}_{ij}(x_i, e_j)$ for only those x_k for which $S(x_k)$ are equal;
(7) Compute $H(x_k) = \sum_{j=1}^{n} \underline{H}_{kj}(x_k, e_j) + \sum_{j=1}^{n} \overline{H}_{kj}(x_k, e_j)$ for $k = k_1, k_2$;
(8) If $H(x_{k_1}) > H(x_{k_2})$, then we select x;
(9) If $H(x_{k_1}) = H(x_{k_2})$, then select any one of x_{k_1} and x_{k_2}.

Algorithm 1: Procedural steps for better decision with the help of score function.

Table 14: Lower approximation of M.

$(\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M})$	m_1	m_2	m_3	m_4	m_5	m_6
$\underline{\sigma}^{\mu_M}(e_1)$	0.4	0	1	0.4	0	0.9
$\underline{\sigma}^{\mu_M}(e_2)$	0.4	0.4	1	0.9	0.9	0.4
$\underline{\sigma}^{\mu_M}(e_3)$	1	0	0.4	1	0	1
$\underline{\sigma}^{\gamma_M}(e_1)$	0.5	0.8	0	0.5	0.8	0
$\underline{\sigma}^{\gamma_M}(e_2)$	0.5	0.5	0	0	0	0.5
$\underline{\sigma}^{\gamma_M}(e_3)$	0	0.8	0.5	0	0.8	0

where $\sigma(e_j)c_i$ represents the model of the color c_i available in the brand e_j.

$$\begin{aligned}
&\text{Define } M = \langle \mu_M, \gamma_M \rangle : U_2 \longrightarrow [0,1], \text{which represents the preference of the} \\
&\qquad \text{colors given by Mr.} X \text{ such that} \\
&\mu_M(c_1) = 0.9, \mu_M(c_2) = 0.8, \mu_M(c_3) = 0.4, \mu_M(c_4) = 0 \\
&\gamma_M(c_1) = 0.0, \gamma_M(c_2) = 0.2, \gamma_M(c_3) = 0.5, \gamma_M(c_4) = 0.8. \\
&\text{Define } N = \langle \mu_N, \gamma_N \rangle : U_1 \longrightarrow [0,1], \text{which represents the preference of the} \\
&\qquad \text{model given by Mr.} X \text{ such that} \\
&\mu_N(m_1) = 1, \mu_N(m_2) = 0.7, \mu_N(m_3) = 0.5, \mu_N(m_4) = 0.1, \\
&\mu_N(m_5) = 0, \mu_N(m_6) = 0.4 \\
&\gamma_N(m_1) = 0, \gamma_N(m_2) = 0.2, \gamma_N(m_3) = 0.5, \gamma_N(m_4) = 0.7, \\
&\gamma_N(m_5) = 1, \gamma_N(m_6) = 0.5.
\end{aligned} \tag{12}$$

Therefore, the lower and upper approximations (with respect to the aftersets as well as with respect to the foresets) are as follows (Tables 14 and 15, respectively):

$$\begin{aligned}
\underline{\sigma}^M &= \left(\underline{\sigma}^{\mu_M}, \underline{\sigma}^{\gamma_M}\right), \\
\overline{\sigma}^M &= \left(\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M}\right).
\end{aligned} \tag{13}$$

The values of score function for car models are given in Table 16.

Table 16 shows that $S(m_4) = S(m_6)$, so we calculate accuracy values for m_4 and m_6.

Hence, the values of accuracy function are given in Table 17.

Table 17 shows that $H(m_4) = H(m_6)$, so we can select any one, m_4 or m_6.

Now, ${}^N\underline{\sigma} = (\mu_N \underline{\sigma}, {}^{\gamma_N}\underline{\sigma})$ (given in Table 18) and ${}^N\overline{\sigma} = (\mu_N \overline{\sigma}, {}^{\gamma_N}\overline{\sigma})$ (given in Table 19).

The values of score function for colors of cars are given in Table 20.

Table 20 shows that $S(c_3) = 0.5$ is maximum, so he will select color c_4.

The flow chart of our decision-making algorithm is given in Figure 2.

Table 15: Upper approximation of M.

$(\overline{\sigma}^{\mu_M}, \overline{\sigma}^{\gamma_M})$	m_1	m_2	m_3	m_4	m_5	m_6
$\overline{\sigma}^{\mu_M}(e_1)$	0.9	0.8	0	0.8	0.4	0.9
$\overline{\sigma}^{\mu_M}(e_2)$	0.4	0.4	0	0.9	0.9	0.8
$\overline{\sigma}^{\mu_M}(e_3)$	0	0	0.9	0	0.4	0
$\overline{\sigma}^{\gamma_M}(e_1)$	0	0.2	1	0.2	0.5	0
$\overline{\sigma}^{\gamma_M}(e_2)$	0.5	0.5	1	0	0	0.2
$\overline{\sigma}^{\gamma_M}(e_3)$	1	0.8	0	1	0.5	1

Table 16: Values of score function for car models.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\overline{S}_{ij}(e_1)$	$\overline{S}_{ij}(e_2)$	$\overline{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\overline{S}(x_i)$	$S(x_i)$
m_1	−0.1	−0.1	1	0.9	−0.1	−1	0.8	−0.2	0.6
m_2	−0.8	−0.1	−0.8	0.6	−0.1	−0.8	−1.7	−0.3	−2
m_3	1	1	−0.1	−1	−1	0.9	1.9	−1.1	0.8
m_4	−0.1	0.9	1	0.6	0.9	−1	1.8	0.5	2.3
m_5	−0.8	0.9	−0.8	−0.1	0.9	0.3	−0.7	1.1	0.4
m_6	0.9	−0.1	1	0.9	0.6	−1	1.8	0.5	2.3

Table 17: Values of accuracy function.

	$\underline{H}_{ij}(e_1)$	$\underline{H}_{ij}(e_2)$	$\underline{H}_{ij}(e_3)$	$\overline{H}_{ij}(e_1)$	$\overline{H}_{ij}(e_2)$	$\overline{H}_{ij}(e_3)$	H
m_4	0.9	0.9	1	1	0.9	1	5.7
m_6	0.9	0.9	1	0.9	1	1	5.7

Table 18: Lower approximation of N.

$(\mu_N \underline{\sigma}, {}^{\gamma_N}\underline{\sigma})$	c_1	c_2	c_3	c_4
${}^{\mu_N}\underline{\sigma}(e_1)$	0.4	0.1	0	0
${}^{\mu_N}\underline{\sigma}(e_2)$	0	0.4	0.7	1
${}^{\mu_N}\underline{\sigma}(e_3)$	0.5	1	0	0
${}^{\gamma_N}\underline{\sigma}(e_1)$	0.5	0.7	1	1
${}^{\gamma_N}\underline{\sigma}(e_2)$	1	0.5	0.2	0
${}^{\gamma_N}\underline{\sigma}(e_3)$	0.5	0	1	1

Table 19: Upper approximation of N.

$(\mu_N \overline{\sigma}, {}^{\gamma_N}\overline{\sigma})$	c_1	c_2	c_3	c_4
${}^{\mu_N}\overline{\sigma}(e_1)$	1	1	1	0.7
${}^{\mu_N}\overline{\sigma}(e_2)$	0.1	0.4	1	0
${}^{\mu_N}\overline{\sigma}(e_3)$	0.5	0	0.5	0.7
${}^{\gamma_N}\overline{\sigma}(e_1)$	0	0	0	0.2
${}^{\gamma_N}\overline{\sigma}(e_2)$	0.7	0.5	0	1
${}^{\gamma_N}\overline{\sigma}(e_3)$	0.5	1	0.5	0.2

Table 20: Values of score function for colors of cars.

	$\underline{S}_{ij}(e_1)$	$\underline{S}_{ij}(e_2)$	$\underline{S}_{ij}(e_3)$	$\overline{S}_{ij}(e_1)$	$\overline{S}_{ij}(e_2)$	$\overline{S}_{ij}(e_3)$	$\underline{S}(x_i)$	$\overline{S}(x_i)$	$S(x_i)$
c_1	−0.1	−1	0	1	−0.6	0	−1.1	0.4	−0.7
c_2	−0.6	−0.1	1	1	−0.1	−1	0.3	−0.1	0.2
c_3	−1	0.5	−1	1	1	0	−1.5	2	0.5
c_4	−1	1	−1	0.5	−1	0.5	−1	0	−1

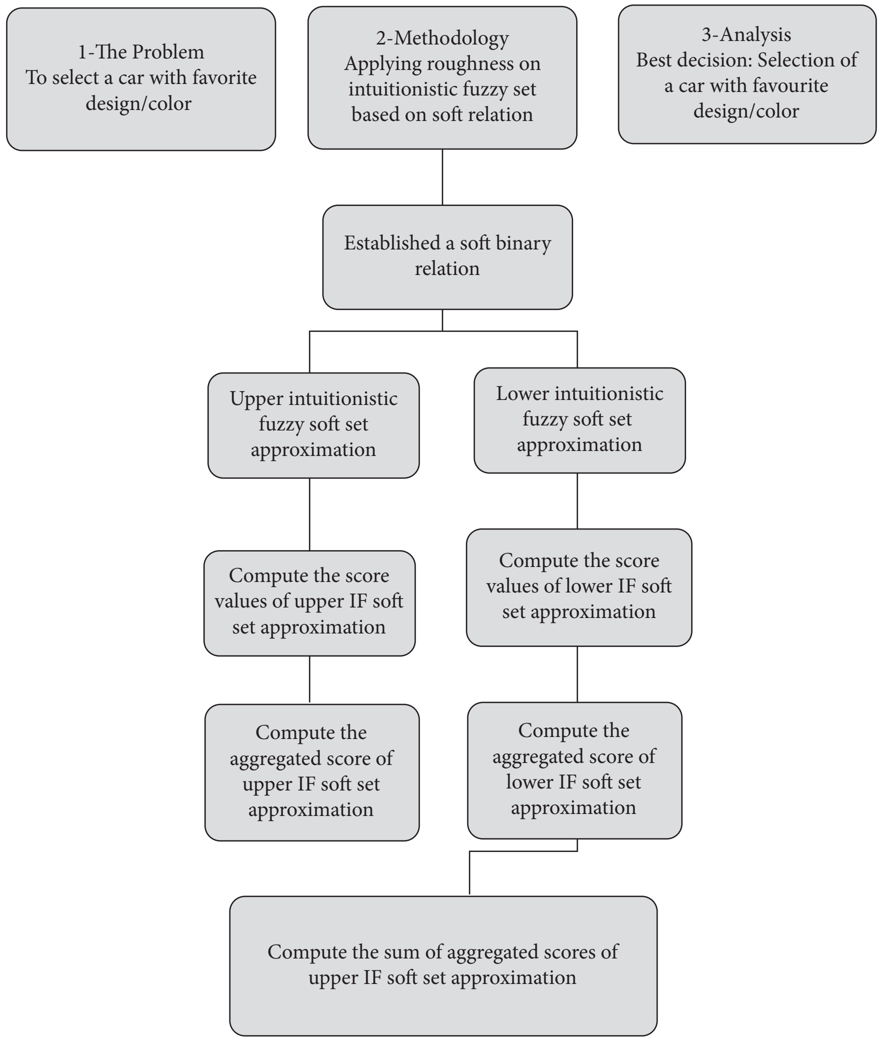

Figure 2: Flow chart of decision-making algorithm of IFRS proposed model.

In Figure 2, the flow chart shows that if a person wants to buy a car with his favourite design and color, then this algorithm helps him to make best and suitable decision according to his choice.

6. Comparison

First, we review existing approaches to intuitionistic fuzzy rough set (IFRS) model-based decision making and then finally we show that our newly proposed IFRS model is very useful than other existing theories. Since the combination of IFS with SS and RS is very helpful to deal with uncertainty and impreciseness, Maji et al. [24] presented a useful model of RS and SS. Chen et al. [20] used SS parameterization reduction and improved SSs-based decision making in [24]. Cagman et al. [63] presented a uni-int decision-making method by using redefined operations of soft sets. But all the above work in decision making is about only crisp soft set. Then, Roy et al. [46] solved recognition problems by using their newly proposed algorithm of fuzzy SSs [64]. Later, Kong et al. [65] modified Roy et al.' algorithm and proved that their algorithm was not able to obtain optimal choice generally. Feng et al. [21] also worked on fuzzy soft set-based algorithm. Later, Jiang et al. [66] discussed intuitionistic fuzzy soft sets with an adjusted approach.

6.1. Maji and Roy's Method and Its Limitation. Maji et al. used concept of knowledge reduction in RSs with SSs to solve decision-making problems. This method consists of two steps. In first step, find one reduct soft set of the original SS based on the knowledge reduction of RSs, and then calculate the choice values of all elements and select the element with the maximum value as the optimum alternative. Chen et al. [20] claimed that soft set reduction in [24] has incorrect results in Step 1.

6.2. Cagman's Method and Its Limitation. Cagman et al. [63] proposed a soft max-min decision-making method. Optimum alternatives are selected from the alternatives set by this method. In this method, the noting point is that this method has its constitutive limitation. An algorithm of this method gets an empty optimum set.

Dubious and Prade presented rough FSs [35] and Feng et al. [1] extended their model in terms of SSs. Feng et al. [1] approximated a FS in a soft approximation space and extended a concept called soft rough FSs. The FSs and IF relations-based models are useful in many fields, but we used soft binary relations in our proposed model. Soft binary relation is the generalization of a binary relation and soft binary relation is the parameterized collection of ordinary binary relations. In soft binary relation, we can use different parameters according to the nature of problem. That is why our proposed model is more useful to manage uncertainty in different types of problems.

6.3. Advantages of Our Proposed IFRS Model

(1) In our IFRS model based on soft binary relations, we also get information about what candidate is optimum alternative and what candidate should not be optimum alternative, whereas other existing theories only get optimum alternatives. That is why our newly proposed IFRS model is more precise and flexible for decision-making problems.

(2) Our proposed model also gives a solution IFRS-based group decision making, whereas other existing approaches have no directions to discuss the intuitionistic fuzzy set group decision making.

(3) This IFRS model based on soft relations can be applied to solve decision-making problems involving intuitionistic fuzzy sets in real life [56].

In 2012, Zhang [56] proposed a RS model based on ordinary binary relation induced by an IF relation over two universes and presented a decision-making algorithm based on RS model with IFSs. In 2020, Shabir et al. [51] proposed a RS model of FSs based on soft relations and presented decision-making algorithm. In comparison of these models, we proposed a RS model of IFSs based on soft relations and presented a decision-making algorithm based on IFRS which is a better technique to manage uncertainty and impreciseness. We used an IFS instead of a crisp set or a FS in our proposed model due to its importance in scientific fields and decision making, such as medical diagnosis, career determination, pattern recognition, and electoral system. An IFS has degree of membership and also degree of nonmembership which is helpful to make better decision in real-life problems.

7. Conclusion

Since the combination of IFS with SS and RS is very helpful to deal with uncertainty, Maji et al. presented a useful model of RS and SS. Chen et al. used SS parameterization reduction and improved SSs-based decision making. Cagman et al. presented a uni-int decision-making method by using redefined operations of SSs. But, all the above work in decision making is about only crisp SS. Then, Roy et al. solved recognition problems by using their newly proposed algorithm of fuzzy SSs. Later, Kong et al. modified Roy et al.'s algorithm and proved that their algorithm was not able to obtain optimal choice generally. Feng et al. also worked on fuzzy SS-based algorithm. Later, Jiang et al. discussed IF soft sets with an adjusted approach. In our paper, we have given a generalization of [51] and we have approximated an IFS by soft binary relations. We used foresets and aftersets to approximate IFS. In this way, we get two pairs of intuitionistic fuzzy soft sets, called the lower approximation and upper approximation. Properties of these approximations are studied. Similarity relations between IFS with respect to this rough set model are also studied. Finally, we developed an algorithm for intuitionistic fuzzy rough sets (IFRS) based on decision making and an example is provided to illustrate the developed algorithm. Further study can be performed to investigate the roughness in interval-valued IFS and multigranulation roughness of IFS by using soft relations.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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