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Research Article

Theta Omega Topological Operators and Some Product Theorems

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We introduce and investigate the concepts of θ_ω-limit points and θ_ω-interior points, and we use them to introduce two new topological operators. For a subset B of a topological space (Y, σ), denote the set of all limit points of B (resp. θ-limit points of B, θ_ω-limit points of B, interior points of B, θ-interior points of B, and θ_ω-interior points of B) by $D(B)$ (resp. $D_\theta(B)$, $D_{\theta_\omega}(B)$, $\mathrm{Int}(B)$, $\mathrm{Int}_\theta(B)$, and $\mathrm{Int}_{\theta_\omega}(B)$). Several results regarding the two new topological operators are given. In particular, we show that $D_{\theta_\omega}(B)$ lies strictly between $D(B)$ and $D_\theta(B)$ and $\mathrm{Int}_{\theta_\omega}(B)$ lies strictly between $\mathrm{Int}_\theta(B)$ and $\mathrm{Int}(B)$. We show that $D(B) = D_{\theta_\omega}(B)$ (resp. $\mathrm{Cl}_\theta(B) = \mathrm{Cl}_{\theta_\omega}(B)$ and $D(B) = D_{\theta_\omega}(B) = D_\theta(B)$) for locally countable topological spaces (resp. antilocally countable topological spaces and regular topological spaces). In addition to these, we introduce several product theorems concerning metacompactness.

1. Introduction

In 1943, Fomin [1] introduced the notion of θ-continuity. For the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases, the notions of θ-open subsets, θ-closed subsets, and θ-closure were introduced by Velicko [2] in 1966, in which he showed that the family of θ-open sets in a topological space (Y, σ) forms a topology on Y denoted by σ_θ (see also [3]). The work of Velicko is continued by [3–26] and others. Hdeib [27] introduced the class of ω-closed sets by which he introduced and investigated the notion of ω-continuity. The family of all ω-open sets in (Y, σ) is denoted by σ_ω. It is known that σ_ω is a topology on Y which is finer than σ. Research related to ω-open sets is still a hot area of research [28–36]. In 2017, Al Ghour and Irshidat [37] introduced θ_ω-open subsets, θ_ω-closed subsets, and θ_ω-closure utilizing the topological spaces (Y, σ_θ) and (Y, σ_ω). It is proved in [37] that σ_{θ_ω} forms a topology on Y which lies between σ_θ and σ, and that $\sigma_{\theta_\omega} = \sigma$ if and only if (Y, σ) is ω-regular. Also, in [37], $\omega - T_2$ topological spaces were characterized via θ_ω-open sets. Authors in [35] introduced θ_ω-connectedness and some new separation axioms. Also, research in [37] was continued by various researchers in [28–31]. The notion of interior operators is important in the axiomatization of modal logics. Judging from the importance of limit points in mathematical analysis, introducing a new limit point notion in any topological structure is still a hot area of research. The first goal of this paper is to introduce and investigate the concepts of θ_ω-limit points and θ_ω-interior points.

In general topology, several topological properties are not finitely productive, such as paracompactness, strong paracompactness, Lindelöfness, and metacompactness. The area of research regarding the problem "What conditions on (Y, σ) and (Z, δ) to insure that their product has property $\mathscr{P}$"is still hot[38–45]. The second goal of this paper is to introduce several product theorems concerning metacompactness.

2. Preliminaries

From now on TS will denote topological space for simplicity. Let (Y, σ) and (Z, δ) be TSs and let $B \subseteq C \subseteq Y$ with C as nonempty. Then, B is called ω-open set in (Y, σ) [27] if for each $y \in B$, there is $M \in \sigma$ and a countable set $F \subseteq Y$ such that $y \in M - F \subseteq B$. The relative topology on C is denoted by σ_C, and the product topology on $Y \times Z$ is denoted by $\sigma \times \delta$. The closure of B in (Y, σ) (resp. (C, σ_C), (Y, σ_ω)) is denoted by $\overline{B}$ (resp. $\overline{B}^C$, $\underline{B}_\omega$). A point $y \in Y$ is in θ-closure of B [2] ($y \in \mathrm{Cl}_\theta(B)$) if for every $G \in \sigma$ with $y \in G$, $\overline{G} \cap B \neq \emptyset$. B is called θ-closed [2] if $\mathrm{Cl}_\theta(B) = B$. The complement of a θ-closed set is called a θ-open set. It is known that $\sigma_\theta = \sigma$ if and only if (Y, σ) is regular. A TS (Y, σ) is called ω-regular [37] if for each closed set C in (Y, σ) and $y \in Y - C$, there exist $G \in \sigma$ and $H \in \sigma_\omega$ such that $y \in G$, $C \subseteq H$, and $G \cap H = \varnothing$. In [37], the author defined θ_ω-closure operator as follows: a point $y \in Y$ is in θ_ω-closure of B ($y \in \mathrm{Cl}_{\theta_\omega}(B)$) if for any $G \in \sigma$ with $y \in G$ we have $\underline{G}_\omega \cap B \neq \varnothing$. G is called θ_ω-closed if $\mathrm{Cl}_{\theta_\omega}(G) = G$. The complement of a θ_ω-closed set is called a θ_ω-open set. A TS (Y, σ) is called metacompact [46] if every open cover of (Y, σ) has a point-finite open refinement.

The following sequence of definitions and theorems will be used in the sequel.

Definition 1 (see [47]). A TS (Y, σ) is called locally countable if for each $y \in Y$, there is $G \in \sigma$ such that G is countable and $y \in G$.

Definition 2 (see [48]). A TS (Y, σ) is called antilocally countable if each $G \in \sigma - \{\varnothing\}$ is uncountable.

Definition 3 (see [9]). Let (Y, σ) be a TS $B \subseteq Y$. A point $y \in Y$ is called θ-limit point of B if for each $G \in \sigma_\theta$ with $y \in G$, $G \cap (B - \{y\}) \neq \varnothing$. The set of all θ-limit points of B is called the θ-derived set of B and is denoted by $D_\theta(B)$.

Definition 4 (see [9]). Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called a θ-interior point of B if there exists $G \in \sigma$ such that $y \in G \subseteq \overline{G} \subseteq B$. The set of all θ-interior points of B is called the θ-interior of B and is denoted by $\mathrm{Int}_\theta(B)$.

Theorem 1 (see [37]). *If (Y, σ) is locally countable and $B \subseteq Y$, then $\overline{B} = Cl_{\theta_\omega}(B)$.*

Theorem 2 (see [37]). *If (Y, σ) is antilocally countable and $B \subseteq Y$, then $Cl_\theta(B) = Cl_{\theta_\omega}(B)$.*

Theorem 3 (see [37]). *For any TS (Y, σ), $\sigma_\theta \subseteq \sigma_{\theta_\omega} \subseteq \sigma$.*

Theorem 4 (see [2]). *A TS (Y, σ) is regular if and only if $\sigma = \sigma_\theta$.*

Theorem 5 (see [37]). *Let (Y, σ) be a TS and $B \subseteq Y$. Then, B is θ_ω-open set if and only if for each $y \in B$, there exists $G \in \sigma$ such that.*

Definition 5. Let (Y, σ) and (Z, δ) be TSs and let $B \subseteq Y$. Then,

(a) (Y, σ) is called C-scattered if every $B \in \sigma^c - \{\varnothing\}$, there is $b \in B$ and a compact set K such that $b \in \mathrm{Int}(K) \subseteq K \subseteq B$ [49]

(b) B is called strongly placed in $Y \times Z$ if for every $z \in Z$ and $H \in \sigma \times \delta$ with $B \times \{z\} \subseteq H$, there are $V \in \sigma$ and $W \in \delta$ such that $B \times \{z\} \subseteq V \times W \subseteq H$ [50]

(c) Y is called scattered relative to $Y \times Z$ if for each $B \in \sigma^c$, there exists $b \in B$ and $V \in \sigma_A$ such that $\overline{V}$ is Lindelöf and strongly placed in $Y \times Z$ [50]

It is well known that if (Y, σ) and (Z, δ) are TSs and Y is C-scattered, then Y is scattered relative to $Y \times Z$ but not conversely.

Definition 6 (see [51]). A Hausdorff TS (Y, σ) is called ultraparacompact if every open cover of Y has a locally finite clopen refinement.

Ellis [51] showed that a Hausdorff space (Y, σ) is ultraparacompact if every open cover has a pairwise disjoint open refinement.

Theorem 6 (see [52]). *Let $f\colon (Y, \sigma) \longrightarrow (Z, \delta)$ be closed and continuous with (Y, σ) regular. If (Z, δ) is metacompact and $f^{-1}(z)$ is Lindelöf for each $z \in Z$, then (Y, σ) is metacompact.*

Theorem 7 (see [50]). *For any two TSs (Y, σ) and (Z, δ), Y is strongly placed in $Y \times Z$ if and only if the projection $\pi\colon (Y \times Z, \sigma \times \delta) \longrightarrow (Z, \delta)$ is closed.*

Theorem 8 (see [35]). *Let (Y, σ) and (Z, δ) be TSs and let $B \subseteq Y$. If B is strongly placed in $Y \times Z$ and $C \in \sigma \cap \sigma^c$, then $B \cap C$ is strongly placed in $Y \times Z$.*

3. Theta Omega Limit Points

In this section, we explore the concept of θ_ω-limit points of a set and study its fundamental properties.

Definition 7. Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called θ_ω-limit point of B if for each $G \in \sigma_{\theta_\omega}$ with $y \in G$, $G \cap (B - \{y\}) \neq \varnothing$.

The set of all θ_ω-limit points of B is called the θ_ω-derived set of B and is denoted by $D_{\theta_\omega}(B)$.

The following result shows that θ_ω-derived set of a set B contains the derived set of B and contained in the θ_ω-derived set of B.

Theorem 9. *Let (Y, σ) be a TS $B \subseteq Y$. The derived set of B is denoted by $D(B)$. Then, $D(B) \subseteq D_{\theta_\omega}(B) \subseteq D_\theta(B)$.*

Proof. To see that $D(B) \subseteq D_{\theta_\omega}(B)$, let $y \notin D_{\theta_\omega}(B)$, then there exists $G \in \sigma_{\theta_\omega}$ such that $y \in G$ and $G \cap (B - \{y\}) = \varnothing$. By Theorem 3, $G \in \sigma$ and so $y \notin D(B)$. Therefore, we have $D(B) \subseteq D_{\theta_\omega}(B)$. To see that $D_{\theta_\omega}(B) \subseteq D_\theta(B)$, let $y \notin D_\theta(B)$, then there exists $G \in \sigma_\theta$ such that $y \in G$ and $G \cap (B - \{y\}) = \varnothing$. By Theorem 3, $G \in \sigma_{\theta_\omega}$ and so $y \notin D_{\theta_\omega}(B)$. Therefore, we have $D_{\theta_\omega}(B) \subseteq D_\theta(B)$.

The following example shows that the equality of each of the inclusions in Theorem 9 does not hold in general. □

Example 1 (Example 2.26 of [37]). Let $X = \mathbb{R}$ and let $\sigma = \{\varnothing, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\varnothing, \mathbb{R}, \mathbb{N}\}$ and $\sigma_\theta = \{\varnothing, \mathbb{R}\}$. Let $B = \{-n\colon n \in \mathbb{N}\} \cup \{0, 1\}$. Then, $D_\theta(B) = \mathbb{R}$, $D_{\theta_\omega}(B) = \mathbb{R} - \{1\}$, and $D(B) = \mathbb{Q} - \{1\}$.

Under the condition "regularity," the θ_ω-derived set, the derived set, and the θ-derived set are all equal.

Theorem 10. *Let (Y, σ) be a regular TS and $B \subseteq Y$. Then, $D(B) = D_{\theta_\omega}(B) = D_\theta(B)$.*

Proof. It follows from Theorems 3, 4, and 9.

"Local countability" is a sufficient condition for the θ_ω-derived set and the derived set to be equal to each other: □

Theorem 11. *Let (Y, σ) be a locally countable TS and $B \subseteq Y$. Then, $D(B) = D_{\theta_\omega}(B)$.*

Proof. By Theorem 9, we have $D(B) = D_{\theta_\omega}(B)$. To see that $D(B) = D_{\theta_\omega}(B)$, suppose to the contrary that there is $y \in D_{\theta_\omega}(B) - D(B)$. Since $x \notin D(B)$, there is $G \in \sigma$ such that $G \cap (B - \{y\}) = \emptyset$. By Theorem 1, $\mathrm{Cl}_{\theta_\omega}(Y - G) = \overline{Y - G} = Y - G$ and so $G \in \sigma_{\theta_\omega}$. We conclude that $y \notin D_{\theta_\omega}(B)$, a contradiction.

"Antilocal countability" is a sufficient condition for the θ_ω-derived set and the θ-derived set to be equal to each other. □

Theorem 12. *Let (Y, σ) be an antilocally countable TS and $B \subseteq Y$. Then, $D_\theta(B) = D_{\theta_\omega}(B)$.*

Proof. By Theorem 9, we have $D_{\theta_\omega}(B) \subseteq D_\theta(B)$. To see that $D_\theta(B) \subseteq D_{\theta_\omega}(B)$, suppose to the contrary that there is $y \in D_\theta(B) - D_{\theta_\omega}(B)$. Since $x \notin D_{\theta_\omega}(B)$, there is $G \in \sigma_{\theta_\omega}$ such that $G \cap (B - \{y\}) = \emptyset$. By Theorem 2, $\mathrm{Cl}_\theta(Y - G) = \mathrm{Cl}_{\theta_\omega}(Y - G) = Y - G$ and so $G \in \sigma_\theta$. We conclude that $y \notin D_\theta(B)$, a contradiction.

In Theorems 13–16, we give some natural properties for θ_ω-derived set. □

Theorem 13. *Let (Y, σ) be a TS. If $B \subseteq C \subseteq Y$, then $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(C)$.*

Proof. Let $y \notin D_{\theta_\omega}(C)$, there exists $G \in \sigma_{\theta_\omega}$ such that $y \in G$ and $G \cap (C - \{y\}) = \emptyset$. Since $B \subseteq C$, then $G \cap (B - \{x\}) = \emptyset$ and hence $y \notin D_{\theta_\omega}(B)$. It follows that $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(C)$. □

Theorem 14. *Let (Y, σ) be a TS, and let A and B be subsets of Y. Then, $D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B) = D_{\theta_\omega}(A \cup B)$.*

Proof. By Theorem 13, $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(A \cup B)$ and $D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(A \cup B)$. Therefore, $D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B) \subseteq D_{\theta_\omega}(A \cup B)$. Now, let $y \notin (D_{\theta_\omega}(A) \cup D_{\theta_\omega}(B))$, then there exist θ_ω-open sets $G, H \in \sigma_{\theta_\omega}$ such that $y \in G \cap H$, $(G - \{y\}) \cap A = \emptyset$, and $(H - \{y\}) \cap B = \emptyset$. Let $W = G \cap H$. Then, $W \in \sigma_{\theta_\omega}$ and

$$
\begin{aligned}
(W - \{y\}) \cap (A \cup B) &= ((W - \{y\}) \cap A) \cup ((W - \{y\}) \cap B) \\
&= \emptyset \cup \emptyset \\
&= \emptyset.
\end{aligned}
\tag{1}
$$

Thus, $y \notin D_{\theta_\omega}(A \cup B)$. □

Theorem 15. *Let (Y, σ) be a TS, and let A and B be subsets of Y. Then, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B)$.*

Proof. By Theorem 13, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(B)$ and $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(B)$. Then, $D_{\theta_\omega}(A \cap B) \subseteq D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B)$.

The following example shows that the inclusion in Theorem 15 can not be replaced by equality in general. □

Example 2 (Example 2.26 of [37]). Let $Y = \mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}$. Let $A = ((3/4), (3/2))$ and $B = ((5/4), (9/4))$. Then, $D_{\theta_\omega}(A) = \mathbb{R} - \{1\}$ and $D_{\theta_\omega}(B) = \mathbb{R} - \{2\}$, so $D_{\theta_\omega}(A) \cap D_{\theta_\omega}(B) = \mathbb{R} - \{1, 2\}$. On the other hand, $D_{\theta_\omega}(A \cap B) = D_{\theta_\omega}(((5/4), (3/2))) = \mathbb{R} - \mathbb{N}$.

Theorem 16. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $D_{\theta_\omega}(D_{\theta_\omega}(B)) - B \subseteq D_{\theta_\omega}(B)$.*

Proof. Let $y \in D_{\theta_\omega}(D_{\theta_\omega}(B)) - B$. Let $G \in \sigma_{\theta_\omega}$ with $y \in G$. Since $y \in D_{\theta_\omega}(D_{\theta_\omega}(B))$, $G \cap (D_{\theta_\omega}(B) - \{y\}) \neq \emptyset$. Choose $z \in G \cap (D_{\theta_\omega}(B) - \{y\})$. Since $z \in D_{\theta_\omega}(B)$ and $z \in G \in \sigma_{\theta_\omega}$, then $G \cap (B - \{z\}) \neq \emptyset$. Choose $w \in G \cap (B - \{z\})$. Since $w \in B$ and $y \notin B$, then $w \neq y$. Thus, $G \cap (B - \{y\}) \neq \emptyset$ and hence $y \in D_{\theta_\omega}(B)$.

The following example shows that the inclusion in Theorem 16 cannot be replaced by equality in general. □

Example 3. Let $Y = \mathbb{R}$ and $\sigma = \{\mathbb{R}, \emptyset, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. Let $B = ((3/4), (3/2))$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\mathbb{R}, \emptyset, \mathbb{N}\}$. By Example 2, $D_{\theta_\omega}(B) = \mathbb{R} - \{1\}$. On the other hand,

$$
\begin{aligned}
D_{\theta_\omega}\left(D_{\theta_\omega}(B)\right) - B &= D_{\theta_\omega}(\mathbb{R} - \{1\}) - \left(\frac{3}{4}, \frac{3}{2}\right) \\
&= \mathbb{R} - \left(\frac{3}{4}, \frac{3}{2}\right).
\end{aligned}
\tag{2}
$$

4. Theta Omega Interior Points

In this section, we explore the concept of θ_ω-interior points of a set and study its fundamental properties.

Definition 8. Let (Y, σ) be a TS and $B \subseteq Y$. A point $y \in Y$ is called a θ_ω-interior point of B if there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. The set of all θ_ω-interior points of B is called the θ_ω-interior of B and is denoted by $\mathrm{Int}_{\theta_\omega}(B)$.

The following result shows that the θ_ω-interior of a set B contains the θ-interior B and contained in the θ_ω-interior of B.

Theorem 17. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $Int_\theta(B) \subseteq Int_{\theta_\omega}(B) \subseteq Int(B)$.*

The following example shows that each of the two inclusions in Theorem 17 cannot be replaced by equality in general.

Example 4 (Example 2.26 of [37]). Let $Y = \mathbb{R}$ and let $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. Let $A = \mathbb{N}$ and $B = \mathbb{Q}^c$. It is proved in [37] that $\sigma_{\theta_\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}$ and $\sigma_\theta = \{\emptyset, \mathbb{R}\}$. We have

$\mathrm{Int}_{\theta_\omega}(A) = \mathbb{N}$ but $\mathrm{Int}_{\theta}(A) = \varnothing$. Also, we have $\mathrm{Int}_{\theta_\omega}(B) = \varnothing$ but $\mathrm{Int}(B) = \mathbb{Q}^c$.

Theorem 18. *Let (Y, σ) be a TS and $B \subseteq Y$. If $G \in \sigma$ such that $G \subseteq \underline{G}_\omega \subseteq B$, then $G \subseteq \mathrm{Int}_{\theta_\omega}(B)$.*

Proof. It follows directly from the definition of$\mathrm{Int}_{\theta_\omega}(B)$.

θ_ω-interior is always open. □

Theorem 19. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $\mathrm{Int}_{\theta_\omega}(B)$ is θ_ω-open.*

Proof. By the definition of $\mathrm{Int}_{\theta_\omega}(B)$ and Theorem 18, for every $y \in \mathrm{Int}_{\theta_\omega}(B)$, there exists $G_y \in \sigma$ such that $y \in G_y \subseteq \underline{G_y}_\omega \subseteq \mathrm{Int}_{\theta_\omega}(B)$. By Theorem 5, it follows that $\mathrm{Int}_{\theta_\omega}(B)$ is θ_ω-open.

The following is a characterization of θ_ω-open via θ_ω-interior. □

Theorem 20. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, B is θ_ω-open if and only if $B = \mathrm{Int}_{\theta_\omega}(B)$.*

Proof. Necessity: suppose that B is a θ_ω-open set. By the definition, we have $\mathrm{Int}_{\theta_\omega}(B) \subseteq B$. To see that $B \subseteq \mathrm{Int}_{\theta_\omega}(B)$, let $y \in B$. By Theorem 5, there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. Then, $y \in \mathrm{Int}_{\theta_\omega}(B)$.

Sufficiency: suppose that $B = \mathrm{Int}_{\theta_\omega}(B)$. Then, by Theorem 19, B is θ_ω-open.

The results in the rest of this section are some natural properties of θ_ω-interior. □

Theorem 21. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $\mathrm{Int}_{\theta_\omega}[\mathrm{Int}_{\theta_\omega}(B)] = \mathrm{Int}_{\theta_\omega}(B)$.*

Proof. Follows from Theorem 20. □

Theorem 22. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $Y - \mathrm{Int}_{\theta_\omega}(B) = \mathrm{Cl}_{\theta_\omega}(Y - B)$.*

Proof. To see that $Y - \mathrm{Int}_{\theta_\omega}(B) \subseteq \mathrm{Cl}_{\theta_\omega}(Y - B)$, let $y \notin \mathrm{Cl}_{\theta_\omega}(Y - B)$. Then, there is $G \in \sigma$ such that $y \in G$ and $\underline{G}_\omega \cap (Y - B) = \varnothing$, so we have $y \in G \subseteq \underline{G}_\omega \subseteq B$. This shows that $y \notin Y - \mathrm{Int}_{\theta_\omega}(B)$. To see that $\mathrm{Cl}_{\theta_\omega}(Y - B) \subseteq Y - \mathrm{Int}_{\theta_\omega}(B)$, let $y \notin Y - \mathrm{Int}_{\theta_\omega}(B)$. Then, $y \in \mathrm{Int}_{\theta_\omega}(B)$, and so there is $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq B$. Therefore, we have $\underline{G}_\omega \cap (Y - B) = \varnothing$, and so $y \notin \mathrm{Cl}_{\theta_\omega}(Y - B)$. □

Theorem 23. *Let (Y, σ) be a TS and $B \subseteq Y$. Then, $Y - \mathrm{Cl}_{\theta_\omega}(B) = \mathrm{Int}_{\theta_\omega}(Y - B)$.*

Proof. By Theorem 22,

$$\begin{aligned} Y - \mathrm{Cl}_{\theta_\omega}(B) &= Y - \left(Y - \mathrm{Int}_{\theta_\omega}(Y - B)\right) \\ &= \mathrm{Int}_{\theta_\omega}(Y - B). \end{aligned} \tag{3}$$

□

Theorem 24. *Let (Y, σ) be a TS and let $A \subseteq B \subseteq Y$. Then, $\mathrm{Int}_{\theta_\omega}(A) \subseteq \mathrm{Int}_{\theta_\omega}(B)$.*

Proof. Let $y \in \mathrm{Int}_{\theta_\omega}(A)$. Then, there exists $G \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq A$. Since $A \subseteq B$, then $G \subseteq \overline{U}^\omega \subseteq B$. Thus, $y \in \mathrm{Int}_{\theta_\omega}(B)$. □

Theorem 25. *Let (Y, σ) be a TS and let A and B be subsets of Y. Then, $\mathrm{Int}_{\theta_\omega}(A) \cup \mathrm{Int}_{\theta_\omega}(B) \subseteq \mathrm{Int}_{\theta_\omega}(A \cup B)$.*

Proof. By Theorem 24, we have $\mathrm{Int}_{\theta_\omega}(A) \subseteq \mathrm{Int}_{\theta_\omega}(A \cup B)$ and $\mathrm{Int}_{\theta_\omega}(B) \subseteq \mathrm{Int}_{\theta_\omega}(A \cup B)$. Thus, $\mathrm{Int}_{\theta_\omega}(A) \cup \mathrm{Int}_{\theta_\omega}(B) \subseteq \mathrm{Int}_{\theta_\omega}(A \cup B)$. □

Theorem 26. *Let (Y, σ) be a TS, and let A and B be subsets of Y. Then, $\mathrm{Int}_{\theta_\omega}(A \cap B) = \mathrm{Int}_{\theta_\omega}(A) \cap \mathrm{Int}_{\theta_\omega}(B)$.*

Proof. By Theorem 24, we have $\mathrm{Int}_{\theta_\omega}(A \cap B) \subseteq \mathrm{Int}_{\theta_\omega}(A)$ and $\mathrm{Int}_{\theta_\omega}(A \cap B) \subseteq \mathrm{Int}_{\theta_\omega}(B)$. Thus, $\mathrm{Int}_{\theta_\omega}(A \cap B) \subseteq \mathrm{Int}_{\theta_\omega}(A) \cap \mathrm{Int}_{\theta_\omega}(B)$. To see that $\mathrm{Int}_{\theta_\omega}(A) \cap \mathrm{Int}_{\theta_\omega}(B) \subseteq \mathrm{Int}_{\theta_\omega}(A \cap B)$, let $y \in \mathrm{Int}_{\theta_\omega}(A) \cap \mathrm{Int}_{\theta_\omega}(B)$. Then, there exist $G, H \in \sigma$ such that $y \in G \subseteq \underline{G}_\omega \subseteq A$ and $y \in G \subseteq \underline{H}_\omega \subseteq B$. Let $W = G \cap H$. Then, $W \in \sigma$ and $y \in W \subseteq \underline{W}_\omega = \underline{G \cap H}_\omega \subseteq \underline{G}_\omega \cap \underline{H}_\omega \subseteq A \cap B$. It follows that $y \in \mathrm{Int}_{\theta_\omega}(A \cap B)$. □

5. Metacompactness Product Theorems

In this section, we introduce several product theorems concerning metacompactness.

The following result will be used in the proof of Theorems 28 and 29.

Theorem 27. *Let (Y, σ) and (Z, δ) be metacompact TSs. If for every $y \in Y$ there exists $W \in \sigma$ such that $y \in W$ and $\overline{W} \times Z$ is metacompact, then $(Y \times Z, \sigma \times \delta)$ is metacompact.*

Proof. Let $\mathscr{A}$ be an open cover of $(Y \times Z, \sigma \times \delta)$. For every $y \in Y$, choose $W_y \in \sigma$ such that $y \in W_y$ and $(\overline{W_y} \times Z, (\sigma \times \delta)_{\overline{W_y} \times Z})$ is metacompact. Since $\{W_y\colon\ y \in Y\}$ is an open cover of the metacompact TS (Y, σ), then it has a point-finite open refinement $\{V_\beta\colon\ \beta \in \Gamma\}$. For each $\beta \in \Gamma$, $\overline{V_\beta} \times Z, (\sigma \times \delta)_{\overline{V_\beta} \times Z}$ is metacompact and has $\mathscr{M}_\beta = \{A \cap (\overline{V_\beta} \times Z)\colon\ A \in \mathscr{A}\}$ as an open cover, and hence $\mathscr{M}_\beta$ has a point-finite open refinement $\mathscr{H}_\beta$. It is not difficult to see that $\{H \cap (V_\beta \times Z)\colon\ H \in \mathscr{H}_\beta, \beta \in \Gamma\}$ is a point-finite open refinement of $\mathscr{A}$. It follows that $(Y \times Z, \sigma \times \delta)$ is metacompact.

The following two product theorems concerning metacompactness will be used in the proof of Theorem 31 which is the main result of this section: □

Theorem 28. *Let (Y, σ) and (Z, δ) be regular metacompact TSs. If for every $y \in Y$ there exists $W \in \sigma$ such that $y \in W$, $\overline{W}$ is strongly placed in $Y \times Z$, and $(\overline{W}, \sigma_{\overline{W}})$ is Lindelöf, then $(Y \times Z, \sigma \times \delta)$ is metacompact.*

Proof. For each $y \in Y$, choose $W_y \in \sigma$ such that $y \in W$, $\overline{W}$ is strongly placed in $Y \times Z$, and $(\overline{W_y}, \sigma_{\overline{W_y}})$ is Lindelöf. For every $y \in Y$, $\overline{W_y}$ is strongly placed in $Y \times Z$ and so by Theorem 6, the projection function $\pi_y\colon (\overline{W_y} \times Z, (\sigma \times \delta)_{\overline{W_y} \times Z}) \longrightarrow (Z, \delta)$ is a closed function. For every $y \in Y$, $(\overline{W_y}, \sigma_{\overline{W_y}})$ is Lindelöf and since $\pi_y^{-1}(z) = \overline{W_y} \times \{z\}$, then

$(\overline{W_y}\times\{z\}, (\sigma\times\delta)_{\overline{W_y}\times\{z\}})$ is Lindelöf. For every $y\in Y$, $(\overline{W}_y, \sigma_{\overline{W_y}})$ is metacompact and so by Theorem 6, $(\overline{W_y}\times Z, (\sigma\times\delta)_{\overline{W_y}\times Z})$ is metacompact. Thus, by Theorem 27, we have $(Y\times Z, \sigma\times\delta)$ is metacompact. □

Theorem 29. *Let (Y,σ) and (Z,δ) be metacompact TSs and let $B \subseteq Y$ such that B is closed in (Y,σ), (B,σ_B) is Lindelöf, and B is strongly placed in $Y\times Z$. If for all $y\in Y-B$ there is $M\in\sigma_{Y-B}$ such that $(\overline{M}^{Y-B}\times Z, (\sigma\times\delta)_{\overline{M}^{Y-B}\times Z})$ is metacompact, then $(Y\times Z,\sigma\times\delta)$ is metacompact.*

Proof. Let $\mathscr{A}$ be an open cover of $(Y\times Z,\sigma\times\delta)$. For each $z\in Z$, $(M\times\{z\}, (\sigma\times\delta)_{M\times\{z\}})$ is Lindelöf with $M\times\{z\}\subset\bigcup\mathscr{A}$, and so there exists $\mathscr{A}_z\subseteq\mathscr{A}$ such that $\mathscr{A}$ is countable and $M\times\{z\}\subseteq\bigcup\mathscr{A}_z$. Since M is strongly places in $Y\times Z$, then for every $z\in Z$, there exist $U_z\in\sigma$ and $V_z\in\delta$ such that $M\times\{z\}\subseteq U_z\times V_z\subseteq\bigcup\mathscr{A}_z$. Since $\{V_z\colon z\in Z\}$ is an open cover of the metacompact TS (Z,δ), then it has a point-finite open refinement $\{G_\beta\colon \beta\in\Gamma\}$. For each $\beta\in\Gamma$, choose $z(\beta)$ such that $G_\beta\subseteq V_{z(\beta)}$. Then, by Theorem 27 and the assumption, it is not difficult to see that $((Y-U_{z(\beta)})\times Z, (\sigma\times\delta)_{(Y-U_{z(\beta)})\times Z})$ is metacompact. Since $\{A\cap((Y-U_{z(\beta)})\times Z)\colon A\in\mathscr{A}\}$ is an open cover of $(Y-U_{z(\beta)})\times Z$, then it has a point-finite open refinement $\mathscr{A}_\beta$. It is not difficult to check that

$$\begin{aligned}&\left\{A\cap\left(U_{z(\beta)}\times G_\beta\right)\colon\quad A\in\mathscr{A}_{z(\beta)},\beta\in\Gamma\right\}\\&\cup\left\{G\cap\left(Y\times G_\beta\right)\colon\quad G\in\mathscr{A}_\beta,\beta\in\Gamma\right\},\end{aligned}\tag{4}$$

is a point-finite open refinement of $\mathscr{A}$. Therefore, $(Y\times Z,\sigma\times\delta)$ is metacompact. □

Theorem 30. *Let (Y,σ) be ultraparacompact and (Z,δ) be metacompact. Suppose there exists $D\subseteq Y$ such that D is closed in (Y,σ) and for every $y\in D$ there exists $W\in\sigma_D$ such that $y\in W$, $\overline{W}$ is strongly placed in $Y\times Z$, and $(\overline{W},\sigma_{\overline{W}})$ is Lindelöf, and for every $y\in Y-D$, there is $K\in\sigma_{Y-D}$ such that $y\in K$ and $\overline{K}^{Y-D}\times Y$ is metacompact. Then, $(Y\times Z,\sigma\times\delta)$ is metacompact.*

Proof. By assumption there exists $\mathscr{A}\subseteq\sigma_D$ such that for all $A\in\mathscr{A}$, $\overline{A}$ is strongly placed in $Y\times Z$ and $(\overline{A},\sigma_{\overline{A}})$ is Lindelöf, and $\bigcup\mathscr{A}=D$. Since (D,σ_D) is ultraparacompact, then $\mathscr{A}$ has a pairwise disjoint open refinement $\{C_\beta\colon \beta\in\Gamma\}\subseteq\sigma_D$. For every $\beta\in\Gamma$, choose $U_\beta\in\sigma$ such that $C_\beta=U_\beta\cap D$. Put $\mathscr{H}=\{U_\beta\colon \beta\in\Gamma\}\cup\{Y-D\}$. Since (Y,σ) is ultraparacompact and $\mathscr{H}$ is an open cover of (Y,σ), then $\mathscr{H}$ has a pairwise disjoint open refinement $\{M_\gamma\colon \gamma\in\Delta\}$. For every $\gamma\in\Delta$, M_γ meets at most one member of $\{C_\beta\colon \beta\in\Gamma\}$. For every $\gamma\in\Delta$, let $(M_\gamma)^*=M_\gamma\cap(\cup\{C_\beta\colon \beta\in\Gamma$ and $C_\beta\cap M_\gamma\neq\varnothing\})$, then $(M_\gamma)^*=\varnothing$ or $(M_\gamma)^*=M_\gamma\cap\overline{A}$; for some $A\in\mathscr{A}$, it follows that $(M_\gamma)^*$ is closed in $(M_\gamma,\sigma_{M_\gamma})$ and $((M_\gamma)^*,\sigma_{(M_\gamma)^*})$ is Lindelöf and by Theorem 8; it is strongly placed in $M_\gamma\times Y$. By the assumption on $Y-D$ and Theorem 29, we conclude that $(M_\gamma\times Z, (\sigma\times\delta)_{M_\gamma\times Y})$ is metacompact. Since $Y\times Z=\cup_{\gamma\in\Delta}M_\gamma\times Z$ is metacompact, then $(Y\times Z,\sigma\times\delta)$ is metacompact.

Now, we are ready to state the main result of this section. □

Theorem 31. *Let (Y,σ) be ultraparacompact and (Z,δ) be regular and metacompact such that Y is scattered relative to the product $Y\times Z$, then $(Y\times Z,\sigma\times\delta)$ is metacompact.*

Proof. Denote by $Y^{(0)}=Y$ and $Y^{(1)}=\{y\in Y$: there is no $U\in\sigma$ such that $y\in U$ and $\overline{U}$ is strongly placed in $Y\times Z$ and closure $(\overline{U},\sigma_{\overline{U}})$ is Lindelöf$\}$. If there is an ordinal $\alpha>1$ such that $Y^{(\alpha)}$ has been defined and $\beta=\alpha+1$, then $Y^{(\beta)}=(Y^{(\alpha)})^{(1)}$. If α is a limit ordinal, then $Y^{(\alpha)}=\cap_{\beta<\alpha}Y^{(\beta)}$. Since Y is scattered relative to $Y\times Z$, then there exists an ordinal α such that $Y^{(\alpha)}=\varnothing$.

The proof proceeds by transfinite induction on α. If $Y^{(1)}=\varnothing$, then for every $y\in Y$ there exists $U_y\in\sigma$ such that $y\in U$ and $\overline{U}$ is strongly placed in $Y\times Z$ and closure $(\overline{U},\sigma_{\overline{U}})$. And by Theorem 28, $(Y\times Z,\sigma\times\delta)$ is metacompact. If $Y^{(\alpha+1)}=\varnothing$, then for every point $y\in Y^{(\alpha)}$ there exists $U_y\in\sigma_{Y^{(\alpha)}}$ such that $y\in U_y$, $\overline{U_y}$ is strongly placed in $Y\times Z$, and $(\overline{U}_y,\sigma_{U_y})$ is Lindelöf and if $y\in Y-Y^{(\alpha)}$, choose a clopen set C_y such that $y\in C_y\subseteq Y-Y^{(\alpha)}$. Since $C_y^{(\alpha)}\subseteq(Y-Y^{(\alpha)})^{(\alpha)}=\varnothing$, then C_y is scattered relative to $C_y\times Z$ and (C_y,σ_{C_y}) is ultraparacompact, and by the inductive assumption, it follows that $(C_y\times Z, (\sigma\times\delta)_{C_y\times Z})$ is metacompact.

If $Y^{(\alpha)}=\varnothing$ for the limit ordinal α, then the open cover $\{Y-Y^{(\beta)}\colon \beta<\alpha\}$ has a pairwise disjoint open refinement $\{O_\gamma\colon \gamma\in\Gamma\}$. For each $\gamma\in\Gamma$, choose $\beta<\alpha$ such that $O_\gamma\subseteq Y-Y^{(\beta)}$. Therefore, $(O_\gamma)^{(\beta)}=\varnothing$, and hence $(O_\gamma\times Z, (\sigma\times\delta)_{O_\gamma\times Z})$ is metacompact. Since $Y\times Z=\cup_{\gamma\in\Gamma}O_\gamma\times Y$, it follows that $(Y\times Z,\sigma\times\delta)$ is metacompact. □

Corollary 1. *The product of an ultraparacompact C-scattered TS with a metacompact regular TS is again metacompact.*

By the end of this paper, the authors found it is suitable to raise the following open question.

Question 1. Let (Y,σ) and (Z,δ) be regular and metacompact TSs such Y is scattered relative to the product $Y\times Z$. Is $(Y\times Z,\sigma\times\delta)$ metacompact?

6. Conclusion

In this work, the research via θ_ω-open sets is continued by introducing the notions of θ_ω-limit points and θ_ω-interior points. Several relationships regarding these two notions are introduced. Moreover, several product theorems concerning metacompactness are given. In future studies, the following topics could be considered: (1) define θ_ω-border, θ_ω-frontier, and θ_ω-exterior of a set using θ_ω-open sets and (2) try to solve Question 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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