Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 9763506, 1 page
https://doi.org/10.1155/2023/9763506

Retraction

Retracted: A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions

Mathematical Problems in Engineering

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] I. Uddin, A. Perveen, H. Işık, and R. Bhardwaj, "A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6579405, 8 pages, 2021.

Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 6579405, 8 pages
https://doi.org/10.1155/2021/6579405

Research Article

A Solution of Fredholm Integral Inclusions via Suzuki-Type Fuzzy Contractions

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Received 28 April 2021; Accepted 25 May 2021; Published 7 June 2021

Academic Editor: Lazim Abdullah

In this study, we introduce fuzzy weak ϕ-contraction and Suzuki-type fuzzy weak ϕ-contraction and employ these to prove some fuzzy fixed point results for fuzzy mappings in the setting of metric spaces, which is followed by an example to support our claim. Next, we deduce some corollaries and fixed point results for multivalued mappings from our main result. Finally, as an application of our result, we provide the existence of a solution for a Fredholm integral inclusion.

1. Introduction and Preliminaries

The idea of fuzzy mapping was inspired by the fuzzy set theory given by Zadeh [1]. It was initiated by Heilpern [2] in 1981, defined to be a mapping from an arbitrary set to a subfamily of fuzzy sets in metric linear spaces. He established a fuzzy expansion of Banach contraction principle. It broadens and develops the concept of fuzzy fixed point theory, and several authors worked in this field afterward ([3–7] and references therein).

We describe some related concepts in short in the successive lines.

Here, (M, d) depicts a metric space. A fuzzy set in M is a function with domain M and codomain $[0, 1]$. If F is a fuzzy set and $\mu \in M$, the function value $F(\mu)$ is called grade of membership of μ in F. The collection of all fuzzy set in M is denoted by $\mathfrak{F}(M)$.

Let $F \in \mathfrak{F}(M)$ and $\alpha \in [0, 1]$. The α-level set of F, which we denote here by F_α, is defined by

$$\begin{aligned} F_\alpha &= \{\mu\colon F(\mu) \geq \alpha\}, \quad \alpha \in (0, 1], \\ F_0 &= \overline{\{\mu\colon F(\mu) > 0\}}, \end{aligned} \tag{1}$$

where $\overline{B}$ denotes closure of set B.

Definition 1 (see [2]). A fuzzy subset F on M is said to be an approximate quantity if and only if its α-level set is a compact convex subset of M, for each $\alpha \in [0, 1]$ and $\sup_{\mu \in M} F(\mu) = 1$.

We denote by $\mathscr{W}(M) \subseteq \mathfrak{F}(M)$, the subcollection of approximate quantities. We also denote $\mathscr{W}_\alpha(M) = \{F \in \mathfrak{F}(M)\colon F_\alpha$ is nonempty compact convex subset of a metric space $(M, d)\}$. If $F \in \mathscr{W}(M)$ and $F(\mu_0) = 1$, for some $\mu_0 \in M$, then F is an approximation of μ_0.

Definition 2 (see [2]). [2] For $F_1, F_2 \in \mathscr{W}(M)$ and $\alpha \in [0,1]$, define

$$\begin{aligned} p_\alpha(F_1, F_2) &= \inf_{\mu \in (F_1)_\alpha, \nu \in (F_2)_\alpha} d(\mu, \nu); \\ p(F_1, F_2) &= \sup_\alpha p_\alpha(F_1, F_2); \\ D_\alpha(F_1, F_2) &= H\left((F_1)_\alpha, (F_2)_\alpha\right) = \max\left\{ \sup_{a \in (F_1)_\alpha} d\left(a, (F_2)_\alpha\right), \sup_{b \in (F_2)_\alpha} d\left(b, (F_1)_\alpha\right) \right\}; \\ D(F_1, F_2) &= \sup_\alpha D_\alpha(F_1, F_2). \end{aligned} \tag{2}$$

Remark 1 (see [2]). D is a metric on $\mathscr{W}(M)$. Let $F_1, F_2 \in \mathscr{W}(M)$. Then, F_1 is more accurate than F_2, denoted by $F_1 \subset F_2$ iff $F_1(\mu) \leq F_2(\mu)$, for each $\mu \in M$.

Definition 3 (see [2]). Let M be a nonempty set and N any metric linear space. A mapping S is called fuzzy mapping if and only if S is a mapping from M into $\mathscr{W}(N)$ (or $\mathscr{W}_\alpha(N)$), i.e., $S\mu \in \mathscr{W}(N)$ (or $\mathscr{W}_\alpha(N)$), for each $\mu \in M$.

Lemma 1 (see [2]). *The following conditions hold for a metric space* (M, d):

(a) *If* $p_\alpha(\mu, F) = 0$, *then* $\mu_\alpha \subset F$

(b) $p_\alpha(\mu, F) \leq d(\mu, \nu) + p_\alpha(\nu, F)$

(c) *If* $\mu_\alpha \subset F_1$, *then* $p_\alpha(\mu, F_2) \leq D_\alpha(F_1, F_2)$

For all $z, \nu \in M$ *and* $F, F_1, F_2 \in \mathscr{W}(M)$.

A fuzzy mapping S is a fuzzy subset on $M \times N$ with membership function $S(\mu, \nu)$. The function value $S(\mu, \nu)$ is the grade of membership of ν in $S(\mu)$.

Definition 4 (see [8]). Let $\alpha \in [0,1]$ and $\mu \in M$. The fuzzy point μ_α of M is the fuzzy set μ_α: $M \longrightarrow [0,1]$ given by

$$\mu_\alpha(\nu) = \begin{cases} \alpha, & \text{if } \mu = \nu, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For $\alpha = 1$, we have

$$\mu_1(\nu) = \{\mu\} = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Definition 5 (see [9]). A fuzzy point μ_α in M is a fixed fuzzy point of the mapping S over M if $\mu_\alpha \subset S\mu$, i.e., $(S\mu)\mu \geq \alpha$ or $\mu \in (S\mu)_\alpha$.

Remark 2 (see [2]). If $\{\mu\} \subset S\mu$, then μ is a fixed point of fuzzy mapping S.

Also, the generalization of Banach contraction principle has been done in many ways along with providing their applications in different fields. One of them is by generalizing the contraction condition, specially using nonlinear contractions, e.g., Suzuki-type contraction, F-contraction, and θ-contraction [10–16]. One of such generalizations, namely, ϕ-weak contraction was done by Alber and Guerre-Delabriere [17] in 1997 to prove fixed point result in the setting of Hilbert space, which was further utilized by Rhoades [18] in metric fixed point theory. Recently, a generalization of the same was furnished by Xue [19]. He used the class of mappings $\Gamma = \{$class of all continuous nondecreasing functions ϕ: $[0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0\}$ and defined generalized ϕ-weak contraction as follows:

$$d(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(S\mu, S\nu)), \quad \forall \mu, \nu \in M, \tag{5}$$

where S is a self-mapping on a metric space (M, d) and $\phi \in \Gamma$. Also, he showed that this contraction condition is more weaker than ϕ-weak contraction condition (viz. $d(S\mu, S\nu) \leq d(\mu, \nu) - \phi(d(\mu, \nu))$). After that, Perveen et al. [20, 21] used his idea and proved results using weaker conditions.

In this study, we utilize the above ideas and define fuzzy weak ϕ-contraction and Suzuki-type fuzzy weak ϕ-contraction, which we use to prove the existence of fuzzy fixed point supported by an example. Last, we furnish an application of our result to prove the existence of a solution of integral inclusion of Fredholm type.

2. Main Result

First, we define the same class of mappings used in [20, 21].

Let Φ denotes the set of all mappings ϕ: $[0, \infty) \longrightarrow [0, \infty)$ satisfying the following:

(a) ($\Phi 1$) ϕ is nondecreasing

(b) ($\Phi 2$) $\phi(\tau) = 0$ iff $\tau = 0$ and $\liminf_{n \longrightarrow \infty} \phi(\tau_n) > 0$, whenever $\lim_{n \longrightarrow \infty} \tau_n > 0$

We have noticed that [19] used the continuity of ϕ. Inspired by [22], we dropped the continuity condition and use a weaker condition, which is given in ($\Phi 2$). In fact, ($\Phi 2$) is also weaker than the condition that ϕ is lower semicontinuous. Indeed, if ϕ is lower semicontinuous, then for a sequence $\{\tau_n\}$ with $\lim_{n \longrightarrow \infty} \tau_n = \tau > 0$, we have $\liminf_{n \longrightarrow \infty} \phi(\tau_n) \geq \phi(\tau) > 0$.

Using the class defined above, we define the following contraction for fuzzy mapping.

Definition 6. Let (M, d) be a metric space. A fuzzy mapping $S\colon M \longrightarrow \mathscr{W}_\alpha(M)$ is

(a) a fuzzy weak ϕ-contraction mapping if

$$D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(S\mu, S\nu)). \tag{6}$$

(b) a Suzuki-type fuzzy weak ϕ-contraction mapping if the following condition is satisfied:

$$\frac{1}{2} p_\alpha(\mu, S\mu) \leq d(\mu, \nu) \Rightarrow D_\alpha(S\mu, S\nu) \leq d(\mu, \nu) - \phi(D_\alpha(S\mu, S\nu)), \tag{7}$$

for all $\mu, \nu \in M$, where $\phi \in \Phi$

Remark 3. If S is fuzzy weak ϕ-contraction, then S is Suzuki-type fuzzy weak ϕ-contraction.

Now, we are ready to commence our main theorem.

Theorem 1. *Let a complete metric space (M, d) and $S\colon M \longrightarrow \mathscr{W}_\alpha(M)$ be a Suzuki-type fuzzy weak ϕ-contraction, such that for every $\mu \in M$, $(S\mu)_\alpha$ is closed. Then, there exists $\mu^* \in M$, such that μ^*_α is a fuzzy fixed point of S, i.e., $\mu^*_\alpha \subset S\mu^*$.*

Proof. Let $\mu_1 \in M$ be any arbitrary point. Since $S\mu_1 \in \mathscr{W}_\alpha(M)$, we can choose $\mu_2 \in (S\mu_1)_\alpha$, such that $d(\mu_1, \mu_2) = p_\alpha(\mu_1, S\mu_1)$. If $\mu_1 = \mu_2 \in (S\mu_1)_\alpha$, then we are done. Suppose that $\mu_1 \neq \mu_2$. Since $S\mu_2 \in \mathscr{W}_\alpha(M)$, there exists $\mu_3 \in (S\mu_2)_\alpha$, such that

$$d(\mu_2, \mu_3) = p_\alpha(\mu_2, S\mu_2) \leq D_\alpha(S\mu_1, S\mu_2). \tag{8}$$

Again, if $\mu_2 = \mu_3$, we are done. Otherwise, we continue this process and obtain a sequence $\{\mu_n\}$ satisfying the following conditions:

$$\begin{aligned} \mu_{n+1} &\in (S\mu_n)_\alpha, \\ \mu_{n+2} &\in (S\mu_{n+1})_\alpha, \\ d(\mu_{n+1}, \mu_{n+2}) &= p_\alpha(\mu_{n+1}, S\mu_{n+1}) \\ &\leq D_\alpha(S\mu_n, S\mu_{n+1}), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{9}$$

Thus, we easily obtain

$$\frac{1}{2} p_\alpha(\mu_n, S\mu_n) < d(\mu_n, \mu_{n+1}), \tag{10}$$

which implies that

$$\begin{aligned} d(\mu_{n+1}, \mu_{n+2}) &\leq D_\alpha(S\mu_n, S\mu_{n+1}) \\ &\leq d(\mu_n, \mu_{n+1}) - \phi(D_\alpha(S\mu_n, S\mu_{n+1})) \\ &\leq d(\mu_n, \mu_{n+1}) - \phi(d(\mu_{n+1}, \mu_{n+2})), \end{aligned} \tag{11}$$

which further implies

$$d(\mu_{n+1}, \mu_{n+2}) \leq d(\mu_n, \mu_{n+1}). \tag{12}$$

Thus, we see that $\{\mu_n\}$ is a nonincreasing sequence of positive real number bounded below by 0. Hence, $\{\mu_n\}$ converges to a point $r \geq 0$. We assert that $r = 0$. Suppose it is not so, then taking limit in (11), we obtain

$$r \leq r - \liminf_{n \longrightarrow \infty} \phi(d(\mu_{n+1}, \mu_{n+2})), \tag{13}$$

which is a contradiction. Therefore, we have

$$\lim_{n \longrightarrow \infty} d(\mu_n, \mu_{n+1}) = 0. \tag{14}$$

Next, we prove that $\{\mu_n\}$ is a Cauchy sequence. Suppose on contrary that it is not so, then there exist two subsequences $\{\mu_{m_k}\}$ and $\{\mu_{n_k}\}$ of $\{\mu_n\}$, such that n_k is the smallest positive integer for which

$$\begin{gathered} n_k > m_k > k, \\ d(\mu_{m_k}, \mu_{n_k}) \geq \varepsilon, \\ d(\mu_{m_k}, \mu_{n_k - 1}) < \varepsilon. \end{gathered} \tag{15}$$

Now, utilizing triangular inequality, we obtain

$$\begin{aligned} \varepsilon \leq d(\mu_{m_k}, \mu_{n_k}) &\leq d(\mu_{m_k}, \mu_{n_k - 1}) + d(\mu_{n_k - 1}, \mu_{n_k}) \\ &< \varepsilon + d(\mu_{n_k - 1}, \mu_{n_k}). \end{aligned} \tag{16}$$

Letting $n \longrightarrow \infty$, we obtain

$$\lim_{n \longrightarrow \infty} d(\mu_{m_k}, \mu_{n_k}) = \varepsilon. \tag{17}$$

Again, by triangular inequality,

$$\begin{aligned} d(\mu_{m_k}, \mu_{n_k}) &\leq d(\mu_{m_k}, \mu_{m_k+1}) + d(\mu_{m_k+1}, \mu_{n_k+1}) + d(\mu_{n_k+1}, \mu_{n_k}), \\ d(\mu_{m_k+1}, \mu_{n_k+1}) &\leq d(\mu_{m_k+1}, \mu_{m_k}) + d(\mu_{m_k}, \mu_{n_k}) + d(\mu_{n_k}, \mu_{n_k+1}), \end{aligned} \tag{18}$$

which on letting $n \longrightarrow \infty$, we obtain

$$\lim_{n \longrightarrow \infty} d(\mu_{m_k+1}, \mu_{n_k+1}) = \varepsilon. \tag{19}$$

Next, by (14) and (17), there exists $n_0 \geq 1$, such that

$$\frac{1}{2} p_\alpha(\mu_{m_k}, S\mu_{m_k}) < \frac{1}{2}\varepsilon < d(\mu_{m_k}, \mu_{n_k}), \quad \forall k \geq n_0. \tag{20}$$

Thus, for $\mu = \mu_{m_k}$ and $\nu = \mu_{n_k}$, by (7), we obtain

$$D_\alpha(S\mu_{m_k}, S\mu_{n_k}) \leq d(\mu_{m_k}, \mu_{n_k}) - \phi(D_\alpha(S\mu_{m_k}, S\mu_{n_k})), \tag{21}$$

which on letting $n \longrightarrow \infty$ yields $\varepsilon \leq \varepsilon - \liminf_{n \longrightarrow \infty} \phi (D_\alpha(S\mu_{m_k}, S\mu_{n_k}))$, a contradiction. Thus, $\{\mu_n\}$ is Cauchy in M. The completeness of (M, d) implies that $\mu_n \longrightarrow \mu^*$, for some $\mu^* \in M$.

Next, we show that $\mu^*_\alpha \subset S\mu^*$. As $\mu_n \longrightarrow \mu^*$, there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$,

$$d(\mu_n, \mu^*) \leq \frac{1}{3} d(\mu, \mu^*), \quad \forall \mu \in M. \tag{22}$$

Using the above inequality, we obtain (for all $n \geq n_1$)

$$\begin{aligned}\frac{1}{2}p_\alpha(\mu_n, S\mu_n) \le p_\alpha(\mu_n, S\mu_n) &\le d(\mu_n, \mu_{n+1})\\ &\le d(\mu_n, \mu^*) + d(\mu^*, \mu_{n+1})\\ &\le d(\mu_n, \mu^*) + p_\alpha(\mu^*, \mu_{n+1})\\ &\le \frac{1}{3}d(\mu, \mu^*) + \frac{1}{3}d(\mu, \mu^*)\\ &= d(\mu, \mu^*) - \frac{1}{3}d(\mu, \mu^*)\\ &\le d(\mu, \mu^*) - d(\mu^*, \mu_n)\\ &\le d(\mu_n, \mu),\end{aligned} \tag{23}$$

i.e.,

$$\frac{1}{2}p_\alpha(\mu_n, S\mu_n) \le d(\mu_n, \mu), \quad \forall n \ge n_1. \tag{24}$$

Thus, by (7), we obtain

$$\begin{aligned}p_\alpha(\mu_{n+1}, S\mu) &\le D_\alpha(S\mu_n, S\mu)\\ &\le d(\mu_n, \mu) - \phi(D_\alpha(S\mu_n, S\mu)),\end{aligned} \tag{25}$$

which implies

$$p_\alpha(\mu_{n+1}, S\mu) \le d(\mu_n, S\mu). \tag{26}$$

Taking limit $n \longrightarrow \infty$, we obtain

$$p_\alpha(\mu^*, S\mu) \le d(\mu^*, \mu). \tag{27}$$

Furthermore, we prove that

$$D_\alpha(S\mu, S\mu^*) \le d(\mu, \mu^*) - \phi(D_\alpha(S\mu, S\mu^*)), \quad \forall \mu \in M. \tag{28}$$

The above equation holds trivially for $\mu = \mu^*$. Suppose $\mu \ne \mu^*$. Then, for every $n \in \mathbb{N}$, there exists $\nu_n \in (S\mu)_\alpha$, such that

$$d(\mu^*, \nu_n) \le p_\alpha(\mu^*, S\mu) + \frac{1}{n}d(\mu, \mu^*). \tag{29}$$

Thus, with the help of the above inequality and (27),

$$\begin{aligned}p_\alpha(\mu, S\mu) &\le d(\mu, \nu_n)\\ &\le d(\mu, \mu^*) + d(\mu^*, \nu_n)\\ &\le d(\mu, \mu^*) + p_\alpha(\mu^*, S\mu) + \frac{1}{n}d(\mu, \mu^*)\\ &\le d(\mu, \mu^*) + d(\mu, \mu^*) + \frac{1}{n}d(\mu, \mu^*)\\ &= \left(2 + \frac{1}{n}\right)d(\mu, \mu^*).\end{aligned} \tag{30}$$

On taking limit $n \longrightarrow \infty$, we obtain

$$\frac{1}{2}p_\alpha(\mu, S\mu) \le d(\mu, \mu^*). \tag{31}$$

So, (28) holds true for all $\mu \in M$. Now, if $\lim_{n\longrightarrow\infty} p_\alpha(\mu_{n+1}, S\mu^*) = 0$, then we are done. Assume that it is not so, then there exists $\varepsilon_0 > 0$, such that for every $k \in \mathbb{N}$, we can choose $n_k \in \mathbb{N}$, such that $p_\alpha(\mu_{n_k+1}, S\mu^*) > \varepsilon_0 > 0$ for all $n_k \ge k$. For $\mu = \mu_{n_k}$, (28) reduces to

$$\begin{aligned}p_\alpha\left(\mu_{n_k+1}, S\mu^*\right) &\le D_\alpha\left(S\mu_{n_k}, S\mu\right)\\ &\le d\left(\mu_{n_k}, \mu^*\right) - \phi\left(D_\alpha\left(S\mu_{n_k}, S\mu^*\right)\right).\end{aligned} \tag{32}$$

Taking $k \longrightarrow \infty$, we obtain

$$\begin{aligned}\lim_{k\longrightarrow\infty} p_\alpha\left(\mu_{n_k+1}, S\mu^*\right) &\le \lim_{k\longrightarrow\infty} D_\alpha\left(S\mu_{n_k}, S\mu^*\right)\\ &\le \lim_{k\longrightarrow\infty}\left[d\left(\mu_{n_k}, \mu^*\right) - \phi\left(D_\alpha\left(S\mu_{n_k}, S\mu^*\right)\right)\right]\\ &\le \lim_{k\longrightarrow\infty} d\left(\mu_{n_k}, \mu^*\right),\end{aligned} \tag{33}$$

i.e., $p_\alpha(\mu^*, S\mu^*) \le 0$, a contradiction. So, $\mu_\alpha^* \subset S\mu^*$, and the proof is completed. □

We present the following example to illustrate the utility of our proven result.

Example 1. Let $M = \{1, 2, 3\}$, and d: $M \times M \longrightarrow [0, \infty)$ is defined by

$$\begin{aligned}&d(1, 3) = \frac{3}{8},\\ &d(1, 2) = \frac{1}{2},\\ &d(2, 3) = \frac{3}{2},\\ &d(\mu, \mu) = 0, \quad \forall \mu \in M,\\ &d(\mu, \nu) = d(\nu, \mu), \quad \forall \mu, \nu \in M.\end{aligned} \tag{34}$$

We define ϕ: $[0, \infty) \longrightarrow [0, \infty)$ by

$$\phi(\tau) = \frac{\tau}{2}, \quad \forall \tau \in [0, \infty), \tag{35}$$

and a fuzzy mapping by

$$S_1(\mu)=\begin{cases}0, & \text{if } \mu=1,\\ \alpha, & \text{if } \mu=2,\\ \dfrac{\alpha}{2}, & \text{if } \mu=3,\end{cases}$$

$$S_2(\mu)=\begin{cases}\dfrac{\alpha}{2}, & \text{if } \mu=1,\\ 2\alpha, & \text{if } \mu=2,\\ \dfrac{\alpha}{4}, & \text{if } \mu=3,\end{cases} \tag{36}$$

$$S_3(\mu)=\begin{cases}2\alpha, & \text{if } \mu=1,\\ \dfrac{\alpha}{3}, & \text{if } \mu=2,\\ 0, & \text{if } \mu=3.\end{cases}$$

Then, $(S_1)_\alpha=(S_2)_\alpha=\{2\}$ and $(S_3)_\alpha=\{1\}$, and

$$\begin{aligned}&\frac{1}{2}p_\alpha(1,S1)\le d(1,\nu), \quad \nu=2,3;\\ &\frac{1}{2}p_\alpha(2,S2)\le d(2,\nu), \quad \forall\nu\in M,\\ &\frac{1}{2}p_\alpha(3,S3)\le d(3,\nu), \quad \nu=1,2.\end{aligned} \tag{37}$$

We consider three cases.

Case 1. If $\mu,\nu\in\{1,2\}$, then we have

$$D_\alpha(S\mu,S\nu)=0, \quad \forall\mu,\nu\in M. \tag{38}$$

Hence, (7) is satisfied for $\mu,\nu\in\{1,2\}$ trivially.

Case 2. If $\mu=3$ and $\nu=1$, then we have

$$\begin{aligned}D_\alpha(S3,S1)&=d(1,2)=\frac{1}{2},\\ d(3,1)&=\frac{3}{8}.\end{aligned} \tag{39}$$

Then,

$$\begin{aligned}D_\alpha(S3,S1)&=\frac{1}{2}\\ &\le\frac{3}{8}-\frac{11}{19}\\ &=\frac{3}{8}-\phi(D_\alpha(S3,S1)).\end{aligned} \tag{40}$$

So condition (7) is satisfied.

Case 3. If $\mu=3$ and $\nu=2$, then we have

$$\begin{aligned}D_\alpha(S3,S2)&=d(1,2)=\frac{1}{2},\\ d(3,2)&=\frac{3}{2}.\end{aligned} \tag{41}$$

Thus, we get

$$\begin{aligned}D_\alpha(S3,S2)&=\frac{1}{2}\\ &\le\frac{3}{2}-\frac{11}{19}\\ &=\frac{3}{2}-\phi(D_\alpha(S3,S1)).\end{aligned} \tag{42}$$

We see that the assumptions of Theorem 1 are fulfilled in all cases, and hence, S has a fuzzy fixed point which is 2.

In view of Remark 3, we deduce the underlying result.

Theorem 2. *Let (M,d) be a complete metric space and $S\colon M\longrightarrow\mathscr{W}_\alpha(M)$ a fuzzy weak ϕ-contraction, such that for every $\mu\in M$, $(S\mu)_\alpha$ is closed. Then, there exists $\mu^*\in M$, such that μ^*_α is a fuzzy fixed point of S, i.e., $\mu^*_\alpha\subset S\mu^*$.*

If the fuzzy mapping S is a Suzuki-type fuzzy weak ϕ-contraction, then it immediately satisfies the following contraction condition:

$$\frac{1}{2}p_\alpha(\mu,S\mu)\le d(\mu,\nu)\Rightarrow D_\alpha(S\mu,S\nu)\le d(\mu,\nu)-\phi(d(\mu,\nu)), \tag{43}$$

where $\mu,\nu\in M$ and $\phi\in\Phi$. But the converse need not be true. We justify this claim by showing that the condition $D_\alpha(S\mu,S\nu)\le d(\mu,\nu)-\phi(D_\alpha(\mu,\nu))$ is weaker than $D_\alpha(S\mu,S\nu)\le d(\mu,\nu)-\phi(d(\mu,\nu))$. For this, we consider the following example.

Let $M=\{a,b,c\}$ with the metric $d\colon M\times M\longrightarrow[0,\infty)$ defined by $d(a,b)=d(a,c)=5$, $d(b,c)=3$; $d(\mu,\mu)=0$, and $d(\mu,\nu)=d(\nu,\mu)$, $\forall\mu,\nu\in M$. Define

$$\phi(\tau)=\frac{\tau}{2}, \quad \forall\tau\in[0,\infty), \tag{44}$$

and a fuzzy mapping S by

$$S_a(\mu)=\begin{cases}\alpha, & \text{if } \mu=a,\\ 2\alpha, & \text{if } \mu\in\{b,c\},\end{cases}$$

$$S_b(\mu)=\begin{cases}\alpha, & \text{if } \mu\in\{a,b\},\\ \dfrac{\alpha}{2}, & \text{if } \mu=c,\end{cases} \tag{45}$$

$$S_c(\mu)=\begin{cases}2\alpha, & \text{if } \mu=a,\\ \alpha, & \text{if } \mu=b,\\ 0, & \text{if } \mu=c.\end{cases}$$

So, we get $(S_a)_\alpha = \{a,b,c\}$ and $(S_b)_\alpha = (S_c)_\alpha = \{a,b\}$, and

$$\begin{aligned} D_\alpha(S_a,S_b) &= D_\alpha(S_a,S_c) = 3, \\ D_\alpha(S_b,S_c) &= 0. \end{aligned} \tag{46}$$

We observe that the condition $D_\alpha(S\mu,S\nu) \le d(\mu,\nu) - \phi(D_\alpha(\mu,\nu))$ is satisfied for all $\mu,\nu \in M$, but for $\mu = a$ and $\nu = b$, the condition $D_\alpha(S\mu,S\nu) \le d(\mu,\nu) - \phi(d(\mu,\nu))$ is not fulfilled.

Hence, we obtain the following result.

Theorem 3. *Let (M,d) be a complete metric space and $S\colon M \longrightarrow \mathscr{W}_\alpha(M)$ a fuzzy mapping satisfying (43). Then, there exists $\mu^* \in M$, such that $\mu^*_\alpha \subset S\mu^*$.*

Taking $\phi(\tau) = (1-k)\tau$, $k \in [0,1)$ in the above result (viz. (43)), we obtain the next result.

Theorem 4. *Let (M,d) be a complete metric space and $S\colon M \longrightarrow \mathscr{W}_\alpha(M)$ a fuzzy mapping satisfying the condition*

$$\frac{1}{2}p_\alpha(\mu,S\mu) \le d(\mu,\nu) \Rightarrow D_\alpha(S\mu,S\nu) \le k\, d(\mu,\nu), \tag{47}$$

where $\mu,\nu \in M$. Then, there exists $\mu^* \in M$, such that $\mu^*_\alpha \subset S\mu^*$.

Remark 4. Let S be a fuzzy mapping from M to $\mathscr{W}_\alpha(M)$ and $T\colon M \longrightarrow K(M)$ a closed mapping (where $K(M)$ denotes the set of all compact subsets of M). Define

$$(S\mu)(\nu) = \begin{cases} \alpha, & \text{if } \nu \in T\mu, \\ 0, & \text{otherwise}, \end{cases} \tag{48}$$

for each $\mu \in M$. Note that,

$$(S\mu)_\alpha = \{\nu\colon (S\mu)\nu \ge \alpha\} = T\mu. \tag{49}$$

In view of above remark, we obtain the fixed point results for multivalued mapping T (defined above) from Theorems 1-4.

Theorem 5. *Let (M,d) be a complete metric space and $T\colon M \longrightarrow K(M)$ a multivalued closed mapping satisfying*

$$\frac{1}{2}d(\mu,T\mu) \le d(\mu,\nu) \Rightarrow H(T\mu,T\nu) \le d(\mu,\nu) - \phi(H(T\mu,T\nu)), \tag{50}$$

$\forall \mu,\nu \in M$ and $\phi \in \Phi$. Then, T has a fixed point.

Theorem 6. *Let (M,d) be a complete metric space and $T\colon M \longrightarrow K(M)$ a multivalued closed mapping satisfying*

$$\frac{1}{2}d(\mu,T\mu) \le d(\mu,\nu) \Rightarrow H(T\mu,T\nu) \le d(\mu,\nu) - \phi(d(\mu,\nu)), \tag{51}$$

$\forall \mu,\nu \in M$ and $\phi \in \Phi$. Then, T has a fixed point.

Theorem 7. *Let (M,d) be a complete metric space and $T\colon M \longrightarrow K(M)$ a multivalued closed mapping satisfying*

$$\frac{1}{2}d(\mu,T\mu) \le d(\mu,\nu) \Rightarrow H(T\mu,T\nu) \le k\, d(\mu,\nu), \tag{52}$$

$\forall \mu,\nu \in M$. Then, T has a fixed point.

Similarly, we can obtain the results corresponding to Theorem 2.

3. An Application to the Fredholm Integral Inclusion

Consider the following Fredholm integral inclusion:

$$\mu(\tau) \in f(\tau) + \int_a^b K(\tau,s,\mu(s))\mathrm{d}s, \quad \tau \in [a,b], \tag{53}$$

where $f \in C[a,b]$ and $K\colon [a,b]\times[a,b]\times\mathbb{R} \longrightarrow P_{CV}(\mathbb{R})$ ($P_{CV}(\mathbb{R})$ denotes the class of all nonempty compact and convex subsets of $\mathbb{R}$), and $\mu \in C[0,1]$ is an unknown function. Consider $M = C[a,b]$ and take the complete metric space (M,d), where

$$d(\mu,\nu) = \max_{\tau\in[a,b]} |\mu(\tau) - \nu(\tau)|, \quad \forall \mu,\nu \in C[a,b]. \tag{54}$$

Before proving our claim, we note down the following lemma.

Lemma 2 (see [23, 24]). *Let (M,d) be a metric space and $P,Q \in P(M)$. If there exists $\eta \in \mathbb{R}$ ($\eta > 0$), such that*

(a) For each $p \in P$, there exists $q \in Q$, such that $d(p,q) \le \eta$

(b) For each $q \in Q$, there exists $p \in P$, such that $d(q,p) \le \eta$

Then, $H(P,Q) \le \eta$.

Theorem 8. *Under the conditions given as follows:*

(A_1) for all $\mu \in C[a,b]$, the operator $K\colon [a,b]\times[a,b]\times\mathbb{R} \longrightarrow P_{CV}(\mathbb{R})$ is such that $K_\mu(\tau,s) = K(\tau,s,\mu(s))$ is lower semicontinuous on $[a,b]\times[a,b]$

(A_2) there exists a continuous function $\lambda\colon [a,b]\times[a,b] \longrightarrow [0,\infty)$, such that

$$H\big(K_\mu(\tau,s),K_\nu(\tau,s)\big) \le \lambda(\tau,s)|\mu(\tau) - \nu(\tau)|, \tag{55}$$

for all $\tau,s \in [a,b]$ and $\mu,\nu \in C[a,b]$ with $\int_a^b \lambda(\tau,s)ds \le 2/3$

The Fredholm integral inclusion (53) has a solution in $C[a,b]$.

Proof. Define the fuzzy mapping $S\colon M \longrightarrow \mathfrak{F}(M)$ in such a way that

$$(S\mu)_\alpha = \left\{\nu \in M\colon \nu(\tau) \in f(\tau) + \int_a^b K_\mu(\tau,s)\mathrm{d}s, \tau \in [a,b]\right\}. \tag{56}$$

It is very obvious that the set of solutions of $(S\mu)_\alpha$ coincides with the set of fixed points of (53). So, we need to prove that $(S\mu)_\alpha$ has at least one fixed point.

For this, we consider an arbitrary fixed point $\mu \in M$ and the set-valued operator $K_\mu\colon [a,b] \times [a,b] \longrightarrow P_{CV}(\mathbb{R})$. Using Michael's theorem, we obtain a continuous function, such that $k_\mu(\tau,s) \in K_\mu(\tau,s)$, for each $\tau, s \in [a,b]$. Thus, $f(\tau) + k_\mu(\tau,s) \in (S\mu)_\alpha$, and so, $(S\mu)_\alpha \neq \varnothing$. Clearly, $(S\mu)_\alpha$ is closed (hence compact) and convex. So, $S \in W_\alpha(M)$.

Now, we will check that

$$D_\alpha(S\mu_1, S\mu_2) \leq d(\mu_1,\mu_2) - \phi(D_\alpha(S\mu_1, S\mu_2)), \quad \forall \mu_1,\mu_2 \in M. \tag{57}$$

Let $\nu_1 \in (S\mu_1)_\alpha$ (arbitrary), such that $\nu_1(\tau) \in f(\tau) + \int_a^b K(\tau,s,\mu_1(s))\mathrm{d}s$, for $\tau \in [a,b]$. This means for all $\tau, s \in [a,b]$, there exists $k_{\mu_1}(\tau,s) \in K_{\mu_1}(\tau,s)$, such that $\nu_1(\tau) = f(\tau) + \int_a^b k_{\mu_1}(\tau,s)\mathrm{d}s$. Now, from (A_2), we have

$$H\left(K_{\mu_1}(\tau,s), K_{\mu_2}(\tau,s)\right) \leq \lambda(\tau,s)\left|\mu_1(\tau) - \mu_2(\tau)\right|. \tag{58}$$

Then, there exists $\mu(\tau,s) \in K_{\mu_2}(\tau,s)$, such that

$$\left|k_{\mu_1}(\tau,s) - \mu(\tau,s)\right| \leq \lambda(\tau,s)\left|\mu_1(\tau) - \mu_2(\tau)\right|. \tag{59}$$

Now, we consider the multivalued operator U defined by

$$\begin{aligned} U(\tau,s) = K_{\mu_2}(\tau,s) \cap \Big\{u \in \mathbb{R}\colon &\left|k_{\mu_1}(\tau,s) - u\right| \\ &\leq \lambda(\tau,s)\left|\mu_1(\tau) - \mu_2(\tau)\right|\Big\} \end{aligned} \tag{60}$$

Hence, by (A_1), U is lower semicontinuous which ensures the existence of a continuous operator $k_{\mu_2}(\tau,s) \in U(\tau,s)$, implying that

$$\nu_2(\tau) = f(\tau) + \int_a^b k_{\mu_2}(\tau,s)\mathrm{d}s, \tag{61}$$

and hence,

$$\nu_2(\tau) \in f(\tau) + \int_a^b K_{\mu_2}(\tau,s)\mathrm{d}s. \tag{62}$$

So, we get $\nu_2 \in (S\mu_2)_\alpha$, and

$$\begin{aligned} \left|\nu_2(\tau) - \nu_1(\tau)\right| &\leq \int_a^b \left|k_{\mu_2}(\tau,s) - k_{\mu_1}(\tau,s)\right|\mathrm{d}s \\ &\leq \int_a^b |\lambda(\tau,s)|\left|\mu_2(s) - \mu_1(s)\right|\mathrm{d}s \\ &\leq \max_{t\in[a,b]}\left|\mu_2(\tau) - \mu_1(\tau)\right| \int_a^b |\lambda(\tau,s)|\mathrm{d}s \\ &< \frac{2}{3} d(\mu_2,\mu_1). \end{aligned} \tag{63}$$

After interchanging the roles of μ_1 and μ_2 and using Lemma 2, we obtain (for each $\mu_1, \mu_2 \in M$)

$$H\left((S\mu_1)_\alpha, (S\mu_2)_\alpha\right) \leq \frac{2}{3} d(\mu_1,\mu_2), \tag{64}$$

and by considering $\phi(\tau) = \tau/2$ (for all $\tau \in [0,\infty)$), all the assumptions of Theorem 1 as well as Theorem 2 are satisfied. Hence, the inclusion problem (53) has a solution. □

4. Conclusion

In this study, inspired by the work of Suzuki [10] and Xue [19], we define two new contractions, i.e., fuzzy weak ϕ-contraction and Suzuki-type fuzzy weak ϕ-contraction and use them to prove the existence of fuzzy fixed point and well exemplify them. Also, we provide an application of our proven result to show the existence of solution of Fredholm integral inclusion problem.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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