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Research Article

Saturation Numbers for Linear Forests $2P_4 \cup tP_2$*

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For a given graph F, graph G is called F-saturated if G does not contain a copy of F, but $G+e$ has a copy of F, where $e \notin E(G)$. The saturation number, denoted by $\mathrm{sat}(n, F)$, is the minimum size of a graph with order n in all F-saturated graphs. For $n \geq 6t+22$, this paper considers the saturation number of linear forests $2P_4 \cup tP_2$ and describes the extremal graph.

1. Introduction

In this article, we only deal with simple graph. Usually, the path and the complete graph with n vertices are denoted by P_n and K_n, respectively. For terminology and notations not undefined in this paper, the reader can refer to [1].

For a given graph F, graph G is called F-saturated if G does not contain a copy of F, but $G+e$ has a copy of F, where $e \notin E(G)$. The famous Turán number [2], denoted by $\mathrm{ex}(n, F)$, is the maximum number size of graphs in all F-saturated graphs with size n. As a complement, saturation number, denoted by $\mathrm{sat}(n, F)$, is the minimum size of graphs in all F-saturated graphs with size n. We use $\underline{\mathrm{SAT}}(n, F)$ which denotes the set of graphs with a minimum number size in $\mathrm{SAT}(n, F)$.

In 1964, Erdős et al. introduced the notion of the saturation number and gave the saturation number of K_t in [3]. Kászonyi et al. gave the general upper bound of $\mathrm{sat}(n, F)$ in [4], when F is a kind of forbidden graphs. Then, for a wider range of graphs F, saturation number of F has been studied by many scholars, for example, k-edge-connected graph [5], cliques [6, 7], complete bipartite graphs [8, 9], nearly complete graphs [10], books [11], cycles [12–17], trees [18, 19], and forests [20–22]. The reader can see summary of known results in [23].

In [24], Bushaw et al. gave the Turán number for the linear forest. Corresponding to that, Chen et al. concentrated on the saturation numbers in [20]. They obtained an interesting set of results; some of those are shown as the following results.

Theorem 1 (see [20])

(i) *Let n be positive integers such that sufficiently large; then, $\mathrm{sat}(n, P_3 \cup tP_2) = 3t$ and the extremal graph is $tK_3 \cup (n-3t)K_1$*
(ii) *Let n and t be positive integers such that n is sufficiently large and $t \geq 1$; then, $\mathrm{sat}(n, P_4 \cup tP_2) = 3t+7$ and the extremal graph is $K_5 \cup (t-1)K_3 \cup (n-3t-2)K_1$*
(iii) *Let n be positive integers such that it is sufficiently large; then, $\mathrm{sat}(n, P_5 \cup P_2) = 15$ and the extremal graph is $K_6 \cup (n-6)K_1$*

Let $T_{p,q}$ be obtained from K_2 by attaching p disjoint triangles to one vertex of K_2 and attaching q disjoint triangles to another vertex of K_2 (see Figure 1).

In this paper, we prove the following theorem.

Theorem 2. *Suppose t and n are positive integers such that $n \geq 6t+12$. Then, $\mathrm{sat}(n, 2P_4 \cup tP_2) = 3t+10$ and $\underline{\mathrm{SAT}}(n, 2P_4 \cup tP_2) = \{T_{p,q} \cup aK_3 \cup (n-2t-a-8)K_1\colon \quad p \geq 2, q \geq 2$ and $p+q+a = t+3\}$ for $n \geq 6t+22$.*

In what follows, Section 2 lists several useful results and study property of $(2P_4 \cup tP_2)$-saturated graphs. In Section 3, we give the proof of Theorem 2.

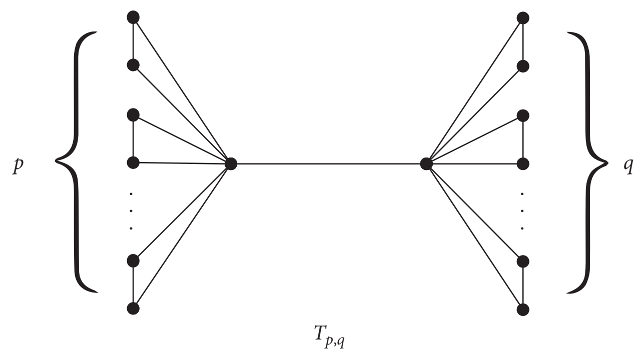

Figure 1: The graph $T_{p,q}$.

2. Preliminaries

Firstly, we introduce some notations. For a graph G, the edge set and vertex set are denoted by $E(G)$ and $V(G)$. The number of edges and the order of G are denoted by $|E(G)|$ and $|V(G)|$, respectively. Let $N_G(x) = \{v \in V(G)\colon\ xv \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$, and $d_G(x)$ be the degree of x, where $x \in V(G)$. Given a subgraph F of G and $X \subseteq V(G)$, let $N_G(X) = \cup_{x\in X} N_G(x) - X$, $N_F(X) = N_G(X) \cap V(F)$, $N_G[X] = \cup_{x\in X} N_G[x]$, and $N_F[X] = N_G[X] \cap V(F)$. We simple denote $d_F(x) = |N_F(x)|$ and $N_F(x) = N_F(\{x\})$ if $X = \{x\}$. When no confusion occurs, we simple identify a subgraph F of G with $V(F)$. For example, $G - V(F)$ is simple and is denoted by $G - F$ sometimes.

For a graph G, let $o(G) = |\{\text{odd components of } G\}|$ and denote the number of edges in the maximum matching of G by $\alpha'(G)$. Suppose $i \geq 0$; let $V_i(G) = \{x \in V(G)\colon\ d_G(x) = i\}$. We list the following useful results.

Lemma 1 (see [25]). *Let G be a graph; then,*

$$\alpha'(G) = \frac{1}{2}\min\{|G| + |S| - o(G - S)\colon\ S \subseteq V(G)\}. \qquad (1)$$

Lemma 2 (see [20]). *Suppose $k_1, \ldots, k_m \geq 2$ are integers. Let G be a $(P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_m})$-saturated graph. If $d_G(x) = 2$ and $N_G(x) = \{u, v\}$, then $uv \in E(G)$.*

Next, we give some properties of $(2P_4 \cup tP_2)$-saturated graphs.

Lemma 3. *Suppose G is $(2P_4 \cup tP_2)$-saturated with $V_0(G) \neq \emptyset$; then, $V_1(G) = \emptyset$. Moreover, if $w \in V_0(G)$ and $x \in V(G) - V_0(G)$, let F be any copy of $2P_4 \cup tP_2$ in $G + xw$; then,*

$$V(F) \supseteq N_G(x) \cup \{x, w\}. \qquad (2)$$

Proof. Suppose for the sake of contradiction that $x \in V_1(G)$. Since $wy \notin E(G)$ and G is $(2P_4 \cup tP_2)$-saturated, $G + wy$ has a $2P_4 \cup tP_2$ containing wy. Let $xy \in E(G)$, with wy replaced by xy, and we can get a new $2P_4 \cup tP_2$ in G, a contradiction. Thus, $V_1(G) = \emptyset$.

Since $xw \notin E(G)$ and G is $(2P_4 \cup tP_2)$-saturated, then $G + xw$ has a $2P_4 \cup tP_2$ containing xw and assumes F. If there exists $x' \in N_G(x) - V(F)$, with xw replaced by xx' in F, we can obtain a new $2P_4 \cup tP_2$ in graph G, a contradiction. Therefore,

$$V(F) \supseteq N_G(x) \cup \{x, w\}. \qquad (3)$$

Lemma 3 is true. □

Lemma 4. *Suppose G is $(2P_4 \cup tP_2)$-saturated such that $V_0(G) \neq \emptyset$ and all the nontrivial components are with at least 4 vertices. Let $x \in V(G)$ such that $d_G(x) = 2$ and $N_G(x) = \{u, v\}$. Then, $d_G(u) + d_G(v) \geq 6$. Moreover, if $d_G(v) = 3$, then $N_G(v) \subseteq N_G(u) \cup \{u\}$.*

Additionally, there does not exist a degree 2 vertex $x' \neq x$ such that $N_G(x') = \{u, v\}$.

Proof. By Lemma 2, we know $uv \in E(G)$. Since all the nontrivial components are with at least 4 vertices, then the triangle induced by $\{x, u, v\}$ is not a component of G. Hence, we can assume $vw \in E(G)$ for some vertex $w \notin \{x, u, v\}$. Consequently, $d_G(v) \geq 3$.

If $d_G(v) = 3$, it is enough to show $uw \in E(G)$. If $uw \notin E(G)$, $G + uw$ has a $2P_4 \cup tP_2$ containing uw, assume F. Assume $v \notin V(F)$, replacing uw by uw, and we obtain a $2P_4 \cup tP_2$ in G. Hence, $v \in V(F)$. If $uv \in F$ or $vw \in F$ or $xu \in F$, then $\{x, u, v, w\}$ are selected as a P_4 in F, and we can replace it with $P_4 = xuvw$, which implies there is a $2P_4 \cup tP_2$ in G. So, we have $uv \notin F$ and $vw \notin F$ and $xu \notin F$. This together with $v \in V(F)$ and $d_G(v) = 3$ implies that $xv \in F$ and xv is selected as a P_2 in F. Now, we can claim that uw is selected as a P_4 in F. Otherwise, both xv and uw are selected as a P_2 in F. Replacing $\{xv, uw\}$ in F with $\{xu, vw\}$, we obtain a copy of $2P_4 \cup tP_2$ in G, a contradiction. If uw is selected as a middle edge in P_4, then $E(F) - uw + xu$ contains a $2P_4 \cup tP_2$, a contradiction. If uw is selected as an end edge in P_4, then $E(F) - uw - xv + vw + xu$ contains a $2P_4 \cup tP_2$, a contradiction.

If there exists a degree 2 vertex $x' \neq x$ such that $N_G(x') = N_G(x)$, then select an isolated vertex $y \in V_0(G)$ and add the edge uy. Let F be a copy of the graph $2P_4 \cup tP_2$ in $G + uy$. No matter uy is selected as an edge in P_2 or P_4 in F, since $d_G(x) = d_G(x') = 2$, we get $|\{x, x'\} \cap V(F)| \leq 1$, which contradicts Lemma 3. □

Lemma 5. *Let G be a $(2P_4 \cup tP_2)$-saturated graph and $V_0(G) \neq \emptyset$. Let G' be an induced subgraph of G. If $G' \cong K_3$ or $G' \cong K_{1,\ell}$ $(\ell \geq 2)$, then $|N_G(G') \cap (V(G) - V(G'))| \neq 1$.*

Proof. Suppose, for the sake of contradiction, that $N_G(G') \cap (V(G) - V(G')) = \{x\}$. Select an isolated vertex $w \in V_0(G)$ and add the edge wx. Since G is $(2P_4 \cup tP_2)$-saturated, then $G + wx$ has a $2P_4 \cup tP_2$ containing wx and assumes F.

First, consider the case $G' \cong K_3$. Suppose $V(G') = \{u_1, u_2, u_3\}$ and $u_1 \in N_G(x)$. For any $u_i \in N_G(x)$, we have $u_i x \notin E(F)$, where $1 \leq i \leq 3$. Otherwise, $|V(G') - V(F)| = 1$ and $\{x, w\} \cup V(G')$ are selected as a P_4. However, this P_4 can be replaced with $xu_1u_2u_3$ in F, and then, we obtain a $2P_4 \cup tP_2$ in G, a contradiction. Hence, $\{u_1, u_2, u_3\}$ is selected as a P_2 in F. We can choose u_2u_3 as P_2 and then replace wx in F with xu_1; we obtain a $2P_4 \cup tP_2$ in G, a contradiction.

Now, consider the case $G' \cong K_{1,\ell}$. Suppose u is the center of the star and $V(K_{1,\ell}) = \{u, u_1, \ldots, u_\ell\}$. Since G is $(2P_4 \cup tP_2)$-saturated with $V_0(G) \neq \emptyset$, by Lemma 3, $V_1(G) = \emptyset$. Then, we have $u_i x \in E(G)$ for $1 \leq i \leq \ell$. Hence, $d_G(u_i) = 2$ and $N_G(u_i) = \{u, x\}$ for all $1 \leq i \leq \ell$, which contradicts Lemma 4. Lemma 5 is true. □

3. Proof of Theorem 2

Before the proof of Theorem 2, we discuss the property of $(2P_4 \cup tP_2)$-saturated graph which has no components of order 2, 3, and 4.

Lemma 6. *Suppose G is $(2P_4 \cup tP_2)$-saturated, but not $(P_4 \cup (t+2)P_2)$-saturated graph. Let $V_0(G) \neq \emptyset$, $H_1, \ldots, H_k$ be the nontrivial components of G and $H = G[\cup\{i=1\}\{i=k\}V(H_i)]$. If $\delta(H) \geq 2$, $|H| \geq 2t+8$ and $|H_i| \geq 5$ $(1 \leq i \leq k)$, we have $|E(G)| \geq 3t+10$.*

Moreover, if the equality $|E(G)| = 3t+10$ holds, then $G \cong T_{p,q} \cup (n-2t-8)K_1$, where $p, q \geq 2$ and $p+q = t+3$.

Proof. Since G is $(2P_4 \cup tP_2)$-saturated, $G+e$ has a $P_4 \cup (t+2)P_2$, where $e \notin E(G)$. On the contrary, G must contain a $P_4 \cup (t+2)P_2$, since G is not $(P_4 \cup (t+2)P_2)$-saturated.

Suppose M is a $P_4 \cup (s+2)P_2$ in G with $E(M) = \{w_1w_2, w_2w_3, w_3w_4\} \cup \{u_1v_1, \ldots, u_{s+2}v_{s+2}\}$. Choose M which satisfies the following:

(i) s is as large as possible
(ii) Subject to (i), $\sum_{i=1}^{s+2}(d_G(u_i) + d_G(v_i))$ is maximality

Observe that $s \geq t$. By the choice of M, $V(H) - V(M)$ is an independent set or an empty set. As G is $(2P_4 \cup tP_2)$-saturated, we have

$$G[V(M) \backslash \{w_1, w_2, w_3, w_4\}] \cong (s+2)P_2, \qquad (4)$$

$$N_{G-V(M)}(\{u_i, v_i\}) \cap N_{G-V(M)}(\{u_j, v_j\}) = \emptyset, \quad \text{for } 1 \leq i \neq j \leq s+2. \qquad (5)$$

Without loss of generality, we may assume $\{w_1, w_2, w_3, w_4\} \subseteq H_1$. □

Claim 1. $k = 1$.

Proof. Suppose, for the sake of contradiction, that $k \geq 2$. Since $|H_i| \geq 5$ and $\delta(H) \geq 2$, we have $\alpha'(H_i) \geq 2$ for every $1 \leq i \leq k$. Without loss of generality, we can assume $\{u_\ell v_\ell, \ldots, u_m v_m\} \subseteq H_2$, where $1 \leq \ell < m \leq s+2$.

As H_2 is connected, there exists a path Q from $\{u_{i_1}, v_{i_1}\}$ to $\{u_{i_2}, v_{i_2}\}$ avoiding all vertices of $V(M) \backslash \{u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}\}$ for some $\ell \leq i_1 < i_2 \leq m$. Again, $G[\{u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}\} \cup V(Q)]$ has a P_4, which, combining the edges in $M - \{u_{i_1}v_{i_1}, u_{i_2}v_{i_2}\}$, creates a $2P_4 \cup sP_2$ in G, a contradiction. Claim 1 is true.

It follows from Claim 1 that $H = H_1$. Hence, in the rest of the proof of Lemma 6, H_1 is denoted by H. Clearly, $|H| \geq 2s+8 \geq 2t+8$. □

Claim 2. For each $1 \leq i \leq s+2$, we have $N_G(\{u_i, v_i\}) \cap \{w_1, w_2, w_3, w_4\} \neq \emptyset$.

Proof. Suppose for the sake of contradiction that there exists $1 \leq i \leq s+2$ such that $N_G(\{u_i, v_i\}) \cap \{w_1, w_2, w_3, w_4\} = \emptyset$. This together with (4) implies that

$$N_G(u_i) \cap V(M) = \{v_i\} \text{ and } N_G(v_i) \cap V(M) = \{u_i\}. \qquad (6)$$

We are now ready to show that

$$d_G(u_i) + d_G(v_i) \geq 5. \qquad (7)$$

Otherwise, $d_G(u_i) = d_G(v_i) = 2$, since $\delta(H) \geq 2$. It follows from (6) that we can assume $w \in N_{H-V(M)}(u_i)$. By Lemma 2, $wv_i \in E(G)$. Then, $M - \{u_iv_i\} + \{u_iw\}$ is a copy of $P_4 \cup (s+2)P_2$ in G. However, by Lemma 4, we have $d_G(w) \geq 4$, and hence, $d_G(u_i) + d_G(w) \geq 6 > d_G(u_i) + d_G(v_i) = 4$, which contradicts the choice of M. Then, we have (7) which holds.

Combining (6) and (7), there exist $z_1 \neq z_2$ such that $z_1 \in N_{H-V(M)}(u_i)$ and $z_2 \in N_{H-V(M)}(v_i)$. Then, the subgraph $G[\{z_1, u_i, v_i, z_2\}]$ will have a P_4, which, combining the edges in $M - \{u_iv_i\}$, creates a $2P_4 \cup (s+1)P_2$ in G. Since $s \geq t$, we obtain a contradiction. Claim 2 is true. □

Claim 3. There does not exist $1 \leq i, j \leq s+2$ such that

$$N_G(\{u_i, v_i\}) \cap \{w_1, w_2\} \neq \emptyset \text{ and } N_G(\{u_j, v_j\}) \cap \{w_3, w_4\} \neq \emptyset. \qquad (8)$$

Proof. Suppose for the sake of contradiction that there exist $1 \leq i, j \leq s+2$ such that $N_G(\{u_i, v_i\}) \cap \{w_1, w_2\} \neq \emptyset$ and $N_G(\{u_j, v_j\}) \cap \{w_3, w_4\} \neq \emptyset$. By Claim 2, we may assume $i \neq j$. However, the subgraphs $G[\{u_i, v_i, w_1, w_2\}]$ and $G[\{u_j, v_j, w_3, w_4\}]$ both contain P_4, which, combining the edges in $M - \{u_iv_i, u_jv_j, w_1w_2, w_2w_3, w_3w_4\}$, creates a $2P_4 \cup sP_2$ in G, a contradiction. Hence, Claim 3 is true. □

Claim 4. For each $1 \leq i \leq s+2$, we have $|N_G(\{u_i, v_i\}) \cap \{w_1, w_2, w_3, w_4\}| = 1$ or $|E(G)| > 3t+10$.

Proof. Suppose, for the sake of contradiction, that Claim 4 is false. Then, by Claim 3 and the symmetry, we may assume $|E(G)| \leq 3t+10$ and

$$N_G(\{u_1, v_1\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1, w_2\}. \qquad (9)$$

Then, by Claim 3, we have

$$N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} \subseteq \{w_1, w_2\}, \quad \text{for each } 1 \leq j \leq s+2. \qquad (10)$$

□

Case 1. $N_G(u_1) \cap \{w_1, w_2\} \neq \emptyset$ and $N_G(v_1) \cap \{w_1, w_2\} \neq \emptyset$.

Combining (9), without loss of generality, in this case, we may assume $u_1w_1 \in E(G)$ and $v_1w_2 \in E(G)$. Now, we consider u_jv_j, where $2 \le j \le s+2$.

If $w_1 \in N_G(\{u_j, v_j\})$, then subgraphs $G[\{u_j, v_j, w_1, u_1\}]$ and $G[\{v_1, w_2, w_3, w_4\}]$ both contain P_4, which, combining the edges in $M - \{u_1v_1, u_jv_j, w_1w_2, w_2w_3, w_3w_4\}$, creates a $2P_4 \cup sP_2$ in G, a contradiction. Hence, we have $w_1 \notin N_G(\{u_j, v_j\})$. Together with Claim 2 and (10), we obtain

$$N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_2\}. \tag{11}$$

Without loss of generality, we may assume $u_jw_2 \in E(G)$.

If $N_{H-V(M)}(v_j) \neq \varnothing$, assume $G' = G[N_{H-V(M)}[\{u_j, v_j\}]]$. In order to avoid a $2P_4 \cup tP_2$ in G, we get that $G' \cong K_3$ or $G' \cong K_{1,\ell}(\ell \ge 2)$ and $N_G(G') \cap (V(G) - V(G')) = \{w_2\}$, which contradicts Lemma 5. Hence, $N_{H-V(M)}(v_j) = \varnothing$. This together with $\delta(H) \ge 2$ and (11) implies that $d_G(v_j) = 2$ and $v_jw_2 \in E(G)$. As the argument of v_j, we have $N_{H-V(M)}(u_j) = \varnothing$. Hence, $d_G(u_j) = 2$. Then, we can conclude that

$$d_G(u_j) = d_G(v_j) = 2 \text{ and } N_G(u_j) \cap N_G(v_j) = \{w_2\}, \quad \text{for any } 1 \le j \le s+2. \tag{12}$$

And, by the choice of M, $d_G(w_3) = d_G(w_4) = 2$. Hence, by Lemma 2, $w_2w_4 \in E(G)$. Combining (12) implies that $d_G(w_2) \ge 2(s+3)$. If $d_G(w_1) = 2$, by Lemma 2, $u_1w_2 \in E(G)$. Hence, $d_G(u_1) + d_G(v_1) + d_G(w_1) \ge 7$. If $d_G(w_1) \ge 3$, we also have $d_G(u_1) + d_G(v_1) + d_G(w_1) \ge 7$. This together with $d_G(w_2) \ge 2(s+3)$, $|H| \ge 2t+8$, $s \ge t$, and $\delta(H) \ge 2$ implies that $2|E(H)| \ge 2(s+3) + 7 + 2(2t+4) > 6t+20$, a contradiction.

Case 2. $N_G(u_1) \cap \{w_1, w_2\} = \varnothing$ or $N_G(v_1) \cap \{w_1, w_2\} = \varnothing$.

Combining (9), without loss of generality, in this case, we may assume $N_G(v_1) \cap \{w_1, w_2, w_3, w_4\} = \varnothing$. This combining $\delta(H) \ge 2$ and (4), we get $N_{H-V(M)}(v_1) \neq \varnothing$ and $|H| \ge 2t+9$. Assume $v_1x \in E(G)$, where $x \in H - V(M)$. By our assumption $N_G(\{u_1, v_1\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1, w_2\}$, we get $u_1w_1 \in E(G)$ and $u_1w_2 \in E(G)$. Now, we consider u_jv_j, where $2 \le j \le s+2$.

If $w_2 \in N_G(\{u_j, v_j\})$, then the subgraphs $G[\{u_j, v_j, w_2, w_3\}]$ and $G[\{x, v_1, u_1, w_1\}]$ both contain P_4, which, combining the edges in $M - \{u_1v_1, u_jv_j, w_1w_2, w_2w_3, w_3w_4\}$, creates a $2P_4 \cup sP_2$ in G, a contradiction. Hence, we have $w_2 \notin N_G(\{u_j, v_j\})$. This together with Claim 2 and (9) implies that $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1\}$, for any $2 \le j \le s+2$. Without loss of generality, we may assume $u_jw_1 \in E(G)$. Then, we have $N_G(x) \cap \{w_1, w_2, w_3, w_4\} = \varnothing$. So, by the choice of M, (5), and $\delta(H) \ge 2$, we get $xu_1 \in E(G)$. Hence,

$$d_G(u_1) \ge 4. \tag{13}$$

In order to avoid a $2P_4 \cup tP_2$ in G, we get that $N_{H-V(M)}(v_j) = \varnothing$. This together with $\delta(H) \ge 2$ and $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1\}$ implies that $v_jw_1 \in E(G)$ and $d_G(v_j) = 2$. Similarly, we have $d_G(u_j) = 2$. Therefore, $u_jw_1 \in E(G)$ and $v_jw_1 \in E(G)$ for any $2 \le j \le s+2$. Hence, combining $s \ge t$ implies that

$$d_G(w_1) \ge 2(s+2) \ge 2t+4. \tag{14}$$

And, by the choice of M, we have $d_G(w_3) = d_G(w_4) = 2$. Then, by Lemma 2, $w_2w_4 \in E(G)$, which implies that

$$d_G(w_2) \ge 4. \tag{15}$$

Recall that $|H| \ge 2t+9$ and $\delta(H) \ge 2$. Together with (13)–(15), we can get that $2|E(H)| \ge 2t+4+4+4+2(2t+6) > 6t+20$, a contradiction. Claim 4 is true.

Claim 5. If there exists $1 \le i \le s+2$ such that $d_G(u_i) + d_G(v_i) \le 4$, then we have $|E(G)| \ge 3t+10$. Moreover, the equality $|E(G)| = 3t+10$ holds if and only if $G \cong T_{p,q} \cup (n-2t-8)K_1$, where $p, q \ge 2$ and $p+q = t+3$.

Proof. Without loss of generality, we may assume $d_G(u_1) + d_G(v_1) \le 4$. Since $\delta(H) \ge 2$, we have $d_G(u_1) = d_G(v_1) = 2$. By Claim 4 and the symmetry, we complete the Claim 5 by following two cases.

Case 1. $N_G(\{u_1, v_1\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1\}$.

Since $d_G(u_1) = d_G(v_1) = 2$, by Lemma 2, $u_1w_1 \in E(G)$ and $v_1w_1 \in E(G)$. By the choice of M, $d_G(w_3) = d_G(w_4) = 2$. Also, by Lemma 2, $w_2w_4 \in E(G)$. Now, we consider u_jv_j, where $2 \le j \le s+2$. It follows from Claim 3 and our assumption that $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} \subseteq \{w_1, w_2\}$, where $2 \le j \le s+2$. This together with Claim 4 implies that $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_i\}$, where $i = 1$ or 2.

If $N_{H-V(M)}(v_j) \neq \varnothing$, assume $G' = G[N_{H-V(M)}[\{u_j, v_j\}]]$. In order to avoid a $2P_4 \cup tP_2$ in G, we get that $G' \cong K_3$ or $G' \cong K_{1,\ell}(\ell \ge 2)$ and $N_G(G') \cap (V(G) - V(G')) = \{w_i\}$, which contradicts Lemma 5. Hence, $N_{H-V(M)}(v_j) = \varnothing$. Since $\delta(H) \ge 2$, we get $v_jw_i \in E(G)$ and $d_G(v_j) = 2$. By the similar argument, we have $N_{H-V(M)}(u_j) = \varnothing$, and hence, $d_G(u_j) = 2$. Therefore, for each $1 \le j \le s+2$, we can conclude that $N_G(u_j) \cap N_G(v_j) = \{w_1\}$ or $N_G(u_j) \cap N_G(v_j) = \{w_2\}$. Together with $w_2w_4 \in E(G)$, we have $d_G(w_1) + d_G(w_2) \ge 2(s+3) + 2 = 2s+8$. Recall that $s \ge t$, $|H| \ge 2t+8$, and $\delta(H) \ge 2$, we can get that $2|E(H)| \ge 2s + 8 + 2(2t+6) \ge 6t+20$. And, if $|E(G)| = 3t+10$, then $G \cong T_{p,q} \cup (n-2t-8)K_1$, where $p+q = t+3$. However, if $\min\{p, q\} \le 1$, $T_{p,q}$ is not a $(2P_4 \cup tP_2)$-saturated graph, since the addition of an edge between an isolated vertex of G and the vertex of degree $\Delta(G)$ does not result in a $2P_4 \cup tP_2$. Hence, $p, q \ge 2$; Claim 5 holds.

Case 2. $N_G(\{u_1, v_1\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_2\}$.

Since $d_G(u_1) = d_G(v_1) = 2$, by Lemma 2, $u_1w_2 \in E(G)$ and $v_1w_2 \in E(G)$. By the choice of M, we have $d_G(w_3) = d_G(w_4) = 2$. Also, by Lemma 2, $w_2w_4 \in E(G)$. Now, we consider u_jv_j, where $2 \le j \le s+2$. It follows from Claim 3 and our assumption that $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} \subseteq \{w_1, w_2\}$.

If there exists $1 \le j \le s+2$ such that $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_1\}$, we can claim that $d_G(u_j) = d_G(v_j) = 2$. Otherwise, assume $G' = G[N_{H-V(M)}[\{u_j, v_j\}]]$. In order to avoid a $2P_4 \cup tP_2$ in G, we get that $G' \cong K_3$ or $G' \cong K_{1,\ell}(\ell \ge 2)$ and $N_G(G') \cap (V(G) - V(G')) = \{w_1\}$, which contradicts Lemma 5. Then, as the similar argument in the Case 1, we can obtain $|E(H)| \ge 3t + 10$, and if $|E(G)| = 3t + 10$, then $G \cong T_{p,q}$, where $p, q \ge 2$ and $p + q = t + 3$. Then, $N_G(\{u_j, v_j\}) \cap \{w_1, w_2, w_3, w_4\} = \{w_2\}$, for all $1 \le j \le s+2$. Without loss of generality, we may assume $u_j w_2 \in E(G)$. If $N_{H-V(M)}(v_j) \ne \varnothing$, assume $G' = G[[N_{H-V(M)}[\{u_j, v_j\}]]]$. In order to avoid a $2P_4 \cup tP_2$ in G, we get that $G' \cong K_3$ or $G' \cong K_{1,\ell}(\ell \ge 2)$ and $N_G(G') \cap (V(G) - V(G')) = \{w_2\}$, which contradicts Lemma 5. Hence, $N_{H-V(M)}(v_j) = \varnothing$. Since $\delta(H) \ge 2$, we get $v_j w_2 \in E(G)$ and $d_G(v_j) = 2$. By the similar argument, we have $N_{H-V(M)}(u_j) = \varnothing$, and hence, $d_G(u_j) = 2$. Therefore, $u_j w_2 \in E(G)$ and $v_j w_2 \in E(G)$ for any $2 \le j \le s+2$. Hence,

$$d_G(w_2) \ge 2(s+3) + 1 \ge 2t + 7. \quad (16)$$

This together with $\delta(H) \ge 2$, $|H| \ge 2t + 8$, and (16) implies that $2|E(H)| \ge 2t + 7 + 2(2t + 7) > 6t + 20$. This completes the proof of Claim 5.

If there exists $1 \le i \le s+2$ such that $d_G(u_i) + d_G(v_i) \le 4$, by Claim 5, the Lemma is proven by Claim 5. Hence, we can assume $d_G(u_i) + d_G(v_i) \ge 5$ for each $1 \le i \le s+2$. On the contrary, by (4) and Claim 4, we can assume $d_{V(M)}(u_i) + d_{V(M)}(v_i) \le 4$. Hence, there exists $x_i \in N_G(\{u_j, v_j\}) \cap (V(H) - V(M))$ for any $1 \le i \le s+2$. And, by (5), $x_i \ne x_j$ for $1 \le i \ne j \le s+2$. Hence, $|H| \ge 3(s+2) + 4 \ge 3t + 10$. Since $\delta(H) \ge 2$, we get $|E(H)| \ge 3t + 10$. And, if the equality $|E(G)| = 3t + 10$ holds, then $\delta(H) = \Delta(H) = 2$ and $|H| = 3t + 10$, which contradicts Lemma 4. Lemma 6 is true. ☐

Lemma 7. *Suppose G is $(2P_4 \cup tP_2)$-saturated and $(P_4 \cup (t+2)P_2)$-saturated with $V_0(G) \ne \varnothing$. Let $H_1, \dots, H_k$ be the nontrivial components of G and $H = G[\cup\{i=1\}\ \{i=k\}V(H_i)]$. If $\delta(H) \ge 2$ and $|H| \ge 2t + 8$ with $|H_i| \ge 5 (1 \le i \le k)$, we have $|E(G)| > 3t + 10$.*

Proof. Suppose, for the sake of contradiction, that $|E(G)| \le 3t + 10$. Since G is $(2P_4 \cup tP_2)$-saturated, there exists a $2P_4 \cup tP_2$ in $G + e$ for any $e \notin E(G)$. Hence, $\alpha'(H) \ge t + 3$. If $\alpha'(H) \ge t + 4$, G must contain a copy of $(t+4)P_2$. Since $\delta(H) \ge 2$ and $|H_i| \ge 5 (1 \le i \le k)$, it is clearly that H has a $P_4 \cup (t+2)P_2$, which contradicts G is $(P_4 \cup (t+2)P_2)$-saturated. So, we have $\alpha'(H) = t + 3$. By Lemma 1, we obtain

$$t + 3 = \frac{1}{2}\min\{|H| + |X| - o(G - X)\colon X \subseteq V(H)\}. \quad (17)$$

Then, we can choose a maximal subset $S \subseteq V(H)$ which satisfies

$$t + 3 = \frac{1}{2}(|H| + |S| - o(H - S)). \quad (18)$$

Suppose $H_1', \dots, H_p'$ are the components of $H - S$. We have the following claims. ☐

Claim 6. $H[S \cup V(H_i')]$ is the complete graph, where $1 \le i \le p$.

Proof. Suppose for the sake of contradiction that $H[S \cup V(H_i')]$ is not a complete graph. Then, there is $x, y \in S \cup V(H_i')$ with $x \ne y$ and $xy \notin E(H)$. Let $H' = H + xy$. We have $|H'| = |H|$ and $o(H' - S) = o(H - S)$. Combining (18), we obtain

$$t + 3 = \frac{1}{2}\Big(|H'| + |S| - o(H\prime - S)\Big). \quad (19)$$

By Lemma 1, we obtain $\alpha'(H') \le t + 3$. Hence, $G + xy$ has no $(t+4)P_2$, contrary to G which is $(2P_4 \cup tP_2)$-saturated. Claim 6 is true. ☐

Claim 7. $S \ne \varnothing$.

Proof. Suppose for the sake of contradiction that $S = \varnothing$. Hence, $H_1', \dots, H_p'$ are all components of H. Using Claim 6, we have H_i' is a complete graph with $|H_i'| \ge 5 (1 \le i \le p)$. Hence, we have $\delta(H) \ge 4$ and

$$2|E(H)| = \sum_{x \in V(H)} d_H(x) \ge 4|H|. \quad (20)$$

Combining $|H| \ge 2t + 8$, we get $|E(H)| > 3t + 10$. Hence, Claim 7 is true.

Let $w \in V_0(G)$ and $x \in S$. By Claim 6, $N_H(x) = V(H) - x$. Let F be a $2P_4 \cup tP_2$ in $G + xw$, using Lemma 3, we get $V(F) \supseteq \{x, w\} \cup N_G(x)$. Hence, we have $|H| + 1 = |N_H(x) \cup \{x, w\}| \le |V(F)| = 2t + 8$, contrary to G is $(2P_4 \cup tP_2)$-saturated. Hence, Lemma 7 is true.

Now, we prove Theorem 2. ☐

Proof. of Theorem 2. Clearly, $T_{p,q} \cup aK_3 \cup (n-\ 2t-\ a-8)K_1 \in \mathrm{SAT}(n, 2P_4 \cup tP_2)$, where $p \ge 2, q \ge 2$, and $p + q + a = t + 3$. Let $G \in \underline{\mathrm{SAT}}(n, 2P_4 \cup tP_2)$; then, $|E(G)| \le 3t + 10$. Since $n \ge 6t + 12$, $|V_0(G)| \ge 2$. Using Lemma 3, we have $V_1(G) \ne \varnothing$. Clearly, in graph G, all components of order 3 or 4 are complete. If there exists one of the components, say $H_1 \cong K_4$, then the addition of an edge e between an isolated vertex of G and a vertex of H_1 results in a $2P_4 \cup tP_2$ in G. However, the edge e can be replaced by the edge in H_1, resulting in a $2P_4 \cup tP_2 \in G$, a contradiction. Hence, component of order 4 in G does not exist.

Now, we consider the graph G', which is obtained by deleting all components of order 3 from G. Let a be the number of components of order 3 in G, and we obtain

$$G \cong G' \cup aK_3. \quad (21)$$

Note $|V_0(G')| = |V_0(G)| \ge 2$. Let e be an edge joining two isolated vertices in $V_0(G)$; then, there exists a $2P_4 \cup tP_2$ in $G + e$. So, G' has $2P_4$. Combining $G \in \underline{\mathrm{SAT}}(n, 2P_4 \cup tP_2)$, we get $a \le t - 1$. Let $t' = t - a$. Then, $t' \ge 1$.

As $G \in \underline{\mathrm{SAT}}(n, 2P_4 \cup tP_2)$, we have $G' \in \mathrm{SAT}(n', 2P_4 \cup tP_2)$, where $n' = n - 3a$. Suppose H' is a graph spanned by all nontrivial components of G'. Using Lemma 3, we have $\delta(H') \geq 2$. Clearly, any component in H' has order at least 5. Moreover, graph G' is $(2P_4 \cup t'P_2)$-saturated such that $|E(G')| = |E(G)| - 3a \leq 3t' + 10$ and $V_0(G') \neq \emptyset$. By Lemma 6 and Lemma 7, we have $|H'| \leq 2t' + 7$ or $H' \cong T_{p,q} \cup (n' - 2t' - 8)K_1$, where $p \geq 2, q \geq 2$ and $p + q = t' + 3$. If $|H'| \leq 2t' + 7$, as $V_0(G') \neq \emptyset$ and $G' \in \mathrm{SAT}(n', 2P_4 \cup t'P_2)$, $H' \cong K_{2t'+7}$. Now, $E(H') = \binom{2t'+7}{2} > 3t' + 21$, and we have $|E(G)| > 3t' + 21 + 3a = 3t + 21 > 3t + 10$, a contradiction. Hence, $H' \cong T_{p,q} \cup (n' - 2t' - 8)K_1$, where $p \geq 2, q \geq 2$, and $p + q = t' + 3$. Combining (21), we get $G \cong T_{p,q} \cup aK_3 \cup (n - 2t - a - 8)K_1$, where $p \geq 2, q \geq 2$, and $p + q + a = t + 3$. Hence, Theorem 2 is true. □

4. Conclusion

Forests and trees have attracted wide attention for several reasons. Firstly, their simplicity has made some more precise results possible. Secondly, they can provide inspiration for some larger results. At present, we only know the saturation number of a few species of trees, but the saturation number of most trees is unknown.

In [20], Chen et al. focused on the saturation numbers for the linear forests. They obtained an interesting set of results and proposed some conjectures. We pave the way for the following conjectures proposed by Chen et al. in [20].

Conjecture 1 (see [20]). *Let $t \geq 2$ be an integer. For n sufficiently large,* $\mathrm{sat}(n, tP_3) = \lfloor (n + 6t - 6)/2 \rfloor$.

Conjecture 2 (see [20]). *Let $t \geq 2$ be an integer. For n sufficiently large,*

$$\mathrm{sat}(n, tP_4) = \begin{cases} \dfrac{n + 12t - 12}{2}, & \text{if } n \text{ is even}, \\ \dfrac{n + 12t - 11}{2}, & \text{if } n \text{ is odd}. \end{cases} \tag{22}$$

Conjecture 3 (see [20]). *For n sufficiently large, $k \geq 4$, and $k \leq \ell \leq \lceil (3k - 2)/2 \rceil$, $\mathrm{sat}(n, P_k \cup P_\ell) = n - \lfloor n/a_k \rfloor + 3$, where*

$$a_k = \begin{cases} 3 \cdot 2^{m-1} - 2, & \text{if } k = 2m, \\ 4 \cdot 2^{m-1} - 2, & \text{if } k = 2m + 1. \end{cases} \tag{23}$$

Data Availability

No data used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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