Research Article

Analytical and Numerical Results on Global Dynamics of the Generalized Boussinesq Equation with Cubic Nonlinearity and External Excitation

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This paper analytically and numerically presents global dynamics of the generalized Boussinesq equation (GBE) with cubic nonlinearity and harmonic excitation. The effect of the damping coefficient on the dynamical responses of the generalized Boussinesq equation is clearly revealed. Using the reductive perturbation method, an equivalent wave equation is then derived from the complex nonlinear equation of the GBE. The persistent homoclinic orbit for the perturbed equation is located through the first and second measurements, and the breaking of the homoclinic structure will generate chaos in a Smale horseshoe sense for the GBE. Numerical examples are used to test the validity of the theoretical prediction. Both theoretical prediction and numerical simulations demonstrate the homoclinic chaos for the GBE.

1. Introduction

The field of global dynamics for infinite-dimensional systems is becoming increasingly attractive for interdisciplinary research. This paper aims to develop a theoretical analysis method of global dynamics for infinite-dimensional dynamical systems. An analytical method is conducted to detect the chaotic waves and nonlinear dynamical behaviors of the generalized Boussinesq equation (GBE) with cubic nonlinearity and external excitation.

As a typical model in physics and mechanics, the Boussinesq equation (BE) possesses both dispersive and nonlinear attributes [1]. BE is a powerful tool to predict coastal hydrodynamics such as wave propagation, wave-

current interaction, wave breaking, and nearshore circulation [2], even in extreme weather conditions such as typhoons and windstorms [3]. Many scholars have theoretically analyzed the characteristics of the BE. Weiss and his coworkers [4–6] explored Lax pairs, Bäcklund transformation, and the Painlevé property of the BE. The homogeneous balance method was extended to study the solution of traveling waves for the Boussinesq–Burgers equation [7]. Yang et al. [8] analyzed the global attractors and asymptotic behaviors for Schrödinger–Boussinesq equations. Based on the Hamiltonian structure, second- and fourth-order energy-preserving wavelet collocation schemes were proposed by Helal et al. [9] for the nonlinear coupled Schrödinger–Boussinesq model. Awais and Ibrahim [10]

carried out studies on water waves in a fourth-order Boussinesq nonlinear model and analyzed its stability for soliton solutions. Considering the impact of diabatic force, the instability of Boussinesq equations was investigated [11]. In [12], a numerical analysis was conducted to investigate the sensitivities of coastal wave attenuation resulting from incident wave height, period, water depth, and vegetation configurations in a nonlinear model of Boussinesq equations. Wang and Su [13] studied the finite time blow-up and solvability of solutions in all space dimensions for the Cauchy problem of a Boussinesq equation with dissipation. Yildirim analyzed, theoretically and numerically, the boundary control problem [14] and active control [15] for the nonlinear Boussinesq system.

Traditionally, infinite-dimensional systems are described as PDEs (partial differential equations), and then the PDEs are truncated into ODEs (ordinary differential equations) [16–22]. Theoretically, ODEs are topologically equivalent to PDEs, but the dynamical phenomena cannot be perfectly exhibited. Hence, how to improve an analytical method of global dynamics for infinite-dimensional systems must be resolved still by scientists and scholars. In recent decades, a level of improvement in this area has been achieved. The first studies of the global perturbation technique for nonlinear PDEs were conducted by Li et al. [23, 24], Zeng [25], and Shatah and Zeng [26]. The persistent homoclinic orbits of nonlinear Schrödinger (NLS) and sine-Gordon (SG) equations were constructed by using invariant manifolds, foliations, and Melnikov analysis. Furthermore, Zhang et al. [27], Wu and Qi [28, 29], and Li et al. [30] improved the analytic theory for infinite-dimensional systems [23–26] to obtain the chaotic threshold of mechanical systems such as truss core sandwich plates, pipes conveying fluid, axial moving beams, and carbon nanotubes. The aforementioned literature focused on solid structure systems; however, analysis on global dynamics for fluid systems, e.g., GBE, remains lacking.

The objective of this study is to derive the analytical chaotic threshold of the GBE and indicate how the damping

coefficient can clearly affect the dynamical responses of the GBE. The reductive perturbation method (RPM) [27, 30] is employed to transform the complicated PDE of the GBE into a topologically equivalent wave equation. The first and second measurements are then applied to explore the homoclinic chaos and the chaotic threshold for the wave equation. Afterward, numerical examples are also carried out using the differential quadrature method [27–29]. The simulating results confirm the validity of the theoretical prediction and illustrate larger damping coefficient resulting in the more complicated dynamical behaviors of the GBE.

2. Motion Equation and Perturbation Analysis

Boussinesq [31] assumed that the horizontal velocity is constant along the direction of water depth, and the vertical velocity is linearly distributed, obtaining the one-dimensional nonlinear water wave governing equation. Based on literature [31], the nonlinear governing equation of the cubic nonlinearity and harmonic excited GBE can be proposed as

$$q_{tt} + \mu q_t - q_{xx} + (q^3 + q_{xx})_{xx} = \alpha \cos(\omega t), \quad (1)$$

with the periodic boundary conditions

$$\begin{aligned} q(t) &= q(t + 2\pi), \\ q_x(t) &= q_x(t + 2\pi), \end{aligned} \quad (2)$$

where μ , α , and ω stand for the damping coefficient, amplitude, and frequency of the external force, respectively.

To carry out the analysis on the complicated dynamical responses of equation (1), the RPM is applied to simplify the formula, and the following formal solution of equation (1) yields:

$$\begin{aligned} q(x, t, \varepsilon) &= \varepsilon u(X, T)e^{i\phi} + c.c., \\ \phi &= kx - \omega t, \end{aligned} \quad (3)$$

where $X = \varepsilon(x - ct)$ and $T = \varepsilon^2 t$.

Thus, the derivatives used in the RPM are given as

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \varepsilon^{1/2} c \frac{\partial}{\partial X} + \varepsilon \frac{\partial}{\partial T}, \quad (4a)$$

$$\frac{\partial^2}{\partial t^2} \longrightarrow \frac{\partial^2}{\partial t^2} - 2\varepsilon^{1/2} c \frac{\partial^2}{\partial X \partial t} + \varepsilon c^2 \frac{\partial^2}{\partial X^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} - 2\varepsilon^{3/2} c \frac{\partial^2}{\partial X \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2}, \quad (4b)$$

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \varepsilon^{1/2} \frac{\partial}{\partial X}, \quad (4c)$$

$$\frac{\partial^2}{\partial x^2} \longrightarrow \frac{\partial^2}{\partial x^2} + 2\varepsilon^{1/2} \frac{\partial^2}{\partial x \partial X} + \varepsilon \frac{\partial^2}{\partial X^2}, \quad (4d)$$

$$\frac{\partial^4}{\partial x^4} \longrightarrow \frac{\partial^4}{\partial x^4} + 4\varepsilon^{1/2} \frac{\partial^4}{\partial x^3 \partial X} + 6\varepsilon \frac{\partial^4}{\partial x^2 \partial X^2} + 4\varepsilon^{3/2} \frac{\partial^4}{\partial x \partial X^3} + \varepsilon^2 \frac{\partial^4}{\partial X^4}. \quad (4e)$$

Moreover, the following scale transformation is introduced:

$$\mu \longrightarrow \varepsilon^2 \mu. \quad (5)$$

By substituting equations (3)–(5) into equation (1) and removing the secular terms in the power of ε , the results are yielded as follows:

$O(\varepsilon)$:

$$(\omega^2 - k^2 - k^4)u = 0. \quad (6)$$

$O(\varepsilon^2)$:

$$(c\omega - k + 4k^3) \frac{\partial u}{\partial X} = 0. \quad (7)$$

$O(\varepsilon^3)$:

$$-2i\omega \frac{\partial u}{\partial T} + (c^2 + 1 - k^2) \frac{\partial^2 u}{\partial X^2} - 3k^2 |u|^2 u - i\mu\omega u = 0. \quad (8)$$

$O(\varepsilon^4)$:

$$2c \frac{\partial^2 u}{\partial X \partial T} + 4ik \frac{\partial^3 u}{\partial X^3} - 6ik \frac{\partial |u|^2 u}{\partial X} - \mu c \frac{\partial u}{\partial X} = 0. \quad (9)$$

In order to derive the perturbed equation possessing the same form in [23, 24], equation (9) is integrated with respect to X , and the integral constant is $-2ic\alpha e^{-i\beta\omega^2 T}$. Letting $u \longrightarrow u e^{-i\beta\omega^2 T}$, equation (9) is transformed into

$$i \frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial X^2} + \beta(|u|^2 - \omega^2)u + i \frac{\mu}{2} u - i\alpha, \quad (10)$$

where $X^* = \sqrt{(2k/c)}X$, $\beta = 3k/c$, and the asterisk is dropped for convenience.

Hence, wave equation (10) has the same form in [23, 24] and is thus topologically equivalent to equation (1). Therefore, by using equation (10), the nonlinear characteristics of the wave equation for the GBE can be explored.

3. Global Analysis

The dynamical characteristics of equation (10) are further studied in this section. A bookkeeping parameter ε is inserted and obtains the following equation:

$$i \frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial X^2} + \beta(|u|^2 - \omega^2)u + i\varepsilon \left(\frac{\mu}{2} u - \alpha \right). \quad (11)$$

Equation (11) contains the invariant subspace

$$\prod = \{u(X, T) | \partial_X u \equiv 0\}, \quad (12)$$

which owns the resonant torus:

$$S = \{u \in \prod | |u| \equiv \omega\}. \quad (13)$$

When $\varepsilon = 0$, we have

$$i \frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial X^2} + \beta(|u|^2 - \omega^2)u, \quad (14)$$

with Hamiltonian

$$H(u) = \int_0^1 \left(|\partial_X u|^2 + \beta\omega^2 |u|^2 - \frac{\beta}{2} |u|^4 \right) dX. \quad (15)$$

Assuming $\beta > 0$, it is observed that equation (14) is completely integrable and has a periodic solution [23–26]

$$u_0(T) = r \exp[-i\beta(r^2 - \omega^2)T + iT_0], \quad (16)$$

as well as an analytical solution [23–26]

$$u_h^\pm(X, T) = \frac{\cos 2p \cosh \bar{T} - i \sin 2p \sinh \bar{T} \pm \sin p \cos X}{\cosh \bar{T} \mp \sin p \cos X} u_0(T), \quad (17)$$

where

$$\begin{aligned} \bar{T} &= \phi(T + T_1), \\ \phi &= \sqrt{2\beta r^2 - \omega^2}, \\ p &= \arctan \sigma. \end{aligned} \quad (18)$$

The symbol $\pm$ represents different parts of solutions $u_h^\pm(X, T)$ being homoclinic to $u_0(T)$. Meanwhile, $u_h^\pm(X, T)$ can also be seen as a heteroclinic orbit connecting different points in periodic orbit $u_0(T)$ due to the phase shifts, as presented in Figure 1. Figure 2 depicts the heteroclinic structure $u_h^\pm(X, T)$ in space $(X, T, |u|)$, respectively. To analyze the dynamic characteristics of equation (11) near S , the following transformations are introduced:

$$\begin{aligned} u &= \sqrt{\omega^2 + \sqrt{\varepsilon} I} \exp(i\theta), \\ \tau &= \sqrt{\varepsilon} T, \end{aligned} \quad (19)$$

and substituting equation (21) into equation (11), we have

$$\frac{\partial I}{\partial \tau} = \frac{\mu}{2} (\omega^2 + \sqrt{\varepsilon} I) - \alpha \sqrt{\omega^2 + \sqrt{\varepsilon} I} \cos \theta, \quad (20a)$$

$$\frac{\partial \theta}{\partial \tau} = -\beta I + \sqrt{\varepsilon} \alpha \frac{\sin \theta}{\sqrt{\omega^2 + \sqrt{\varepsilon} I}}, \quad (20b)$$

with a Hamilton energy

$$E(I, \theta) = \frac{\beta}{2} I^2 + \frac{\mu}{2} \omega^2 \theta - \omega \alpha \sin \theta. \quad (21)$$

FIGURE 1: Schematic diagram of the homoclinic tubes.

FIGURE 2: The exact form of the heteroclinic orbit for equation (17).

3.1. First Measurement. Note that the first measurement, the Melnikov method [22–25], shown in Figure 3(a), can be expressed as

$$\begin{aligned} \Delta &= \operatorname{Re} \int_{-\infty}^{+\infty} \int_0^1 i \left(\frac{\mu}{2} \frac{\partial p_h}{\partial T} p_h - \alpha \frac{\partial p_h}{\partial T} \right) dX dT \\ &= 2\omega \sqrt{2\beta - 1} \left(\frac{\mu}{2} \phi \omega^3 \left(1 - \frac{2}{3} \phi^2 \right) + \alpha \sin 2\phi \cos 2\theta \right). \end{aligned} \quad (22)$$

To ensure the transversality of SM $W_\varepsilon^s(\Pi)$ and USM $W^u(Q_\varepsilon)$ holds, we have

$$\frac{\mu}{2} \phi \omega^3 \left(1 - \frac{2}{3} \phi^2 \right) + \alpha \sin 2\phi \cos 2\theta = 0. \quad (23)$$

Thus, the persistence of the heteroclinic structure in $W^u(Q_\varepsilon) \cup W_\varepsilon^s(\Pi)$ is confirmed.

3.2. Second Measurement. According to [27–30], the second measurement, called as the geometric analysis (Figure 3(b)), is described as

$$\frac{\mu}{2} \phi \omega^3 + \alpha \sin 2\phi \cos 2\theta = 0. \quad (24)$$

Inserting equation (23) into equation (24) and using $|\cos 2\theta| \leq 1$, the chaotic threshold can be derived as

$$\frac{\alpha}{\mu} \leq \frac{1}{2\omega^2} \left| \frac{3 \sin \left(2 \arctan \sqrt{2\beta\omega^2 - \omega^2} \right)}{\omega \arctan \sqrt{2\beta\omega^2 - \omega^2} \left(3 - 2 \arctan^2 \sqrt{2\beta\omega^2 - \omega^2} \right)} \right|, \quad (25)$$

where $\beta > (1/2)$. The chaotic threshold curves, i.e., black, red, and blue curves, denote the threshold curve (25) as $\beta = 1.0$, 2.0, and 3.0 in Figure 4, respectively. The chaotic motions of the GBE may be generated as the ratio α/μ locates below the curve.

4. Numerical Simulation

To test the aforementioned chaotic threshold (25), the differential quadrature method [27–29] in MATLAB is employed using equation (1) to study the influence of the damping coefficient μ on the nonlinear dynamical behaviors

FIGURE 3: (a) The first measurement. (b) The second measurement.

FIGURE 4: The chaotic threshold curve is plotted with the parameter ratio α/μ via the parameter ω .

of the GBE with cubic nonlinearity. Letting $\alpha = 1.5$ and $\omega = 1.0$, Figure 5 displays the bifurcation diagram, where the abscissa and ordinate represent the damping coefficient $\mu \in [0, 1]$ and displacement of the GBE, respectively. The bifurcation behaviors show that the increase of damping coefficient can enhance the complicated dynamics of the GBE in comparison to the results in [27].

The chaotic wave in space (x, t, q) is presented in Figure 6 as damping coefficient $\mu = 0.3$. Figure 7 illustrates the chaotic motion for the GBE when damping coefficient $\mu = 0.3$. Figure 7(a) shows the phase trajectory on the plane $(q_7, \dot{q}_7)$, Figure 7(b) presents the time response on the plane (t, q_7) , Figure 7(c) is the phase trajectory on the plane $(q_{13}, \dot{q}_{13})$, Figure 7(d) denotes the time response on the plane

(t, q_{13}) , and Figures 7(e) and 7(f) are the phase trajectories in 3D spaces $(q_7, \dot{q}_7, q_{13})$ and $(q_{13}, \dot{q}_{13}, q_7)$, respectively.

The chaotic behavior in 3D space (x, t, q) as $\mu = 0.63$ is presented in Figure 8. Figure 9 displays the chaotic phenomenon when the damping coefficient $\mu = 0.63$. When μ is increased to 0.71, Figures 10 and 11 illustrate the chaotic motion and chaotic wave for the GBE, respectively. When the damping coefficient $\mu = 0.80$, the chaotic motion and chaotic wave are depicted, respectively, in Figures 12 and 13. The observed results from Figures 6 to 13 show that the amplitude of the GBE is larger for smaller μ . However, the higher peak value of the truss core sandwich plate system is exhibited when the damping coefficient is sufficiently small or large [27].

FIGURE 5: The bifurcation diagram of the composite laminated plate is plotted as q via the damping coefficient μ .

FIGURE 6: The chaotic wave in space (x, t, q) when damping coefficient $\mu = 0.3$.

FIGURE 7: Continued.

FIGURE 7: The chaotic motion when damping coefficient $\mu = 0.3$. (a) The phase trajectory on plane $(q_7, \dot{q}_7)$. (b) The phase trajectory on plane $(q_{13}, \dot{q}_{13})$. (c) The time response on plane (t, q_7) . (d) The time response on plane (t, q_{13}) . (e) The phase trajectory in 3D space $(q_7, \dot{q}_7, q_{13})$. (f) The phase trajectory in 3D space $(q_{13}, \dot{q}_{13}, q_7)$.

FIGURE 8: The chaotic wave in space (x, t, q) when damping coefficient $\mu = 0.63$.

FIGURE 9: The chaotic motion when damping coefficient $\mu = 0.63$.

FIGURE 10: The chaotic wave in space (x, t, q) when damping coefficient $\mu = 0.71$.

FIGURE 11: The chaotic motion when damping coefficient $\mu = 0.71$. FIGURE 12: The chaotic wave in space (x, t, q) when damping coefficient $\mu = 0.80$.

FIGURE 13: The chaotic motion when damping coefficient $\mu = 0.80$.

5. Conclusions

The global perturbation method for the infinite-dimensional system was used to discuss global dynamics for the GBE in this work. Using the RPM, an analytical wave equation was obtained by simplifying the complicated nonlinear equation of the GBE. Afterward, the persistent homoclinic structure for wave equations was constructed by using the first and second measurements. Finally, quantitative analysis was conducted to examine the theoretical results using the differential quadrature method.

The numerical studies illustrate that the damping coefficient μ is essential to determine the dynamical responses of the GBE. Different bifurcation phenomena are shown with changing damping coefficient μ . Therefore, nonlinear wave responses for the GBE can be controlled by adjusting the damping coefficient.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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