Research Article

A Nonconstant Shape Parameter-Dependent Competing Risks' Model in Accelerate Life Test Based on Adaptive Type-II Progressive Hybrid Censoring

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In this paper, the dependent competing risks' model is considered in the constant-stress accelerated life test under the adaptive type-II progressive hybrid-censored scheme. The dependency between failure causes is modeled by Marshall–Olkin bivariate Gompertz distribution. The scale and shape parameters in the model both change with the stress levels, and the failure causes of some test units are unknown. Then, the maximum likelihood estimations and approximation confidence intervals of the unknown parameters are considered. And, the necessary and sufficient condition is established for the existence and uniqueness of the maximum likelihood estimations for unknown parameters. The Bayes approach is also employed to estimate the unknown parameters under suitable prior distributions. The Bayes estimations and highest posterior credible intervals of the unknown parameters are obtained. Finally, a simulation experiment has been performed to illustrate the methods proposed in this paper.

1. Introduction

The accelerated life test (ALT) has been proposed as an effective method to obtain failure information of the highly reliable units, which frequently appears in electronics, automation, or aerospace fields, and its test units are subjected to the higher-than-use stress levels. In the ALT, it is generally assumed that the scale parameters of life distribution change with the stress levels, while the other shape parameter is constant during the test. However, some research results have shown that the shape parameters change with the stress levels. Researchers such as Li et al. [1] discovered that the shape parameter of life distribution for electrolyte changes with the increase of voltage, when electrolyte is subjected to an electric test. Hiergeist et al. [2] discovered that the shape parameter of life distribution changes with the corresponding temperature in the capacitor test.

In the competing risks' model, there are more than one failure causes, and each cause possibly leads to the final failure

of the unit. The unit operation may cause correlation among the failure causes, and failure causes may make the unit fail early. So, a failure cause is commonly associated with the other failure causes, and it is not usually possible to study the units with the isolated failure cause, and it is necessary to access each failure cause in the presence of other failure causes. To construct the dependence between failure causes, it is natural to consider a bivariate or multivariate distribution for the lifetimes. Among the multivariate distribution, the Marshall–Olkin family distributions in [3] are important bivariate distributions, and they have a parameter that describes the correlation structure between marginal distributions. Thus, they are used to describe the dependence between the failure causes. For example, Guan et al. [4] discussed the Bayes estimations for Marshall–Olkin bivariate exponential (MOBE) distribution. Shen and Xu [5] analyzed the statistical inference of dependent competing risks' model using the Marshall–Olkin bivariate Weibull (MOBW) distribution. Bai et al. [6] considered the maximum likelihood estimations (MLEs) and the

confidence intervals (CIs) for MOBW distribution under the type-I progressive interval censoring scheme. Kundu and Gupta [7] discussed the Bayes estimations (BEs) of the MOBW distribution. Feizjavadian and Hashemi [8] investigated the MOBW distribution under progressive hybrid censoring.

In the competing risks' model, the failure data of each unit is represented by a bivariate form data, which is composed of the failure time and corresponding failure cause. In the previous studies on competing risks' models, it is usually assumed that the failure cause of unit can be observed. However, in some cases, because of the high cost or the other reasons, the determination cause of failure may be difficult to obtain [9, 10]. Thus, the failure time is observed, while no corresponding failure occurs.

Censoring has become a part of the life test where the test terminates before all of the undergoing units fail. This is done to save and reduce duration of the experiment and cost incurred in the test. Among the censoring schemes, the adaptive type-II progressive hybrid censored scheme (AT-II PHCS), introduced by Ng et al. [11], is one of the popular schemes, where one can obtain effective failure numbers for the statistical inference and also speed up the test when the test reaches the given time. The AT-II PHCS can be described as follows. Suppose n identical units are placed on the life test. The observed failure number m ($m < n$) is fixed in advance, and the experiment time is allowed to run over the prefixed time τ , $\tau \in (0, \infty)$. The progressive censoring scheme $(R_1, R_2, \dots, R_m)$ is allowed to change according to the m failure times $T_{1:m:n}, T_{2:m:n}, \dots, T_{m:m:n}$. For convenience, let $T_i = T_{i:m:n}$, $i = 1, 2, \dots, m$. At the time of the first failure T_1 , R_1 units are randomly removed from the remaining $n - 1$ units. Similarly, at the second failure T_2 , R_2 units from the remaining $n - 2 - R_1$ units are randomly removed and so on. If the m th observed failure occurs before time τ (i.e., $T_m < \tau$), the test stops at time T_m , and all remaining R_m units are removed, where $R_m = n - m - \sum_{i=1}^m R_i$. Otherwise, if $T_J < \tau < T_{J+1}$, where $J + 1 < m$, and we adapt the number of units removed from the test upon failures by setting $R_{J+1} = R_{J+2} = \dots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^J R_i$. In this case, the actually applied progressive censored scheme is $(R_1, \dots, R_J, 0, \dots, 0, n - m - \sum_{i=1}^J R_i)$. Presently, some scholars discussed the AT-II PHCS, i.e., Cramer and Iliopoulos [12] derived some distributional results and showed that maximum likelihood estimators coincide with those in deterministic progressive type-II censoring. Hemmati and Khorram [13] discussed the AT-II PHCS in the presence of the competing risk. Nassar and Abo-Kasem [14] adopted the frequent and Bayse methods to obtain the estimations of the unknown parameters for inverse Weibull distribution under AT-II PHCS.

Different from the previous studies, the dependent competing risks' model using Marshall–Olkin bivariate Gompertz (MOBG) with nonconstant shape parameter in constant stress accelerated life test (CSALT) is considered under AT-IIPHCS. In addition, we assume that the failure causes of some units are unknown. The Gompertz distribution is first introduced by Gompertz [15] to describe human mortality and establish actuarial tables. This distribution is a generalization of the exponential distribution and has many real life applications, especially in medical and actuarial studies. To quote a select

few, Doblhammer [16], Perozek [17], and Willemse and Koppelaar [18] adopted this distribution to describe the adult lifespan by demographers and actuaries. Finch and Pike [19] and Ricklefs and Scheuerlein [20] used the Gompertz distribution to fit the mortality data of birds and mammals. In addition, this distribution which is used as a statistical model is fitted the tumor growth in survival analysis and reliability. Statistical inference for the Gompertz distribution has received many authors' attention, such as Wu and Shi [21] considered the BEs of the Gompertz distribution in the competing risks' model under progressively hybrid censoring. Liu et al. [22] studied the estimation for stress-strength reliability of the system with multiple types of components based on the Gompertz distribution. Mohie El-Din et al. [23] investigated the MLEs and BEs of the Gompertz distribution based on the generalized progressively hybrid censored scheme. Dey et al. [24] considered the properties and the estimation of unknown parameters of Gompertz distribution and so on.

This paper aims to investigate statistical analysis of the dependent competing risk model by using the MOBG distribution under CSALT with AT-II PHCS, where the shape and scale parameters are both changed with the stress levels. The rest of this paper is organized as follows. In Section 2, we describe the MOBG distribution and the accelerated dependent competing risks' model in which the shape and scale parameters are both changed with the stress levels and the failure causes of some units are unknown. In Section 3, MLEs and BEs of the unknown parameters are obtained based on the discrete distribution and gamma distribution. The CIs of the unknown parameters based on MLEs and BEs are considered in Section 4. The simulation results and real data analysis are showed in Section 5. Conclusions are provided in Section 6.

2. Model Description

2.1. The Marshall–Olkin Gompertz Distribution. The probability density function (PDF) and the cumulative distribution function (CDF) of the Gompertz distribution with parameters λ and α can be represented as follows, respectively:

$$f(t) = \alpha \exp(\lambda t) \exp\left\{-\frac{\alpha}{\lambda} (\exp(\lambda t) - 1)\right\}, \quad t > 0, \alpha > 0, \lambda > 0, \quad (1)$$

$$F(t) = 1 - \exp\left\{-\frac{\alpha}{\lambda} (\exp(\lambda t) - 1)\right\}, \quad t > 0, \alpha > 0, \lambda > 0. \quad (2)$$

The reliability function of Gompertz distribution can be expressed as follows:

$$R(t) = \exp\left\{-\frac{\alpha}{\lambda} (\exp(\lambda t) - 1)\right\}, \quad t > 0, \alpha > 0, \lambda > 0, \quad (3)$$

where λ is the shape parameter and α is the scale parameter. We denote the Gompertz distribution with parameters λ and α as $GP(\lambda, \alpha)$.

Suppose that the random variable U_i , $i = 0, 1$, and 2 , are independent of each other and $U_i \sim GP(\lambda, \alpha_i)$. Let $X = \min(U_0, U_1)$ and $Y = \min(U_0, U_2)$, then the bivariate

random variable (X, Y) is followed by the MOBG distribution with shape parameter λ and scale parameters α_0, α_1 , and α_2 , and it is denoted by $\text{MOBG}(\lambda, \alpha_0, \alpha_1, \alpha_2)$. Thus, the joint survival function of (X, Y) is shown as follows:

$$\begin{aligned} S_{X,Y}(x, y) &= P(X > x, Y > y), \\ &= P(U_0 > \max(x, y), U_1 > x, U_2 > y), \\ &= S(\max(x, y)|\lambda, \alpha_0)S(x|\lambda, \alpha_1)S(y|\lambda, \alpha_2), \\ &= \begin{cases} S(x|\lambda, \alpha_1)S(y|\lambda, \alpha_0 + \alpha_2), & \text{if } x < y, \\ S(x|\lambda, \alpha_0 + \alpha_1)S(y|\lambda, \alpha_2), & \text{if } x > y, \\ S(z|\lambda, \alpha_0 + \alpha_1 + \alpha_2), & \text{if } x = y = z. \end{cases} \end{aligned} \quad (4)$$

And, the joint PDF of (X, Y) can be obtained as the following expression:

$$f_{X,Y}(x, y) = \begin{cases} f_1(x, y), & \text{if } x < y, \\ f_2(x, y), & \text{if } x > y, \\ f_0(x, y), & \text{if } x = y = z, \end{cases} \quad (5)$$

where

$$\begin{aligned} f_1(x, y) &= f(x|\lambda, \alpha_1)f(y|\lambda, \alpha_0 + \alpha_2), \\ f_2(x, y) &= f(x|\lambda, \alpha_2)f(y|\lambda, \alpha_0 + \alpha_1), \\ f_0(z) &= \left(\frac{\alpha_0}{\alpha_0 + \alpha_1 + \alpha_2} \right) f(z|\lambda, \alpha_0 + \alpha_1 + \alpha_2). \end{aligned} \quad (6)$$

2.2. Model Description. In this section, the dependent competing risks' model in CSALT under AT-II PHCS is considered.

Let $S_1 < \dots < S_l$ be the accelerated stress levels and S_0 be the normal stress level. The AT-II PHCS under CSALT can be run as follows. Under the i th stress level S_i , $i = 1, 2, \dots, l$, and suppose n_i identical units are put on the life test, the observed failure number m_i ($m_i < n_i$), terminal time τ , and progressive censoring scheme $(R_{i1}, R_{i2}, \dots, R_{im_i})$ are given in advance. For the given τ , if $T_{ij} < \tau < T_{ij+1}$, where $J_i + 1 < m_i$, the removed units from the test upon failures is $R_{ij}^* = R_{ij}I(T_{ij} < \tau)$, $j = 1, 2, \dots, m_i - 1$, and $R_{im_i}^* = \sum_{j=J_i+1}^{m_i} R_{ij}$. Otherwise, $R_{ij}^* = R_{ij}$, $j = 1, 2, \dots, m_i$.

The following basic assumptions are made for the statistical inference of the dependent competing risks' model.

A1: under the stress level S_i , $i = 1, 2, \dots, l$; for the j th unit, there are two random variables X_{ij} and Y_{ij} , which are the failure time under first and second failure causes, respectively. Thus, the failure time of the unit is $T_{ij} = \min(X_{ij}, Y_{ij})$.

A2: under the stress level S_i , $i = 1, 2, \dots, l$; the pair of random variable (X_{ij}, Y_{ij}) follows a MOBG($\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}$). X_{ij} follows GP(λ_i, α_{i01}), Y_{ij} follows GP(λ_i, α_{i02}), and $T_{ij} = \min(X_{ij}, Y_{ij})$ follows GP(λ_i, α_i) where, $\alpha_{i01} = \alpha_{i0} + \alpha_{i1}$, $\alpha_{i02} = \alpha_{i0} + \alpha_{i2}$, and $\alpha_i = \alpha_{i0} + \alpha_{i1} + \alpha_{i2}$.

A3: the accelerated function of the scale parameter is log-linear:

$$\log \alpha_{ik} = a_{ik} + b_{ik}\varphi(S_i), \quad i = 1, 2, \dots, l, k = 0, 1, \text{ and } 2. \quad (7)$$

The accelerated function of the shape parameter is log-linear:

$$\log \lambda_i = a + b\varphi(S_i), \quad i = 1, 2, \dots, l, \quad (8)$$

where a, b, a_{ik} , and b_{ik} are the accelerated factors and the unknown parameters and $\varphi(S_i)$ is the known function of the stress level S_i .

Based on the model description, we can obtain the adaptive type-II progressive hybrid censored competing risks' data $(T_{i1}, c_{i1}, R_{i1}^*), (T_{i2}, c_{i2}, R_{i2}^*), \dots, (T_{im_i}, c_{im_i}, R_{im_i}^*)$, where $c_{ij} \in \{0, 1, 2, *\}$ is failure cause of the i th unit, and it is defined as follows:

$$c_{ij} = \begin{cases} 0, & \text{if } X_{ij} = Y_{ij}, \\ 1, & \text{if } X_{ij} < Y_{ij}, \\ 2, & \text{if } X_{ij} > Y_{ij}, \end{cases} \quad j = 1, 2, \dots, m_i, \quad (9)$$

while $c_{ij} = *$ indicates that the failure cause is uncertain.

Under the stress level S_i , $i = 1, 2, \dots, l$, let $m_{ik} = \sum_{j=1}^{m_i} \delta_k(c_{ij})$, $k = 0, 1, 2$ represent the failure numbers of those caused by both risks, first risk and second risk, where $\delta_k(\cdot)$, $k = 0, 1$, and 2 , is the indicator function; $\delta_k(c_{ij}) = 1$, if $c_{ij} = k$; $\delta_k(c_{ij}) = 0$, if $c_{ij} \neq k$. In addition, $m_{i3} = \sum_{j=1}^{m_i} \delta(c_{ij} = *)$ represents the numbers of the failure units which have the failure time, while not having the corresponding failure cause. Let $m_{i0} + m_{i1} + m_{i2} = m_i^*$ and $m_i^* + m_{i3} = m_i$, and we can obtain the following result.

Theorem 1. Under the stress level S_i , $i = 1, 2, \dots, l$,

$$(m_{i0}, m_{i1}, m_{i2}) \sim \text{multinomial}\left(m_i^*, \frac{\alpha_{i0}}{\alpha_i}, \frac{\alpha_{i1}}{\alpha_i}, \frac{\alpha_{i2}}{\alpha_i}\right), \quad (10)$$

where $\alpha_i = \alpha_{i0} + \alpha_{i1} + \alpha_{i2}$.

Proof. Under the stress level S_i , $i = 1, 2, \dots, l$, each unit failure is caused by one of $X_{ij} > Y_{ij}$, $X_{ij} < Y_{ij}$, or $X_{ij} = Y_{ij}$. Probability of each case is

$$P(X_{ij} < Y_{ij}) = \int_0^{+\infty} \int_0^{y_{ij}} f_1(x_{ij}, y_{ij}) dx_{ij} dy_{ij} = \frac{\alpha_{i1}}{\alpha_i}, \quad (11)$$

$$P(Y_{ij} < X_{ij}) = \int_0^{+\infty} \int_0^{x_{ij}} f_1(x_{ij}, y_{ij}) dy_{ij} dx_{ij} = \frac{\alpha_{i2}}{\alpha_i}.$$

For the given m_i^*, m_{i0}, m_{i1} , and m_{i2} , we can obtain $P(Y_{ij} = X_{ij}) = \alpha_{i0}/\alpha_i$ by

$$\begin{aligned} & \int_0^{+\infty} \int_0^{y_{ij}} f_1(x_{ij}, y_{ij}) dx_{ij} dy_{ij} \\ & + \int_0^{+\infty} \int_0^{x_{ij}} f_2(x_{ij}, y_{ij}) dy_{ij} dx_{ij} + \int_0^{+\infty} f_0(x) dx = 1. \end{aligned} \quad (12)$$

Then, the theorem is completed. $\square$

3. Point Estimations

3.1. Maximum Likelihood Estimations. Let $\mathbf{T} = (T_{ij}, i = 1, 2, \dots, l, j = 1, 2, \dots, m_i)$ is the censored sample in CSALT and $\mathbf{t} = (t_{ij}, i = 1, 2, \dots, l, j = 1, 2, \dots, m_i)$ is the sample

observation. Let $\theta = (\lambda_1, \dots, \lambda_l, a_{10}, a_{11}, a_{12}, \dots, a_{l0}, a_{l1}, a_{l2})$; under the stress level $S_i, i = 1, 2, \dots, l$, based on the dependent competing risks' data, the likelihood function can be expressed as follows:

$$l_i(t|\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \propto \prod_{j=1}^{m_i} (f_0(t_{ij}, t_{ij}))^{\delta_0(c_{ij})} \left[\frac{-S_{X,Y}(x, y)}{\partial x} \Big|_{(t_{ij}, t_{ij})} \right]^{\delta_1(c_{ij})} \left[\frac{-S_{X,Y}(x, y)}{\partial y} \Big|_{(t_{ij}, t_{ij})} \right]^{\delta_2(c_{ij})} \cdot \left(f_0(t_{ij}, t_{ij}) + \frac{-S_{X,Y}(x, y)}{\partial x} \Big|_{(t_{ij}, t_{ij})} + \frac{-S_{X,Y}(x, y)}{\partial y} \Big|_{(t_{ij}, t_{ij})} \right)^{\delta(c_{ij}=*)} \left(S_{X,Y}(x, y) \Big|_{(t_{ij}, t_{ij})} \right)^{R_{ij}^*}, \quad (13)$$

where

$$\begin{aligned} f_0(t_{ij}, t_{ij}) &= \alpha_{i0} \exp(\lambda_i t_{ij}) \exp \left\{ -\frac{(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})}{\lambda_i} (\exp(\lambda_i t_{ij}) - 1) \right\}, \\ \frac{-S_{X,Y}(x, y)}{\partial x} \Big|_{(t_{ij}, t_{ij})} &= \alpha_{i1} \exp(\lambda_i t_{ij}) \exp \left\{ -\frac{(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})}{\lambda_i} (\exp(\lambda_i t_{ij}) - 1) \right\}, \\ \frac{-S_{X,Y}(x, y)}{\partial y} \Big|_{(t_{ij}, t_{ij})} &= \alpha_{i2} \exp(\lambda_i t_{ij}) \exp \left\{ -\frac{(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})}{\lambda_i} (\exp(\lambda_i t_{ij}) - 1) \right\}, \\ -S_{X,Y}(x, y) \Big|_{(t_{ij}, t_{ij})} &= \exp \left\{ -\frac{(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})}{\lambda_i} (\exp(\lambda_i t_{ij}) - 1) \right\}. \end{aligned} \quad (14)$$

So, we can obtain

$$l_i(t|\lambda, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \propto \alpha_{i0}^{m_{i0}} \alpha_{i1}^{m_{i1}} \alpha_{i2}^{m_{i2}} \alpha_i^{m_i - m_i^*} \cdot \exp \left\{ \lambda_i \sum_{j=1}^{m_i} t_{ij} - \lambda_i^{-1} \alpha_i A_i \right\}, \quad (15)$$

where $A_i = \sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1)$.

The full likelihood function is

$$L = \prod_{i=1}^l l_i. \quad (16)$$

Based on equation (16), the log-likelihood function can be transformed into the following result:

$$\ln L = \sum_{i=1}^l \left\{ \left(\sum_{k=0}^2 m_{ik} \ln \alpha_{ik} \right) + (m_i - m_i^*) \ln \alpha_i + \lambda_i \sum_{j=1}^{m_i} t_{ij} - \lambda_i^{-1} \alpha_i A_i \right\}. \quad (17)$$

The MLEs of unknown parameters can be obtained by the maximum function in equation (17). By setting the first partial

derivative of $\ln L$ with respect to λ_i and $\alpha_{ik}, k = 0, 1$, and 2, to zero, we can obtain following results:

$$\frac{\partial \ln L}{\partial \alpha_{ik}} = \frac{m_{ik}}{\alpha_{ik}} + \frac{m_i - m_i^*}{\alpha_{i0} + \alpha_{i1} + \alpha_{i2}} - \frac{1}{\lambda_i} A_i = 0, \quad k = 0, 1, \text{ and } 2, \quad (18)$$

$$\frac{\partial \ln L}{\partial \lambda_i} = \sum_{j=1}^{m_i} t_{ij} + \lambda_i^{-2} \alpha_i A_i - \lambda_i^{-1} \alpha_i \sum_{j=1}^{m_i} t_{ij} (R_{ij}^* + 1) \exp(\lambda_i t_{ij}) = 0. \quad (19)$$

By solving equation (18), we can get the MLE of α_{ik} , $k = 0, 1$, and 2 , as the following form:

$$\hat{\alpha}_{ik} = \frac{m_{ik} m_i \lambda_i}{m_i^* A_i}, \quad k = 0, 1, \text{ and } 2. \quad (20)$$

Substituting $\hat{\alpha}_{ik}$, $k = 0, 1$, and 2 , into equation (19), we can obtain the equation for λ_i :

$$H(\lambda_i) = \sum_{j=1}^{m_i} t_{ij} + \lambda_i^{-1} m_i - \frac{m_i \sum_{j=1}^{m_i} t_{ij} (R_{ij}^* + 1) \exp(\lambda_i t_{ij})}{A_i} = 0. \quad (21)$$

Theorem 2. Based on the censored data $t_{i1} \leq t_{i2} \leq \dots \leq t_{ij} \leq \dots \leq t_{im_i}$ and the progressive scheme R_{ij}^* , $j = 1, 2, \dots, m_i$, the MLEs $\hat{\lambda}_i$ and $\hat{\alpha}_{ik}$, $k = 0, 1$, and 2 , of the parameters λ_i and α_{ik} , $k = 0, 1$, and 2 , exist and are unique if and only if

$$2 \sum_{j=1}^{m_i} t_{ij} \sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij} > m_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij}^2. \quad (22)$$

Proof. From equations (20) and (21), we can obtain that $\hat{\lambda}_i$ and $\hat{\alpha}_{ik}$, $k = 0, 1$, and 2 , exist and are unique by showing that $\hat{\lambda}_i$ exists and is unique if and only if condition (22) holds. The derivative of $H(\lambda_i)$ can be expressed as follows:

$$H'(\lambda_i) = -\frac{m_i}{\lambda_i^2} - \left[\frac{m_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(\lambda_i t_{ij}) t_{ij}^2}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1)} - \frac{m_i (\sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(\lambda_i t_{ij}) t_{ij})^2}{(\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1))^2} \right]. \quad (23)$$

Let $z_j = z_j(\lambda_i) = \lambda_i t_{ij}$, we have

$$\begin{aligned} H'(\lambda_i) &= -\frac{m_i}{\lambda_i^2} \left[1 + \frac{\sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(z_j) z_j^2}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(z_j) - 1)} - \frac{(\sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(z_j) z_j)^2}{(\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(z_j) - 1))^2} \right], \\ &= -\frac{m_i}{\lambda_i^2} \left[\frac{H_1(\lambda_i)}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(z_j) - 1)} + \frac{H_2(\lambda_i)}{(\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(z_j) - 1))^2} \right], \end{aligned} \quad (24)$$

where

$$\begin{aligned} H_1(\lambda_i) &= \sum_{j=1}^{m_i} (R_{ij}^* + 1) \frac{1}{(\exp(z_j) - 1)} \left[(\exp(z_j) - 1)^2 - z_j^2 \exp(z_j) \right], \\ H_2(\lambda_i) &= \sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(z_j) - 1) \sum_{j=1}^{m_i} (R_{ij}^* + 1) \frac{(z_j \exp(z_j))^2}{(\exp(z_j) - 1)} - \left(\sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(z_j) z_j \right)^2. \end{aligned} \quad (25)$$

Since $(\exp(z_j) - 1)^2 > z_j^2 \exp(z_j)$ for all $z_j > 0$, we have $H_1(\lambda_i) > 0$.

Now, let

$$\begin{aligned} h_j &= [(R_{ij}^* + 1) (\exp(z_j) - 1)]^{1/2}, \\ g_j &= \left[\frac{(R_{ij}^* + 1)}{(\exp(z_j) - 1)} \right]^{1/2} z_j \exp(z_j). \end{aligned} \quad (26)$$

Then, we have

$$H_2(\lambda_i) = \sum_{j=1}^{m_i} h_j^2 \sum_{j=1}^{m_i} g_j^2 - \left(\sum_{j=1}^{m_i} h_j g_j \right)^2. \quad (27)$$

By using the Cauchy-Schwarz inequality, we can obtain $H_2(\lambda_i) \geq 0$. Then, it follows that $H'(\lambda_i) < 0$ which implies that $H(\lambda_i)$ is a decrease function. Therefore, equation (21) has the unique root $\hat{\lambda}_i \in (0, \infty)$ if and only if $H(0) > 0$ and $H(\infty) < 0$:

$$\begin{aligned}
H(0) &= \sum_{j=1}^{m_i} t_{ij} + m_i \lim_{\lambda_i \rightarrow 0} \left[\frac{1}{\lambda_i} - \frac{\sum_{j=1}^{m_i} t_{ij} (R_{ij}^* + 1) \exp(\lambda_i t_{ij})}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1)} \right], \\
&= \sum_{j=1}^{m_i} t_{ij} - \frac{m_i}{2} \frac{\sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij}^2}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij}}, \\
&= \frac{2 \sum_{j=1}^{m_i} t_{ij} \sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij} - m_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij}^2}{2 \sum_{j=1}^{m_i} (R_{ij}^* + 1) t_{ij}}.
\end{aligned} \tag{28}$$

If and only if equation (22) is satisfied, $H(0) > 0$:

$$\begin{aligned}
H(\infty) &= \sum_{j=1}^{m_i} t_{ij} - m_i \lim_{\lambda_i \rightarrow \infty} \frac{\sum_{j=1}^{m_i} t_{ij} (R_{ij}^* + 1) \exp(\lambda_i t_{ij})}{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1)}, \\
&= \sum_{j=1}^{m_i} t_{ij} - m_i \lim_{\lambda_i \rightarrow \infty} \frac{\sum_{j=1}^{m_i} t_{ij}^2 (R_{ij}^* + 1) \exp \lambda_i (t_{ij} - t_{im_i})}{\sum_{j=1}^{m_i} t_{ij} (R_{ij}^* + 1) \exp \lambda_i (t_{ij} - t_{im_i})}, \\
&= \sum_{j=1}^{m_i} (t_{ij} - t_{im_i}) < 0.
\end{aligned} \tag{29}$$

Then, the theorem is completed. $\square$

Theorem 3. Using the MLE of the parameter $\hat{\alpha}_{ik}$, $k = 0, 1$, and 2, we can get the MLE $\hat{\vartheta} = \hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}$ of the parameter $\vartheta = \alpha_{i0} + \alpha_{i1} + \alpha_{i2}$ having the following properties:

- (1) $\hat{\vartheta}$ is the asymptotic unbiased estimator of ϑ
- (2) $\hat{\vartheta}$ is a consistent estimator of ϑ
- (3) $\hat{\vartheta}$ is a sufficient statistics of ϑ

Proof. From A2, T_{ij} follows the Gompertz distribution with the shape parameter λ_i and scale parameter α_i . By the variable transformed method, the random variable $Y_{ij} = (\exp(\lambda_i T_{ij}) - 1)/\lambda_i$ follows the exponential distribution with mean $1/\alpha_i$.

Let

$$\begin{aligned}
Z_{i1} &= n_i Y_{i1}, \\
Z_{i2} &= (n_i - R_{i1}^* - 1)(Y_{i2} - Y_{i1}), \\
&\dots \\
Z_{im_i} &= (n_i - R_{i1}^* - \dots - R_{im_i}^* - m_i - 1)(Y_{im_i} - Y_{i, m_i-1}).
\end{aligned} \tag{30}$$

We can obtain that $Z_{i1}, Z_{i2}, \dots, Z_{im_i}$ are independent and identical distribution as the exponential distribution with mean $1/\alpha_i$. According to the relationship between the exponential distribution and gamma distribution, we have

$$Z_{ij} \sim E(\alpha_i) = \alpha_i e^{-\alpha_i z_{ij}} = \frac{\alpha_i^1}{\Gamma(1)} (z_{ij})^{1-1} e^{-\alpha_i z_{ij}} \sim \text{Gamma}(1, \alpha_i). \tag{31}$$

Let $W_i = Z_{i1} + Z_{i2} + \dots + Z_{im_i}$, then $W_i \sim \text{Gamma}(m_i, \alpha_i)$:

$$W_i = Z_{i1} + Z_{i2} + \dots + Z_{im_i} = \frac{\sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\lambda_i t_{ij}) - 1) / \lambda_i}{\lambda_i} = A_i. \tag{32}$$

Then, λ_i/A_i follows the inverse gamma distribution with the parameters m_i and α_i , and it is denoted by $\lambda_i/A_i \sim \text{IGa}(m_i, \alpha_i)$.

Using equation (20),

$$E(\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}) = E\left(\frac{m_i \lambda_i}{A_i}\right) = \frac{(m_i (\alpha_{i0} + \alpha_{i1} + \alpha_{i2}))}{(m_i - 1)}. \tag{33}$$

Then, we have

$$\lim_{m_i \rightarrow \infty} E(\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}) = \alpha_{i0} + \alpha_{i1} + \alpha_{i2}. \tag{34}$$

The proof of statement 1 is completed.

Using equation (20), we can obtain

$$\text{Var}(\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}) = E\left(\frac{m_i \lambda_i}{A_i}\right)^2 = \frac{m_i^2 (\alpha_{i0} + \alpha_{i1} + \alpha_{i2})^2}{((m_i - 1)(m_i - 2))}. \tag{35}$$

Then,

$$\begin{aligned}
\lim_{m_i \rightarrow \infty} \text{Var}(\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}) &= \frac{(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})^2 \lim_{m_i \rightarrow \infty} m_i^2}{((m_i - 1)^2 (m_i - 2))} \\
&= 0.
\end{aligned} \tag{36}$$

The proof of statement 2 is completed.

Finally, under the stress level S_i , $i = 1, 2, \dots, l$, the likelihood function can be written as follows:

$$l_i \propto \alpha_{i0}^{m_{i0}} \alpha_{i1}^{m_{i1}} \alpha_{i2}^{m_{i2}} \alpha_i^{m_i - m_i^*} \exp \left\{ \lambda_i \sum_{j=1}^{m_i} t_{ij} - \lambda_i^{-1} (\alpha_{i0} + \alpha_{i1} + \alpha_{i2}) A_i \right\}. \tag{37}$$

Based on equation (20), we can obtain

$$A_i = \frac{\lambda_i m_i}{(\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2})}. \tag{38}$$

Then, the likelihood function can be written as follows:

$$\begin{aligned}
l_i &\propto \alpha_{i0}^{m_{i0}} \alpha_{i1}^{m_{i1}} \alpha_{i2}^{m_{i2}} \alpha_i^{m_i - m_i^*} \exp \left\{ \lambda_i \sum_{j=1}^{m_i} t_{ij} - m_i (\alpha_{i0} + \alpha_{i1} + \alpha_{i2}) / (\hat{\alpha}_{i0} + \hat{\alpha}_{i1} + \hat{\alpha}_{i2}) \right\} \\
&\propto G(t_{i1}, \dots, t_{im_i}) H(\alpha_{i0}, \alpha_{i1}, \alpha_{i2}, \hat{\alpha}_{i0}, \hat{\alpha}_{i1}, \hat{\alpha}_{i2}).
\end{aligned} \tag{39}$$

The likelihood function can be decomposed into two functions $G(t_{i1}, \dots, t_{im_i})$ and $H(\alpha_{i0}, \alpha_{i1}, \alpha_{i2}, \hat{\alpha}_{i0}, \hat{\alpha}_{i1}, \hat{\alpha}_{i2})$, where G is only dependent on the data and H is dependent on the parameters. This completes the proof of statement 3 and the theorem. $\square$

3.2. The Bayes Estimations. We assume that the parameters λ_i , α_{i0} , α_{i1} , and α_{i2} are independent of each other. According to Wu and Shi [21], for the parameter λ_i , we select a finite number of values on the interval $(0, \infty)$, which is denoted by $\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ir}$. The PDF of the parameters λ_i is $\pi(\lambda_i = \lambda_{ir}) = \pi_{ir}$, where $\sum_{r=1}^q \pi_{ir} = 1$, $0 \leq \pi_{ir} \leq 1$, $i = 1, 2, \dots, l$.

Based on equation (16), we can obtain the conditional probability density function of parameter α_{ik} , $k = 0, 1$, and 2:

$$\begin{aligned} p(\alpha_{ik}|\lambda) &\propto \alpha_{ik}^{m_{ik}} \alpha_i^{m_i - m_i^*} \exp\{\lambda_i^{-1} \alpha_{ik} A_i\} \\ &\propto \sum_{q=1}^{m_i - m_i^*} \alpha_{ik}^{m_{ik} + q} \exp\{\lambda_i^{-1} \alpha_{ik} A_i\}. \end{aligned} \quad (40)$$

When $m_i - m_i^* = 0$, the conditional probability density function of parameter α_{ik} , $k = 0, 1$, and 2, can be expressed as follows:

$$(\alpha_{i0} + \alpha_{i1} + \alpha_{i2})^{m_i - m_i^*} = \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \alpha_{i0}^u \alpha_{i1}^v \alpha_{i2}^{m_i - m_i^* - u - v}. \quad (44)$$

The likelihood function in equation (15) can be written as the following form:

$$\begin{aligned} l_i(t|\lambda, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \\ \propto \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \alpha_{i0}^{m_{i0} + u} \alpha_{i1}^{m_{i1} + v} \alpha_{i2}^{m_{i2} + m_i - m_i^* - u - v} \exp\left\{\lambda_i \sum_{j=1}^{m_i} t_{ij} - \lambda_i^{-1} \alpha_i A_i\right\}. \end{aligned} \quad (45)$$

Then, the condition posterior density function of the parameter α_{ik} , $k = 0, 1$, and 2, can be obtained as follows:

$$\begin{aligned} \pi(\alpha_{i0}|\lambda_i = \lambda_{ir}, t) &= B_{ir}^{-1} \exp(-\alpha_{i0}(\lambda_{ir} A_{ir} + d_{ir0})) \\ &\times \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \frac{\Gamma(m_{i1} + c_{ir1} + v)}{(\lambda_{ir} A_{ir} + d_{ir1})^{(m_{i1} + c_{ir1} + v)}} \right. \\ &\times \left. \frac{\Gamma(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}{(\lambda_{ir} A_{ir} + d_{ir2})^{(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}} \alpha_{i0}^{m_{i0} + c_{ir0} + u - 1} \right\}, \end{aligned} \quad (46)$$

$$p(\alpha_{ik}|\lambda) \propto \alpha_{ik}^{m_{ik}} \exp\{\lambda_i^{-1} \alpha_{ik} A_i\}. \quad (41)$$

Then, we can obtain that $p(\alpha_{ik}|\lambda)$ belongs to the gamma distribution family.

Under the condition $\lambda_i = \lambda_{ir}$, the condition prior distribution of the parameter α_{ik} , $k = 0, 1$, and 2, is Gamma (a_{ikr}, b_{ikr}) , which is the conjugate distribution for the parameter α_{ik} , $k = 0, 1$, and 2. The PDF can be expressed as follows:

$$\pi(\alpha_{ik}|\lambda_i = \lambda_{ir}) = \frac{d_{ikr}^{c_{ikr}} \alpha_{ik}^{a_{ikr} - 1} \exp(-d_{ikr} \alpha_{ik})}{\Gamma(c_{ikr})}. \quad (42)$$

The condition posterior density function of the parameter α_{ik} , $k = 0, 1$, and 2, is

$$\pi(\alpha_{ik}|t, \lambda_i = \lambda_{ir}) = \frac{l_i(\theta) \pi(\alpha_{ik}|\lambda_i = \lambda_{ir}) \pi(\lambda_i = \lambda_{ir})}{\int l_i(\theta) \pi(\alpha_{ik}|\lambda_i = \lambda_{ir}) \pi(\lambda_i = \lambda_{ir}) d\alpha_{ik}}. \quad (43)$$

Using the binomial expansion,

$$\begin{aligned} \pi(\alpha_{i1}|\lambda_i = \lambda_{ir}, t) &= B_{ir}^{-1} \exp(-\alpha_{i1}(\lambda_{ir}A_{ir} + d_{ir1})) \\ &\times \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \frac{\Gamma(m_{i0} + c_{ir0} + u)}{(\lambda_{ir}A_{ir} + d_{ir0})^{(m_{i0} + a_{i0} + u)}} \right. \\ &\times \left. \frac{\Gamma(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}{(\lambda_{ir}A_{ir} + d_{ir2})^{(m_{i2} + a_{i2} + m_i - m_i^* - u - v)}} \alpha_{i1}^{m_{i0} + c_{ir0} + v - 1} \right\}, \end{aligned} \quad (47)$$

$$\begin{aligned} \pi(\alpha_{i2}|t, \lambda_i = \lambda_{ir}) &= B_{ir}^{-1} \exp(-\alpha_{i2}(\lambda_{ir}A_{ir} + b_{ir2})) \\ &\times \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \frac{\Gamma(m_{i0} + a_{ir0} + u)}{(\lambda_{ir}A_{ir} + b_{ir0})^{(m_{i0} + a_{ir0} + u)}} \right. \\ &\times \left. \frac{\Gamma(m_{i1} + a_{ir1} + v)}{(\lambda_{ir}A_{ir} + b_{ir1})^{(m_{i1} + a_{ir1} + v)}} \alpha_{i2}^{m_{i2} + a_{ir2} + m_i - m_i^* - u - v} \right\}, \end{aligned} \quad (48)$$

where $A_{ir} = \sum_{j=1}^{m_i} (R_{ij}^* + 1)(\exp(\lambda_{ir}t_{ij}) - 1)$.

The marginal posterior density distribution of the parameter λ_i , $i = 1, 2, \dots, l$, can be obtained as follows:

$$\begin{aligned} P_{ir} &= P(\lambda_i = \lambda_{ir}) = \frac{\int \int \int \prod_{j=1}^{m_i} l_i(t|\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \pi(\alpha_{i0}, \alpha_{i1}, \alpha_{i2}|\lambda_i = \lambda_{ir}) \pi(\lambda_i = \lambda_{ir}) d\alpha_{i0} d\alpha_{i1} d\alpha_{i2}}{\sum_{r=1}^{q_i} \int \int \int \prod_{j=1}^{m_i} l_i(t|\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \pi(\alpha_{i0}, \alpha_{i1}, \alpha_{i2}|\lambda_i = \lambda_{ir}) \pi(\lambda_i = \lambda_{ir}) d\alpha_{i0} d\alpha_{i1} d\alpha_{i2}}, \\ &= \frac{\pi_{ir} \prod_{i=1}^l (\prod_{k=0}^2 d_{irk}^{a_{irk}} / \Gamma(c_{irk})) B_{ir}}{\sum_{r=1}^{q_i} \pi_{ir} \prod_{i=1}^l (\prod_{k=0}^2 d_{irk}^{a_{irk}} / \Gamma(c_{irk})) B_{ir}}, \end{aligned} \quad (49)$$

where

$$\begin{aligned} B_{ir} &= \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \frac{\Gamma(m_{i0} + c_{ir0} + u)}{(\lambda_{ir}A_{ir} + d_{ir0})^{(m_{i0} + a_{ir0} + u)}} \right. \\ &\times \left. \frac{\Gamma(m_{i1} + c_{ir1} + v)}{(\lambda_{ir}A_{ir} + d_{ir1})^{(m_{i1} + c_{ir1} + v)}} \frac{\Gamma(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}{(\lambda_{ir}A_{ir} + d_{ir2})^{(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}} \right\}. \end{aligned} \quad (50)$$

Under the squared error loss function, the BEs of λ_i , α_{i0} , α_{i1} , and α_{i2} can be written as follows:

$$\begin{aligned} \hat{\lambda}_{i(BS)} &= E^{\lambda_i|t}(\lambda_i) = \sum_{r=1}^{q_i} P_{ir} \lambda_{ir}, \\ \hat{\alpha}_{i0(BS)} &= E^{\alpha_{i0}|\lambda_i, t}(\alpha_{i0}) = \sum_{r=1}^{q_i} P_{ir} E(\alpha_{i0}) = \sum_{r=1}^{q_i} P_{ir} \frac{1}{B_{ir}} \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \frac{\Gamma(m_{i0} + c_{ir0} + u + 1)}{(\lambda_{ir}A_{ir} + d_{ir0})^{(m_{i0} + c_{ir0} + u + 1)}} \right. \\ &\times \left. \frac{\Gamma(m_{i1} + c_{ir1} + v)}{(\lambda_{ir}A_{ir} + d_{ir1})^{(m_{i1} + c_{ir1} + v)}} \frac{\Gamma(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}{(\lambda_{ir}A_{ir} + d_{ir2})^{(m_{i2} + c_{ir2} + m_i - m_i^* - u - v)}} \right\}, \end{aligned}$$

$$\begin{aligned}
\hat{\alpha}_{i1(BS)} &= E^{\alpha_{i1}|\lambda,t}(\alpha_{i1}) = \sum_{r=1}^{q_i} P_{ir} E(\alpha_{i1}) = \sum_{r=1}^{q_i} P_{ir} \frac{1}{B_{ir}} \left\{ \sum_{u=0}^{m_i-m_i^*} \binom{m_i-m_i^*}{u} \sum_{v=0}^{m_i-m_i^*-u} \binom{m_i-m_i^*-u}{v} \frac{\Gamma(m_{i0}+c_{ir0}+u)}{(\lambda_{ir}A_{ri}+d_{ir0})^{(m_{i0}+c_{ir0}+u)}} \right. \\
&\quad \times \left. \frac{\Gamma(m_{i1}+c_{ir1}+v+1)}{(\lambda_{ir}A_{ri}+d_{ir1})^{(m_{i1}+c_{ir1}+v+1)}} \frac{\Gamma(m_{i2}+c_{ir2}+m_i-m_i^*-u-v)}{(\lambda_{ir}A_{ir}+d_{ir2})^{(m_{i2}+c_{ir2}+m_i-m_i^*-u-v)}} \right\}, \\
\hat{\alpha}_{i2(BS)} &= E^{\alpha_{i2}|\lambda,t}(\alpha_{i2}) = \sum_{r=1}^{q_i} P_{ir} E(\alpha_{i2}) = \sum_{r=1}^{q_i} P_{ir} \frac{1}{B_{ir}} \left\{ \sum_{u=0}^{m_i-m_i^*} \binom{m_i-m_i^*}{u} \sum_{v=0}^{m_i-m_i^*-u} \binom{m_i-m_i^*-u}{v} \frac{\Gamma(m_{i0}+c_{ir0}+u)}{(\lambda_{ir}A_{ri}+d_{ir0})^{(m_{i0}+c_{ir0}+u)}} \right. \\
&\quad \times \left. \frac{\Gamma(m_{i1}+c_{ir1}+v)}{(\lambda_{ir}A_{ri}+d_{ir1})^{(m_{i1}+c_{ir1}+v)}} \frac{\Gamma(m_{i2}+c_{ir2}+m_i-m_i^*-u-v+1)}{(\lambda_{ir}A_{ir}+d_{ir2})^{(m_{i2}+c_{ir2}+m_i-m_i^*-u-v+1)}} \right\}.
\end{aligned} \tag{51}$$

3.3. *The Estimation of Reliability.* The reliability estimation of the unit at any time t under the normal stress level is

$$\hat{R}(t) = \exp \left\{ -\frac{(\hat{\alpha}_{00} + \hat{\alpha}_{01} + \hat{\alpha}_{02})}{\hat{\lambda}_0} (\exp(\hat{\lambda}_0 t) - 1) \right\}. \tag{52}$$

The estimation of the shape parameter λ_0 under normal stress level S_0 can be obtained from the equation

$\log \lambda_0 = a + b\varphi(S_0)$, $i = 1, 2, \dots, l$. And, the estimation of the scale parameter α_{0k} under normal stress level S_0 can be obtained from the equation $\log \alpha_{0k} = a_{0k} + b_{0k}\varphi(S_0)$, $i = 1, 2, \dots, l, k = 0, 1$, and 2. Putting the estimations of the parameters $\hat{\lambda}_i$ and $\hat{\alpha}_{ik}$, $i = 1, 2, \dots, l, k = 0, 1$, and 2, into equations (5) and (6), we can obtain the estimations of a, b, a_{0k} , and b_{0k} , $k = 0, 1$, and 2, by the least square method:

$$\begin{aligned}
\hat{a} &= \frac{\sum_{i=1}^l \ln \hat{\lambda}_i - \hat{b} \sum_{i=1}^l \varphi(S_i)}{l}, \quad \hat{b} = \frac{l \sum_{i=1}^l \ln \hat{\lambda}_i \varphi(S_i) - \sum_{i=1}^l \ln \hat{\lambda}_i \sum_{i=1}^l \varphi(S_i)}{l \sum_{i=1}^l \varphi^2(S_i) - \left(\sum_{i=1}^l \varphi(S_i) \right)^2}, \\
\hat{a}_{ik} &= \frac{\sum_{i=1}^l \ln \hat{\alpha}_{ik} - \hat{b}_{ik} \sum_{i=1}^l \varphi(S_i)}{l}, \quad \hat{b}_{ik} = \frac{l \sum_{i=1}^l \varphi(S_i) \ln \hat{\alpha}_{ik} - \sum_{i=1}^l \ln \hat{\alpha}_{ik} \sum_{i=1}^l \varphi(S_i)}{l \sum_{i=1}^l \varphi^2(S_i) - \left(\sum_{i=1}^l \varphi(S_i) \right)^2}.
\end{aligned} \tag{53}$$

4. Interval Estimation

4.1. *Approximation Confidence Intervals.* For the parameters λ_i and α_{ik} , $i = 1, 2, \dots, l, k = 0, 1$, and 2, it is impossible to get the exact confidence intervals. So, we use the asymptotic property of the MLEs to construct the approximation confidence intervals (ACIs), and the observed Fisher information matrix of the parameters λ_i and α_{ik} , $i = 1, 2, \dots, l, k = 0, 1$, and 2 is denoted by

$$\hat{I}(\theta) = \text{diag}(\hat{I}^1, \hat{I}^2, \dots, \hat{I}^l), \tag{54}$$

where the submatrix can be expressed as follows:

$$\hat{I}^i(\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) = \begin{bmatrix} \hat{I}_{11}^i & \hat{I}_{12}^i & \hat{I}_{13}^i & \hat{I}_{14}^i \\ \hat{I}_{21}^i & \hat{I}_{22}^i & \hat{I}_{23}^i & \hat{I}_{24}^i \\ \hat{I}_{31}^i & \hat{I}_{32}^i & \hat{I}_{33}^i & \hat{I}_{34}^i \\ \hat{I}_{41}^i & \hat{I}_{42}^i & \hat{I}_{43}^i & \hat{I}_{44}^i \end{bmatrix}, \tag{55}$$

where

$$\begin{aligned}
\hat{I}_{11}^i &= -\frac{\partial^2 \ln L}{\partial \lambda_i^2} \Big|_{\lambda_i=\hat{\lambda}_i, \alpha_i=\hat{\alpha}_i} = -\hat{\lambda}_i^{-1} \hat{\alpha}_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(\hat{\lambda}_i t_{ij}) t_{ij}^2 + 2\hat{\lambda}_i^{-2} \hat{\alpha}_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(\hat{\lambda}_i t_{ij}) t_{ij} \\
&\quad - 2\hat{\lambda}_i^{-3} \hat{\alpha}_i \sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\hat{\lambda}_i t_{ij}) - 1),
\end{aligned}$$

$$\hat{I}_{1(k+2)}^i = \hat{I}_{(k+2)1}^i = -\frac{\partial^2 \ln L}{\partial \lambda_i \partial \alpha_{ik}} \Big|_{\lambda_i=\hat{\lambda}_i, \alpha_{ik}=\hat{\alpha}_{ik}} = -\hat{\lambda}_i^{-1} \sum_{j=1}^{m_i} (R_{ij}^* + 1) \exp(\hat{\lambda}_i t_{ij}) t_{ij} - \hat{\lambda}_i^{-2} \sum_{j=1}^{m_i} (R_{ij}^* + 1) (\exp(\hat{\lambda}_i t_{ij}) - 1),$$

$$\begin{aligned}\hat{I}_{(k+2)(k+2)}^i &= -\frac{\partial^2 \ln L}{\partial \alpha_{ik}^2} \Big|_{\alpha_{ik}=\hat{\alpha}_{ik}} = \frac{-m_{ik}}{\hat{a}_{ik}^2} + \frac{-(m_i - m_i^*)}{\hat{a}_i^2}, \\ \hat{I}_{23}^i &= \hat{I}_{32}^i = \hat{I}_{24}^i = \hat{I}_{42}^i = \hat{I}_{34}^i = \hat{I}_{43}^i = -\frac{\partial^2 \ln L}{\partial \alpha_{ik} \partial \alpha_{ih}} \Big|_{\alpha_{ik}=\hat{\alpha}_{ik}, \alpha_{ih}=\hat{\alpha}_{ih}} = -\frac{(m_i - m_i^*)}{\hat{a}_i^2}, \quad k \text{ and } h = 0, 1, \text{ and } 2, k \neq h.\end{aligned}\quad (56)$$

The variance-covariance matrix of the parameters λ_i and α_{ik} , $i = 1, 2, \dots, l$, $k = 0, 1$, and 2 , can be obtained as follows:

$$\hat{V}(\theta) = (\hat{I}(\theta))^{-1} = \text{diag}\left(\left(\hat{I}^1\right)^{-1}, \left(\hat{I}^2\right)^{-1}, \dots, \left(\hat{I}^l\right)^{-1}\right). \quad (57)$$

Let

$$\begin{aligned}\hat{V}^i(\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}) &= \left(\hat{I}^i(\lambda_i, \alpha_{i0}, \alpha_{i1}, \alpha_{i2})\right)^{-1} \\ &= \begin{bmatrix} \hat{V}_{11}^i & \hat{V}_{12}^i & \hat{V}_{13}^i & \hat{V}_{14}^i \\ \hat{V}_{21}^i & \hat{V}_{22}^i & \hat{V}_{23}^i & \hat{V}_{24}^i \\ \hat{V}_{31}^i & \hat{V}_{32}^i & \hat{V}_{33}^i & \hat{V}_{34}^i \\ \hat{V}_{41}^i & \hat{V}_{42}^i & \hat{V}_{43}^i & \hat{V}_{44}^i \end{bmatrix} \\ &= \begin{bmatrix} \hat{I}_{11}^i & \hat{I}_{12}^i & \hat{I}_{13}^i & \hat{I}_{14}^i \\ \hat{I}_{21}^i & \hat{I}_{22}^i & \hat{I}_{23}^i & \hat{I}_{24}^i \\ \hat{I}_{31}^i & \hat{I}_{32}^i & \hat{I}_{33}^i & \hat{I}_{34}^i \\ \hat{I}_{41}^i & \hat{I}_{42}^i & \hat{I}_{43}^i & \hat{I}_{44}^i \end{bmatrix}^{-1}.\end{aligned}\quad (58)$$

For the given confidence level γ , the $100(1 - \gamma)\%$ ACIs of parameters λ_i and α_{ik} , $k = 0, 1$, and 2 , can be expressed as follows:

$$\begin{aligned}(\hat{\lambda}_i^L, \hat{\lambda}_i^U) &= \left[\hat{\lambda}_i - Z_{\gamma/2} \sqrt{\hat{V}_{11}^i}, \hat{\lambda}_i + Z_{\gamma/2} \sqrt{\hat{V}_{11}^i}\right], \\ (\hat{\alpha}_{ik}^L, \hat{\alpha}_{ik}^U) &= \left[\hat{\alpha}_{ik} - Z_{\gamma/2} \sqrt{\hat{V}_{(k+1)(k+1)}^i}, \hat{\alpha}_{ik} + Z_{\gamma/2} \sqrt{\hat{V}_{(k+1)(k+1)}^i}\right], \\ k &= 0, 1, 2,\end{aligned}\quad (59)$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$ percentile point of the standard normal distribution.

4.2. Highest Posterior Density Credible Intervals. The $100(1 - \gamma)\%$ two-side Bayes credible interval (BCI) of α_{i0} which is denoted by $[\hat{\alpha}_{i0}^{BL}, \hat{\alpha}_{i0}^{BU}]$ can be obtained by solving the following two equations:

$$\frac{\gamma}{2} = \int_0^{\hat{\alpha}_{i0}^{BL}} \pi(\alpha_{i0} | \lambda, t) d\alpha_{i0}, \quad (60)$$

$$1 - \frac{\gamma}{2} = \int_0^{\hat{\alpha}_{i0}^{BU}} \pi(\alpha_{i0} | \lambda, t) d\alpha_{i0}. \quad (61)$$

Substituting equation (46) into (60) and (61), we have

$$\gamma/2 = \frac{1}{B_i} \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \Gamma(m_{i1} + c_{i1} + v) \right. \quad (62)$$

$$\left. \times \Gamma(m_{i2} + c_{i2} + m_i - m_i^* - u - v) C_{uv}(\lambda_i A_i + d_{i0})^{-(m_{i0} + c_{i0} + u)} \times \Gamma(m_{i0} + c_{i0} + u, (\lambda_i A_i + d_{i0}) \hat{\alpha}_{i0}^{BL}) \right\},$$

$$\begin{aligned}1 - \frac{\gamma}{2} &= \frac{1}{B_i} \left\{ \sum_{u=0}^{m_i - m_i^*} \binom{m_i - m_i^*}{u} \sum_{v=0}^{m_i - m_i^* - u} \binom{m_i - m_i^* - u}{v} \Gamma(m_{i1} + c_{i1} + v) \right. \\ &\quad \left. \times \Gamma(m_{i2} + c_{i2} + m_i - m_i^* - u - v) C_{uv}(\lambda_i A_i + d_{i0})^{-(m_{i0} + c_{i0} + u)} \times \Gamma(m_{i0} + c_{i0} + u, (\lambda_i A_i + d_{i0}) \hat{\alpha}_{i0}^{BU}) \right\},\end{aligned}\quad (63)$$

where

$$\begin{aligned}C_{uv} &= (\lambda_i A_i + d_{i0})^{(m_{i0} + c_{i0} + u)} (\lambda_i A_i + d_{i1})^{(m_{i1} + c_{i1} + v)} \\ &\quad \cdot (\lambda_i A_i + d_{i2})^{(m_{i2} + c_{i2} + m_i - m_i^* - u - v)}.\end{aligned}\quad (64)$$

And, $\Gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$ is an incomplete gamma function.

Equations (62) and (63) cannot yield the explicit solutions for $\hat{\alpha}_{i0}^{BL}$ and $\hat{\alpha}_{i0}^{BU}$. Thus, we can only obtain the numerical solution. Similarly, we can get $[\hat{\alpha}_{ik}^{BL}, \hat{\alpha}_{ik}^{BU}]$, $k = 1$ and 2 .

When $m_i - m_i^* = 0$, the parameter α_{ik} , $k = 0, 1$, and 2 , follows the gamma distribution with parameters $m_{ik} + c_{ik}$ and $\lambda_i A_i + d_{ik}$, and it is denoted by $\alpha_{ik} \sim \text{Gamma}(m_{ik} + c_{ik}, \lambda_i A_i + d_{ik})$, $k = 0, 1$, and 2 . The $100(1 - \gamma)\%$ two-side

BCI of α_{ik} , $k = 0, 1$, and 2 , can be acquired by the following equations:

$$\begin{aligned} \frac{\gamma}{2} &= \int_0^{\hat{\alpha}_{ik}^{BU}} \frac{(\lambda_i A_i + d_{ik})^{(m_{ik} + c_{ik})}}{\Gamma(m_{ik} + c_{ik})} \alpha_{ik}^{m_{ik} + c_{ik} - 1} \exp(-\alpha_{ik}(\lambda_i A_i + d_{ik})) d\alpha_{ik}, \\ 1 - \frac{\gamma}{2} &= \int_0^{\hat{\alpha}_{ik}^{BU}} \frac{(\lambda_i A_i + d_{ik})^{(m_{ik} + c_{ik})}}{\Gamma(m_{ik} + c_{ik})} \alpha_{ik}^{m_{ik} + c_{ik} - 1} \exp(-\alpha_{ik}(\lambda_i A_i + d_{ik})) d\alpha_{ik}. \end{aligned} \quad (65)$$

In this case, if α_{ik} , $k = 0, 1$, and 2 , is an integer, the $100(1 - \gamma)\%$ BCI is

$$\left[\frac{\chi^2_{1-\gamma/2}(2(m_{ik} + c_{ik}))}{2(\lambda_i A_i + d_{ik})}, \frac{\chi^2_{\gamma/2}(2(m_{ik} + c_{ik}))}{2(\lambda_i A_i + d_{ik})} \right]. \quad (66)$$

The highest posterior density (HPD) CIs of α_{ik} , $k = 0, 1$, and 2 , say $[\hat{\alpha}_{ik}^{HL}, \hat{\alpha}_{ik}^{HU}]$, is the shortest interval and can be obtained from $\hat{\alpha}_{ik}^{HL} - \hat{\alpha}_{ik}^{HU} = \min(\hat{\alpha}_{ik}^{BL} - \hat{\alpha}_{ik}^{BU})$, $k = 0, 1$, and 2 .

5. Numerical Simulation

5.1. Simulation Study. Consider a two-level CSALT under AT-II PHCS. The subjected units have two dependent failure causes where the lifetime of the two failure causes are assumed to follow a MOBG distribution. Suppose the temperature is the accelerate stress, the accelerated stress levels are $S_1 = 50^\circ\text{C} = 323\text{K}$ and $S_2 = 80^\circ\text{C} = 353\text{K}$, and the normal stress level is $S_0 = 25^\circ\text{C} = 298\text{K}$.

Under the stress level S_i , $i = 1$ and 2 , the sample size is $n_1 = n_2 = 30, 40$, and 50 , and the predetermined failure numbers are $m_1 = (10, 15, 20, 25)$ and $m_2 = (15, 20, 25, 30)$. The life tests are allowed to run over the time $\tau = (0.5, 1)$. It is assumed that the proportion of failure data which has unknown failure cause is 10%. Three different progressive censoring schemes are used in the CSALT, which are denoted as follows.

Scheme 1: $R_1 = R_2 = \dots = R_{m-1} = 0$ and $R_m = n - m$

Scheme 2: $R_1 = n - m$ and $R_2 = \dots = R_{m-1} = R_m = 0$

Scheme 3: $R_1 = R_2 = \dots = R_{[m/2]} = 1$, $R_{[(m+1)/2]} = \dots = R_{m-1} = 0$ and $R_m = n - m - [m/2]$

The parameters λ_i , α_{i0} , α_{i1} , and α_{i2} are given by the accelerated life functions as follows:

$$\begin{aligned} \log \lambda_i &= -6 + 1500/s_i, \quad \log \alpha_{i0} = -3 + 800/s_i, \\ \log \alpha_{i1} &= -4 + 1000/s_i, \quad \log \alpha_{i2} = -4 + 1200/s_i. \end{aligned} \quad (67)$$

Then, the initial values of the parameters are $(\lambda_1, \alpha_{10}, \alpha_{11}, \alpha_{12}) = (0.2577, 0.5926, 0.4049, 0.7521)$ and $(\lambda_2, \alpha_{20}, \alpha_{21}, \alpha_{22}) = (0.1733, 0.4801, 0.3113, 0.5484)$. To compute the BEs and HPD CIs of the parameters, the hyperparameters of the priors are given by $(c_{i0}, c_{i1}, c_{i2}) = (1, 1, 3)$ and $(d_{i0}, d_{i1}, d_{i2}) = (1, 1, 1)$. The prior distribution of λ_i , $i = 1$ and 2 , is given in Table 1.

Under different progressive censoring schemes, the process is repeated 1000 times, and then, the expected values of MLEs and BEs and mean squared errors (MSEs) of the unknown parameters are computed, and the results are shown in Tables 2 and 3. The average interval lengths (AIL) and coverage percentages (CPs) of ACIs and HPD CIs with the given significant level (0.05) are also computed. The results are shown in Tables 4 and 5.

From Tables 2 and 3, it can be observed that the MSEs of the MLEs and BEs for the unknown parameters are decreased with the sample size n increase or the effective sample size m increases, and the MLEs and BEs of the unknown parameters are closer to the true values. For the bigger τ , the MSEs for the MLEs and Bayes of unknown parameters both are the smaller. In addition, the difference under three difference progressive censored schemes is not sensible. Furthermore, the BEs of the unknown parameters are better than MLEs, and its MSEs are always less than the MLEs.

From Tables 4 and 5, the CPs of ACIs and HPD CIs for the unknown parameters are increased, and meanwhile, the AILs of the ACIs and the HPD CIs are decreased with the sample size n or the effective sample size m increase. For the fixed n , m , and τ , the CPs of HPD CIs are more stable than the ACIs for the unknown parameters.

5.2. Real Data Analysis. To illustrate the proposed methods, a real data is selected from Ebrahimi [25], of which the accelerated stress is voltage, the stress levels are $S_1 = 10\text{V}$, $S_2 = 15\text{V}$, and $S_3 = 20\text{V}$, and the normal stress level is $S_0 = 5\text{V}$. Furthermore, for each stress level, the life tests are allowed to run over the time $(\tau_1, \tau_2, \tau_3) = (2, 1.5, 1)$, the numbers under the test were $(n_1, n_2, n_3) = (15, 15, 15)$, the expected failure number are $(m_1, m_2, m_3) = (11, 12, 13)$, and the progressive censoring scheme is $R_1 = R_2 = \dots = R_{m_i-1} = 0$ and $R_{m_i} = n_i - m_i$, $i = 1, 2$, and 3 .

Assume that the unknown failure cause of the failure data proportion is 20%, and then, the failure data and the corresponding failure causes under each stress level are presented in Table 6.

For the Bayes analysis, we select small values of the hyperparameters $c_{i0} = c_{i1} = c_{i2} = d_{i0} = d_{i1} = d_{i2} = 0.0001$, which gives noninformation priors [26]. We make the data in Table 6 be brought to the above mode and obtain the point estimation and interval estimation of the unknown parameters. The results are shown in Table 7.

TABLE 1: The prior distribution of the parameter λ_i , $i = 1$ and 2.

λ_1	0.2	0.3	λ_2	0.15	0.2
$P(\lambda_1 = \lambda_{1r})$	1/2	1/2	$P(\lambda_2 = \lambda_{2r})$	1/2	1/2

TABLE 2: The MLEs and MSEs for the unknown parameters ($\tau = 0.5$).

(n, m_1, m_2)	Scheme	Estimator (MSE)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(30,10,15)	1	MLE	0.3702 (0.0998)	0.7014 (0.1089)	0.5158 (0.1067)	0.8842 (0.1274)	0.2689 (0.0873)	0.5969 (0.1081)	0.4376 (0.0967)	0.6534 (0.1093)
		Bayes	0.3498 (0.0884)	0.6912 (0.0987)	0.5036 (0.0957)	0.8775 (0.1054)	0.2517 (0.0782)	0.5784 (0.0975)	0.4128 (0.0897)	0.6232 (0.0972)
	2	MLE	0.3668 (0.0987)	0.7012 (0.1047)	0.5178 (0.1017)	0.8834 (0.1276)	0.2687 (0.0879)	0.5975 (0.1077)	0.4377 (0.0999)	0.6513 (0.1099)
		Bayes	0.3484 (0.0825)	0.6841 (0.0989)	0.4878 (0.0935)	0.6279 (0.1053)	0.2512 (0.0785)	0.5765 (0.0978)	0.4086 (0.0899)	0.6235 (0.0974)
	3	MLE	0.3677 (0.1017)	0.7005 (0.1089)	0.5175 (0.1019)	0.8832 (0.1259)	0.2669 (0.0957)	0.5999 (0.1076)	0.4372 (0.0972)	0.6511 (0.1075)
		Bayes	0.3489 (0.0822)	0.6817 (0.0988)	0.4976 (0.0932)	0.8536 (0.1061)	0.2511 (0.0791)	0.5766 (0.0977)	0.4095 (0.0895)	0.6231 (0.0979)
	1	MLE	0.3515 (0.0879)	0.6895 (0.0992)	0.4996 (0.0972)	0.6333 (0.1162)	0.2574 (0.0783)	0.5873 (0.0989)	0.4232 (0.0891)	0.4513 (0.0978)
		Bayes	0.3378 (0.0795)	0.6789 (0.0907)	0.4879 (0.0889)	0.8489 (0.0992)	0.2486 (0.0699)	0.5703 (0.0838)	0.4036 (0.0817)	0.6177 (0.0897)
	2	MLE	0.3484 (0.0875)	0.6884 (0.0992)	0.5018 (0.0977)	0.8699 (0.1164)	0.2543 (0.0791)	0.5884 (0.0985)	0.4259 (0.0888)	0.6417 (0.0981)
		Bayes	0.3355 (0.0784)	0.6747 (0.0906)	0.4857 (0.0888)	0.8481 (0.0987)	0.2453 (0.0697)	0.5654 (0.0832)	0.4003 (0.0818)	0.6158 (0.0894)
	3	MLE	0.3477 (0.0878)	0.6876 (0.0998)	0.5024 (0.0975)	0.8696 (0.1164)	0.2565 (0.0787)	0.5886 (0.0997)	0.4257 (0.0895)	0.6413 (0.0982)
		Bayes	0.3355 (0.0794)	0.6725 (0.0905)	0.4787 (0.0884)	0.8452 (0.0987)	0.2457 (0.0695)	0.5655 (0.0844)	0.3988 (0.0819)	0.6139 (0.0893)
(40,15,20)	1	MLE	0.3379 (0.0816)	0.6712 (0.0939)	0.4916 (0.0871)	0.8559 (0.1027)	0.2416 (0.0725)	0.5753 (0.0894)	0.4118 (0.0829)	0.6313 (0.0882)
		Bayes	0.3255 (0.0711)	0.6689 (0.0847)	0.4685 (0.0768)	0.8354 (0.0915)	0.2398 (0.0604)	0.5596 (0.0775)	0.3975 (0.0753)	0.6166 (0.0811)
	2	MLE	0.337 (0.0808)	0.6775 (0.0936)	0.4886 (0.0883)	0.8594 (0.1031)	0.2487 (0.0727)	0.5764 (0.0881)	0.4119 (0.0831)	0.6327 (0.0871)
		Bayes	0.3273 (0.0716)	0.6672 (0.0852)	0.4694 (0.0769)	0.8376 (0.0917)	0.2369 (0.0609)	0.5581 (0.0772)	0.3876 (0.0772)	0.6165 (0.0817)
	3	MLE	0.3378 (0.0816)	0.6786 (0.0932)	0.4878 (0.0894)	0.8593 (0.1029)	0.2468 (0.0728)	0.5763 (0.0895)	0.4101 (0.0821)	0.6368 (0.0897)
		Bayes	0.3288 (0.0714)	0.6691 (0.0859)	0.4691 (0.0762)	0.8378 (0.0919)	0.2371 (0.0612)	0.5568 (0.0778)	0.3893 (0.0768)	0.6045 (0.0813)
	1	MLE	0.325 (0.0764)	0.6611 (0.0849)	0.4768 (0.0779)	0.8438 (0.0945)	0.2359 (0.0679)	0.5667 (0.0812)	0.3964 (0.0785)	0.6269 (0.0811)
		Bayes	0.3143 (0.0656)	0.6571 (0.0776)	0.4623 (0.0682)	0.8294 (0.0897)	0.2252 (0.0587)	0.5508 (0.0705)	0.3758 (0.0711)	0.6044 (0.0732)
	2	MLE	0.3255 (0.0762)	0.669 (0.0843)	0.4758 (0.0771)	0.8493 (0.0951)	0.2375 (0.0682)	0.5674 (0.0815)	0.4044 (0.0794)	0.6277 (0.0809)
		Bayes	0.3155 (0.0657)	0.6549 (0.0779)	0.4589 (0.0687)	0.8283 (0.0892)	0.2275 (0.0583)	0.5496 (0.0702)	0.3779 (0.0713)	0.6036 (0.0735)
	3	MLE	0.3235 (0.0764)	0.6673 (0.0858)	0.4686 (0.0773)	0.8392 (0.0953)	0.2388 (0.0681)	0.5647 (0.0811)	0.3988 (0.0789)	0.6249 (0.0809)
		Bayes	0.3153 (0.0653)	0.6585 (0.0778)	0.4578 (0.0684)	0.8254 (0.0893)	0.2369 (0.0585)	0.5486 (0.0708)	0.3764 (0.0712)	0.5984 (0.0736)

TABLE 2: Continued.

(n, m_1, m_2)	Scheme	Estimator (MSE)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(50,20,25)	1	MLE	0.3053 (0.0635)	0.6538 (0.0758)	0.4655 (0.0661)	0.8281 (0.0863)	0.2268 (0.0584)	0.5466 (0.0718)	0.3861 (0.0681)	0.6129 (0.0711)
		Bayes	0.2944 (0.0558)	0.6406 (0.0675)	0.4438 (0.0578)	0.8192 (0.0783)	0.2181 (0.0511)	0.5324 (0.0625)	0.3666 (0.0609)	0.5893 (0.0633)
	2	MLE	0.3049 (0.0631)	0.6596 (0.0757)	0.4739 (0.0663)	0.8295 (0.0859)	0.2254 (0.0587)	0.5542 (0.0714)	0.3913 (0.0682)	0.6128 (0.0717)
		Bayes	0.2963 (0.0554)	0.6393 (0.0681)	0.4475 (0.0576)	0.8186 (0.0787)	0.2188 (0.0513)	0.5337 (0.0626)	0.3651 (0.0607)	0.585 (0.0631)
	3	MLE	0.3099 (0.0633)	0.6545 (0.0754)	0.4669 (0.0664)	0.8288 (0.0862)	0.2283 (0.0586)	0.5546 (0.0716)	0.3889 (0.0683)	0.6137 (0.0719)
		Bayes	0.2989 (0.0552)	0.6379 (0.0678)	0.4461 (0.0575)	0.8187 (0.0785)	0.2185 (0.059)	0.5348 (0.0624)	0.3666 (0.0608)	0.5881 (0.0632)
(50,25,30)	1	MLE	0.2892 (0.0512)	0.6438 (0.0642)	0.4551 (0.0518)	0.8189 (0.0778)	0.2164 (0.0487)	0.5377 (0.0614)	0.3747 (0.0528)	0.6085 (0.0684)
		Bayes	0.2876 (0.0419)	0.6368 (0.0565)	0.4329 (0.0509)	0.8029 (0.0695)	0.2089 (0.0409)	0.5248 (0.0512)	0.3586 (0.0482)	0.5784 (0.0537)
	2	MLE	0.2902 (0.0511)	0.6437 (0.0639)	0.4548 (0.0517)	0.8173 (0.0776)	0.2174 (0.0485)	0.5393 (0.0618)	0.3773 (0.0527)	0.6072 (0.0681)
		Bayes	0.2851 (0.0411)	0.6369 (0.0567)	0.4356 (0.0502)	0.8063 (0.0697)	0.2074 (0.0408)	0.5245 (0.0513)	0.3592 (0.0485)	0.5788 (0.0535)
	3	MLE	0.2874 (0.0513)	0.6487 (0.0641)	0.4548 (0.0514)	0.8186 (0.0777)	0.2187 (0.0488)	0.5383 (0.0612)	0.3773 (0.0525)	0.6087 (0.0686)
		Bayes	0.2736 (0.0418)	0.6295 (0.0561)	0.4373 (0.0508)	0.8059 (0.0695)	0.2077 (0.0411)	0.5259 (0.0516)	0.3603 (0.0484)	0.5778 (0.0536)

TABLE 3: The MLEs and MSEs for the unknown parameters ($\tau = 1$).

(n, m_1, m_2)	Scheme	Estimator (MSE)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(30,10,15)	1	MLE	0.3686 (0.0954)	0.7009 (0.1027)	0.5129 (0.1015)	0.8813 (0.1257)	0.2615 (0.0864)	0.5946 (0.1074)	0.4346 (0.0953)	0.6516 (0.1078)
		Bayes	0.3488 (0.0818)	0.6897 (0.0985)	0.4913 (0.0916)	0.627 (0.1051)	0.2472 (0.0749)	0.5751 (0.0905)	0.4053 (0.0895)	0.6218 (0.0971)
	2	MLE	0.3617 (0.0946)	0.7011 (0.1033)	0.5124 (0.0977)	0.8816 (0.1245)	0.2646 (0.0872)	0.5975 (0.1077)	0.4334 (0.0973)	0.6469 (0.1032)
		Bayes	0.3472 (0.0812)	0.6814 (0.0984)	0.4923 (0.0917)	0.6262 (0.1049)	0.2496 (0.0775)	0.5742 (0.0918)	0.4054 (0.0898)	0.6205 (0.0968)
	3	MLE	0.3662 (0.0996)	0.6977 (0.1034)	0.5117 (0.0966)	0.8792 (0.1247)	0.2668 (0.0899)	0.5984 (0.1073)	0.4346 (0.0959)	0.6493 (0.1013)
		Bayes	0.3439 (0.0811)	0.6798 (0.0981)	0.4933 (0.0914)	0.8524 (0.1056)	0.2493 (0.0765)	0.5741 (0.0969)	0.4051 (0.0899)	0.6218 (0.0964)
(30,15,20)	1	MLE	0.3482 (0.0877)	0.6854 (0.0982)	0.4979 (0.0923)	0.8698 (0.1155)	0.2567 (0.0759)	0.5854 (0.0981)	0.4209 (0.0877)	0.6385 (0.0971)
		Bayes	0.3341 (0.0775)	0.6774 (0.0902)	0.483 (0.0876)	0.844 (0.0989)	0.2457 (0.0664)	0.5695 (0.0821)	0.3962 (0.0812)	0.6149 (0.0886)
	2	MLE	0.3476 (0.0873)	0.6871 (0.984)	0.4939 (0.0926)	0.8679 (0.1157)	0.2532 (0.0782)	0.5845 (0.0981)	0.4206 (0.0876)	0.6395 (0.0978)
		Bayes	0.3317 (0.0772)	0.6715 (0.0903)	0.4819 (0.0875)	0.8475 (0.0985)	0.2439 (0.0667)	0.5642 (0.0822)	0.3959 (0.0819)	0.6139 (0.0883)
	3	MLE	0.3466 (0.0875)	0.6853 (0.0989)	0.4958 (0.0924)	0.8688 (0.1163)	0.2549 (0.0784)	0.5835 (0.0987)	0.4208 (0.0874)	0.6393 (0.0973)
		Bayes	0.3325 (0.0778)	0.6703 (0.0901)	0.4769 (0.0867)	0.8395 (0.0987)	0.2427 (0.0667)	0.5624 (0.0821)	0.3955 (0.0818)	0.6129 (0.0886)

TABLE 3: Continued.

(n, m_1, m_2)	Scheme	Estimator (MSE)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}	
(40,15,20)	1	MLE	0.3355 (0.0812)	0.679 (0.0924)	0.4864 (0.0856)	0.8526 (0.1021)	0.2508 (0.0719)	0.5712 (0.0887)	0.4083 (0.0815)	0.6296 (0.0876)	
		Bayes	0.3255 (0.0702)	0.6671 (0.0833)	0.4681 (0.0756)	0.8358 (0.0914)	0.2417 (0.0603)	0.5546 (0.0767)	0.386 (0.0744)	0.6025 (0.0807)	
	2	MLE	0.3358 (0.0814)	0.6755 (0.0914)	0.4822 (0.0851)	0.8588 (0.1026)	0.2445 (0.0718)	0.573 (0.0878)	0.4077 (0.0814)	0.6305 (0.0881)	
		Bayes	0.3254 (0.0706)	0.6644 (0.0839)	0.4669 (0.0756)	0.8354 (0.0914)	0.2312 (0.0605)	0.5539 (0.0766)	0.3856 (0.0748)	0.6022 (0.0809)	
	3	MLE	0.3368 (0.0818)	0.6768 (0.0915)	0.4865 (0.0855)	0.8577 (0.1027)	0.2448 (0.0713)	0.5721 (0.0873)	0.4091 (0.0812)	0.6317 (0.0891)	
		Bayes	0.3256 (0.0708)	0.668 (0.0837)	0.4686 (0.0758)	0.833 (0.0911)	0.2348 (0.0607)	0.5537 (0.0763)	0.3879 (0.0748)	0.6027 (0.0811)	
	(40,20,25)	1	MLE	0.3229 (0.0748)	0.6655 (0.0836)	0.4721 (0.0768)	0.6633 (0.0941)	0.246 (0.0661)	0.4029 (0.0803)	0.2338 (0.0777)	0.471 (0.0802)
			Bayes	0.3123 (0.0644)	0.6523 (0.0765)	0.4579 (0.0675)	0.8257 (0.0885)	0.2335 (0.0579)	0.5483 (0.0696)	0.3727 (0.0705)	0.6018 (0.0731)
		2	MLE	0.3213 (0.0746)	0.6656 (0.0833)	0.473 (0.0766)	0.8479 (0.0949)	0.2365 (0.0669)	0.5621 (0.0801)	0.3973 (0.0772)	0.6262 (0.0808)
Bayes			0.3127 (0.0642)	0.6502 (0.0762)	0.456 (0.0674)	0.8259 (0.0888)	0.2247 (0.0577)	0.5489 (0.0697)	0.3737 (0.0701)	0.6019 (0.0729)	
3		MLE	0.3215 (0.0746)	0.6625 (0.0839)	0.4767 (0.0767)	0.6692 (0.0952)	0.2361 (0.0662)	0.5628 (0.0809)	0.2324 (0.0774)	0.6212 (0.0801)	
		Bayes	0.3141 (0.0641)	0.651 (0.0761)	0.4563 (0.0679)	0.8232 (0.0887)	0.2241 (0.0573)	0.5479 (0.0699)	0.3727 (0.0703)	0.5931 (0.0731)	
(50,20,25)		1	MLE	0.3016 (0.0621)	0.6596 (0.0751)	0.4641 (0.0656)	0.8265 (0.0852)	0.2343 (0.0576)	0.5522 (0.0709)	0.3852 (0.0675)	0.6114 (0.0707)
			Bayes	0.2924 (0.0549)	0.6395 (0.0674)	0.4429 (0.0567)	0.8187 (0.0779)	0.2271 (0.0509)	0.5319 (0.0623)	0.364 (0.0605)	0.5807 (0.0629)
		2	MLE	0.3017 (0.0627)	0.6587 (0.0748)	0.3418 (0.0657)	0.8281 (0.0851)	0.2248 (0.0582)	0.5515 (0.0706)	0.3879 (0.0674)	0.6114 (0.0711)
	Bayes		0.2916 (0.0549)	0.6387 (0.0679)	0.4457 (0.0562)	0.8152 (0.0778)	0.2174 (0.0505)	0.5329 (0.0619)	0.3644 (0.0607)	0.5832 (0.0621)	
	3	MLE	0.3093 (0.0626)	0.6537 (0.0749)	0.4651 (0.0655)	0.8274 (0.0859)	0.2269 (0.0577)	0.5518 (0.0701)	0.3852 (0.0679)	0.6103 (0.0712)	
		Bayes	0.297 (0.0544)	0.6359 (0.0677)	0.4455 (0.0564)	0.8176 (0.0781)	0.2169 (0.0507)	0.5325 (0.0621)	0.3628 (0.0603)	0.5846 (0.0624)	
	(50,25,30)	1	MLE	0.2828 (0.0508)	0.6414 (0.0639)	0.4527 (0.0511)	0.8164 (0.0761)	0.2251 (0.0471)	0.5362 (0.0607)	0.3724 (0.0511)	0.6059 (0.0681)
			Bayes	0.2752 (0.0407)	0.6345 (0.0563)	0.4318 (0.0507)	0.8033 (0.0688)	0.2169 (0.0404)	0.4379 (0.0508)	0.3552 (0.0479)	0.5742 (0.0524)
		2	MLE	0.2892 (0.0509)	0.6423 (0.0637)	0.4517 (0.0507)	0.8151 (0.0768)	0.2163 (0.0478)	0.5391 (0.0611)	0.3747 (0.0518)	0.4871 (0.0678)
Bayes			0.273 (0.0403)	0.6233 (0.0561)	0.4312 (0.0509)	0.8016 (0.0685)	0.2046 (0.0405)	0.5216 (0.0509)	0.3585 (0.0482)	0.5762 (0.0531)	
3		MLE	0.2851 (0.0509)	0.6443 (0.0635)	0.4508 (0.0505)	0.815 (0.0769)	0.2152 (0.0477)	0.5377 (0.0609)	0.3755 (0.0517)	0.6072 (0.0682)	
		Bayes	0.2716 (0.0406)	0.6276 (0.0558)	0.4352 (0.0506)	0.803 (0.0687)	0.2062 (0.0407)	0.5224 (0.0507)	0.3595 (0.0481)	0.5766 (0.0527)	

From Table 7, we can obtain that there is little difference between the MLEs and BEs of the unknown parameters. The lengths for the HPD CIs are shorter than the ACIs in most cases. For the given significance value, the accuracy of the HPD CIs is better than the ACIs.

Here, aiming at the above data analysis, there exists a question on how good the proposed MOBG model fits the data. That is, there have been multiple goodness-of-fit testing methods for univariate distributions. However, for bivariate distribution, the goodness-of-fit testing methods are less.

Thus, to verify the above fit results of data and distribution, we mainly used the method which is from the Kundu and Gupta [27], and we verified two margins and the minimum lifetime by using the Kolmogorov–Smirnov test (K–S test).

Give the significance value $\gamma = 5\%$, the K–S statistic and the corresponding p values under different failure numbers are obtained, and the result is presented in Table 8. The comparison charts for empirical distribution and its fitted distribution of the sample under each stress are shown in Figures 1–3.

TABLE 4: The CPs and AILs of the confidence intervals for unknown parameters ($\tau = 0.5$).

(n, m_1, m_2)	Scheme	CP (AIL)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(30,10,15)	1	MLE	84.7 (0.2892)	84.7 (0.3312)	84.4 (0.2763)	85.1 (0.3762)	85.2 (0.2195)	84.6 (0.2953)	84.7 (0.2682)	85.8 (0.3152)
		Bayes	88.1 (0.2768)	87.1 (0.3198)	88.1 (0.2684)	88.7 (0.3638)	89.1 (0.2079)	90.2 (0.2861)	88.4 (0.2586)	89.3 (0.3055)
	2	MLE	84.3 (0.2868)	84.5 (0.3237)	85.9 (0.2748)	85.2 (0.3724)	85.1 (0.2174)	84.2 (0.2972)	84.5 (0.2684)	85.3 (0.3154)
		Bayes	87.9 (0.2732)	88.2 (0.3158)	89.2 (0.2681)	88.7 (0.3628)	89.5 (0.2738)	90.1 (0.2822)	88.2 (0.2558)	89.1 (0.3037)
	3	MLE	84.2 (0.2872)	84.1 (0.3254)	85.1 (0.2775)	85 (0.3713)	85.2 (0.2174)	84.9 (0.2961)	84.4 (0.2653)	85.3 (0.3171)
		Bayes	87.2 (0.2752)	88.1 (0.3149)	89.1 (0.2669)	88.1 (0.3652)	89.4 (0.2044)	89.9 (0.2846)	88.2 (0.2393)	89.2 (0.3024)
	1	MLE	86 (0.2675)	86.3 (0.3084)	85.2 (0.2569)	86.5 (0.3546)	87 (0.1989)	85.2 (0.2773)	86.8 (0.2489)	87 (0.2965)
		Bayes	90.3 (0.2546)	89.2 (0.2969)	89.6 (0.2478)	89.2 (0.3412)	90.3 (0.1891)	91.1 (0.2684)	89.9 (0.2372)	91.1 (0.2858)
	(30,15,20)	2	MLE	85.6 (0.2672)	85.7 (0.3058)	86.3 (0.2545)	87.2 (0.3553)	86.7 (0.1987)	86.2 (0.2785)	85.8 (0.2487)
Bayes			89.6 (0.2549)	90.3 (0.2964)	89.4 (0.2469)	89.3 (0.3454)	90.2 (0.1871)	91.8 (0.2684)	89.2 (0.2379)	90.2 (0.2845)
3		MLE	85.4 (0.2689)	85.6 (0.3074)	86.4 (0.2582)	86 (0.3591)	87 (0.2025)	86.2 (0.2778)	85.6 (0.2483)	86.8 (0.2954)
		Bayes	89.1 (0.2571)	89.7 (0.2956)	90.1 (0.2448)	89.3 (0.3447)	90.4 (0.1887)	90.1 (0.2676)	89.3 (0.2368)	90.2 (0.2822)
1		MLE	87.4 (0.2527)	87.7 (0.2873)	86.1 (0.2366)	86.8 (0.3362)	88 (0.1831)	87.3 (0.2569)	88 (0.2287)	89 (0.2778)
		Bayes	91.4 (0.2376)	92.1 (0.2765)	92.9 (0.2264)	91.1 (0.3254)	91.6 (0.1683)	92.6 (0.2488)	92 (0.2185)	92.3 (0.2687)
2		MLE	87 (0.2545)	86.6 (0.2853)	88.4 (0.2347)	88.5 (0.3376)	88.3 (0.1826)	88.8 (0.2595)	87.4 (0.2273)	89.4 (0.2795)
		Bayes	91.5 (0.2423)	92.3 (0.2733)	91.6 (0.2276)	91.6 (0.3262)	91.6 (0.1667)	92.2 (0.2457)	91.5 (0.2177)	92.8 (0.2693)
3		MLE	87.1 (0.2579)	86.9 (0.2893)	87.9 (0.2376)	88.1 (0.3368)	88 (0.1852)	88.1 (0.2548)	87.2 (0.2297)	88.7 (0.2762)
	Bayes	91.1 (0.2425)	92 (0.2758)	91.2 (0.2269)	91.1 (0.3245)	91.5 (0.1685)	92.2 (0.2481)	91.3 (0.2183)	92.3 (0.2679)	
(40,15,20)	1	MLE	89 (0.2333)	89 (0.2736)	88 (0.2149)	88.3 (0.3258)	90 (0.1684)	89.6 (0.2335)	90 (0.2142)	90.2 (0.2611)
		Bayes	94.2 (0.2273)	94.1 (0.2543)	94.5 (0.2076)	93.5 (0.3062)	94.1 (0.1482)	94.2 (0.2246)	93.3 (0.1964)	93.2 (0.2563)
	2	MLE	89.5 (0.2362)	89 (0.2698)	89.5 (0.2182)	89.5 (0.3221)	89.2 (0.1651)	89.8 (0.2361)	89 (0.2138)	91 (0.2578)
		Bayes	94.4 (0.2256)	94.9 (0.2577)	94.8 (0.2086)	94.3 (0.3076)	93.9 (0.1463)	94.8 (0.2265)	93.5 (0.1967)	94.4 (0.2467)
	3	MLE	89.3 (0.2344)	88.2 (0.2756)	89.2 (0.2226)	89.3 (0.3221)	89.7 (0.1692)	89.7 (0.2368)	88.7 (0.2139)	89.9 (0.2572)
		Bayes	94.3 (0.2272)	94.8 (0.2566)	94 (0.2077)	94.3 (0.3089)	93.3 (0.1469)	94.1 (0.2344)	93.3 (0.1987)	94.1 (0.2452)
	1	MLE	90.9 (0.2166)	89.9 (0.2547)	89.9 (0.2051)	89.5 (0.3089)	90.8 (0.1474)	90.8 (0.2148)	91.8 (0.1989)	91.3 (0.248)
		Bayes	95.7 (0.2066)	96.1 (0.2342)	95.6 (0.1878)	95 (0.2854)	96.8 (0.1324)	95.9 (0.2078)	95.2 (0.1732)	95.1 (0.2283)
	2	MLE	90.8 (0.2185)	89.2 (0.2532)	90.7 (0.2025)	90.2 (0.3097)	90.9 (0.1442)	90.1 (0.2176)	90.8 (0.1942)	91.7 (0.2423)
Bayes		95.2 (0.2068)	96.2 (0.2338)	95.8 (0.1853)	97.8 (0.2887)	96.2 (0.1292)	96.7 (0.2083)	97.8 (0.1784)	96.4 (0.2256)	
3	MLE	90.5 (0.2169)	89.1 (0.2541)	90.2 (0.2015)	90.1 (0.3076)	90.3 (0.1458)	90.1 (0.2234)	90.1 (0.1935)	91.1 (0.2373)	
	Bayes	95.1 (0.2089)	96.1 (0.2429)	95.2 (0.1871)	97 (0.2895)	96.4 (0.1282)	96.3 (0.2143)	96.8 (0.1763)	96.3 (0.2245)	

TABLE 4: Continued.

(n, m_1, m_2)	Scheme	CP (AIL)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(50,25,30)	1	MLE	92.7 (0.1967)	91.7 (0.2357)	91.6 (0.1879)	92.1 (0.2865)	91.5 (0.1263)	91.6 (0.1971)	92.4 (0.1784)	93 (0.2266)
		Bayes	97.1 (0.1866)	97.8 (0.2151)	97.7 (0.1657)	97.9 (0.2648)	98.3 (0.1132)	97.9 (0.1843)	98.4 (0.1579)	
	2	MLE	91.6 (0.1971)	91.4 (0.2356)	92.6 (0.1857)	92.2 (0.2859)	92.5 (0.1287)	92.4 (0.1954)	92.8 (0.1784)	93.5 (0.2283)
		Bayes	97.6 (0.1832)	97.3 (0.2152)	97.5 (0.1669)	98 (0.2666)	98.6 (0.1187)	97.5 (0.1824)	98.7 (0.1576)	97.8 (0.2073)
	3	MLE	91.4 (0.1978)	91.2 (0.2352)	92.1 (0.1834)		92.2 (0.2858)	92.1 (0.1266)	92.2 (0.2026)	92.1 (0.1728)
		Bayes	97.5 (0.1852)	97.1 (0.2276)	97.9 (0.1653)	98.1 (0.2673)	97.5 (0.1142)	97.1 (0.1945)	98 (0.1575)	97.8 (0.2065)

TABLE 5: The CPs and AILs of the confidence intervals for unknown parameters ($\tau = 1$).

(n, m_1, m_2)	Scheme	CP (AIL)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(30,10,15)	1	MLE	84.7 (0.2886)	84.3 (0.3294)	84.3 (0.2739)	84.9 (0.3737)	85.2 (0.2186)	84.5 (0.2922)	84.3 (0.2662)	85.1 (0.3135)
		Bayes	88.1 (0.2737)	87.3 (0.3177)	87.9 (0.2675)	88.4 (0.3619)	88.9 (0.2056)	90.2 (0.2832)	88.4 (0.2563)	89.1 (0.3027)
	2	MLE	84.5 (0.2875)	84.2 (0.3246)	84.3 (0.2723)	84.6 (0.3732)	85.1 (0.2168)	84.2 (0.2952)	84.1 (0.269)	84.8 (0.3172)
		Bayes	88.1 (0.2755)	87.1 (0.3168)	87.6 (0.2659)	88.2 (0.3625)	88.2 (0.2027)	89.7 (0.2821)	88.2 (0.2528)	89.2 (0.3018)
	3	MLE	84.3 (0.2881)	84.2 (0.3239)	84.2 (0.2751)	84.4 (0.3702)	85 (0.2189)	84.1 (0.297)	84.2 (0.2611)	84.6 (0.3138)
		Bayes	88.1 (0.2748)	87.3 (0.3127)	87.4 (0.2666)	88.1 (0.3622)	88.1 (0.1989)	89.5 (0.2812)	88.3 (0.2532)	89.1 (0.3015)
	1	MLE	85.7 (0.2694)	86.2 (0.3076)	85.8 (0.2577)	86.2 (0.3544)	86.5 (0.1999)	86.2 (0.2785)	85.3 (0.2478)	86.9 (0.2948)
		Bayes	90.5 (0.2569)	89.1 (0.2965)	89.6 (0.2485)	89.1 (0.3426)	90.1 (0.1885)	90.1 (0.2676)	89.4 (0.2388)	90.1 (0.2869)
	(30,15,20)	2	MLE	85.5 (0.2684)	86.1 (0.3065)	85.7 (0.2588)	86.4 (0.3589)	86.4 (0.1997)	86.1 (0.2797)	86 (0.2481)
Bayes			90.2 (0.2596)	89.3 (0.2974)	89.6 (0.2486)	89 (0.3456)	90 (0.1864)	90.1 (0.2679)	89.3 (0.2356)	90.1 (0.2852)
3		MLE	85.4 (0.2702)	86.1 (0.3096)	85.7 (0.2571)	86.2 (0.3605)	89.3 (0.2041)	86.2 (0.2798)	85.8 (0.2495)	86.2 (0.2962)
		Bayes	90 (0.2583)	89.2 (0.2977)	89.3 (0.2463)	88.8 (0.3414)	89.9 (0.1887)	90.1 (0.2675)	89.2 (0.2369)	90.2 (0.283)
1		MLE	87.2 (0.2513)	87.3 (0.2888)	86.7 (0.2369)	86.7 (0.3377)	87.9 (0.1812)	87.3 (0.2589)	87.6 (0.2277)	88.5 (0.2765)
		Bayes	91.2 (0.2426)	91.6 (0.2713)	92.6 (0.2296)	91 (0.3249)	91.4 (0.1697)	92.4 (0.2484)	91.9 (0.2197)	92.2 (0.2696)
2		MLE	87.2 (0.2537)	87.2 (0.2876)	86.3 (0.2318)	87.8 (0.3399)	87.8 (0.1807)	87.2 (0.2596)	87.5 (0.2298)	88.2 (0.2787)
		Bayes	91.1 (0.2419)	91.2 (0.2737)	92.7 (0.2278)	91 (0.3232)	91.2 (0.1687)	92.4 (0.2437)	91.7 (0.2189)	92 (0.2683)
3		MLE	87 (0.2562)	87.1 (0.2908)	86.1 (0.2396)	87.7 (0.3356)	87.8 (0.1874)	87.2 (0.2585)	87.5 (0.2303)	88.2 (0.2785)
	Bayes	90.9 (0.2412)	91.2 (0.2765)	92.1 (0.2257)	90.8 (0.3275)	91.2 (0.1667)	92.2 (0.2493)	91.7 (0.2141)	92 (0.2625)	

TABLE 5: Continued.

(n, m_1, m_2)	Scheme	CP (AIL)	λ_1	α_{10}	α_{11}	α_{12}	λ_2	α_{20}	α_{21}	α_{22}
(40,20,25)	1	MLE	88.6 (0.2346)	88.9 (0.2724)	87.9 (0.2184)	88.2 (0.3212)	89.5 (0.1686)	89.4 (0.2342)	89.1 (0.2139)	90.1 (0.2601)
		Bayes	93.8 (0.2265)	94 (0.2584)	94.3 (0.2093)	93.2 (0.3086)	94.1 (0.1499)	94.1 (0.2253)	93.7 (0.1997)	94 (0.2579)
	2	MLE	88.4 (0.2369)	88.7 (0.2692)	87.9 (0.2193)	88.3 (0.3215)	89.2 (0.1677)	89.3 (0.2372)	89.1 (0.2104)	90 (0.2597)
		Bayes	93.8 (0.2273)	93.9 (0.2595)	94.4 (0.2091)	93.1 (0.3093)	93.9 (0.1476)	94 (0.2295)	93.4 (0.1999)	93.8 (0.2448)
	3	MLE	88.4 (0.235)	88.7 (0.2719)	87.8 (0.2219)	88 (0.3241)	89.2 (0.1685)	89.3 (0.2393)	89 (0.2123)	89.6 (0.2587)
		Bayes	93.4 (0.2286)	93.9 (0.2598)	94.2 (0.2085)	93.3 (0.3108)	93.8 (0.1439)	93.8 (0.2314)	93.3 (0.1979)	93.6 (0.2436)
(50,20,25)	1	MLE	90.5 (0.2164)	89.6 (0.2561)	89.6 (0.2048)	90.1 (0.3099)	90.3 (0.1474)	90.5 (0.2186)	91.4 (0.1964)	91.3 (0.2486)
		Bayes	95.4 (0.2072)	96.1 (0.2337)	95.6 (0.1853)	95.1 (0.2876)	96.3 (0.1314)	95.4 (0.2089)	95.1 (0.1776)	95 (0.2243)
	2	MLE	90.3 (0.2195)	89.5 (0.2517)	89.5 (0.2015)	90 (0.3093)	90.1 (0.1481)	90.1 (0.2192)	91.4 (0.1947)	91.2 (0.2401)
		Bayes	95.4 (0.2095)	95.8 (0.2359)	95.3 (0.1877)	95 (0.2892)	96.2 (0.1287)	95.3 (0.2107)	95.1 (0.1795)	95 (0.2261)
	3	MLE	90.2 (0.2167)	89.5 (0.2538)	89.4 (0.2029)	89.5 (0.3085)	89.7 (0.1497)	90.1 (0.2213)	91.3 (0.1942)	91.2 (0.2389)
		Bayes	94.3 (0.2104)	95.3 (0.2413)	95.2 (0.1895)	95 (0.2886)	96.1 (0.1274)	95.2 (0.2171)	95.1 (0.1778)	94.9 (0.2216)
(50,25,30)	1	MLE	92.7 (0.1987)	91.4 (0.2356)	91.3 (0.1864)	92.6 (0.2885)	91.4 (0.1237)	92.3 (0.1993)	92.3 (0.1763)	93 (0.224)
		Bayes	97.1 (0.1857)	97.3 (0.2134)	97.2 (0.1646)	97.2 (0.2661)	98.2 (0.1114)	97.5 (0.1856)	98.4 (0.1575)	97.8 (0.2037)
	2	MLE	92.6 (0.1972)	91.4 (0.2364)	91.1 (0.1876)	92.4 (0.2869)	91.4 (0.1261)	92.2 (0.1927)	92.2 (0.1769)	92.8 (0.2222)
		Bayes	97.1 (0.1829)	97.3 (0.2155)	97.2 (0.1655)	97.1 (0.2658)	98.2 (0.1142)	97.4 (0.1887)	98.3 (0.1586)	97.8 (0.2062)
	3	MLE	92.6 (0.1957)	91.4 (0.2357)	91.1 (0.1835)	92.4 (0.2886)	91.4 (0.1274)	92.2 (0.2013)	92.2 (0.1733)	92.8 (0.2162)
		Bayes	97.8 (0.1848)	97.3 (0.2258)	97.2 (0.1642)	97.1 (0.2679)	98.2 (0.1145)	97.4 (0.1923)	98.3 (0.1566)	97.8 (0.2045)

TABLE 6: The failure data with the dependent competing risks' model.

S_1	(0.04, 0) (1.05, 1)	(0.10, 1) (1.50, 2)	(0.16, 0) (1.80, 0)	(0.17, *)	(0.42, 2)	(0.46, 0)	(0.47, *)	(0.47, 1)
S_2	(0.02, 1) (0.80, 2)	(0.06, 2) (0.80, 0)	(0.27, *) (0.92, *)	(0.30, 2) (1.06, 2)	(0.30, 2)	(0.34, 1)	(0.36, *)	(0.70, 2)
S_3	(0.07, 2) (0.33, *)	(0.13, 1) (0.40, 0)	(0.20, *) (0.40, 1)	(0.29, 1) (0.63, *)	(0.30, 2) (0.82, 0)	(0.30, *)	(0.30, 1)	(0.33, 2)

TABLE 7: Estimations and 95% confidence and incredible intervals.

Estimator	MLE	ACI	Bayes	HPD
λ_1	0.3133	(0.0911, 0.5982)	0.3027	(0.0953, 0.5976)
α_{10}	1.7241	(0.9851, 2.4724)	1.7163	(0.9735, 2.3853)
α_{11}	0.8616	(0.3953, 1.1285)	0.8473	(0.4075, 1.1153)
α_{12}	0.8247	(0.4261, 1.1176)	0.8193	(0.4378, 1.1104)
λ_2	0.8359	(0.3956, 1.1432)	0.8287	(0.3918, 1.1427)
α_{20}	1.4937	(0.6853, 2.2589)	1.4517	(0.6978, 2.2667)
α_{21}	0.7745	(0.4137, 1.0134)	0.7689	(0.4166, 1.0089)
α_{22}	1.1201	(0.6176, 1.6792)	1.0816	(0.6186, 1.6811)
λ_3	1.1024	(0.5983, 1.5925)	1.0673	(0.6017, 1.5937)
α_{30}	2.3691	(1.4327, 3.2896)	2.2164	(1.4673, 3.2576)
α_{31}	1.4151	(0.8221, 2.1074)	1.4091	(0.8226, 2.0974)
α_{32}	1.5121	(0.8831, 2.2146)	1.4973	(0.8875, 2.2121)

TABLE 8: The K-S statistics and the corresponding p value.

Distribution	$F(t; \lambda_1, \alpha_{101})$	$F(t; \lambda_1, \alpha_{102})$	$F(t; \lambda_1, \alpha_1)$	$F(t; \lambda_2, \alpha_{201})$	$F(t; \lambda_2, \alpha_{202})$	$F(t; \lambda_2, \alpha_2)$	$F(t; \lambda_3, \alpha_{301})$	$F(t; \lambda_3, \alpha_{302})$	$F(t; \lambda_3, \alpha_3)$
K-S	0.2443	0.1435	0.1689	0.1515	0.1715	0.2035	0.1737	0.2089	0.2388
p	0.5562	0.9536	0.8620	0.9076	0.8157	0.6325	0.7676	0.6324	0.5873

FIGURE 1: Comparison chart for empirical distribution and its fitted distribution under S_1 level. (a) The comparison chart when $\alpha = \alpha_{101}$. (b) The comparison chart when $\alpha = \alpha_{102}$. (c) The comparison chart when $\alpha = \alpha_1$. FIGURE 2: Comparison chart for empirical distribution and its fitted distribution under S_2 level. (a) The comparison chart when $\alpha = \alpha_{201}$. (b) The comparison chart when $\alpha = \alpha_{202}$. (c) The comparison chart when $\alpha = \alpha_2$. FIGURE 3: Comparison chart for empirical distribution and its fitted distribution under S_3 level. (a) The comparison chart when $\alpha = \alpha_{301}$. (b) The comparison chart when $\alpha = \alpha_{302}$. (c) The comparison chart when $\alpha = \alpha_3$.

From Table 8, we can obtain that all the corresponding p values are bigger than the given significance value. It indicates that the failure data comes from the MOBG distribution.

6. Conclusion

Under the AT-II PHCS, we consider statistical analysis of the dependent competing risk model by using the MOBG distribution under CSALT in which the shape parameters of life distribution are changed with the stress levels. Moreover, we also consider the failure causes of some units are unknown in the dependent competing risks' model, and then, both the ML method and Bayes approach are used to obtain the estimations of the unknown parameters. Based on these works, the expected values and MSEs of MLEs and BEs are derived, and the AILs and CPs of the unknown parameters based on ACIs and HPD CIs are also obtained. Finally, the performances of all procedures are evaluated. The simulation results show that the MSEs, AIL, and CPs of the unknown parameters perform well, which are in the bigger sample size n or effective sample size m . Furthermore, compared with the ML method, the Bayes approach is superior according to the MSEs, AIL, and CPs of the unknown parameters.

Data Availability

The data used to support the findings of this paper are included within the supplementary information file.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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