Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 6646231, 19 pages
https://doi.org/10.1155/2021/6646231

Research Article

LMI-Based Analysis and Stabilization of Nonlinear Descriptors with Multiple Delays via Delayed Nonlinear Controller Schemes

Javier Adrián Romero-Vega,[1] **Raúl Villafuerte-Segura**,[1] **and Víctor Estrada-Manzo**[2]

[1]*Research Center on Information Technology and Systems, Hidalgo State University, Pachuca-Hidalgo C. P. 42184, Mexico*
[2]*Department of Mechatronics of the Universidad Politécnica de Pachuca, Zempoala, C.P 43830, Mexico*

Correspondence should be addressed to Raúl Villafuerte-Segura; villafuerte@uaeh.edu.mx

Received 17 December 2020; Revised 31 January 2021; Accepted 10 February 2021; Published 8 March 2021

Academic Editor: Maria Patrizia Pera

This paper presents a convex approach for nonlinear descriptor systems with multiple delays; it allows designing delayed nonlinear controllers such that the closed-loop system holds exponential estimates for convergence. The proposal takes advantage of an equivalent convex representation of the given descriptor model together with Lyapunov-Krasovskii functionals; thus, the conditions are in the form of linear matrix inequalities, which can be efficiently solved by commercially available software. To avoid possible saturation in the actuators, conditions for bounding the control input are also given. Numerical and academic examples illustrate the performance of the proposal.

1. Introduction

In the last decades, a large number of results concerning the analysis and stabilization of systems by means of the direct Lyapunov method [1], since the publication of the book [2], linear matrix inequalities (LMIs) have become a preferred solution to many control problems [3, 4], as they can be effectively solved by means of convex optimization techniques already implemented in commercially available software [5, 6]. These ideas have been extended to the analysis of time-delay systems (TDSs) via LyapunovKrasovskii (L-K) functionals [7] or LyapunovRazumikhin (L-R) functions [8]. In this context, there are several results that provide sufficient stability conditions using LMI-based approaches for different classes of TDS, such as linear time-delay systems [9–13], uncertain linear time-delay systems [14–16], neutral linear systems [17–20], systems with uncertain time-invariant delays [21], descriptor system approach for TDS [22], linear parameter-varying (LPV) time-delay systems [23], systems with time-varying delays [24–29], exponential estimates for TDS [30, 31], systems with polytopic-type uncertainties [32], singular systems [33], neural networks with time delay [34, 35], and genetic regulatory networks with probabilistic time delays [36]. Recently, in [37] convex approaches are employed to provide robust stability conditions based on quasi-polynomials.

In general, delays are undesirable phenomena, because they can destabilize or produce a poor performance in the system response. However, in recent years, it has been shown that delays can also stabilize and improve the close-loop performance of a system. Moreover, the deliberate induction of delays by means of the control law is an efficient alternative to stabilize systems [38, 39]; these types of controllers are known as delayed ones. For example, in [40–42], a proportional control with an appropriate delay replaces a traditional proportional-derivative one; thus, the system response is fast and insensitive to high-frequency noise. In [43], a scheme called time-delayed feedback control (TDFC) is proposed, originating different investigations [44–55].

As mentioned above, LMI-based approaches have become important in the control community; however, in the context of TDS, there is an inherent conservatism for stability and stabilization conditions even for linear setups [37]; this leaves room for improvements. Moreover, a problem little explored by the community is obtaining stability conditions for nonlinear TDS. Although, originally, LMI-based stability conditions were given for linear time-invariant (LTI) systems, these have also been used on LPV [3] and nonlinear setups via exact Takagi-Sugeno (TS) [56]. The latter case employs the sector of no linearity approach [57] which allows rewriting the original nonlinear model as a convex one by means of scalar convex functions that capture uncertainties and nonlinearities. This technique has also been applied to a class of nonlinear TDS; for instance, in [58, 59], sufficient LMI conditions are proposed; in [60], uncertain TS systems are considered; in [61], sufficient LMI conditions have been given for a class of nonlinear systems. A larger family of functionals is explored in [62]. Nonetheless, none of these previous works deal with nonlinear descriptor systems; they appear when using the EulerLagrange formalism for modeling plants [63]. In the context of convex descriptor models without delays, there are some recent works [64, 65]; time-delay nonlinear descriptor systems are a few works in the literature; for instance, in [66], LMI stability and stabilization conditions have been developed for systems with only one time-delay.

Contribution: this paper proposes an LMI methodology for analysis and stability of nonlinear descriptor systems with multiple delays, thus overcoming recent results in the literature. For example, the work [4] only considers linear in standard form systems with multiple delays, [52] only studies nonlinear systems in standard form, and [66] treats nonlinear systems in descriptor with one delay. Additionally, to avoid possible saturation in the actuators, LMI conditions for bounding the control signal are established. Numerical and academic examples illustrate that including delays in the controller can reduce noise in the control signal, which increases the useful life of the actuators.

The paper is organized as follows: the problem statement and preliminary results are shown in Section 2. LMI-based stability analysis and delayed nonlinear controller design conditions for a class of nonlinear descriptors systems with multiple delays are given in Section 3, and additionally conditions for input constraints are also given. In Section 4, the implementation and numerical validation of the previous theoretical results are provided. Concluding remarks are stated in Section 5.

2. Problem Statement and Preliminary Results

2.1. Problem Statement. Let us consider a nonlinear descriptor system under multiple delays of the following form:

$$E(x)\dot{x}(t) = A(x)x(t) + \sum_{h=1}^{d} A_{\tau_h}(x)x(t-\tau_h) + B(x)u,$$
$$x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $E(x)$, $A(x)$, $B(x)$, and $A_{\tau_h}(x)$ are matrix functions assumed to be smooth and bounded, $0 < \tau_1 < \tau_2 < \cdots < \tau_d = \tau$ are time delays, and $\phi \in \mathscr{C}([-\tau, 0], \mathbb{R}^n)$ are the initial functions, where $\mathscr{C}([-\tau, 0], \mathbb{R}^n)$ is the Banach space of real continuous functions on the intervals $[-\tau, 0]$ with the following norm:

$$\|\phi\|_\tau = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|, \tag{2}$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$. It is assumed that for each initial condition $\phi \in \mathscr{C}([-\tau, 0], \mathbb{R}^n)$, $t > 0$ there exists a unique solution $x(t, \phi)$ of the system; moreover, $x_t(\phi) = \{x(t+\theta, \phi): \theta \in [-\tau, 0]\}$. In this work, matrix $E(x)$ is assumed to be invertible, at least in a region including the origin; this is a common assumption when studying systems (1) derived from the EulerLagrange formalism [63, 65].

In order to obtain LMI conditions, the sector nonlinearity approach [57] is employed to compute an exact convex representation of (1). This methodology begins by defining a premise vector $z(x) \in \mathbb{R}^p$ whose entries are different nonconstant terms in $A(x)$, $A_{\tau_h}(x)$, $B(x)$, $h \in \{1, 2, \ldots, d\}$; similarly, $\zeta(x) \in \mathbb{R}^q$ is the premise vector with nonconstant terms in $E(x)$. It is assumed that each entry of the vectors $z(x)$ and $\zeta(x)$ is bounded in the compact set Ω_x that includes the origin, that is, $z_i(x) \in [z_i^0, z_i^1]$ and $\zeta_k(x) \in [\zeta_k^0, \zeta_k^1]$. Thus, each of them can be expressed as convex sums of their bounds:

$$z_i(x) = z_i^0 w_0^i(x) + z_i^1 w_1^i(x), \quad i \in \{1, 2, \ldots, p\},$$
$$\zeta_k(x) = \zeta_k^0 \omega_0^k(x) + \zeta_k^1 \omega_1^k(x), \quad k \in \{1, 2, \ldots, q\}, \tag{3}$$

where

$$w_0^i(z) = \frac{z_i^1 - z_i(x)}{z_i^1 - z_i^0},$$
$$w_1^i(x) = 1 - w_0^i(z),$$
$$\omega_0^k(\zeta) = \frac{\zeta_k^1 - \zeta_k(x)}{\zeta_k^1 - \zeta_k^0},$$
$$\omega_1^k(\zeta) = 1 - \omega_0^k(\zeta), \tag{4}$$

are scalar convex functions holding the convex sum property for all $x \in \Omega_x$, i.e., $0 \le w_i(z) \le 1$, $w_1^i + w_0^i = 1$, $0 \le \omega_k(\zeta) \le 1$, and $\omega_1^k + \omega_0^k = 1$. Then, the so-called scheduling (membership) functions can be computed:

$$\mathbf{w}_i(z) = w_{i_1}^1(x) w_{i_2}^2(x) \cdots w_{i_r}^p(x), \quad i_j \in \{0, 1\},$$
$$\boldsymbol{\omega}_k(\zeta) = \omega_{k_1}^1(x) \omega_{k_2}^2(x) \cdots \omega_{k_\rho}^q(x), \quad k_j \in \{0, 1\}, \tag{5}$$

where $i \in \{1, 2, \ldots, r\}$, $r = 2^p$, and indexes $[i_1 i_2 \cdots i_p]$ are chosen as a p-digit binary representation of $(i-1)$; similarly, $k \in \{1, 2, \ldots, \rho\}$, $i_j \in \{0, 1\}$, $\rho = 2^q$; and the set $[k_1 k_2 \cdots k_q]$ is a q-digit binary representation of $(k-1)$. The scheduling functions also hold the convex sum property in Ω_x. Finally, an equivalent convex representation of (1) is [67]

$$\sum_{k=1}^{\rho} \boldsymbol{\omega}_k(\zeta) E_k \dot{x}(t) = \sum_{i=1}^{r} \mathbf{w}_i(z) \cdot \left(A_i x(t) + \sum_{h=1}^{d} A_{\tau_h i} x(t-\tau_h) + B_i u \right), \tag{6}$$

where $E_k = E(x)|_{\omega_k(\zeta)=1}$, $A_i = A(x)|_{\mathbf{w}_i(z)=1}$, $A_{\tau_h i} = A_{\tau_h}(x)|_{w_i(z)=1}$, $h \in \{1, 2, \ldots, d\}$, and $B_i = B(x)|_{w_i(z)=1}$ are constants matrices; $r = 2^p$ and $\rho = 2^q$ are the number of vertices for the right and left side of (6), respectively. It is important to notice that (6) is a convex rewriting of (1); thus, all the conclusions derived from the former directly apply to the latter.

2.2. Notation and Properties. In the following, convex sums of matrices will be shortly represented by

$$\Upsilon_{\mathbf{w}} = \sum_{i=1}^{r} \mathbf{w}_i(z)\Upsilon_i, \Upsilon_{\boldsymbol{\omega}} = \sum_{k=1}^{\rho} \boldsymbol{\omega}_k(\zeta)\Upsilon_k, \Upsilon_{\mathbf{w}}^{-1} = \left(\sum_{i=1}^{r} \mathbf{w}_i(z)\Upsilon_i\right)^{-1},$$
$$\Upsilon_{\mathbf{ww\omega}} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{\rho} \mathbf{w}_i(z)\mathbf{w}_j(z)\boldsymbol{\omega}_k(\zeta)\Upsilon_{ijk}. \tag{7}$$

Thus, (6) is expressed as $E_\omega \dot{x}(t) = A_{\mathbf{w}}x(t) + \sum_{h=1}^{m} A_{\tau_h \mathbf{w}} x(t-\tau_h) + B_{\mathbf{w}} u(t)$. Additionally, an asterisk $(*)$ will be employed in matrix expressions to denote the transpose of the symmetric element; for in-line ones, it indicates the transpose of the terms in its left-hand side, that is, $A + B + A^T + B^T + C = A + B + (*) + C$.

Usually, when deriving LMI conditions for convex descriptor models, the designer is faced to inequalities of the form $\Upsilon_{\mathbf{ww}\omega} < 0$; the scheduling functions are dropped off by means of the following relaxation lemma:

Lemma 1 (see [68]). *Let $\Upsilon_{ijk} = \Upsilon_{ijk}^T$, $(i, j) = \{1, 2, \ldots, r\}^2$, and $k \in \{1, 2, \ldots, \rho\}$ be matrices of adequate sizes. Then,*

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{\rho} \mathbf{w}_i(z)\mathbf{w}_j(z)\boldsymbol{\omega}_k(\zeta)\Upsilon_{ijk} < 0, \tag{8}$$

holds if the following LMIs,

$$\frac{2}{r-1}\Upsilon_{iik} + \Upsilon_{ijk} + \Upsilon_{jik} < 0, \tag{9}$$

are satisfied for all $(i, j) = \{1, 2, \ldots, r\}^2$, $k \in \{1, 2, \ldots, \rho\}$.

The following results establish the exponential estimates for time-delay nonlinear systems:

Lemma 2 (see [30]). *Consider system (1). If there exists a functional $V(\cdot)$ and positive constants c_1, c_2, and α, such that*

(1) $c_1\|x\|^2 \le V(x_t) \le c_2\|x_t\|_\tau^2$,

(2) $\dot{V}(x_t) + 2\alpha V(x_t) < 0$,

then, the solutions $x(t, \phi)$ of the system (1) satisfy the exponential estimates:

$$\|x(t, \phi)\| \le \sqrt{\frac{c_2}{c_1}} e^{-\alpha t}\|\phi\|_\tau. \tag{10}$$

As customary, for the analysis and design of convex descriptor models, the so-called descriptor redundancy is employed [69]; in our case, the augmented vectors $\overline{x}(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix}^T$ and $\overline{x}(t-\tau_h) = \begin{bmatrix} x^T(t-\tau_h) & \dot{x}^T(t-\tau_h) \end{bmatrix}^T$, $h \in \{1, 2, \ldots, d\}$, are employed to rewrite (6) as follows:

$$\overline{E}\dot{\overline{x}}(t) = \overline{A}_{\mathbf{w\omega}}\overline{x}(t) + \sum_{h=1}^{d} \overline{A}_{\tau_h \mathbf{w}}\overline{x}(t-\tau_h) + \overline{B}_{\mathbf{w}}u, \tag{11}$$

with

$$\overline{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$
$$\overline{A}_{\mathbf{w\omega}} = \begin{bmatrix} 0 & I \\ A_{\mathbf{w}} & -E_{\boldsymbol{\omega}} \end{bmatrix},$$
$$\overline{A}_{\tau_h \mathbf{w}} = \begin{bmatrix} 0 & 0 \\ A_{\tau_h \mathbf{w}} & 0 \end{bmatrix},$$
$$\overline{B}_{\mathbf{w}} = \begin{bmatrix} 0 \\ B_{\mathbf{w}} \end{bmatrix}, \quad h \in \{1, 2, \ldots, d\}. \tag{12}$$

In what follows, the stability and stabilization conditions are derived from the augmented system (11); nevertheless, it is important to stress that the system under study has the form (1).

Let us recall previous works on the subject. The work [66] studies the stability and stabilization of a nonlinear descriptor system with only one delay, that is (1) with $d = 1$. For stabilization purposes, the following control law is proposed:

$$u = K_{\mathbf{w\omega}}x(t) + F_{\mathbf{w\omega}}x(t-\tau), \tag{13}$$

with $K_{\mathbf{w}\omega}x(t) = \sum_{j=1}^{r}\sum_{k=1}^{\rho} \mathbf{w}_j(z(x))\omega_k(\zeta(x))K_{jk}$ and $F_{\mathbf{w}\omega} = \sum_{j=1}^{r}\sum_{k=1}^{\rho} \mathbf{w}_j(z(x))\omega_k(\zeta(x))F_{jk}$; it is a nonlinear control law with nonlinearities of both sides of the nonlinear descriptor model. In the Section 3, a generalization of this controller will be presented.

3. LMI Conditions for Descriptor Systems with Multiple Delays

In this section, the developments are based on the following LyapunovKrasovskii functional candidate:

$$V(x_\tau) = \overline{x}^T(t)\overline{E}^T\overline{P}_{\mathbf{w}}\overline{x}(t) + \sum_{h=1}^{d}\int_{-\tau_h}^{0} \overline{x}^T(t+\theta)\overline{E}^T\overline{Q}_h e^{2\alpha\theta}\overline{x}(t+\theta)\mathrm{d}\theta, \tag{14}$$

with

$$\overline{P}_{\mathbf{w}} = \begin{bmatrix} P_1 & 0 \\ P_{2\mathbf{w}} & P_{3\mathbf{w}} \end{bmatrix},$$
$$\overline{Q}_h = \begin{bmatrix} Q_h & 0 \\ 0 & 0 \end{bmatrix},$$
$$\overline{E}^T\overline{P}_{\mathbf{w}} = \overline{P}_{\mathbf{w}}^T\overline{E} \geq 0, \quad \alpha > 0, P_1 > 0, Q_h > 0, h \in \{1, 2, \ldots, d\}. \tag{15}$$

Note that, the functional (14) is a valid L-K functional candidate as it reduces to

$$V\left(x_\tau\right) = x^T(t)P_1x(t) + \sum_{h=1}^{d}\int_{-\tau_h}^{0} x^T(t+\theta)Q_he^{2\alpha\theta}x(t+\theta)\mathrm{d}\theta, \tag{16}$$

which is clearly positive definite and bounded by $c_1\|x\|^2 \leq V(x_t) \leq c_2\|x_t\|_\tau^2$, with $c_1 = \lambda_{\min}(P_1)$ and $c_2 = \lambda_{\max}(P_1) + \sum_{h=1}^{d}\tau_h\lambda_{\max}(Q_h)$, thus fulfilling conditions (1) in Lemma 2.

For the analysis of system (1), i.e., when $u = 0$, we have the following result.

Theorem 1. *The origin of system (1), with $u = 0$ and an exact convex representation (6), is exponentially stable if the exist matrices $P_1 > 0$, P_{2j}, P_{3j}, $Q_h > 0$ with $h \in \{1, 2, \ldots, d\}$, $j \in \{1, 2, \ldots, r\}$, and a scalar $\alpha > 0$ such that LMIs (9) hold with*

$$\Upsilon_{ijk} \coloneqq \begin{bmatrix} P_{2j}^TA_i + A_i^TP_{2j} + 2\alpha P_1 + \sum_{h=1}^{d}Q_h & (*) & (*) & (*) & (*) \\ P_1 - E_k^TP_{2j} + P_{3j}^TA_i & -E_k^TP_{3j} - P_{3j}^TE_k & \cdots & (*) & (*) \\ P_{2j}^TA_{\tau_1 i} & A_{\tau_1 i}^TP_{3j} & -e^{-2\alpha\tau_1}Q_1 & \cdots & (*) \\ \vdots & \vdots & \cdots & \ddots & (*) \\ P_{2j}^TA_{\tau_d i} & A_{\tau_d i}^TP_{3j} & 0 & \cdots & -e^{-2\alpha\tau_d}Q_d \end{bmatrix}. \tag{17}$$

where $(i, j) = \{1, 2, \ldots, r\}^2$ and $k \in \{1, 2, \ldots, \rho\}$. Additionally, the solution $x(t, \phi)$ of (1) satisfies the exponential estimates:

$$\|x(t, \phi)\| \leq \sqrt{\frac{c_2}{c_1}}e^{-\alpha t}\|\phi\|_\tau, \tag{18}$$

with $c_1 = \lambda_{\min}(P_1)$ and $c_2 = \lambda_{\max}(P_1) + \sum_{h=1}^{d}\tau_h\lambda_{\max}(Q_h)$, $h \in \{1, 2, \ldots, d\}$.

Proof. The time derivative of (14) along the trajectories of (11) is $\dot{V}(x_\tau) = V_1 + V_2$ with

$$\begin{aligned} V_1 &= \overline{x}^T(t)\overline{P}_{\mathbf{w}}^T\overline{E}\dot{\overline{x}}(t) + \dot{\overline{x}}^T(t)\overline{E}^T\overline{P}_{\mathbf{w}}\overline{x}(t), \\ V_2 &= \sum_{h=1}^{m}\frac{d}{\mathrm{d}t}\int_{-\tau_h}^{0}\overline{x}^T(t+\theta)\overline{E}^T\overline{Q}_he^{2\alpha\theta}\overline{x}(t+\theta)\mathrm{d}\theta, \end{aligned} \tag{19}$$

which once the dynamics of (11) are substituted and using Leibniz's rule yield

$$\begin{aligned} V_1 &= \overline{x}^T(t)\overline{P}_{\mathbf{w}}^T\left(\overline{A}_{\mathbf{w}\boldsymbol{\omega}}\overline{x}(t) + \sum_{h=1}^{d}\overline{A}_{\tau_h\mathbf{w}}\overline{x}\left(t-\tau_h\right)\right) + \left(\overline{A}_{\mathbf{w}\boldsymbol{\omega}}\overline{x}(t) + \sum_{h=1}^{d}\overline{A}_{\tau_h\mathbf{w}}\overline{x}\left(t-\tau_h\right)\right)^T\overline{P}_{\mathbf{w}}\overline{x}(t), \\ V_2 &= \sum_{h=1}^{d}\left\{\overline{x}^T(t)\overline{E}^T\overline{Q}_h\overline{x}(t) - \overline{x}^T\left(t-\tau_h\right)e^{-2\alpha\tau_h}\overline{E}^T\overline{Q}_h\overline{x}(t-\tau_h)\right\} - 2\alpha\int_{-\tau_h}^{0}\overline{x}^T(t+\theta)\overline{E}^T\overline{Q}_he^{2\alpha\theta}\overline{x}(t+\theta)\mathrm{d}\theta\Bigg\}. \end{aligned} \tag{20}$$

From the latter, we have that $\dot{V}(x_t) + 2\alpha V(x_t)$ is equal to

$$\begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-\tau_1) \\ \vdots \\ \overline{x}(t-\tau_d) \end{bmatrix}^T \begin{bmatrix} \overline{P}_{\mathbf{w}}^T\overline{A}_{\mathbf{w}\omega} + \overline{A}_{\mathbf{w}\omega}^T\overline{P}_{\mathbf{w}} + 2\alpha\overline{E}^T\overline{P}_{\mathbf{w}} + \sum_{h=1}^{d}\overline{E}^T\overline{Q}_h & (*) & \cdots & (*) \\ \overline{A}_{\tau_1\mathbf{w}}^T\overline{P}_{\mathbf{w}} & -e^{-2\alpha\tau_1}\overline{E}\,\overline{Q}_1 & \cdots & (*) \\ \vdots & \vdots & \ddots & (*) \\ \overline{A}_{\tau_d\mathbf{w}}^T\overline{P}_{\mathbf{w}} & 0 & \cdots & -e^{-2\alpha\tau_d}\overline{E}\,\overline{Q}_d \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-\tau_1) \\ \vdots \\ \overline{x}(t-\tau_d) \end{bmatrix}. \quad (21)$$

Thus, in order to fulfill condition (2) in Lemma 2, $\dot{V}(x_t) + 2\alpha V(x_t) < 0$ can be established after some algebraic manipulations when developing vectors $\overline{x}(t)$ and $\overline{x}(t-\tau_h), h \in \{1, 2, \ldots, d\}$, by the following inequality:

$$\begin{bmatrix} P_{2\mathbf{w}}^T A_{\mathbf{w}} + A_{\mathbf{w}}^T P_{2\mathbf{w}} + 2\alpha P_1 + \sum_{h=1}^{d} Q_h & (*) & (*) & (*) & (*) \\ P_1 - E_{\omega}^T P_{2\mathbf{w}} + P_{3\mathbf{w}}^T A_{\mathbf{w}} & -E_{\omega}^T P_{3\mathbf{w}} - P_{3\mathbf{w}}^T E_{\omega} & \cdots & (*) & (*) \\ P_{2\mathbf{w}}^T A_{\tau_1\mathbf{w}} & A_{\tau_1\mathbf{w}}^T P_{3\mathbf{w}} & -e^{-2\alpha\tau_1}Q_1 & \cdots & (*) \\ \vdots & \vdots & \cdots & \ddots & (*) \\ P_{2\mathbf{w}}^T A_{\tau_d\mathbf{w}} & A_{\tau_d\mathbf{w}}^T P_{3\mathbf{w}} & 0 & \cdots & -e^{-2\alpha\tau_d}Q_d \end{bmatrix} < 0. \quad (22)$$

Finally, in order to drop off the scheduling functions $\mathbf{w}$ and ω via the relaxation scheme in Lemma 1, the proof is concluded.

Let us consider system (1). Now, the task is to design a multiple delayed PDC control law of the following form:

$$u = K_{\mathbf{w}\omega}x(t) + \sum_{h=1}^{d} F_{h\mathbf{w}\omega}x(t-\tau_h), \quad (23)$$

where $K_{\mathbf{w}\omega} = \sum_{j=1}^{r}\sum_{k=1}^{\rho}\mathbf{w}_j(z(x))\omega_k(\zeta(x))K_{jk}$ and $F_{h\mathbf{w}\omega} = \sum_{j=1}^{r}\sum_{k=1}^{\rho}\mathbf{w}_j(z(x))\omega_k(\zeta(x))F_{hjk}, h \in \{1, 2, \ldots, d\}$ are nonlinear gains to be designed via the augmented system (11); thus, (23) can be expressed as $u(t) = \overline{K}_{\mathbf{w}\omega}\overline{x}(t) + \sum_{h=1}^{d}\overline{F}_{h\mathbf{w}\omega}\overline{x}(t-\tau_h)$ with $\overline{K}_{\mathbf{w}\omega} = [\,K_{\mathbf{w}\omega}\ \ 0\,]$ and $\overline{F}_{h\mathbf{w}\omega} = [\,F_{h\mathbf{w}\omega}\ \ 0\,]$. The following result provides LMI conditions for the design of the control law (23). It is based on a slightly modification of the L-K functional candidate (14), that is,

$$V(x_\tau) = \overline{x}^T(t)\overline{E}^T\overline{P}_{\mathbf{w}}^{-1}\overline{x}(t) + \sum_{h=1}^{d}\int_{-\tau_h}^{0}\overline{x}^T(t+\theta)\overline{E}^T\overline{Q}_h e^{2\alpha\theta}\overline{x}(t+\theta)\mathrm{d}\theta, \quad (24)$$

with

$$\overline{E}^T\overline{P}_{\mathbf{w}}^{-1} = \overline{P}_{\mathbf{w}}^{-T}\overline{E} \geq 0,$$
$$\overline{P}_{\mathbf{w}}^{-1} = \begin{bmatrix} P_1 & 0 \\ P_{2\mathbf{w}} & P_{3\mathbf{w}} \end{bmatrix}^{-1},$$
$$\overline{Q}_h = \begin{bmatrix} Q_h & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1 > 0, Q_h > 0, \alpha > 0, h \in \{1, 2, \ldots, d\}. \quad (25)$$

□

Theorem 2. *The origin of system (1) with an exact convex representation (6), under the law of control (23), is exponentially stable if existing matrices $P_1 > 0$, P_{2j}, P_{3j}, $R_h > 0$, M_{jk}, N_{hjk}, $h \in \{1, 2, \ldots, d\}$, $j \in \{1, 2, \ldots, r\}$, and a scalar $\alpha > 0$ if the LMIs (9) hold with the following:*

$$\Upsilon_{ijk} := \begin{bmatrix} P_{2j}^T + P_{2j} + 2\alpha P_1 + \sum_{h=1}^{d} R_h & (*) & (*) & (*) & (*) \\ A_iP_1 + B_iM_{jk} - E_kP_{2j} + P_{3j}^T & -E_kP_{3j} - P_{3j}^TE_k^T & \cdots & (*) & (*) \\ 0 & P_1A_{\tau_1 i}^T + N_{1jk}^TB_i^T & -e^{-2\alpha\tau_1}R_1 & \cdots & (*) \\ \vdots & \vdots & \cdots & \ddots & (*) \\ 0 & P_1A_{\tau_d i}^T + N_{djk}^TB_i^T & 0 & \cdots & -e^{-2\alpha\tau_d}R_d \end{bmatrix}. \quad (26)$$

Then, the vertex control gains are computed as $K_{jk} = M_{jk}P_1^{-1}$ and $F_{hjk} = N_{hjk}P_1^{-1}$, $j \in \{1, 2 \dots, r\}$, $k \in \{1, 2, \dots, \rho\}$, and $h \in \{1, 2, \dots, d\}$. Moreover, $Q_h = P_1^{-1}R_hP_1^{-1}$, $h \in \{1, 2, \dots, d\}$ and the solution satisfies the following exponential estimates:

$$\|x(t, \phi)\| \leq \sqrt{\frac{c_2}{c_1}} e^{-\alpha t} \|\phi\|_\tau, \tag{27}$$

where $c_1 = \lambda_{\min}(P_1^{-1})$ and $c_2 = \lambda_{\max}(P_1^{-1}) + \sum_{h=1}^{d} \tau_h \lambda_{\max}(Q_h)$.

Proof. Using the augmented system (11) and its corresponding control law, the closed-loop system is

$$\overline{E}\dot{\overline{x}}(t) = (\overline{A}_{\mathbf{w}\omega} + \overline{B}_{\mathbf{w}}\overline{K}_{\mathbf{w}\omega})\overline{x}(t) + \sum_{h=1}^{d} \left(\overline{A}_{\mathbf{w}\tau_h} + \overline{B}_{\mathbf{w}}\overline{F}_{h\mathbf{w}\omega}\right)\overline{x}(t - \tau_h). \tag{28}$$

Similar to the proof of Theorem 1, consider the functional (24) and its time derivative $\dot{V}(x_t) = V_1 + V_2$ with

$$\begin{aligned} V_1 &= \dot{\overline{x}}^T(t)\overline{E}^T\overline{P}_{\mathbf{w}}^{-1}\overline{x}(t) + \overline{x}^T(t)\overline{P}_{\mathbf{w}}^{-T}\overline{E}\dot{\overline{x}}(t), \\ V_2 &= \sum_{h=1}^{d} \frac{d}{\mathrm{d}t} \int_{-\tau_h}^{0} \overline{x}^T(t + \theta)\overline{E}^T\overline{Q}_h e^{2\alpha\theta}\overline{x}(t + \theta)\mathrm{d}\theta. \end{aligned} \tag{29}$$

Substituting the dynamics of (28) in V_1 while using Leibniz's rule in V_2, we have

$$\begin{aligned} \dot{V}_1 &= \left((\overline{A}_{\mathbf{w}\omega} + \overline{B}_{\mathbf{w}}\overline{K}_{\mathbf{w}\omega})\overline{x}(t) + \sum_{h=1}^{d} \left(\overline{A}_{\mathbf{w}\tau_h} + \overline{B}_{\mathbf{w}}\overline{F}_{h\mathbf{w}\omega}\right)\overline{x}(t - \tau_h) \right)^T \overline{P}_{\mathbf{w}}^{-1}\overline{x}(t) + \overline{x}^T(t)\overline{P}_{\mathbf{w}}^{-T} \\ &\quad \cdot \left((\overline{A}_{\mathbf{w}\omega} + \overline{B}_{\mathbf{w}}\overline{K}_{\mathbf{w}\omega})\overline{x}(t) + \sum_{h=1}^{d} \left(\overline{A}_{\mathbf{w}\tau_h} + \overline{B}_{\mathbf{w}}\overline{F}_{h\mathbf{w}\omega}\right)\overline{x}(t - \tau_h) \right), \\ V_2 &= \sum_{h=1}^{d} \left\{ \overline{x}^T(t)\overline{E}^T\overline{Q}_h\overline{x}(t) - \overline{x}^T(t - \tau_h)e^{-2\alpha\tau_h}\overline{E}^T\overline{Q}_h\overline{x}(t - \tau_h) - 2\alpha \int_{-\tau_h}^{0} \overline{x}^T(t + \theta)\overline{E}^T\overline{Q}_h e^{2\alpha\theta}\overline{x}(t + \theta)\mathrm{d}\theta \right\}. \end{aligned} \tag{30}$$

Therefore, $\dot{V}(x_t) + 2\alpha V(x_t)$ is equivalent to

$$\begin{bmatrix} \overline{x}(t) \\ \overline{x}(t - \tau_1) \\ \vdots \\ \overline{x}(t - \tau_d) \end{bmatrix}^T \begin{bmatrix} \overline{P}_{\mathbf{w}}^{-T}(\overline{A}_{\mathbf{w}\omega} + \overline{B}_{\mathbf{w}}\overline{K}_{\mathbf{w}\omega}) + (*) + 2\alpha\overline{E}^T\overline{P}_{\mathbf{w}}^{-1} + \sum_{h=1}^{d} \overline{E}^T\overline{Q}_h & (*) & \dots & (*) \\ \left(\overline{A}_{\tau_1\mathbf{w}} + \overline{B}_{\mathbf{w}}\overline{F}_{1\mathbf{w}\omega}\right)^T\overline{P}_{\mathbf{w}}^{-1} & -e^{-2\alpha\tau_1}\overline{E}^T\overline{Q}_1 & \dots & (*) \\ \vdots & \dots & \ddots & (*) \\ \left(\overline{A}_{\tau_d\mathbf{w}} + \overline{B}_{\mathbf{w}}F_{d\mathbf{w}\omega}\right)^T\overline{P}_{\mathbf{w}}^{-1} & 0 & \dots & -e^{-2\alpha\tau_d}\overline{E}^T\overline{Q}_d \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{x}(t - \tau_1) \\ \vdots \\ \overline{x}(t - \tau_d) \end{bmatrix}. \tag{31}$$

After some simplifications, $\dot{V}(x_t) + 2\alpha V(x_t) < 0$ (condition (2) in Lemma 2 holds if

$$\begin{bmatrix} -P_1^{-T}P_{2\mathbf{w}}P_{3\mathbf{w}}^{-T}(A_{\mathbf{w}} + B_{\mathbf{w}}K_{\mathbf{w}\omega}) + (*) + 2\alpha P_1^{-1} + \sum_{h=1}^{d} Q_h & (*) & (*) & (*) & (*) \\ P_{3\mathbf{w}}^{-T}(A_{\mathbf{w}} + B_{\mathbf{w}}K_{\mathbf{w}\omega}) + P_i^{-1} + E_\omega^T P_{3\mathbf{w}}^{-1}P_1^{-1} & -E_\omega^T P_{3\mathbf{w}}^{-1} - P_{3\mathbf{w}}^{-T}E_\omega & \dots & (*) & (*) \\ -\left(A_{\tau_1\mathbf{w}}^T + F_{1\mathbf{w}\omega}^T B_W^T\right)P_{3\mathbf{w}}^{-1}P_{2\mathbf{w}}P_1^{-1} & \left(A_{\tau_1\mathbf{w}}^T + F_{1\mathbf{w}\omega}^T\right)P_{3\mathbf{w}}^{-1} & -e^{-2\alpha\tau_1}Q_1 & \dots & (*) \\ \vdots & \vdots & \dots & \ddots & (*) \\ -\left(A_{\tau_d\mathbf{w}}^T + F_{d\mathbf{w}\omega}^T B_W^T\right)P_{3\mathbf{w}}^{-1}P_{2\mathbf{w}}P_1^{-1} & \left(A_{\tau_d\mathbf{w}}^T + F_{d\mathbf{w}\omega}^T B_W^T\right)P_{3\mathbf{w}}^{-1} & 0 & \dots & -e^{-2\alpha\tau_d}Q_d \end{bmatrix} < 0, \tag{32}$$

holds too; however, from the previous inequality, one cannot directly obtain LMI conditions. Thus, by means of the congruence property, that is, pre- and postmultiplying by the matrix block – diag$\left[\begin{bmatrix} P_1 & 0 \\ P_{2\mathbf{w}} & P_{3\mathbf{w}} \end{bmatrix}, \underbrace{P_1, \ldots, P_1}_{d}\right]$ gives

$$\Upsilon_{ww\omega} := \begin{bmatrix} P_{2\mathbf{w}}^T + P_{2\mathbf{w}} + 2\alpha P_1 + \sum_{h=1}^{d} R_h & (*) & (*) & (*) & (*) \\ A_{\mathbf{w}} P_1 + B_{\mathbf{w}} M_{\mathbf{w}\boldsymbol{\omega}} - E_{\boldsymbol{\omega}} P_{2\mathbf{w}} + P_{3\mathbf{w}}^T & -E_{\boldsymbol{\omega}} P_{3\mathbf{w}} - P_{3\mathbf{w}}^T E_{\boldsymbol{\omega}}^T & \cdots & (*) & (*) \\ 0 & P_1 A_{\tau_1 \mathbf{w}}^T + N_{1\mathbf{w}\boldsymbol{\omega}}^T B_{\mathbf{w}}^T & -e^{-2\alpha\tau_1} R_1 & \cdots & (*) \\ \vdots & \vdots & \cdots & \ddots & (*) \\ 0 & P_1 A_{\tau_d \mathbf{w}}^T + N_{d\mathbf{w}\boldsymbol{\omega}}^T B_{\mathbf{w}}^T & 0 & \cdots & -e^{-2\alpha\tau_d} R_d \end{bmatrix} < 0, \tag{33}$$

with the definitions $M_{\mathbf{w}\omega} = K_{\mathbf{w}\omega} P_1$ and $N_{h\mathbf{w}\omega} = F_{h\mathbf{w}\omega} P_1$, $h \in \{1, 2, \ldots, d\}$. Now, the previous inequality can be translated into the LMI conditions in the theorem once the relaxation scheme in Lemma 1 is applied.

Recall that, by hypothesis matrix, $E(x)$ in (1) is invertible in a region around the origin; thus, it is always possible to calculate a standard state-space form as follows:

$$\dot{x}(t) = \widetilde{A}(x)x(t) + \sum_{h=1}^{d} \widetilde{A}_{\tau_h}(x)x(t - \tau_h) + \widetilde{B}(x)u, \tag{34}$$

where $\widetilde{A}(x) = E^{-1}(x)A(x)$, $\widetilde{B}(x) = E^{-1}(x)B(x)$, and $\widetilde{A}_{\tau_h}(x) = E^{-1}(x)A_{\tau_h}(x), h \in \{1, 2, \ldots, d\}$. Naturally, it is possible to obtain a convex representation of system (34), that is,

$$\dot{x}(t) = \widetilde{A}_{\mathbf{w}}(x)x(t) + \sum_{h=1}^{d} \widetilde{A}_{\tau_h \mathbf{w}}(x)x(t - \tau_h) + \widetilde{B}_{\mathbf{w}}(x)u, \tag{35}$$

with $\widetilde{A}_{\mathbf{w}} = \sum_{i=1}^{\widetilde{r}} \mathbf{w}_i(z)\widetilde{A}_i$, $\widetilde{A}_{\tau_h \mathbf{w}} = \sum_{i=1}^{\widetilde{r}} \mathbf{w}_i(z)\widetilde{A}_{\tau_h i}$, and $\widetilde{B}_{\mathbf{w}} = \sum_{i=1}^{\widetilde{r}} \mathbf{w}_i(z)\widetilde{B}_i$, where $\widetilde{r}$ is the number of vertex models, $h \in \{1, 2, \ldots, d\}$. Even though nonlinear systems (34) and (1) and their convex forms are equivalent, establishing exponential stability of them via LMIs may lead to different feasibility set solution. Keeping the original descriptor form (1) results in a convex representation with less vertex matrices; this, in general, yields less conservative results [64].

Thus, the following result provides stability and stabilization conditions for systems of the form (34) by means of the L-K functional (14) (for stability) and (24) (for stabilization), respectively.

Corollary 1. *Stability: the origin of system (1), with $u = 0$ and an exact convex representation (35), is exponentially stable if the exist matrices $P_1 > 0$, P_{2j}, P_{3j}, and $Q_h > 0$ with $h \in \{1, 2, \ldots, d\}$, $j \in \{1, 2, \ldots, \widetilde{r}\}$, and a scalar $\alpha > 0$ such that LMIs,*

$$\frac{2}{\widetilde{r} - 1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i, j) = \{1, 2, \ldots, \widetilde{r}\}^2, \tag{36}$$

hold with

$$\Upsilon_{ij} := \begin{bmatrix} P_{2j}^T \widetilde{A}_i + \widetilde{A}_i^T P_{2j} + 2\alpha P_1 + \sum_{h=1}^{d} Q_h & (*) & (*) & (*) & (*) \\ P_1 - P_{2j} + P_{3j}^T \widetilde{A}_i & -P_{3j} - P_{3j}^T & \cdots & (*) & (*) \\ P_{2j}^T \widetilde{A}_{\tau_1 i} & \widetilde{A}_{\tau_1 i}^T P_{3j} & -e^{-2\alpha\tau_1} Q_1 & \cdots & (*) \\ \vdots & \vdots & \cdots & \ddots & (*) \\ P_{2j}^T \widetilde{A}_{\tau_d i} & \widetilde{A}_{\tau_d i}^T P_{3j} & 0 & \cdots & -e^{-2\alpha\tau_d} Q_d \end{bmatrix}. \tag{37}$$

Moreover, the solution $x(t, \phi)$ satisfies the following exponential estimates:

$$\|x(t, \phi)\| \leq \sqrt{\frac{c_2}{c_1}} e^{-\alpha t} \|\phi\|_\tau. \tag{38}$$

Stabilization: the origin of system (1) with an exact convex representation (35), under the law of control $u = \widetilde{K}_{\mathbf{w}\omega} x(t) + \sum_{h=1}^{d} \widetilde{F}_{h\mathbf{w}\omega} x(t - \tau_h)$, is exponentially stable if existing matrices $P_1 > 0$, P_{2j}, P_{3j}, $R_h > 0$, $\widetilde{M}_j$, $\widetilde{N}_{hj}$, $h \in \{1, 2, \ldots, d\}$, $j \in \{1, 2, \ldots, \widetilde{r}\}$, and a scalar $\alpha > 0$ if the LMIs (36) hold with

$$\Upsilon_{ij} := \begin{bmatrix} P_{2j}^T + P_{2j} + 2\alpha P_1 + \sum_{h=1}^{d} R_h & (*) & (*) & (*) & (*) \\ \widetilde{A}_i P_1 + \widetilde{B}_i \widetilde{M}_j - P_{2j} + P_{3j}^T & -P_{3j} - P_{3j}^T & \dots & (*) & (*) \\ 0 & P_1 \widetilde{A}_{\tau_1 i}^T + \widetilde{N}_{1j}^T \widetilde{B}_i^T & -e^{-2\alpha\tau_1} R_1 & \dots & (*) \\ \vdots & \vdots & \dots & \ddots & (*) \\ 0 & P_1 \widetilde{A}_{\tau_d i}^T + \widetilde{N}_{dj}^T \widetilde{B}_i^T & 0 & \dots & -e^{-2\alpha\tau_d} R_d \end{bmatrix}. \tag{39}$$

Then, the vertex control gains are computed as $\widetilde{K}_j = \widetilde{M}_j P_1^{-1}$ and $\widetilde{F}_{hj} = \widetilde{N}_{hj} P_1^{-1}$, $\widetilde{F}_{hj} = \widetilde{N}_{hj} P_1^{-1}$, $j \in \{1, 2 \dots, \widetilde{r}\}$, $h \in \{1, 2, \dots, d\}$. Then, the solution satisfies the following exponential estimates:

$$\|x(t,\phi)\| \le \sqrt{\frac{c_2}{c_1}} e^{-\alpha t} \|\phi\|_\tau. \tag{40}$$

Proof. It follows a similar path than results in Theorems 1 and 2, respectively.

The result above employs the same L-K functional of the descriptor approach, and thus the slack matrices $P_{2\mathrm{w}}$ and $P_{3\mathrm{w}}$ are also considered into the LMI conditions. Another set of LMI conditions for establishing the exponential estimates of systems in standard form (34) and its convex representation (35) can be done via a L-K functional without slack matrices, namely,

$$V(x_\tau) = x^T(t) P x(t) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) Q_h e^{2\alpha\theta} x(t+\theta) \mathrm{d}\theta, \tag{41}$$

whose positiveness is inferred by $P > 0$, $Q_h > 0$, $h \in \{1, 2, \dots, d\}$ and its boundedness $c_1\|x\|^2 \le V(x_t) \le c_2 \|x_t\|_\tau^2$, with $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P) + \sum_{h=1}^{d} \tau_h \lambda_{\max}(Q_h)$. This is summarized in the following result. □

Corollary 2. *The origin of the system (1) with an exact convex representation (35) and $u(t) = 0$ is exponentially stable if there exists matrices $P > 0$, $Q_h > 0$ with $h \in \{1, 2, \dots, d\}$ and a scalar $\alpha > 0$ such that,*

$$\begin{bmatrix} P\widetilde{A}_i + \widetilde{A}_i^T P + 2\alpha P + \sum_{h=1}^{d} Q_h & (*) & \cdots & (*) \\ \widetilde{A}_{\tau_1 i}^T P & -e^{-2\alpha\tau_1} Q_1 & \cdots & (*) \\ \vdots & \vdots & \ddots & (*) \\ \widetilde{A}_{\tau_d i}^T P & \cdots & \cdots & -e^{-2\alpha\tau_d} Q_d \end{bmatrix} < 0, \tag{42}$$

holds for $i \in \{1, 2, \dots, \widetilde{r}\}$. Then, the solution satisfies the following exponential estimates:

$$\|x(t,\phi)\| \le \sqrt{\frac{c_2}{c_1}} e^{-\alpha t} \|\phi\|_\tau, \tag{43}$$

with $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P) + \sum_{h=1}^{d} \tau_h Q_h$.

Proof. It follows a similar path than previous results. □

Remark 1. Theorem 1 establishes LMI conditions for the exponential estimates for the origin of system (1), these conditions include results in [66, Theorem 1] are always included when $d = 1$ (system (1) with only one delay). Moreover, conditions in Corollary 2 always include those for linear systems in [4, Theorem 2], to see this set $r = 1$.

Remark 2. The numerical complexity of the LMI problems in the above results can be approximated by $\log_{10} = (n_d^3 n_l)$, where n_l is the number of total LMI rows and n_d is the number of scalar decision variables [70]. For Theorem 1 we have $n_l = n(d+2)r^2 \rho + n(d+1)$ and $n_d = 0.5n(n+1)(1+d) + 2rn^2$; as for Theorem 2, $n_l = n(d+2)r^2\rho + n(d+1)$ and $n_d = 0.5n(n+1)(1+d) + 2rn^2 + nmr\rho(1+d)$; as for standard systems, Corollary 1 (stability) is $n_l = n(d+2)\widetilde{r}^2 + n(d+1)$ and $n_d = 0.5n(n+1)(1+d) + 2\widetilde{r}n^2$; Corollary 1 (stabilization) is $n_l = n(d+2)\widetilde{r}^2 + n(d+1)$ and $n_d = 0.5n(n+1)(1+d) + 2\widetilde{r}n^2 + nm\widetilde{r}(1+d)$; while Corollary 2 is $n_l = n(d+1)\widetilde{r} + n(d+1)$ and $n_d = 0.5n(n+1)(1+d)$.

Results in Theorem 2 can be directly applied for real-world setups; nevertheless, the LMIs might render controller gains whose magnitude cannot be applied in practice. To alleviate this issue as well as to avoid damages in the actuators, the following result provides conditions for bounding the control input (23); they can be combined with those of Theorem 2.

Theorem 3. *Consider the delayed nonlinear controller given in (23); then, this controller satisfies that $\|u\| < \mu$, for any $\mu > 0$, if the following inequalities hold:*

$$\begin{bmatrix} P_1 & M_{\mathbf{w}\boldsymbol{\omega}}^T \\ M_{\mathbf{w}\boldsymbol{\omega}} & \dfrac{\mu^2}{2} I \end{bmatrix} > 0, \tag{44}$$

$$\begin{bmatrix} e^{-2\alpha\tau_h} R_h & N_{h\mathbf{w}\boldsymbol{\omega}}^T \\ N_{h\mathbf{w}\boldsymbol{\omega}} & \dfrac{\mu^2}{2} I \end{bmatrix} > 0, \quad h \in \{1, 2, \ldots, d\}, \tag{45}$$

$$\begin{bmatrix} \tau^{-1} & \phi^T(\theta) & \phi^T(\theta) & \cdots & \phi^T(\theta) \\ \phi(\theta) & P_1 & 0 & \cdots & 0 \\ \phi(\theta) & 0 & R_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \phi(\theta) & 0 & 0 & \cdots & R_d \end{bmatrix} > 0, \quad \forall \theta \in [-\tau, 0]. \tag{46}$$

Proof. Observe that $\|u\|^2 = u^T u$ with the delayed PDC (23) yields

$$\begin{aligned} \|u\|^2 &= \left(K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{d} F_{h\mathbf{w}\boldsymbol{\omega}} x(t-\tau_h) \right)^T \left(K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{d} F_{h\mathbf{w}\boldsymbol{\omega}} x(t-\tau_h) \right) \\ &= x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T K_{\mathbf{w}\boldsymbol{\omega}} x(t) + 2 \sum_{h=1}^{d} x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T F_{h\mathbf{w}\boldsymbol{\omega}} x(t-\tau_h) + \sum_{h=1}^{d} x^T(t-\tau_h) F_{h\mathbf{w}\boldsymbol{\omega}}^T F_{h\mathbf{w}\boldsymbol{\omega}} x(t-\tau_h) \\ &\leq 2 \left(x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{d} x^T(t-\tau_h) F_{h\mathbf{w}\boldsymbol{\omega}}^T F_{h\mathbf{w}\boldsymbol{\omega}} x(t-\tau_h) \right) \\ &\leq 2 \left(x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) F_{h\mathbf{w}\boldsymbol{\omega}}^T F_{h\mathbf{w}\boldsymbol{\omega}} x(t+\theta) \mathrm{d}\theta \right) \leq \mu^2, \end{aligned} \tag{47}$$

which is satisfied if the following holds:

$$\begin{aligned} & x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2} K_{\mathbf{w}\boldsymbol{\omega}} x(t) \\ & + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) F_{h\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2} F_{h\mathbf{w}\boldsymbol{\omega}} x(t+\theta) \mathrm{d}\theta \leq 1. \end{aligned} \tag{48}$$

On the other hand, let us consider the following inequality on functional (14):

$$\begin{aligned} V(x_t) &= x^T(t) P_1^{-1} x(t) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) e^{2\alpha\theta} Q_h x(t+\theta) \mathrm{d}\theta \\ &\leq \phi^T(0) P_1^{-1} \phi(0) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} \phi^T(\theta) e^{2\alpha\theta} Q_h \phi(\theta) \mathrm{d}\theta \leq \phi^T(0) P_1^{-1} \phi(0) + \sum_{h=1}^{d} \int_{-\tau}^{0} \phi^T(\theta) e^{2\alpha\theta} Q_h \phi(\theta) \mathrm{d}\theta \leq 1. \end{aligned} \tag{49}$$

Now, combining (48) and (49), it follows that

$$\begin{aligned} 0 &< x^T(t) K_{\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2} K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) F_{h\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2} F_{h\mathbf{w}\boldsymbol{\omega}} x(t+\theta) \mathrm{d}\theta \\ &\leq x^T(t) P_1^{-1} x(t) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} x^T(t+\theta) e^{2\alpha\theta} Q_h x(t+\theta) \mathrm{d}\theta \\ &\leq \phi^T(0) P_1^{-1} \phi(0) + \sum_{h=1}^{d} \int_{-\tau_h}^{0} \phi^T(\theta) e^{2\alpha\theta} Q_h \phi(\theta) \mathrm{d}\theta \\ &< \phi^T(0) P_1^{-1} \phi(0) + \sum_{h=1}^{d} \int_{-\tau}^{0} \phi^T(\theta) \left(P_1^{-1} Q_h^{-1} \right) Q_h \left(Q_h^{-1} P_1^{-1} \right) \phi(\theta) \mathrm{d}\theta \leq 1, \end{aligned} \tag{50}$$

or equivalently

$$x^T(t)\left(P_1^{-1}-K_{\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2}K_{\mathbf{w}\boldsymbol{\omega}}\right)x(t)+\sum_{h=1}^{d}\int_{-\tau_h}^{0}x^T(t+\theta)\left(e^{2\alpha\theta}Q_h-F_{h\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2}F_{h\mathbf{w}\boldsymbol{\omega}}\right)x(t+\theta)\mathrm{d}\theta>0, \tag{51}$$

$$\int_{-\tau}^{0}\left(\tau^{-1}-\phi^T(\theta)P_1^{-1}\phi(\theta)-\sum_{h=1}^{d}\phi^T(\theta)X_1Q_h^{-1}X_1\phi(\theta)\right)\mathrm{d}\theta>0. \tag{52}$$

Thus, (51) is satisfied if

$$\begin{aligned}&P_1^{-1}-K_{\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2}K_{\mathbf{w}\boldsymbol{\omega}}>0,\\ &e^{-2\alpha\tau_h}Q_h-F_{h\mathbf{w}\boldsymbol{\omega}}^T 2\mu^{-2}F_{h\mathbf{w}\boldsymbol{\omega}}>0,\quad h\in\{1,\dots,d\}\end{aligned} \tag{53}$$

hold too. From the latter inequalities and using the Schur complement together with congruence property with block – diag$[P_1, I]$, conditions (44) and (45) follow with $M_{\mathbf{w}\omega}=K_{\mathbf{w}\omega}P_1^{-1}$, $N_{h\mathbf{w}\omega}=F_{h\mathbf{w}\omega}P_1^{-1}$, and $R_h=P_1Q_hP_1$, $h\in\{1,2,\dots,d\}$. Once again, employing the Schur complement on (52) gives

$$\begin{bmatrix}\tau^{-1} & \phi^T(\theta) & \phi^T(\theta) & \cdots & \phi^T(\theta)\\ \phi(\theta) & P_1 & 0 & \cdots & 0\\ \phi(\theta) & 0 & P_1Q_1P_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ \phi(\theta) & 0 & 0 & \cdots & P_1Q_dP_1\end{bmatrix}>0,\quad \forall\theta\in[-\tau,0], \tag{54}$$

which yields (46) after the substitution of $R_h=P_1Q_hP_1, h\in\{1,2,\dots,d\}$.

4. Examples

Next, a numerical example as well as the well-known inverted pendulum on a car is employed in order to illustrate the effectiveness of the proposed results. The LMI conditions have been checked with the LMIToolbox [6] within MATLAB 2109a.

4.1. Systems in Standard Form versus Descriptor Form. The following numerical example illustrates the advantages of the descriptor structure over standard state-space representations. Firstly, it compares stability at the origin via the LMIs in Theorem 1 and those in Corollary 2 by means of their feasibility sets. Secondly, a delayed nonlinear control law is designed via Theorem 2.

Consider a nonlinear system with two delays ($d=2$) in the descriptor form (1):

$$\begin{aligned}E(x)\dot{x}(t)&=A(x)x(t)+A_{\tau_1}(x)x(t-\tau_1)\\&\quad+A_{\tau_2}(x)x(t-\tau_2)+Bu,\\ x(\theta)&=\phi(\theta),\quad \theta\in[-\tau,0],\end{aligned} \tag{55}$$

where the time delays are $\tau_1=0.1$ y $\tau_2=0.3=\tau$, and matrices are as follows:

$$\begin{aligned}E(x)&=\begin{bmatrix}0.8 & 0.1-\dfrac{3}{5x_2^2+5}\\ 0.08 & 0.97\end{bmatrix},\\ A(x)&=\begin{bmatrix}-0.5\cos x_2-7.5 & -1\\ \dfrac{0.083(7x_1-7\sin x_1)}{x_1}+a & -5.5\end{bmatrix},\\ B&=\begin{bmatrix}0\\1\end{bmatrix},\\ A_{\tau_1}(x)&=\begin{bmatrix}9.5+b & 14\\ -6.6 & -\dfrac{0.067(155x_1-10\sin x_1)}{x_1}\end{bmatrix},\\ A_{\tau_2}(x)&=\begin{bmatrix}\cos x_2-4.2 & -6.2\\ 3.6 & 5.1\end{bmatrix}.\end{aligned} \tag{56}$$

Descriptor form: following the sector nonlinearity approach, the following nonlinear terms have been identified: $\zeta_1=(x_2^2+1)^{-1}\in[0,1]$, $z_1=\cos x_2\in[-1,1]$, and $z_2=(\sin(x_1)/x_1)\in[-0.2,1]$; their bounds have been calculated in the region $\Omega_x=\mathbb{R}^2$. Thus the vertex matrices are

$$A_1=\begin{bmatrix}-7 & -1\\ 0.5+a & -5.5\end{bmatrix},$$

$$A_2=\begin{bmatrix}-7 & -1\\ -0.2+a & -5.5\end{bmatrix},$$

$$A_3=\begin{bmatrix}-8 & -1\\ 0.5+a & -5.5\end{bmatrix},$$

$$A_4=\begin{bmatrix}-8 & -1\\ -0.2+a & -5.5\end{bmatrix},$$

$$A_{\tau_1 1}=\begin{bmatrix}9.4+b & 14.2\\ -6.6 & -10.2\end{bmatrix},$$

$$A_{\tau_1 2}=\begin{bmatrix}9.4+b & 14.2\\ -6.6 & -9.4\end{bmatrix},$$

$$A_{\tau_1 3}=\begin{bmatrix}9.4+b & 14.2\\ -6.6 & -10.2\end{bmatrix},$$

$$A_{\tau_1 4}=\begin{bmatrix}9.4+b & 14.2\\ -6.6 & -9.4\end{bmatrix},$$

$$A_{\tau_2 1}=\begin{bmatrix}-5.2 & -6.2\\ 3.6 & 5.1\end{bmatrix},$$

$$\begin{gathered}A_{\tau_2 2}=\begin{bmatrix}-5.2 & -6.2\\ 3.6 & 5.1\end{bmatrix},\\ A_{\tau_2 3}=\begin{bmatrix}-3.2 & -6.2\\ 3.6 & 5.1\end{bmatrix},\\ A_{\tau_2 4}=\begin{bmatrix}-3.2 & -6.2\\ 3.6 & 5.1\end{bmatrix},\\ B_1=B_2=B_3=B_4=\begin{bmatrix}0\\ 1\end{bmatrix},\\ E_1=\begin{bmatrix}0.8 & 0.1\\ 0.08 & 0.97\end{bmatrix},\\ E_2=\begin{bmatrix}0.8 & -0.5\\ 0.08 & 0.97\end{bmatrix}.\end{gathered} \tag{57}$$

Commonly, systems of the form (55) are analyzed in the following standard form:

$$\dot{x}=\widetilde{A}(x)x(t)+\widetilde{A}_{\tau_1}(x)x(t-\tau_1)+\widetilde{A}_{\tau_2}(x)x(t-\tau_2)+\widetilde{B}(x)u, \tag{58}$$

where

$$\begin{gathered}\widetilde{A}(x)=E^{-1}(x)A(x)=\frac{x_2^2+1}{0.768x_2^2+0.816}\begin{bmatrix}0.97 & -0.1+\dfrac{3}{5x_2^2+5}\\ -0.08 & 0.8\end{bmatrix}\begin{bmatrix}-0.5\cos x_2-7.5 & -1\\ \dfrac{0.083(7x_1-7\sin x_1)}{x_1}+a & -5.5\end{bmatrix},\\ \widetilde{A}_{\tau_1}(x)=E^{-1}(x)A_{\tau_1}(x)=\frac{x_2^2+1}{0.768x_2^2+0.816}\begin{bmatrix}0.97 & -0.1+\dfrac{3}{5x_2^2+5}\\ -0.08 & 0.8\end{bmatrix}\begin{bmatrix}9.5+b & 14\\ -6.6 & -\dfrac{0.067(155x_1-10\sin x_1)}{x_1}\end{bmatrix},\\ \widetilde{A}_{\tau_2}(x)=E^{-1}(x)A_{\tau_2}(x)=\frac{x_2^2+1}{0.768x_2^2+0.816}\begin{bmatrix}0.97 & -0.1+\dfrac{3}{5x_2^2+5}\\ -0.08 & 0.8\end{bmatrix}\begin{bmatrix}\cos x_2-4.2 & -6.2\\ 3.6 & 5.1\end{bmatrix},\\ \widetilde{B}(x)=E^{-1}(x)B=\frac{x_2^2+1}{0.768x_2^2+0.816}\begin{bmatrix}0.97 & -0.1+\dfrac{3}{5x_2^2+5}\\ -0.08 & 0.8\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}.\end{gathered} \tag{59}$$

In this case, four nonlinear terms are defined as follows: $z_1 = \cos x_2 \in [-1, 1]$, $z_2 = \sin x_1/x_1 \in [-0.2, 1]$, $z_3 = (x_2^2 + 1)^{-1} \in [0, 1]$, and $z_4 = (x_2^2 + 1)/(0.768x_2^2 + 0.816) \in [1.2255, 1.3021]$; their bounds hold within Ω_x. Some of the vertex matrices are given below:

$$\begin{aligned}
\widetilde{A}_1 &= \begin{bmatrix} -8.3824 & -0.5147 \\ 1.1765 & -5.2942 \end{bmatrix}, \\
\widetilde{A}_{10} &= \begin{bmatrix} -10.1694 & -0.5469 \\ 1.3542 & -5.6251 \end{bmatrix}, \\
\widetilde{A}_{15} &= \begin{bmatrix} -9.6324 & -4.5589 \\ 0.5882 & -5.2942 \end{bmatrix}, \\
\widetilde{A}_{\tau_1 1} &= \begin{bmatrix} 11.9829 & 18.13 \\ -7.3922 & -11.3922 \end{bmatrix}, \\
\widetilde{A}_{\tau_1 10} &= \begin{bmatrix} 12.732 & 19.263 \\ -7.8543 & -12.104 \end{bmatrix}, \\
\widetilde{A}_{\tau_1 15} &= \begin{bmatrix} 7.13 & 11.12 \\ -7.3922 & -10.608 \end{bmatrix}, \\
\widetilde{A}_{\tau_2 1} &= \begin{bmatrix} -6.6226 & -7.9952 \\ 4.0392 & 5.6079 \end{bmatrix}, \\
\widetilde{A}_{\tau_2 10} &= \begin{bmatrix} -4.5105 & -8.4949 \\ 4.0834 & 5.9584 \end{bmatrix}, \\
\widetilde{A}_{\tau_2 15} &= \begin{bmatrix} -1.5981 & -4.2451 \\ 3.8432 & 5.6079 \end{bmatrix}, \\
\widetilde{B}_1 &= \begin{bmatrix} -0.12255 \\ 0.9804 \end{bmatrix}, \\
\widetilde{B}_{10} &= \begin{bmatrix} -0.13021 \\ 1.0417 \end{bmatrix}, \\
\widetilde{B}_{15} &= \begin{bmatrix} 0.61275 \\ 0.9804 \end{bmatrix}.
\end{aligned} \tag{60}$$

Note that for this example, we have the following:

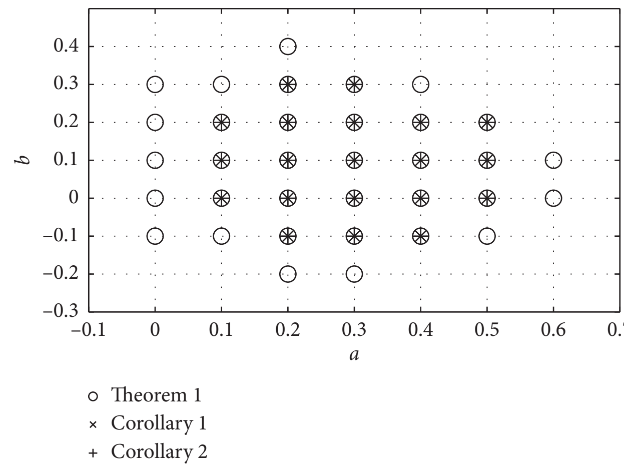


Figure 1: Feasibility sets for Theorem 1, Corollary 1, and Corollary 2 are applied to the example given in Subsection 4.1.

(i) With respect to its exact convex representation, the descriptor one has $(E_k, A_i, A_{\tau_i}, B), k = 1, 2, i = 1, 2, 3, 4$, i.e., 8 vertexes, while the standard has $(\widetilde{A}_i, \widetilde{A}_{\tau_i}, \widetilde{B}_i), i = 1, 2, \ldots, 2^4$, i.e., 16 vertexes. Additionally, the descriptor keeps a constant matrix B; thus, descriptor form requires less computational resources [64]. Indeed, in this case with $n = 2, m = 1$, $r = 4$, $\rho = 2$, $\widetilde{r} = 16$, and $d = 2$, the computational complexity using Theorem 1 is 7.2567 with 35 LMIs, using Corollary 1 is 9.7228 with 35 LMIs, and using Corollary 2 is 4.8713 with 19 LMIs.

(ii) With respect to feasibility sets for Theorem 1, and Corollaries 1 and 2, when $u = 0$ and parameter values as $a \in [0, 0.8]$, $b \in [-0.2, 0.5]$ for convex representations of systems (55) and (58), respectively. In Figure 1, the regions marked with a circle ($\circ$) correspond to the feasibility sets using Theorem 1, while the regions marked by ($\times$) and ($+$) are the feasibility sets obtained using Corollaries 1 and 2, respectively. It can be seen that by using the descriptor form the feasibility set is larger, i.e., the descriptor approach provides more relaxed results than the standard approach.

Thus, results given in Theorem 1 improve the previously classic results found in the literature.

On the other hand, Figure 2 shows that the system response (55) does not converge to the trivial equilibrium point when $a = -2.5$, $b = 6.2$, $u = 0$, $\tau_1 = 0.1$, $\tau_2 = 0.3$, and $\phi(\theta) = \begin{bmatrix} -5 & 5 \end{bmatrix}^T$, $\theta \in [-0.3, 0]$.

Next, a delayed nonlinear controller of the form (23) is given by

$$u = K_{\mathbf{w}\boldsymbol{\omega}} x(t) + \sum_{h=1}^{2} F_{h\mathbf{w}\boldsymbol{\omega}} x(t - \tau_h), \tag{61}$$

which is employed to stabilize this system. To this end, LMI conditions in Theorem 2 with an exponential decay $\alpha = 0.5$ together with those from in Theorem 3 for $u < 48 = \mu$ render feasible solution providing the following values:

$$P_1^{-1} = \begin{bmatrix} 0.20603 & 0.19458 \\ 0.19458 & 0.21717 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 3.5866 & 3.3247 \\ 3.3247 & 3.1135 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 1.0972 & 1.215 \\ 1.215 & 1.4539 \end{bmatrix},$$
$$\begin{aligned}
K_{11} &= [\,-11.0453 \;\; -13.9169\,], \\
F_{111} &= [\,-11.7060 \;\; -6.5843\,], \\
F_{211} &= [\,2.5606 \;\; 2.2870\,], \\
K_{12} &= [\,-7.3722 \;\; -12.1755\,], \\
F_{112} &= [\,-13.9524 \;\; -8.5677\,], \\
F_{212} &= [\,3.2894 \;\; 3.1315\,], \\
K_{21} &= [\,-10.4266 \;\; -13.6762\,], \\
F_{121} &= [\,-10.9789 \;\; -6.6378\,], \\
F_{221} &= [\,2.2735 \;\; 1.9269\,], \\
K_{22} &= [\,-6.7172 \;\; -11.7399\,], \\
F_{122} &= [\,-13.2816 \;\; -8.7126\,], \\
F_{222} &= [\,3.0448 \;\; 2.8193\,], \\
K_{31} &= [\,-9.8536 \;\; -13.3457\,], \\
F_{131} &= [\,-9.7933 \;\; -4.8426\,], \\
F_{231} &= [\,-0.2040 \;\; 1.5048\,], \\
K_{32} &= [\,-6.7529 \;\; -11.8054\,], \\
F_{132} &= [\,-13.8964 \;\; -8.4844\,], \\
F_{232} &= [\,0.6296 \;\; 3.0749\,], \\
K_{41} &= [\,-9.0894 \;\; -12.9706\,], \\
F_{141} &= [\,-9.2846 \;\; -5.0947\,], \\
F_{241} &= [\,-0.3599 \;\; 1.2441\,], \\
K_{42} &= [\,-5.7661 \;\; -11.1366\,], \\
F_{142} &= [\,-13.6875 \;\; -9.0667\,], \\
F_{242} &= [\,0.5619 \;\; 2.9637\,].
\end{aligned} \tag{62}$$

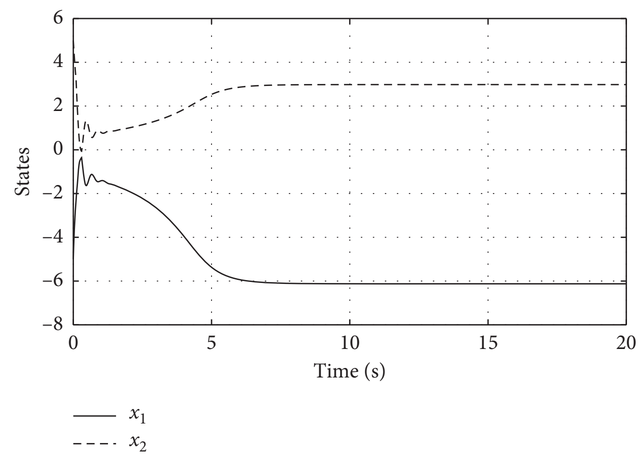


Figure 2: System response of the example given in Subsection 4.1 (descriptor form) when $a = -2.5$, $b = 6.2$, and $u = 0$.

Effectively, system (55) is stabilized at the origin as shown in Figure 3. The evolution in time of the system state is shown in Figure 3(a), while the guaranteed exponential decay in the system response is depicted in Figure 3(b).

4.2. Nonlinear Controller versus Delayed Nonlinear Controller. The following example is to illustrate the advantages of the use of artificial delays in controllers when there is the presence of noise, as mentioned in the introduction and its corroboration by various results found in the literature.

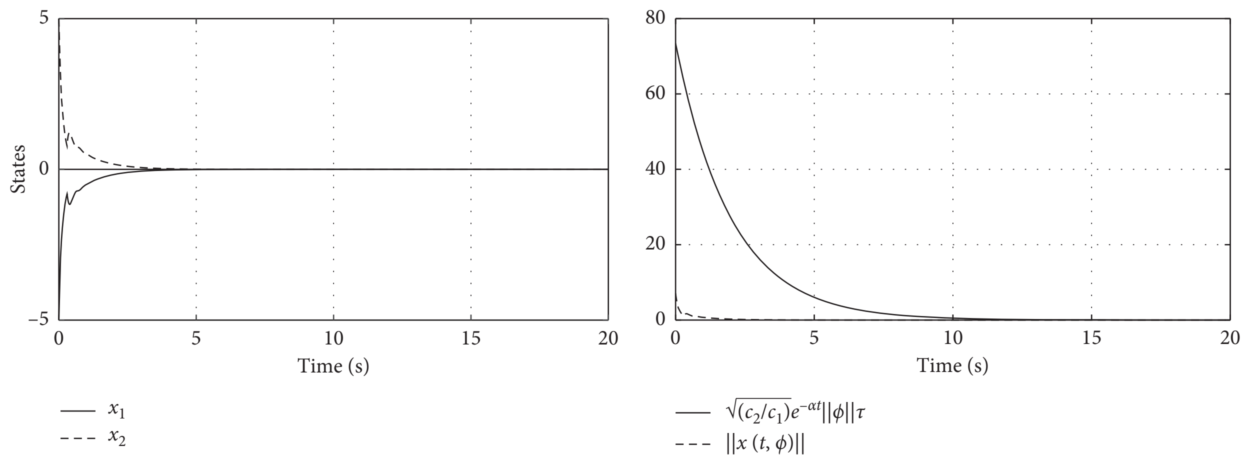
Consider the system known as the car-pendulum, whose scheme is shown in Figure 4, a mathematical model is given by

$$\begin{aligned}
&\begin{bmatrix} M_1 + M_2 & -M_2 l \cos\theta \\ -M_2 l \cos\theta & J + M_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}}(t) \\ \ddot{\theta}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} c\dot{\bar{x}} + M_2 l \dot{\theta}^2 \sin\theta \\ \gamma\dot{\theta} - M_2 g l \sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.
\end{aligned} \tag{63}$$

For illustrative purposes, a delayed measurement $\tau > 0$ in the positions of the car and pendulum are intentionally added. Also, we define $x_1 = \bar{x}$, $x_2 = \dot{\bar{x}}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, and $u = F$. Thus, for $g = 9.81\,\mathrm{m/s^2}$, $l = 0.304\,\mathrm{m}$, $M_2 = 0.2\,\mathrm{kg}$, $M_1 = 1.3282\,\mathrm{kg}$, $J = (M_2 l^2)/3$, $c = 0.001$, $\gamma = 0.001$, and $\tau = 0.05$, the system (63) can be rewritten as

$$E(x)\dot{x}(t) = A(x)x(t) + A_\tau(x)x(t-\tau) + Bu(t). \tag{64}$$

where



(a) (b)

Figure 3: System response of the example given in Section 4.1 (descriptor form) using delayed controller (61). (a) Time evolution of the closed-loop system. (b) Guaranteed exponential decay $\alpha = 0.5$ for the closed-loop system.

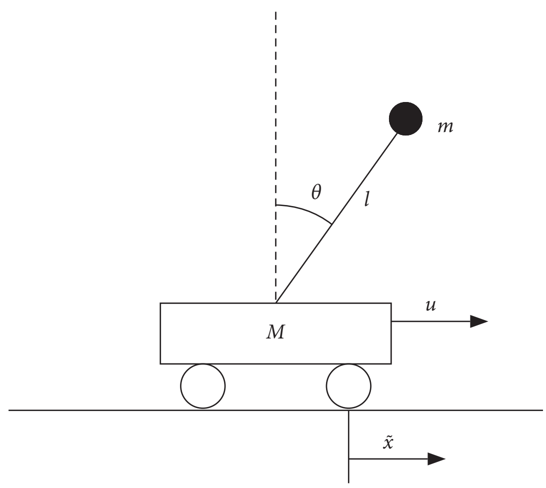


Figure 4: Schematic of the inverted pendulum system.

$$E(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.5482 & 0 & -0.0669\cos x_3 \\ 0 & 0 & 1 & 0 \\ 0 & -0.0669\cos x_3 & 0 & 0.0271 \end{bmatrix}, \quad A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.001 & 0 & -0.0669 x_4 \sin x_3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{0.6561\sin x_3}{x_3} & -0.001 \end{bmatrix},$$

$$A_\tau(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{65}$$

Considering the region $\Omega_x = \{x\colon |x_1| \le 2\,\text{m}, |x_2| \le 3\,\text{m/s}, |x_3| \le \pi/3\,\text{rad}, |x_4| \le 4\,\text{rad/s}\}$, the nonconstant terms and their bounds are $\zeta_1 = \cos x_3 \in [0.5, 1]$, $z_1 = x_4 \sin x_3 \in [-0.4238, 0.4238]$, and $z_2 = \sin x_3 / x_3 \in [0.827, 1]$; the vertex matrices are

$$A_1 = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -0.001 & 0 & -0.0258 \\ 0 & 0 & 0 & 1.0 \\ 0 & 0 & -0.4933 & -0.001 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.001 & 0 & -0.025 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.596 & -0.001 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.001 & 0 & 0.0258 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.4933 & -0.001 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.001 & 0 & 0.0258 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5964 & -0.001 \end{bmatrix},$$
$$A_{\tau 1} = A_{\tau 2} = A_{\tau 3} = A_{\tau 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}, \tag{66}$$
$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.5282 & 0 & -0.0304 \\ 0 & 0 & 1 & 0 \\ 0 & -0.0304 & 0 & 0.0370 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.5482 & 0 & -0.0608 \\ 0 & 0 & 1 & 0 \\ 0 & -0.0608 & 0 & 0.0370 \end{bmatrix}.$$

As mentioned above, with the purpose of showing the advantages of using a controller with delayed action in the presence of noise; for this example, two controllers are used: a nonlinear controller of the following form is

$$u = K_{\mathbf{w}\boldsymbol{\omega}} x(t), \tag{67}$$

and a delayed nonlinear controller of the form (23) is given by

$$u = K_{\mathbf{w}\boldsymbol{\omega}} x(t) + F_{\mathbf{w}\boldsymbol{\omega}} x(t-\tau). \tag{68}$$

To illustrate the effectiveness of Theorem 3, the controllers (68) are conditioned to satisfy that $\|u(t)\| < 12$. For the controller (67), the same condition is requested, for which Theorem 3 can be used, after some simple adjustments when considering free-delays controller. For the controller (67), the corresponding LMI conditions in Theorem 2 with an exponential decay $\alpha = 0.003$ are found feasible with the following values:

$$P_1 = \begin{bmatrix} 0.02024 & 0.04411 & 0.1428 & -0.005929 \\ 0.04411 & 0.1874 & 0.6166 & -0.02698 \\ 0.1428 & 0.6166 & 6.036 & 0.1574 \\ -0.005929 & -0.02698 & 0.1574 & 0.2537 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.0077 & 0.0225 & 0.1455 & 0.00005 \\ 0.02258 & 0.09603 & 0.4836 & 0.0042 \\ 0.1455 & 0.4836 & 7.879 & 0.0778 \\ 0.00005 & 0.0042 & 0.0778 & 0.1026 \end{bmatrix},$$
$$K_{11} = \begin{bmatrix} -0.3685 & -1.71 & -8.965 & -3.468 \end{bmatrix},$$
$$K_{12} = \begin{bmatrix} -0.4382 & -1.979 & -8.867 & -2.879 \end{bmatrix},$$
$$K_{21} = \begin{bmatrix} -0.3567 & -1.626 & -5.913 & -3.452 \end{bmatrix}, \tag{69}$$
$$K_{22} = \begin{bmatrix} -0.4368 & -1.95 & -6.884 & -2.852 \end{bmatrix},$$
$$K_{31} = \begin{bmatrix} -0.3619 & -1.668 & -8.823 & -3.516 \end{bmatrix},$$
$$K_{32} = \begin{bmatrix} -0.435 & -1.955 & -8.748 & -2.915 \end{bmatrix},$$
$$K_{41} = \begin{bmatrix} -0.3483 & -1.573 & -5.728 & -3.505 \end{bmatrix},$$
$$K_{42} = \begin{bmatrix} -0.4333 & -1.924 & -6.765 & -2.889 \end{bmatrix}.$$

On the other hand, for the delayed controller (68), LMI conditions in Theorem 2 with an exponential decay $\alpha = 0.003$ are also feasible with the following values:

$$
\begin{aligned}
P_1^{-1} &= \begin{bmatrix} 0.0020 & 0.0062 & 0.02261 & -0.0005 \\ 0.0062 & 0.0422 & 0.1531 & -0.003 \\ 0.0226 & 0.1531 & 2.366 & 0.07454 \\ -0.0005 & -0.0030 & 0.0745 & 0.1213 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 0.0003 & 0.0016 & 0.0134 & 0.0002 \\ 0.0016 & 0.0114 & 0.0789 & 0.002 \\ 0.0134 & 0.0789 & 2.284 & 0.031 \\ 0.0002 & 0.002 & 0.031 & 0.0418 \end{bmatrix}, \\
K_{11} &= \begin{bmatrix} -0.1389 & -0.9876 & -4.926 & -2.323 \end{bmatrix}, \\
F_{11} &= \begin{bmatrix} -0.092 & -0.0006 & -5.691 & 0.0014 \end{bmatrix}, \\
K_{12} &= \begin{bmatrix} -0.1457 & -0.99 & -4.341 & -2.052 \end{bmatrix}, \\
F_{12} &= \begin{bmatrix} -0.0867 & -0.0001 & -3.54 & 0.0004 \end{bmatrix}, \\
K_{21} &= \begin{bmatrix} -0.1368 & -0.9425 & -2.012 & -2.263 \end{bmatrix}, \\
F_{21} &= \begin{bmatrix} -0.0932 & -0.001 & -5.956 & 0.0011 \end{bmatrix}, \\
K_{22} &= \begin{bmatrix} -0.1479 & -0.9792 & -2.502 & -2.001 \end{bmatrix}, \\
F_{22} &= \begin{bmatrix} -0.0870 & -0.0001 & -3.634 & 0.0004 \end{bmatrix}, \\
K_{31} &= \begin{bmatrix} -0.1394 & -0.9793 & -4.867 & -2.345 \end{bmatrix}, \\
F_{31} &= \begin{bmatrix} -0.0928 & -0.0008 & -5.829 & 0.0017 \end{bmatrix}, \\
K_{32} &= \begin{bmatrix} -0.1459 & -0.9801 & -4.278 & -2.078 \end{bmatrix}, \\
F_{32} &= \begin{bmatrix} -0.0870 & -0.0001 & -3.628 & 0.0004 \end{bmatrix}, \\
K_{41} &= \begin{bmatrix} -0.1373 & -0.9317 & -1.924 & -2.284 \end{bmatrix}, \\
F_{41} &= \begin{bmatrix} -0.0944 & -0.00135 & -6.136 & 0.0019 \end{bmatrix}, \\
K_{42} &= \begin{bmatrix} -0.1485 & -0.9693 & -2.432 & -2.025 \end{bmatrix}, \\
F_{42} &= \begin{bmatrix} -0.0873 & -0.0002 & -3.74 & 0.0005 \end{bmatrix}.
\end{aligned}
\tag{70}
$$

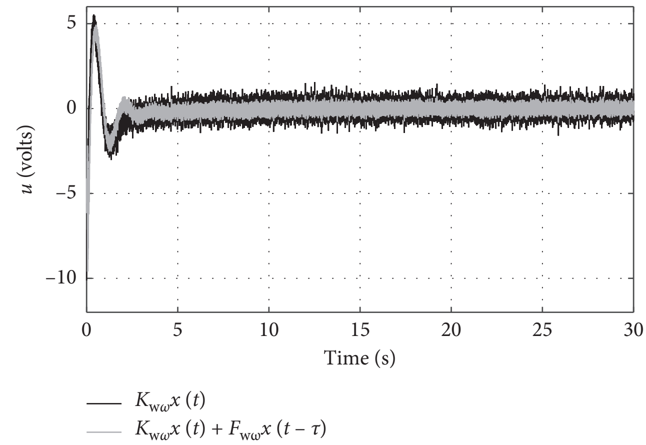


FIGURE 5: Time evolution of control laws (67) and (68).

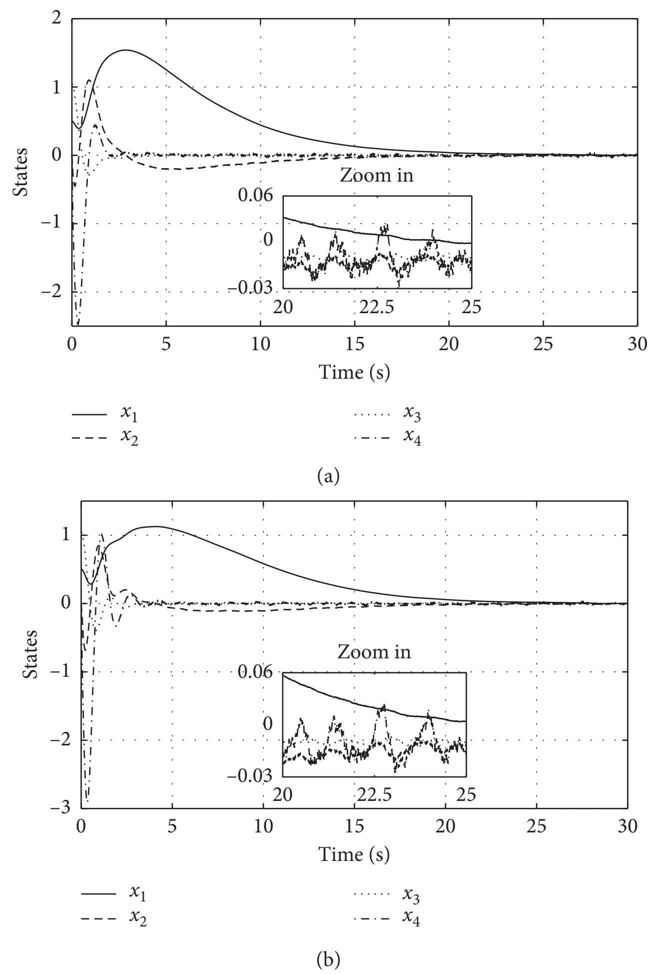


FIGURE 6: System response of the example given in Section 4.2. Time evolution of the closed-loop system with (a) (28) and (b) (68).

To simulate the noise present in the sensors, a random signal with variance 0.001 and a step 0.001 is introduced at the system input. In Figure 5, the applied control signals are plotted, and it can be seen that the control signal presents less noisy when using the delayed controller than the other.

In Figure 6, the evolution in time of the state of the system (64), with $\phi(\theta) = \begin{bmatrix} 0.5 & 0 & \pi/3 & 0 \end{bmatrix}^T$, $\theta \in [-0.05, 0]$, under the control laws (67) and (68), is shown. It can be seen that the overshoot is greater when using a controller without a delayed action.

Remark 3. Systems of the form (63) can be stabilized by a free-delay controller of the form (67). However, using delayed controllers of the form (68) or (23) to stabilize this system class may be a better option when systems have inherent noise.

5. Conclusions

In this paper, analysis and design using a convex approach for nonlinear descriptor systems with multiple delays have been presented. This analysis allows synthesizing delayed nonlinear controllers to ensure convergence of the system trajectories with a guaranteed exponential decay; moreover, conditions for bounding the control input avoid possible saturation in the actuators have been provided. It also has been shown that keeping the descriptor form increases the possibility of obtaining feasibility in the LMI conditions, unlike the use of standard forms. Also, it is observed that including deliberately delays in the controller can reduce noise in the control signal, thus avoiding mechanical wear of the actuators. As future work, an extension of the proposed results in nonlinear descriptor systems with multiple time-varying delays is in course, since it will allow the synthesis of controllers for a larger class of systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interests regarding the publication of this paper.

Acknowledgments

This work was partially supported by the CONACYT scholarship 998399 (J. A. Romero-Vega).

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