Research Article

Numerical Characterizations of Topological Reductions of Covering Information Systems in Evidence Theory

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The reductions of covering information systems in terms of covering approximation operators are one of the most important applications of covering rough set theory. Based on the connections between the theory of topology and the covering rough set theory, two kinds of topological reductions of covering information systems are discussed in this paper, which are characterized by the belief and plausibility functions from the evidence theory. The topological spaces by two pairs of covering approximation operators in covering information systems are pseudo-discrete, which deduce partitions. Then, using plausibility function values of the sets in the partitions, the definitions of significance and relative significance of coverings are presented. Hence, topological reduction algorithms based on the evidence theory are proposed in covering information systems, and an example is adopted to illustrate the validity of the algorithms.

1. Introduction

The theory of rough set is a new mathematical method to deal with imprecision, vagueness, and uncertainty in data analysis [1]. The basic notions in the rough set theory are the lower and upper approximation operators. The attribute reduction of datasets in terms of the lower and upper approximation operators is one of most important applications of the rough set theory. The covering rough set is one of the most important extensions of the classical Pawlak rough sets, which has received increasing concern in recent years [2–21]. By using the covering approximation operators, attribute reductions of covering information systems may be unraveled [22–32].

The theory of topology has many applications in almost all branches of mathematics and many real life applications. The topological interior and closure operators are two basic definitions in the topological theory. The theory of topology has a close contact with the theory of rough set based on the connections between the topological interior and closure operators and the lower and upper approximation operators,

and there exists much result on relationships between topology and covering rough sets [20, 33–39]. Moreover, topological reductions for three types of covering rough sets in covering information systems have been discussed [40].

The Dempster–Shafer theory of evidence or the theory of belief function [41, 42] is an important method to deal with uncertainty in information systems. The plausibility and belief functions construct a dual pair of uncertainty measures in Dempster–Shafer theory of evidence. There exist strong connections between the Dempster–Shafer theory of evidence and the rough set theory. For example, the relationships between the belief functions and covering rough sets are discussed [23, 43–45]. Furthermore, the evidence theory was used to characterize knowledge reductions for covering rough sets in covering information systems [23, 46–48].

The purpose of this paper is to characterize two types of topological reductions of covering information systems by evidence theory. In Section 2, we review basic definitions of covering rough sets, topology, and evidence theory. Properties of two pairs of covering approximation operators and

the topologies induced by the two pairs of covering approximation operators are also presented. In Section 3, topological reductions in covering information systems are characterized by the belief and plausibility functions from the evidence theory. Using plausibility function values of the sets in the partitions, the definitions of significance and relative significance of coverings are also developed. Then, topological reduction algorithms based on the evidence theory are proposed in covering information systems, and an example is adopted to illustrate the validity of the algorithms. In Section 4, we compare a type of topological reduction with a kind of reduction proposed in [23].

2. Preliminaries

In this section, we introduce some basic definitions of topology, covering rough sets, and evidence theory. Throughout this paper, we always assume that the universe of discourse U is a finite and nonempty set unless other statements. The class of all subsets of U will be denoted by $\mathcal{P}(U)$.

2.1. Basic Concepts in Topology. In this section, some basic concepts of topological spaces are reviewed. For the other basic topological concepts, we refer to [49].

Definition 1 (see [49]). Let U be a nonempty set. A topology on U is a collection τ of subsets of U having the following properties:

- (1) $\emptyset$ and U are in τ
- (2) The union of the elements of any subcollection of τ is in τ
- (3) The intersection of the elements of any finite subcollection of τ is in τ

Then, (U, τ) is called a topological space, each element in τ is called an open set, and the complement of an open set is called a closed set. In a topological space (U, τ) , if $A \subseteq U$ is open in U and if and only if A is closed in U , then (U, τ) is called a pseudo-discrete space.

Definition 2 (see [49]). Let (U, τ) be a topological space and $X \in \mathcal{P}(U)$. Then, the topological interior and closure of X are, respectively, defined by $\text{int}_\tau(X) = \cup \{G \mid G \text{ is an open set and } G \subseteq X\}$, $\text{cl}_\tau(X) = \cap \{K \mid K \text{ is a closed set and } X \subseteq K\}$. int_τ and cl_τ are, respectively, called the topological interior operator and the topological closure operator of τ .

It can be shown that $\text{cl}_\tau(X)$ is a closed set and $\text{int}_\tau(X)$ is an open set in (U, τ) . X is an open set in (U, τ) if and only if $\text{int}_\tau(X) = X$, and X is a closed set in (U, τ) if and only if $\text{cl}_\tau(X) = X$. The topological interior and closure operators can be also defined by Kuratowski interior and closure axioms.

Definition 3 (see [49, 50]). Let U be a nonempty set, $\text{int} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$, and $\text{cl} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$. For any $X, Y \subseteq U$,

- (i1) $\text{int}(U) = U$.
- (i2) $\text{int}(X) \subseteq X$.
- (i3) $\text{int}(X \cap Y) = \text{int}(X) \cap \text{int}(Y)$.
- (i4) $\text{int}(\text{int}(X)) = \text{int}(X)$.
- (c1) $\text{cl}(\emptyset) = \emptyset$.
- (c2) $X \subseteq \text{cl}(X)$.
- (c3) $\text{cl}(X \cup Y) = \text{cl}(X) \cup \text{cl}(Y)$.
- (c4) $\text{cl}(\text{cl}(X)) = \text{cl}(X)$.

If int satisfies (i1)–(i3), then int is called an interior operator, and (U, int) is called an interior space [50]. If int satisfies (i1)–(i4), that is, int satisfies Kuratowski interior axiom, then int is called a topological interior operator [49]. If cl satisfies (c1)–(c3), then cl is called a closure operator, and (U, cl) is called a closure space [50]. If cl satisfies (c1)–(c4), that is, cl satisfies Kuratowski closure axiom, then cl is called a topological closure operator [49].

In an interior space (U, int) , it is easy to prove that $\tau(\text{int}) = \{X \mid \text{int}(X) = X\}$ is a topology. In a closure space (U, cl) , it is easy to verify that $\tau(\text{cl}) = \{U - X \mid \text{cl}(X) = X\}$ is a topology.

2.2. Basic Definitions of Covering Rough Sets. We present definitions of two pairs of covering approximation operators.

Definition 4 (see [7, 8, 18, 24]). Let $\mathcal{C}$ be a covering of the universe U . For any $x \in U$, $(x)_\mathcal{C} = \cap \{K \in \mathcal{C} \mid x \in K\}$ is called a neighborhood of x by $\mathcal{C}$. Define two pairs of covering approximation operators $(FL_\mathcal{C}, FH_\mathcal{C})$ and $(LL_\mathcal{C}, LH_\mathcal{C})$ as follows: $\forall X \subseteq U$,

$$FL_\mathcal{C}(X) = \{x \mid \forall K \in \mathcal{C} \ x \in K \iff K \subseteq X\} \quad [7].$$

$$FH_\mathcal{C}(X) = \cup \{K \mid K \in \mathcal{C} \ K \cap X \neq \emptyset\} \quad [18].$$

$$LL_\mathcal{C}(X) = \{x \mid \forall y \in U, x \in (y)_\mathcal{C} \iff (y)_\mathcal{C} \subseteq X\} \quad [8].$$

$$LH_\mathcal{C}(X) = \cup \{(x)_\mathcal{C} \mid (x)_\mathcal{C} \cap X \neq \emptyset\} \quad [8, 24].$$

The pair of covering approximation operators $(FL_\mathcal{C}, FH_\mathcal{C})$ was discussed in [7, 18, 21, 51, 52], and the pair of covering approximation operators $(LL_\mathcal{C}, LH_\mathcal{C})$ was explored in [8, 24, 53].

For any $x \in U$, define $\text{st}(x, \mathcal{C}) = \cup \{K \in \mathcal{C} \mid x \in K\}$. Then, $FL_\mathcal{C}(X) = \{x \in U \mid \text{st}(x, \mathcal{C}) \subseteq X\}$. It is easy to obtain that for any $x, y \in U$, (1) $x \in (x)_\mathcal{C}$, (2) if $x \in (y)_\mathcal{C}$, then $(x)_\mathcal{C} \subseteq (y)_\mathcal{C}$, (3) $x \in \text{st}(x, \mathcal{C})$, and (4) $x \in \text{st}(y, \mathcal{C}) \iff y \in \text{st}(x, \mathcal{C})$.

Some basic properties of the pairs of approximation operators $(FL_\mathcal{C}, FH_\mathcal{C})$ and $(LL_\mathcal{C}, LH_\mathcal{C})$ are presented in Proposition 1.

Proposition 1 (see [7, 18, 53, 54]). Let $\mathcal{C}$ be a covering of the universe U . Then, for any $X, Y \subseteq U$, we get

- (1) $FL_\mathcal{C}(\emptyset) = FH_\mathcal{C}(\emptyset) = \emptyset$,
 $FL_\mathcal{C}(U) = FH_\mathcal{C}(U) = U$.
- (2) $FL_\mathcal{C}(X \cap Y) = FL_\mathcal{C}(X) \cap FL_\mathcal{C}(Y)$,
 $FH_\mathcal{C}(X \cup Y) = FH_\mathcal{C}(X) \cup FH_\mathcal{C}(Y)$.

- (3) $FL_{\mathcal{C}}(X) \subseteq X \subseteq FH_{\mathcal{C}}(X)$.
- (4) $FL_{\mathcal{C}}(X) = X \Leftrightarrow FH_{\mathcal{C}}(X) = X$.
- (5) $FH_{\mathcal{C}}(U - X) = U - FL_{\mathcal{C}}(X)$.
- (6) $LL_{\mathcal{C}}(\emptyset) = LH_{\mathcal{C}}(\emptyset) = \emptyset$, $LL_{\mathcal{C}}(U) = LH_{\mathcal{C}}(U) = U$.
- (7) $LL_{\mathcal{C}}(X \cap Y) = LL_{\mathcal{C}}(X) \cap LL_{\mathcal{C}}(Y)$,
 $LH_{\mathcal{C}}(X \cup Y) = LH_{\mathcal{C}}(X) \cup LH_{\mathcal{C}}(Y)$.
- (8) $LL_{\mathcal{C}}(X) \subseteq X \subseteq LH_{\mathcal{C}}(X)$.
- (9) $LL_{\mathcal{C}}(X) = X \Leftrightarrow LH_{\mathcal{C}}(X) = X$.
- (10) $LH_{\mathcal{C}}(U - X) = U - LL_{\mathcal{C}}(X)$.

According to Definition 3 and Proposition 1, it is easy to get the following.

Corollary 1 (see [40, 54]). *Let $\mathcal{C}$ be a covering of the universe U . Then,*

- (1) $FL_{\mathcal{C}}$ and $LL_{\mathcal{C}}$ are interior operators.
- (2) $FH_{\mathcal{C}}$ and $LH_{\mathcal{C}}$ are closure operators.
- (3) $\tau(FH_{\mathcal{C}}) = \{X | FH_{\mathcal{C}}(U - X) = U - X\} = \tau(FL_{\mathcal{C}})$,
 $\tau(LH_{\mathcal{C}}) = \{X | LH_{\mathcal{C}}(U - X) = U - X\} = \tau(LL_{\mathcal{C}})$.
- (4) $(U, \tau(FH_{\mathcal{C}}))$ and $(U, \tau(LH_{\mathcal{C}}))$ are pseudo-discrete spaces.

By Corollary 1, $(U, \tau(FH_{\mathcal{C}}))$ and $(U, \tau(LH_{\mathcal{C}}))$ are pseudo-discrete spaces; then, $\tau(FH_{\mathcal{C}}) = \{X | FH_{\mathcal{C}}(X) = X\}$ and $\tau(LH_{\mathcal{C}}) = \{X | LH_{\mathcal{C}}(X) = X\}$. There also exists a conclusion about a pseudo-discrete space.

Lemma 1 (see [40]). *Let (U, τ) be a pseudo-discrete space; then, $\{(x)_{\tau} | x \in U\}$ is a partition of U .*

Proof. Suppose that there exist $x, y \in U$ such that $(x)_{\tau} \cap (y)_{\tau} \neq \emptyset$ and $(x)_{\tau} \neq (y)_{\tau}$. Then, $x \notin (y)_{\tau}$ or $y \notin (x)_{\tau}$. Otherwise, $(x)_{\tau} \subseteq (y)_{\tau}$ and $(y)_{\tau} \subseteq (x)_{\tau}$, and it follows that $(x)_{\tau} \subseteq (y)_{\tau}$, which contradicts that $(x)_{\tau} \neq (y)_{\tau}$. Without loss of generality, we assume that $x \notin (y)_{\tau}$. Since (U, τ) is pseudo-discrete, $(x)_{\tau} \cap (y)_{\tau}$ is open and closed. It implies that $x \in (x)_{\tau} - ((x)_{\tau} \cap (y)_{\tau})$ and $(x)_{\tau} - ((x)_{\tau} \cap (y)_{\tau})$ is an open and closed set. Since $(x)_{\tau} \cap (y)_{\tau} \neq \emptyset$, $(x)_{\tau} - ((x)_{\tau} \cap (y)_{\tau}) \subsetneq (x)_{\tau}$, which contradicts the definition of $(x)_{\tau}$. Consequently, $\{(x)_{\tau} | x \in U\}$ is a partition of U .

By Corollary 1(4) and Lemma 1, it is easy to get the following. $\square$

Corollary 2 (see [40]). *Let $\mathcal{C}$ be a covering of the universe U . Then, $\{(x)_{\tau(FH_{\mathcal{C}})} | x \in U\}$ and $\{(x)_{\tau(LH_{\mathcal{C}})} | x \in U\}$ are two partitions of U .*

2.3. Basic Notions Related to Evidence Theory. This section will recall some basic definitions about evidence theory.

Definition 5 (see [41, 42]). A set function $\text{Bel}: \mathcal{P}(U) \rightarrow [0, 1]$ is referred to as a belief function if

- (1) $\text{Bel}(\emptyset) = 0$, $\text{Bel}(U) = 1$.
- (2) For every collection of subsets $X_1, X_2, \dots, X_n \subseteq U$, $n \geq 1$,

$$\text{Bel}\left(\bigcup_{i=1}^n X_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1, 2, \dots, n\}} (-1)^{|I|+1} \text{Bel}\left(\bigcap_{i \in I} X_i\right), \quad (1)$$

where $|I|$ is the cardinality of the set I . A set function $Pl: \mathcal{P}(U) \rightarrow [0, 1]$ is referred to as a plausibility function if

- (1) $Pl(\emptyset) = 0$, $Pl(U) = 1$.
- (2) For every collection of subsets $X_1, X_2, \dots, X_n \subseteq U$, $n \geq 1$,

$$Pl\left(\bigcap_{i=1}^n X_i\right) \leq \sum_{\emptyset \neq I \subseteq \{1, 2, \dots, n\}} (-1)^{|I|+1} Pl\left(\bigcup_{i \in I} X_i\right). \quad (2)$$

Belief and plausibility functions based on the same belief structure are connected by the dual property $Pl(X) = 1 - \text{Bel}(-X)$. Furthermore, $\text{Bel}(X) \leq Pl(X)$ for all $X \subseteq U$.

Definition 6. Let Ω be a sample space and $\mathcal{F}$ be a σ -algebra on Ω . Then, a real-valued function $P: \mathcal{F} \rightarrow [0, 1]$ is referred to as a probability on $(\Omega, \mathcal{F})$ if it satisfies

- (1) For any $X \in \mathcal{F}$, $0 \leq P(X) \leq 1$.
- (2) $P(\Omega) = 1$.
- (3) For any $X_i \in \mathcal{F}$ ($i = 1, 2, \dots$), if $X_i \cap X_j = \emptyset$ ($i \neq j$), then $P(\bigcup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} P(X_i)$.

Moreover, $(\Omega, \mathcal{F}, P)$ is a probability space.

The probability of covering lower approximate operator and covering upper approximate operator are, respectively, belief and plausibility functions.

Proposition 2 (see [44, 47]). *Let $(U, \mathcal{P}(U), P)$ be a probability space and $\mathcal{C}$ be a covering on U . For any $A \subseteq U$, define*

$$\begin{aligned} \text{Bel}_{\mathcal{C}}^F(X) &= P(FL_{\mathcal{C}}(X)), \\ Pl_{\mathcal{C}}^F(X) &= P(FH_{\mathcal{C}}(X)), \\ \text{Bel}_{\mathcal{C}}^L(X) &= P(LL_{\mathcal{C}}(X)), \\ Pl_{\mathcal{C}}^L(X) &= P(LH_{\mathcal{C}}(X)), \end{aligned} \quad (3)$$

where $P(X) = (|X|/|U|)$. Then, $\text{Bel}_{\mathcal{C}}^F$ and $\text{Bel}_{\mathcal{C}}^L$ are belief functions and $Pl_{\mathcal{C}}^F$ and $Pl_{\mathcal{C}}^L$ are plausibility functions.

3. Evidence-Theory-Based Numerical Characterization of Topological Reductions in Covering Information Systems

A covering on U can be induced from a family of coverings $\mathbf{C}$ on U as follows.

Definition 7 (see [55]). Let $\mathbf{C}$ be a family of coverings on U . For any $\mathbf{B} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\} \subseteq \mathbf{C}$, define $\bigwedge \mathbf{B} \triangleq \mathcal{C}_1 \wedge \mathcal{C}_2 \wedge \dots \wedge \mathcal{C}_n = \{K_1 \cap K_2 \cap \dots \cap K_n \neq \emptyset | K_i \in \mathcal{C}_i, i = 1, 2, \dots, n\}$.

It is easy to obtain that $\bigwedge \mathbf{B}$ is a covering of U . For simplicity, $FL_{\bigwedge \mathbf{B}}$, $FH_{\bigwedge \mathbf{B}}$, $LL_{\bigwedge \mathbf{B}}$, $LH_{\bigwedge \mathbf{B}}$, $(x)_{\bigwedge \mathbf{B}}$, $\text{Bel}_{\bigwedge \mathbf{B}}^F$, $Pl_{\bigwedge \mathbf{B}}^F$, $\text{Bel}_{\bigwedge \mathbf{B}}^L$, and $Pl_{\bigwedge \mathbf{B}}^L$ are written as $FL_{\mathbf{B}}$, $FH_{\mathbf{B}}$, $LL_{\mathbf{B}}$, $LH_{\mathbf{B}}$, $(x)_{\mathbf{B}}$, $\text{Bel}_{\mathbf{B}}^F$, $Pl_{\mathbf{B}}^F$, $\text{Bel}_{\mathbf{B}}^L$, and $Pl_{\mathbf{B}}^L$, respectively.

It is clear that for any $\mathbf{B} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n\} \subseteq \mathbf{C}$, $X \subseteq U$,

- (1) $st(x, \wedge \mathbf{B}) = st(x, \mathcal{C}_1) \cap st(x, \mathcal{C}_2) \cap \dots \cap st(x, \mathcal{C}_n)$.
- (2) $(x)_{\mathbf{B}} = \cap_{\mathcal{C} \in \mathbf{B}} (x)_{\mathcal{C}}$.
- (3) $FL_{\mathbf{B}}(X) \subseteq FL_{\mathbf{C}}(X) \subseteq X \subseteq FH_{\mathbf{C}}(X) \subseteq FH_{\mathbf{B}}(X)$.
- (4) $LL_{\mathbf{B}}(X) \subseteq LL_{\mathbf{C}}(X) \subseteq X \subseteq LH_{\mathbf{C}}(X) \subseteq LH_{\mathbf{B}}(X)$.

Remark 1. From Corollary 2, we can know that $\{(x)_{\tau(FH_{\mathbf{C}})} | x \in U\}$ and $\{(x)_{\tau(LH_{\mathbf{C}})} | x \in U\}$ are two partitions of U . We denote $\{(x)_{\tau(FH_{\mathbf{C}})} | x \in U\} = \{K_1, K_2, \dots, K_m\} (m \in N)$ and $\{(x)_{\tau(LH_{\mathbf{C}})} | x \in U\} = \{G_1, G_2, \dots, G_n\} (n \in N)$. Let $\mathbf{C}$ be a family of coverings on U ; then, $(U, \mathbf{C})$ is called a covering information system in [23]. Two kinds of reductions of covering information systems are defined as follows.

Definition 8 (see [40]). Let $(U, \mathbf{C})$ be a covering information system.

- (1) For any $\mathbf{B} \subseteq \mathbf{C}$, if $\tau(FH_{\mathbf{B}}) = \tau(FH_{\mathbf{C}})$ (or $\tau(FL_{\mathbf{B}}) = \tau(FL_{\mathbf{C}})$), then $\mathbf{B}$ is called as an F topological consistent set of $\mathbf{C}$. If $\mathbf{B}$ is an F topological consistent set of $\mathbf{C}$ and no proper subset of $\mathbf{B}$ is an F topological consistent set of $\mathbf{C}$, then $\mathbf{B}$ is referred to as an F topological reduct of $\mathbf{C}$. The intersection of all F topological reduct of $\mathbf{C}$ is called F topological core of $\mathbf{C}$, which is denoted by $\text{Core}^F(\mathbf{C})$.
- (2) For any $\mathbf{B} \subseteq \mathbf{C}$, if $\tau(LH_{\mathbf{B}}) = \tau(LH_{\mathbf{C}})$ (or $\tau(LL_{\mathbf{B}}) = \tau(LL_{\mathbf{C}})$), then $\mathbf{B}$ is called as an L topological consistent set of $\mathbf{C}$. If $\mathbf{B}$ is an L topological consistent set of $\mathbf{C}$ and no proper subset of $\mathbf{B}$ is an L topological consistent set of $\mathbf{C}$, then $\mathbf{B}$ is referred to as an L topological reduct of $\mathbf{C}$. The intersection of all L topological reduct of $\mathbf{C}$ is called L topological core of $\mathbf{C}$, which is denoted by $\text{Core}^L(\mathbf{C})$.

We employ Example 1 below to state Definition 8, which is a modified example in [23, 24].

Example 1. Consider the problem of evaluating credit card applicants. Let $U = \{x_1, x_2, \dots, x_5\}$ be a set of five applicants and $E = \{\text{education; salary; assets}\}$ be a set of three attributes, where the values of “education” are {higher; secondary; primary}, the values of “salary” are {high; middle; low}, and the values of “assets” are {high; middle; low}. Suppose we have three specialists $\{A, B, C\}$ to evaluate the attribute values for these applicants, and their evaluation results of the same attribute may not be the same. The results are listed below.

For the attribute “education”:

- A: higher = $\{x_1, x_2\}$, secondary = $\{x_3, x_4\}$, low = $\{x_5\}$.
 B: higher = $\{x_1\}$, secondary = $\{x_2, x_3\}$, low = $\{x_4, x_5\}$.
 C: higher = $\{x_1, x_2\}$, secondary = $\{x_3\}$, low = $\{x_4, x_5\}$.

For the attribute “salary”:

- A: higher = $\{x_1, x_2\}$, middle = $\{x_3, x_4\}$, low = $\{x_5\}$.
 B: higher = $\{x_1, x_2, x_3\}$, middle = $\{x_4\}$, low = $\{x_5\}$.

C: higher = $\{x_1, x_2\}$, middle = $\{x_3, x_4\}$, low = $\{x_5\}$.

For the attribute “assets”:

- A: higher = $\{x_1, x_2\}$, middle = $\{x_3\}$, low = $\{x_4, x_5\}$.
 B: higher = $\{x_1, x_2\}$, middle = $\{x_3, x_4\}$, low = $\{x_5\}$.
 C: higher = $\{x_1, x_2\}$, middle = $\{x_3, x_4\}$, low = $\{x_5\}$.

Suppose that the weights of the specialists $\{A, B, C\}$ are equal. To combine the evaluations without losing information, the evaluations provided by each specialist for every attribute value should be union. Then, we obtain three coverings from the attribute set E :

$$\begin{aligned}\mathcal{C}_1 &= \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}\}. \\ \mathcal{C}_2 &= \{\{x_1, x_2, x_3\}, \{x_3, x_4\}, \{x_5\}\}. \\ \mathcal{C}_3 &= \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_4, x_5\}\}.\end{aligned}$$

Let $\mathbf{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, and hence $(U, \mathbf{C})$ is a covering information system.

- (1) By Definition 7, we obtain

$$\begin{aligned}\mathcal{C}_1 \wedge \mathcal{C}_2 &= \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4\}, \{x_5\}\}. \\ \mathcal{C}_1 \wedge \mathcal{C}_3 &= \{\{x_1, x_2\}, \{x_2\}, \{x_3, x_4\}, \{x_4\}, \{x_4, x_5\}\}. \\ \mathcal{C}_2 \wedge \mathcal{C}_3 &= \{\{x_1, x_2\}, \{x_3\}, \{x_3, x_4\}, \{x_4\}, \{x_5\}\}. \\ \wedge \mathbf{C} &= \mathcal{C}_1 \wedge \mathcal{C}_2 \wedge \mathcal{C}_3 = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_3, x_4\}, \{x_4\}, \{x_4, x_5\}\}.\end{aligned}$$

We can see that $\{x_1, x_2\}$ in $\mathcal{C}_1 \wedge \mathcal{C}_2$ is the set of applicants, whose education is higher and salary is higher. The other sets in $\mathcal{C}_1 \wedge \mathcal{C}_2$, $\mathcal{C}_1 \wedge \mathcal{C}_3$, $\mathcal{C}_2 \wedge \mathcal{C}_3$, $\wedge \mathbf{C}$ have the same meanings.

By Definition 4 and Corollary 1, $\tau(FH_{\mathbf{C}}) = \{\emptyset, \{x_1, x_2\}, \{x_1, x_2, x_5\}, \{x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, U\}$. Thus, $(x_1)_{\tau(FH_{\mathbf{C}})} = (x_2)_{\tau(FH_{\mathbf{C}})} = \{x_1, x_2\}$, $(x_3)_{\tau(FH_{\mathbf{C}})} = (x_4)_{\tau(FH_{\mathbf{C}})} = \{x_3, x_4\}$, $(x_5)_{\tau(FH_{\mathbf{C}})} = \{x_5\}$, that is, the partition induced by $\tau(FH_{\mathbf{C}})$ is $\{K_1 = \{x_1, x_2\}, K_2 = \{x_3, x_4\}, K_3 = \{x_5\}\}$. According to Definition 8, the F topological reduct of $\mathbf{C}$ is $\{\mathcal{C}_2, \mathcal{C}_3\}$.

- (2) The neighborhoods of elements are presented in Table 1. $(x_1)_{\{\mathcal{C}_1, \mathcal{C}_2\}}$ contains the elements whose values of “education” and “salary” are the same with x_1 . The other neighborhoods of elements have the same meanings.

By Definition 4 and Corollary 1, $\tau(LH_{\mathbf{C}}) = \{\emptyset, \{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, U\}$. It follows that $(x_1)_{\tau(LH_{\mathbf{C}})} = (x_2)_{\tau(LH_{\mathbf{C}})} = \{x_1, x_2\}$, $(x_3)_{\tau(LH_{\mathbf{C}})} = \{x_3\}$, $(x_4)_{\tau(LH_{\mathbf{C}})} = \{x_4\}$, $(x_5)_{\tau(LH_{\mathbf{C}})} = \{x_5\}$. Thus, the partition induced for $\tau(LH_{\mathbf{C}})$ is $\{G_1 = \{x_1, x_2\}, G_2 = \{x_3\}, G_3 = \{x_4\}, G_4 = \{x_5\}\}$. According to Definition 8, the L topological reducts of $\mathbf{C}$ are $\{\mathcal{C}_1, \mathcal{C}_2\}$ and $\{\mathcal{C}_2, \mathcal{C}_3\}$.

Theorem 1 (see [40]). Let $(U, \mathbf{C})$ be a covering information system, $\mathbf{B} \subseteq \mathbf{C}$.

- (1) The following are equivalent:

TABLE 1: The neighborhoods of elements in Example 1.

x	x_1	x_2	x_3	x_4	x_5
$(x)_{\mathcal{E}_1}$	$\{x_1, x_2\}$	$\{x_2\}$	$\{x_2, x_3, x_4\}$	$\{x_4\}$	$\{x_4, x_5\}$
$(x)_{\mathcal{E}_2}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_3\}$	$\{x_3, x_4\}$	$\{x_5\}$
$(x)_{\mathcal{E}_3}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_3, x_4\}$	$\{x_4\}$	$\{x_4, x_5\}$
$(x)_{\{\mathcal{E}_1, \mathcal{E}_2\}}$	$\{x_1, x_2\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
$(x)_{\{\mathcal{E}_1, \mathcal{E}_3\}}$	$\{x_1, x_2\}$	$\{x_2\}$	$\{x_3, x_4\}$	$\{x_4\}$	$\{x_4, x_5\}$
$(x)_{\{\mathcal{E}_2, \mathcal{E}_3\}}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
$(x)_{\mathcal{C}}$	$\{x_1, x_2\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$

(1a) $\mathbf{B}$ is an F topological consistent set.

(1b) $FH_{\mathbf{B}}(K_i) = K_i$ for all $i \in \{1, 2, \dots, m\}$.

(1c) $FL_{\mathbf{B}}(K_i) = K_i$ for all $i \in \{1, 2, \dots, m\}$.

(2) The following are equivalent:

(2a) $\mathbf{B}$ is an L topological consistent set.

(2b) $LH_{\mathbf{B}}(G_j) = G_j$ for all $j \in \{1, 2, \dots, n\}$.

(2c) $LL_{\mathbf{B}}(G_j) = G_j$ for all $j \in \{1, 2, \dots, n\}$.

Proof. (1). (1a) $\iff$ (1b). $\mathbf{B}$ is an F topological consistent set; then, $\tau(FH_{\mathbf{B}}) = \tau(FH_{\mathbf{C}})$. It follows that for any $x \in U$, $(x)_{\tau(FH_{\mathbf{B}})} = (x)_{\tau(FH_{\mathbf{C}})}$. By Corollary 1, $(U, \tau(FH_{\mathbf{B}}))$ is pseudo-discrete. Then, $(x)_{\tau(FH_{\mathbf{B}})}$ is a closed set in $(U, \tau(FH_{\mathbf{C}}))$, which implies that $FH_{\mathbf{B}}((x)_{\tau(FH_{\mathbf{C}})}) = (x)_{\tau(FH_{\mathbf{C}})}$. That is, $FH_{\mathbf{B}}(K_i) = K_i$ for all $i \in \{1, 2, \dots, m\}$. (1b) $\iff$ (1c). By Proposition 1 (4), $FH_{\mathbf{B}}(K_i) = K_i \iff FL_{\mathbf{B}}(K_i) = K_i$ for all $i \in \{1, 2, \dots, m\}$. (1b) $\iff$ (1a). For any $A \in \tau(FH_{\mathbf{B}})$, $FH_{\mathbf{B}}(A) = A$. By Proposition 1, $A \subseteq FH_{\mathbf{C}}(A) \subseteq FH_{\mathbf{B}}(A)$. It follows that $FH_{\mathbf{C}}(A) = A$. Then, $A \in \tau(FH_{\mathbf{C}})$. It implies that $\tau(FH_{\mathbf{B}}) \subseteq \tau(FH_{\mathbf{C}})$. Conversely, for any $A \in \tau(FH_{\mathbf{C}})$, $A = \cup_{x \in A} (x)_{\tau(FH_{\mathbf{C}})}$. Then, $FH_{\mathbf{B}}(A) = FH_{\mathbf{B}}(\cup_{x \in A} (x)_{\tau(FH_{\mathbf{C}})}) = \cup_{x \in A} FH_{\mathbf{B}}((x)_{\tau(FH_{\mathbf{C}})}) = \cup_{x \in A} (x)_{\tau(FH_{\mathbf{C}})} = A$. Thus, $A \in \tau(FH_{\mathbf{B}})$. Consequently, $\tau(FH_{\mathbf{C}}) \subseteq \tau(FH_{\mathbf{B}})$. $\square$

Theorem 2. Let $(U, \mathbf{C})$ be a covering information system, $\mathbf{B} \subseteq \mathbf{C}$.

(1) The following are equivalent:

(1a) $\mathbf{B}$ is an F topological consistent set.

(1b) $\sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) = 1$.

(1c) $\sum_{i=1}^m Bel_{\mathbf{B}}^F(K_i) = 1$.

(2) The following are equivalent:

(2a) $\mathbf{B}$ is an L topological consistent set.

(2b) $\sum_{j=1}^n Pl_{\mathbf{B}}^L(G_j) = 1$.

(2c) $\sum_{j=1}^n Bel_{\mathbf{B}}^L(G_j) = 1$.

Proof. (1) (1a) $\implies$ (1b). Since $\mathbf{B}$ is an F topological consistent set, by Theorem 1, $FH_{\mathbf{B}}(K_i) = K_i$ for all $i \in \{1, 2, \dots, m\}$. Then,

$$\begin{aligned} \sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) &= \sum_{i=1}^m P(FH_{\mathbf{B}}(K_i)) \\ &= \sum_{i=1}^m P(K_i) = P(\cup_{i=1}^m K_i) = P(U) = 1. \end{aligned} \quad (4)$$

(1b) $\implies$ (1c). For any $i \in \{1, 2, \dots, m\}$, $K_i \subseteq FH_{\mathbf{B}}(K_i)$. According to Definition 6 and Proposition 2,

$$\begin{aligned} 1 &= \sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) = \sum_{i=1}^m P(FH_{\mathbf{B}}(K_i)) \\ &\geq \sum_{i=1}^m P(K_i) = P(\cup_{i=1}^m K_i) = P(U) = 1. \end{aligned} \quad (5)$$

Since $P(K_i) \leq P(FH_{\mathbf{B}}(K_i))$ for all $i \in \{1, 2, \dots, m\}$, we get $P(K_i) = P(FH_{\mathbf{B}}(K_i))$, that is, $P(K_i) = (|K_i|/|U|) = (|FH_{\mathbf{B}}(K_i)|/|U|) = P(FH_{\mathbf{B}}(K_i))$. Then, for any $i \in \{1, 2, \dots, m\}$, $|K_i| = |FH_{\mathbf{B}}(K_i)|$. Since $K_i \subseteq FH_{\mathbf{B}}(K_i)$, $K_i = FH_{\mathbf{B}}(K_i)$. By Proposition 1 (4), $K_i = FL_{\mathbf{B}}(K_i)$. Hence,

$$\begin{aligned} \sum_{i=1}^m Bel_{\mathbf{B}}^F(K_i) &= \sum_{i=1}^m P(FL_{\mathbf{B}}(K_i)) \\ &= \sum_{i=1}^m P(K_i) = P(\cup_{i=1}^m K_i) = P(U) = 1. \end{aligned} \quad (6)$$

(1c) $\implies$ (1a). For any $i \in \{1, 2, \dots, m\}$, $FL_{\mathbf{B}}(K_i) \subseteq K_i$. By Definition 2 and Proposition 2,

$$\begin{aligned} 1 &= \sum_{i=1}^m Bel_{\mathbf{B}}^F(K_i) = \sum_{i=1}^m P(FL_{\mathbf{B}}(K_i)) \\ &\leq \sum_{i=1}^m P(K_i) = P(\cup_{i=1}^m K_i) = P(U) = 1. \end{aligned} \quad (7)$$

Since $P(FL_{\mathbf{B}}(K_i)) \leq P(K_i)$ for all $i \in \{1, 2, \dots, m\}$, we get $P(K_i) = P(FL_{\mathbf{B}}(K_i))$, that is, $P(K_i) = (|K_i|/|U|) = (|FL_{\mathbf{B}}(K_i)|/|U|) = P(FL_{\mathbf{B}}(K_i))$. Then, for any $i \in \{1, 2, \dots, m\}$, $|K_i| = |FL_{\mathbf{B}}(K_i)|$. It follows from $FL_{\mathbf{B}}(K_i) \subseteq K_i$ that $FL_{\mathbf{B}}(K_i) = K_i$. Therefore, according to Theorem 1, $\mathbf{B}$ is an F topological consistent set.

(2) It is similar to the proof of (1). $\square$

Theorem 3. Let $(U, \mathbf{C})$ be a covering information system, $\mathbf{B} \subseteq \mathbf{C}$.

(1) The following are equivalent:

(1a) $\mathbf{B}$ is an F topological reduct.

(1b) $\sum_{i=1}^m Pl_{\mathbf{B}}^L(K_i) = 1$, and for any nonempty subset $\mathbf{B}' \subset \mathbf{B}$, $\sum_{i=1}^m Pl_{\mathbf{B}'}^L(K_i) > 1$.

(1c) $\sum_{i=1}^m Bel_{\mathbf{B}}^L(K_i) = 1$, and for any nonempty subset $\mathbf{B}' \subset \mathbf{B}$, $\sum_{i=1}^m Bel_{\mathbf{B}'}^L(K_i) < 1$.

(2) The following are equivalent:

(2a) $\mathbf{B}$ is an L topological reduct.

(2b) $\sum_{j=1}^n Pl_{\mathbf{B}}^L(G_j) = 1$, and for any nonempty proper subset $\mathbf{B}' \subset \mathbf{A}$, $\sum_{j=1}^n Pl_{\mathbf{B}'}^L(G_j) > 1$.

(2c) $\sum_{j=1}^n Bel_{\mathbf{B}}^L(G_j) = 1$, and for any nonempty proper subset $\mathbf{B}' \subset \mathbf{A}$, $\sum_{j=1}^n Bel_{\mathbf{B}'}^L(G_j) < 1$.

- (1) Let $\text{Core}^F(\mathbf{C}) = \emptyset$, $\sum_{i=1}^m Pl_{\emptyset}^F(K_i) = m$;
- (2) For each $\mathcal{C} \in \mathbf{C}$, calculate $\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) = \sum_{i=1}^m Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^F(K_i) - \sum_{i=1}^m Pl_{\mathbf{C}}^F(K_i)$;
- (3) If $\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0$, then $\text{Core}^F(\mathbf{C}) = \text{Core}^F(\mathbf{C}) \cup \{\mathcal{C}\}$;
- (4) If $\text{Core}^F(\mathbf{C}) = \emptyset$, then go to step (6), else go to step (5);
- (5) If $\sum_{i=1}^m Pl_{\text{Core}^F(\mathbf{C})}^F(K_i) = 1$, return $\text{Core}^F(\mathbf{C})$ as a reduct, else go to step (6);
- (6) Let $\mathbf{B} = \text{Core}^F(\mathbf{C})$;
- (7) For each $\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}$, calculate $\text{Sig}_{\mathbf{B}}^F(\mathcal{C}) = \sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) - \sum_{i=1}^m Pl_{\mathbf{B} \cup \{\mathcal{C}\}}^F(K_i)$;
- (8) If $\text{Sig}_{\mathbf{B}}^F(\mathcal{C}_0) = \max_{\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}} \text{Sig}_{\mathbf{B}}^F(\mathcal{C})$, then $\mathbf{B} = \mathbf{B} \cup \{\mathcal{C}_0\}$;
- (9) If $\sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) = 1$, stop and output $\mathbf{B}$ as a reduct, else go to step (7).

ALGORITHM 1: Computing the F topological core and F topological reduct of $(U, \mathbf{C})$.

- (1) Let $\text{Core}^L(\mathbf{C}) = \emptyset$, $\sum_{j=1}^n Pl_{\emptyset}^L(G_j) = n$;
- (2) For each $\mathcal{C} \in \mathbf{C}$, calculate $\text{Sig}_{\mathbf{C}}^L(\mathcal{C}) = \sum_{j=1}^n Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^L(G_j) - \sum_{j=1}^n Pl_{\mathbf{C}}^L(G_j)$;
- (3) If $\text{Sig}_{\mathbf{C}}^L(\mathcal{C}) > 0$, then $\text{Core}^L(\mathbf{C}) = \text{Core}^L(\mathbf{C}) \cup \{\mathcal{C}\}$;
- (4) If $\text{Core}^L(\mathbf{C}) = \emptyset$, then go to step (6), else go to step (5);
- (5) If $\sum_{j=1}^n Pl_{\text{Core}^L(\mathbf{C})}^L(G_j) = 1$, return $\text{Core}^L(\mathbf{C})$ as a reduct, else go to step (6);
- (6) Let $\mathbf{B} = \text{Core}^L(\mathbf{C})$;
- (7) For each $\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}$, calculate $\text{Sig}_{\mathbf{B}}^L(\mathcal{C}) = \sum_{j=1}^n Pl_{\mathbf{B}}^L(G_j) - \sum_{j=1}^n Pl_{\mathbf{B} \cup \{\mathcal{C}\}}^L(G_j)$;
- (8) If $\text{Sig}_{\mathbf{B}}^L(\mathcal{C}_0) = \max_{\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}} \text{Sig}_{\mathbf{B}}^L(\mathcal{C})$, then $\mathbf{B} = \mathbf{B} \cup \{\mathcal{C}_0\}$;
- (9) If $\sum_{j=1}^n Pl_{\mathbf{B}}^L(G_j) = 1$, stop and output $\mathbf{B}$ as a reduct, else go to step (7).

ALGORITHM 2: Computing the L topological core and L topological reduct of $(U, \mathbf{C})$.

Proof. It is immediately obtained from Theorem 2 and Definition 8. $\square$

3.1. Algorithms for Computing Topological Reducts of a Covering Information System. To give algorithms for finding topological reducts, we define the significance of a covering in a covering information system.

Definition 9. Let $(U, \mathbf{C})$ be a covering information system. Define the F significance of the covering $\mathcal{C} \in \mathbf{C}$ by

$$\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) = \sum_{i=1}^m Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^F(K_i) - \sum_{i=1}^m Pl_{\mathbf{C}}^F(K_i). \quad (8)$$

Define the L significance of the covering $\mathcal{C} \in \mathbf{C}$ by

$$\text{Sig}_{\mathbf{C}}^L(\mathcal{C}) = \sum_{j=1}^n Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^L(G_j) - \sum_{j=1}^n Pl_{\mathbf{C}}^L(G_j). \quad (9)$$

By the definition of F significance of a covering (or L significance of a covering), the $\text{Core}^F(\mathbf{C})$ (or $\text{Core}^L(\mathbf{C})$) can be characterized.

Proposition 3. Let $(U, \mathbf{C})$ be a covering information system. Then,

$$\begin{aligned} \text{Core}^F(\mathbf{C}) &= \{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0\}, \\ \text{Core}^L(\mathbf{C}) &= \{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^L(\mathcal{C}) > 0\}. \end{aligned} \quad (10)$$

Proof. For any $\mathcal{C} \in \text{Core}^F(\mathbf{C})$, $\mathbf{C} \setminus \{\mathcal{C}\}$ is not an F topological consistent set. Otherwise, there exists an F topological reduct $\mathbf{B} \subseteq \mathbf{C} \setminus \{\mathcal{C}\}$. Then, $\mathcal{C} \notin \mathbf{B}$, which contradicts the fact $\mathcal{C} \in \text{Core}^F(\mathbf{C})$. By Theorem 2, $\sum_{i=1}^m Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^F(K_i) > 1$. It follows that $\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0$. Therefore, $\text{Core}^F(\mathbf{C}) \subseteq \{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0\}$. Conversely, for any $\mathcal{C} \in \{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0\}$, $\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0$. Hence, $\text{Sig}_{\mathbf{C}}^F(\mathcal{C}) = \sum_{i=1}^m Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^F(K_i) - \sum_{i=1}^m Pl_{\mathbf{C}}^F(K_i) > 0$. Since $\sum_{i=1}^m Pl_{\mathbf{C}}^F(K_i) = 1$, $\sum_{i=1}^m Pl_{\mathbf{C} \setminus \{\mathcal{C}\}}^F(K_i) > 1$. By Theorem 2, $\mathbf{C} \setminus \{\mathcal{C}\}$ is not an F topological consistent set, which implies that $\mathbf{C} \setminus \{\mathcal{C}\}$ is not an F topological reduct. Hence, $\mathcal{C}$ belongs to each F topological reduct of $\mathbf{C}$. It follows that $\mathcal{C} \in \text{Core}^F(\mathbf{C})$. Then, $\{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^F(\mathcal{C}) > 0\} \subseteq \text{Core}^F(\mathbf{C})$.

Similarly, we can get $\text{Core}^L(\mathbf{C}) = \{\mathcal{C} \in \mathbf{C} \mid \text{Sig}_{\mathbf{C}}^L(\mathcal{C}) > 0\}$.

From Proposition 3, the significance of each covering in the core of $\mathbf{C}$ is larger than zero. Now we present a concept of the significance of a covering $\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}$ relative to the family of coverings $\mathbf{B}$. $\square$

Definition 10. Let $(U, \mathbf{C})$ be a covering information system, $\mathbf{B} \subseteq \mathbf{C}$. Define the F significance of $\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}$ relative to $\mathbf{B}$ by

$$\text{Sig}_{\mathbf{B}}^F(\mathcal{C}) = \sum_{i=1}^m Pl_{\mathbf{B}}^F(K_i) - \sum_{i=1}^m Pl_{\mathbf{B} \cup \{\mathcal{C}\}}^F(K_i). \quad (11)$$

Define the L significance of $\mathcal{C} \in \mathbf{C} \setminus \mathbf{B}$ relative to $\mathbf{B}$ by

$$\text{Sig}_{\mathbf{B}}^L(\mathcal{C}) = \sum_{j=1}^n Pl_{\mathbf{B}}^L(G_j) - \sum_{j=1}^n Pl_{\mathbf{B} \cup \{\mathcal{C}\}}^L(G_j). \quad (12)$$

Let $\sum_{i=1}^m Pl_{\mathcal{C}}^F(K_i) = m$ and $\sum_{j=1}^n Pl_{\mathcal{C}}^L(G_j) = n$. The relative significance $Sig_{\mathcal{B}}^F(\mathcal{C})$ (or $Sig_{\mathcal{B}}^L(\mathcal{C})$) can measure importance degree of the covering $\mathcal{C}$ relative to $\mathbf{B}$.

Now we design algorithms to find an F topological reduct or an L topological reduct of the covering information system $(U, \mathbf{C})$.

The time complexity of Algorithms 1 and 2 is $O(|\mathbf{C}|^3 \times |U|^2)$. In the following, an example is given to illustrate the mechanism of Algorithms 1 and 2.

Example 2. Continued from Example 1.

- (1) Computing F topological reducts: first, $FH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(K_1) = \{x_1, x_2\}$, $FH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(K_2) = \{x_3, x_4\}$, and $FH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(K_3) = \{x_5\}$. Thus, we get

$$\begin{aligned} Sig_{\mathcal{C}}^F(\mathcal{E}_1) &= \sum_{i=1}^3 Pl_{\mathbf{C} \setminus \{\mathcal{E}_1\}}^F(K_i) - \sum_{i=1}^3 Pl_{\mathbf{C}}^F(K_i) \\ &= \left(\left(\frac{2}{5} \right) + \left(\frac{2}{5} \right) + \left(\frac{1}{5} \right) \right) - 1 = 0. \end{aligned} \quad (13)$$

By $FH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(K_1) = \{x_1, x_2\}$, $FH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(K_2) = \{x_3, x_4, x_5\}$, and $FH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(K_3) = \{x_4, x_5\}$, we have

$$\begin{aligned} Sig_{\mathcal{C}}^F(\mathcal{E}_2) &= \sum_{i=1}^3 Pl_{\mathbf{C} \setminus \{\mathcal{E}_2\}}^F(K_i) - \sum_{i=1}^3 Pl_{\mathbf{C}}^F(K_i) \\ &= \left(\left(\frac{2}{5} \right) + \left(\frac{3}{5} \right) + \left(\frac{2}{5} \right) \right) - 1 = \left(\frac{2}{5} \right) > 0. \end{aligned} \quad (14)$$

Since $FH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(K_1) = \{x_1, x_2, x_3\}$, $FH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(K_2) = \{x_2, x_3, x_4\}$, and $FH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(K_3) = \{x_5\}$, we obtain

$$\begin{aligned} Sig_{\mathcal{C}}^1(\mathcal{E}_3) &= \sum_{i=1}^3 Pl_{\mathbf{C} \setminus \{\mathcal{E}_3\}}^F(K_i) - \sum_{i=1}^3 Pl_{\mathbf{C}}^F(K_i) \\ &= \left(\left(\frac{3}{5} \right) + \left(\frac{3}{5} \right) + \left(\frac{1}{5} \right) \right) - 1 = \left(\frac{2}{5} \right) > 0. \end{aligned} \quad (15)$$

Hence, $Core^F(\mathbf{C}) = \{\mathcal{E}_2, \mathcal{E}_3\}$.

Second, since $\sum_{i=1}^3 Pl_{Core^F(\mathbf{C})}^F(K_i) = 1$, $Core^F(\mathbf{C}) = \{\mathcal{E}_2, \mathcal{E}_3\}$ is an F topological reduct.

- (2) Computing L topological reducts: first, $LH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(G_1) = \{x_1, x_2\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(G_2) = \{x_3\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(G_3) = \{x_4\}$, and $LH_{\mathbf{C} \setminus \{\mathcal{E}_1\}}(G_4) = \{x_5\}$; then,

$$\begin{aligned} Sig_{\mathcal{C}}^L(\mathcal{E}_1) &= \sum_{j=1}^4 Pl_{\mathbf{C} \setminus \{\mathcal{E}_1\}}^L(G_j) - \sum_{j=1}^4 Pl_{\mathbf{C}}^L(G_j) \\ &= \left(\left(\frac{2}{5} \right) + \left(\frac{1}{5} \right) + \left(\frac{1}{5} \right) + \left(\frac{1}{5} \right) \right) - 1 = 0. \end{aligned} \quad (16)$$

By $LH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(G_1) = \{x_1, x_2\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(G_2) = \{x_3, x_4\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(G_3) = \{x_3, x_4, x_5\}$, and $LH_{\mathbf{C} \setminus \{\mathcal{E}_2\}}(G_4) = \{x_4, x_5\}$, we obtain

$$\begin{aligned} Sig_{\mathcal{C}}^L(\mathcal{E}_2) &= \sum_{j=1}^4 Pl_{\mathbf{C} \setminus \{\mathcal{E}_2\}}^L(G_j) - \sum_{j=1}^4 Pl_{\mathbf{C}}^L(G_j) \\ &= \left(\left(\frac{2}{5} \right) + \left(\frac{2}{5} \right) + \left(\frac{3}{5} \right) + \left(\frac{2}{5} \right) \right) - 1 = \left(\frac{4}{5} \right) > 0. \end{aligned} \quad (17)$$

Since $LH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(G_1) = \{x_1, x_2\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(G_2) = \{x_3\}$, $LH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(G_3) = \{x_4\}$, and $LH_{\mathbf{C} \setminus \{\mathcal{E}_3\}}(G_4) = \{x_5\}$, we have

$$\begin{aligned} Sig_{\mathcal{C}}^L(\mathcal{E}_3) &= \sum_{j=1}^4 Pl_{\mathbf{C} \setminus \{\mathcal{E}_3\}}^L(G_j) - \sum_{j=1}^4 Pl_{\mathbf{C}}^L(G_j) \\ &= \left(\left(\frac{2}{5} \right) + \left(\frac{1}{5} \right) + \left(\frac{1}{5} \right) + \left(\frac{1}{5} \right) \right) - 1 = 0. \end{aligned} \quad (18)$$

Hence, $Core^L(\mathbf{C}) = \{\mathcal{E}_2\}$.

Second, it follows from $\sum_{j=1}^4 Pl_{Core^L(\mathbf{C})}^L(G_j) = (10/5) \neq 1$ that $Core^L(\mathbf{C}) = \{\mathcal{E}_2\}$ is not an L topological reduct.

Since

$$\begin{aligned} Sig_{Core^L(\mathbf{C})}^L(\mathcal{E}_1) &= \sum_{j=1}^4 Pl_{\{\mathcal{E}_2\}}^L(G_j) - \sum_{j=1}^4 Pl_{\{\mathcal{E}_1, \mathcal{E}_2\}}^L(G_j) \\ &= \left(\frac{10}{5} \right) - 1 = 1, \end{aligned}$$

$$\begin{aligned} Sig_{Core^L(\mathbf{C})}^L(\mathcal{E}_3) &= \sum_{j=1}^4 Pl_{\{\mathcal{E}_2\}}^L(G_j) - \sum_{j=1}^4 Pl_{\{\mathcal{E}_2, \mathcal{E}_3\}}^L(G_j) \\ &= \left(\frac{10}{5} \right) - 1 = 1, \end{aligned} \quad (19)$$

let $\mathbf{B} = Core^L(\mathbf{C}) \cup \{\mathcal{E}_1\} = \{\mathcal{E}_1, \mathcal{E}_2\}$. We get $\sum_{j=1}^4 Pl_{\mathbf{B}}^L(G_j) = 1$; then, $\mathbf{B} = \{\mathcal{E}_1, \mathcal{E}_2\}$ is an L topological reduct.

Similarly, we can obtain that $\{\mathcal{E}_2, \mathcal{E}_3\}$ is an L topological reduct.

Therefore, $Core^L(\mathbf{C}) = \{\mathcal{E}_2\}$, and $\{\mathcal{E}_1, \mathcal{E}_2\}$ and $\{\mathcal{E}_2, \mathcal{E}_3\}$ are L topological reducts.

4. Comparing L Topological Reduction with a Kind of Reduction in Covering Information Systems

Chen et al. presented a definition of reduction in a covering information system [23].

Definition 11 (see [23]). Let $(U, \mathbf{C})$ be a covering information system. For any $\mathbf{B} \subseteq \mathbf{C}$, denote $Cov(\mathbf{B}) = \{(x)_{\mathbf{B}} | x \in U\}$. If $Cov(\mathbf{B}) = Cov(\mathbf{C})$ and for any $\mathcal{C} \in \mathbf{B}$, $Cov(\mathbf{B}) \neq Cov(\mathbf{B} - \{\mathcal{C}\})$, then $\mathbf{B}$ is called a reduct of $(U, \mathbf{C})$.

Example 3. Let U and C be the same as those in Example 1. By Definition 11 and Table 1, $\text{Cov}(C) = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$, $\text{Cov}(\{\mathcal{C}_1, \mathcal{C}_2\}) = \text{Cov}(C)$, $\text{Cov}(\{\mathcal{C}_1, \mathcal{C}_2\}) \neq \text{Cov}(\{\mathcal{C}_1\})$, and $\text{Cov}(\{\mathcal{C}_1, \mathcal{C}_2\}) \neq \text{Cov}(\{\mathcal{C}_2\})$. Then, $\{\mathcal{C}_1, \mathcal{C}_2\}$ is a reduct of (U, C) . Furthermore, $\{\mathcal{C}_1, \mathcal{C}_2\}$ is the only reduct of (U, C) .

Proposition 2. Let (U, C) be a covering information system, $B \subseteq C$. If B is a reduct, then B is an L topological consistent set.

Proof. Since B is a reduct, by Definition 11, $(x)_C = (x)_B$ for all $x \in U$. It follows that for any $X \subseteq U$, $LH_C(X) = LH_B(X)$. Thus, $\tau(LH_C) = \tau(LH_B)$, which implies that B is an L topological consistent set. $\square$

Remark 2. (1) F topological reducts of (U, C) are not necessary correlated with the reducts of (U, C) .

From Example 1 and Example 3, we can know that the F topological reduct of (U, C) is $\{\mathcal{C}_2, \mathcal{C}_3\}$, and the reduct of (U, C) is $\{\mathcal{C}_1, \mathcal{C}_2\}$.

(2) L topological reduct of (U, C) could not be a reduct of (U, C) .

For example, $\{\mathcal{C}_2, \mathcal{C}_3\}$ is an L topological reduct of the covering information system (U, C) in Example 1. However, $\{\mathcal{C}_2, \mathcal{C}_3\}$ is not a reduct of (U, C) .

5. Conclusion

In this paper, the L topological reduction and the F topological reduction of covering information systems have been characterized by the belief and plausibility functions from the evidence theory. The topological spaces by the two pairs of covering approximation operators in covering information systems are pseudo-discrete, which deduce partitions. Then, using plausibility function values of sets in the partitions, the definitions of significance and relative significance of coverings in covering information systems have been also developed. Hence, topological reduction algorithms based on the evidence theory have been proposed in covering information systems, and an example has been adopted to illustrate the validity of the algorithms. We have also compared the L topological reduction with a kind of reduction proposed in [23].

Data Availability

The underlying data supporting the results of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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