

Research Article

An Inventory Model for Growing Items with Imperfect Quality When the Demand Is Price Sensitive under Carbon Emissions and Shortages

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Nowadays, it is well known that global warming is a great hazard to the planet, and the carbon emissions are a principal source of global warming. For this reason, the customers have become more environment and quality conscious than before, and as a result, they request the firms to be ecofriendly. In this context, it is desirable that companies develop and implement inventory models which consider sustainability issues. Furthermore, the companies face problems of shortages and setting prices in order to persist in a competitive and challenging business. Besides, there exists a kind of items different than the traditional products that it is necessary to feed them until a target weight is reached in order to slaughter and sell to customers. These are named as growing items. In this sense, this research work proposes an inventory model for growing items with imperfect quality when the demand is price sensitive under carbon emissions and shortages. The shortages are fully backordered. The demand is price sensitive according to a polynomial function. The proposed inventory model determines jointly the optimal policy for the selling price of perfect-quality growing items, the order quantity, and the backordering quantity which maximize the expected total profit per unit of time. Some numerical examples are resolved in order to illustrate the use and the applicability of the inventory model. Finally, a sensitivity analysis is conducted and some managerial insights are given.

1. Introduction

Inventory management is a valued function for all firms. The main purpose of the inventory management department is to control the materials from the acquisition to sale, taking decisions of how much and when to buy the items avoiding excess or unplanned stockout.

Since the introduction of the classical economic order quantity (EOQ) inventory model proposed by Harris [1], industries have shown great interest in the development of inventory control models with new features. For example, Ghosh et al. [2] built an inventory model for a deteriorating production system when there are random machine breakdowns considering that there exists a selling price discount. Manna et al. [3] studied the impacts of inspection

errors on an imperfect production-inventory model when the demand rate depends on both selling-price discount and warranty time. Khan et al. [4] and Shaikh et al. [5] introduced a two-warehouse inventory model for instantaneously deteriorating goods considering an advance payment agreement. In the same research direction, Khan et al. [6] investigated the effects of noninstantaneous deterioration on the ordering decisions for a two-warehouse inventory system taking into consideration an advance payment scheme. Shaikh et al. [7] developed an EOQ inventory model for deteriorating goods with stock-dependent demand when there exists a price discount facility.

It is well known that the EOQ determines the optimal order quantity to minimize total inventory costs. Although this inventory model has the groundwork for inventory

systems, it has unrealistic assumptions. For that reason, many researchers have incorporated different characteristics in order to model real world applications such as imperfect quality, carbon emissions reduction, growing items, shortages, and pricing, among others.

Harris's [1] inventory model assumes that all items are of perfect quality. However, this assumption not always occurs because the majority of the manufacturing processes generate a portion of imperfect quality items. So, Salameh and Jaber [8] formulated an EOQ inventory model considering that the ordered lot contains imperfect-quality items and these items are sold as a single batch at a lower price. Cárdenas-Barrón [9] identified and corrected an error in the inventory model presented by Salameh and Jaber [8]. Wee et al. [10] extended Salameh and Jaber's [8] inventory model by incorporating shortages with full backordering. Afterwards, Maddah and Jaber [11] detected and rectified one flaw in Salameh and Jaber's [8] inventory model. The flaw is in the manner of determining the expected total profit per unit of time ($E[TPU]$). Salameh and Jaber [8] utilized the equation $E[TPU] = E[TP/T]$ which is not exact. The research work of Wee et al. [10] contains the same flaw. It is important to remark that the process of producing the profit is a renewal process with renewal points at order placement periods. Therefore, in order to calculate the expected total profit per unit of time, it is needed to apply the well-known elementary renewal reward theorem given by Ross [12]. For this reason, the expected total profit per unit of time must be computed with the following equation: $E[TPU] = E[TP]/E[T]$. Chang and Ho [13] revisited and rescued Wee et al.'s [10] inventory model by applying the well-known elementary renewal reward theorem to get the expected total profit per unit time. Cárdenas-Barrón [14] derived closed-form expressions to determining the optimal solution to an EOQ inventory model considering items with imperfect quality. Ghiami and Beullens [15] developed a production-inventory model for a deteriorating product with partial backordering using a cash-flow net present value analysis. They generated managerial insights related if it is convenient financially to have planned shortages with partial backordering and lost sales. Zhou et al. [16] developed an economic-order quantity model considering shortages, imperfect quality, and inspection errors. Taleizadeh et al. [17] studied an inventory model when the lot contains imperfect-quality items, and these cannot be replaced with perfect-quality ones immediately, but these are repaired and then they are sold at lower price. In this line of research, Rezaei [18] introduced an inventory model considering imperfect-quality items. Sampling inspection plans are designed so the buyer would be able to decide regarding the next step.

At the present time, as we know, global warming is a big threat to the world and carbon emissions are a leading cause of global warming. For this reason, the sustainability is an extremely significant concern for all people that desire to maintain the planet healthy. It is well known that industries generate large amounts of carbon emissions. Furthermore, the transportation of goods through the supply chains produces carbon emissions too. Consequently, these damage the environment and cause global warming at the same time.

In this context, the governments of the countries have imposed strict policies that limit the amounts of carbon emissions with the aim of having a sustainable environment free of pollution. The regulations related to avoid pollution impose additional costs to firms; thus, it is convenient for these to adapt and obey the regulations. In this direction, several companies have adopted inventory models that consider reducing carbon emissions. For example, Hua et al. [19] investigated how companies manage carbon emissions in an inventory management. Arslan and Turkay [20] integrated sustainability features into the design of an inventory model and proposed different policies that include environmental and social criteria. Battini et al. [21] introduced a sustainable EOQ inventory model integrating factors that have a great impact on the environment. Lin and Sarker [22] presented an inventory model that considers carbon tax policy and imperfect-quality items avoiding shortages and incorporated and evaluated some carbon tax systems. Tiwari et al. [23] investigated an integrated vendor-buyer inventory model for deteriorating goods with imperfect quality, taking into account that all processes in the supply chain cause carbon emissions. The target of their inventory model is to decrease the ecological effects. Kazemi et al. [24] proposed an inventory model with imperfect quality from a sustainable point of view, and the aim is to know the impact of emission costs into the total profit. Modak et al. [25] addressed a manufacturer-retailer supply chain inventory model by considering that there is a cost of GHG emission of the production system and determined the optimal pricing policies. Afterwards, Sinha and Modak [26] presented an economic production quantity (EPQ) inventory model taking into account aspects of carbon emissions and carbon trading. Li and Hai [27] studied an inventory system for a warehouse with multiple retailers considering carbon emissions. Manna et al. [28] constructed a production-inventory model for controlling the GHG emissions when the pollution parameters are fuzzy. Later, Manna et al. [29] studied the effects of carbon emissions on an imperfect production-inventory model. Huang et al. [30] proposed an inventory model for a two-echelon supply chain considering that all operations within the chain produce carbon emissions. Their inventory model helps companies to determine the optimal production quantity and delivery quantity which minimize the total costs under different carbon emission policies. Recently, Mishra et al. [31] revised the standard EOQ inventory model to integrate sustainability matters and developed three inventory models with and without shortage situations. Medina-Santana and Cárdenas-Barrón [32] formulated an inventory model considering a discontinuous transportation cost function and carbon emissions function. The most recent investigation about carbon emissions belongs to the authors Modak and Kelle [33]. They proposed to apply the social work donation as a scheme of corporate social responsibility into a closed-loop supply chain, taking into consideration the carbon emission tax and that demand is uncertain.

In recent years, a new topic, that is, inventory management for growing items, has also been included in the derivation of inventory models. A growing item is a kind of

product which is capable of growth during time like farmyard animals, unlike inert products that do not increase in weight during storage. Growing items are incorporated for first time in an EOQ inventory model by Rezaei [34]. Here, the weight increment of items is the main difference between the inventory models proposed by Rezaei [34] and Harris [1]. Rezaei [34] developed an EOQ for growing items for different types of animals, specifically for a kind of poultry. Zhang et al. [35] revisited and extended the research work of Rezaei [34] by creating an inventory model for growing items reducing the carbon emissions in operations. Sebatjane [36] presented three inventory models for growing items. Afterwards, Khalilpourazari and Pasandideh [37] worked with growing items too. They formulated a mathematical model for multi-item economic order quantity subjected to some operational constraints. Nobil et al. [38] derived an EOQ inventory model for growing items when the shortages are allowed, and these are fully backordered. They argued that their inventory model helps poultry industries. Under an environment of a two-echelon supply chain composed by one supplier and one farmer, Malekitabar et al. [39] developed an inventory model for growing mortal items considering both the growth function and the mortality rate for the items. Recently, Sebatjane and Adetunji [40] presented an EOQ inventory model for growing items with imperfect quality. It is assumed that a certain fraction of the growing items has lower quality than desired. They mentioned that it is necessary to define the growth function of the items in order to calculate the feeding cost. In this context, Sebatjane and Adetunji [40] considered three growth functions: logistic, linear, and split linear. In the same year, Sebatjane and Adetunji [41] introduced an economic-order quantity inventory model for growing items with incremental quantity discounts. They proposed an optimal inventory policy to minimize the total inventory cost in both the owned and rented facilities. Sebatjane and Adetunji [42] continued working with growing items, but in this case, they formulated an inventory control model more realistically, considering that the items need to be transformed and packaged into a consumable form before customer demand is met. The next year, Sebatjane and Adetunji [43] built a model for managing inventory in a perishable food products supply chain that begins with farming operations where growing items live and finishes with the consumption of processed inventory. Sebatjane and Adetunji [44] derived an inventory model for a four-echelon supply chain with farming, processing, screening, and retail operations. Gharraei and Almehdawe [45] provided an economic growing quantity (EGQ) inventory model to determine the optimal economic growth and slaughter period and the economic growing quantity to minimize the total cost of the inventory system. Hidayat et al. [46] presented an EOQ inventory model with a capacitated warehouse facility and limited budget for growing products when the seller provides an incremental quantity discounts scheme. Mokhtari et al. [47] addressed a production-inventory model for growing goods which deteriorate through time. Nishandhi [48] studied an EOQ inventory model with budget-capacity constraint for growing items when a

portion of the items are of imperfect quality. Pourmohammad-Zia and Karimi [49] determined the optimal replenishment and breeding policies for growing products. Afzal and Alfares [50] and Alfares and Afzal [51] developed EOQ inventory models for growing items considering shortages with full backordering. Recently, Sebatjane and Adetunji [52] created an inventory model for growing items under a three-echelon supply chain environment taking into account farming, processing, and retail echelons. Table 1 shows the inventory models related with growing items.

The organizations need to determine the optimal selling prices for the products in order to encourage that the customers buy more and more products, and therefore, these organizations can survive in the competitive business environment. In this context, the academicians and researchers are modelling the demand rate as dependent of price. For example, Khan et al. [53] analyzed two supply chain inventory systems when the demand rate depends on price. Khan et al. [54] proposed an inventory model for deteriorating merchandises when the demand is price sensitive, and there exists a discount policy according all-units arrangement. Khan et al. [55] developed inventory models for perishable products when the demand rate is dependent on both price and advertisement. On the one hand, Khan et al. [56] examined the effects of an advance payment with discount facility on ordering decisions for perishable items taking into account that the demand rate is both price and stock dependent. On the other hand, Panda et al. [57] dealt with a two-warehouse system for deteriorating items when the demand rate is dependent on both price and stock; however, the price is not optimized. Sinha et al. [58] formulated an entropic-order quantity inventory model when the demand of the product is dependent on selling price and there is an inspection process to split the imperfect-quality products from the perfect ones. Modak [59] and Modak and Kelle [60] introduced omni- and dual-channel supply chain models, respectively, by considering that there exists a price- and delivery-time-sensitive stochastic demand.

Nowadays, the firms face problems of shortages and setting the selling price in order to survive in a competitive business, which is becoming more challenging day by day. Moreover, the customers are more quality and environmental conscious than before, and therefore, they demand products of perfect quality and request firms to minimize the carbon emissions to the environment. However, the firms always face problems of process quality and contaminate the planet. Besides, there exists a type of goods different than the conventional products which requires feeding until a target weight is reached in order to slaughter and sell to customers. These are named as growing items (chickens, cows, pigs, goats, fish, shrimps, etc.). In this direction, this research work develops an inventory model for growing items with imperfect quality when the demand is price sensitive under carbon emissions and shortages. The shortages are fully backordered. The demand is price sensitive according to a polynomial function. The proposed inventory model determines jointly the optimal policy for the selling price of perfect-quality growing items, the order quantity, and

TABLE 1: Inventory models with growing items.

Authors	Price-dependent demand	Type of price-dependent demand	Allowed shortages	Type of backordering	Imperfect quality	Carbon tax	Type of objective function	Optimize
Rezaei [34]	No	—	No	—	No	No	Max. profit	Order quantity and slaughter time
Zhang et al. [35]	No	—	No	—	No	Yes	Min. cost	Order quantity and slaughter time
Sebatjane [36]	No	—	No	—	Yes	No	Max. profit	Order quantity and cycle time
	No	—	No	—	No	No	Min. cost	Order quantity and cycle time
Khalilpourazari and Pasandideh [37]	No	—	No	—	No	No	Min. cost	Order quantity and cycle time
	No	—	No	—	No	No	Max. profit	Order quantity and slaughter time
Nobil et al. [38]	No	—	Yes	Full	No	No	Min. cost	Order quantity, backordering quantity, and cycle time
Malekitabar et al. [39]	Yes	Linear	No	—	No	No	Max. profit	Selling price and cycle time
Sebatjane and Adetunji [40]	No	—	No	—	Yes	No	Max. profit	Order quantity and cycle time
Sebatjane and Adetunji [41]	No	—	No	—	No	No	Min. cost	Order quantity and cycle time
Sebatjane and Adetunji [42]	No	—	No	—	No	No	Min. cost	Order quantity, cycle time, and number of shipments
Sebatjane and Adetunji [43]	Yes	Exponential	No	—	No	No	Max. profit	Selling price, order quantity, cycle time, and number of shipments
Sebatjane and Adetunji [44]	No	—	No	—	Yes	No	Max. profit	Order quantity, cycle time, and number of shipments
Gharaei and Almehdawe [45]	No	—	No	—	No	No	Min. cost	Order quantity and cycle time
Hidayat et al. [46]	No	—	No	—	No	No	Min. cost	Order quantity and cycle time
Mokhtari et al. [47]	No	—	No	—	No	No	Max. profit	Order quantity and slaughter time
Nishandhi [48]	No	—	Yes	Full	Yes	No	Min. cost	Order quantity, backordering quantity, and cycle time
Pourmohammad-Zia and Karimi [49]	No	—	No	—	Yes	No	Min. cost	Order quantity and cycle time
Afzal and Alfares [50]	No	—	Yes	Full	Yes	No	Min. cost	Order quantity, backordering quantity, and cycle time
Alfares and Afzal [51]	No	—	Yes	Full	Yes	No	Min. cost	Order quantity, backordering quantity, and cycle time
Sebatjane and Adetunji [52]	No	—	No	—	No	No	Max. profit	Order quantity, cycle time, and number of shipments
This paper	Yes	Polynomial	Yes	Full	Yes	Yes	Max. profit	Selling price, order quantity, backordering quantity, and cycle time

backordering quantity which maximize the total profit per unit of time.

The remaining parts of this research work are described as follows: Section 2 introduces the notation defining symbols for the parameters, decision variables, decision-dependent variables, and functions. Section 3 states the assumptions under which the inventory model is built. Section 4 formulates the inventory model with shortages for growing items with imperfect quality when the demand is price sensitive considering carbon emissions. Section 5 determines some theoretical results, develops the solution procedure to determine the optimal solution to the inventory problem, and identifies some inventory models as special cases. Section 6 solves some numerical examples. Section 7 provides a sensitivity analysis and some managerial insights. Section 8 gives some conclusions and summaries several research points that can be addressed in the near future.

2. Notations

With the aim of having a standard notation for the inventory models with growing items, the nomenclature of Sebatjane and Adetunji [40] is used and extra symbols are defined too. Therefore, the following notations are adopted in order to develop the inventory model for growing items with imperfect quality, carbon emissions, and planned shortages.

2.1. *Parameters.* π : Scale parameter for the price-dependent demand

ρ : Sensitivity parameter for the price-dependent demand

n : Demand power index

v : Selling price of imperfect items (currency symbol/unit of weight)

K : Setup cost (currency symbol/cycle)

h : Holding cost (currency symbol/unit of weight/unit of time)

b : Shortage cost (currency symbol/unit of weight/unit of time)

c : Feeding cost (currency symbol/unit of weight)

p : Purchasing cost (currency symbol/unit of weight)

z : Inspection cost (currency symbol/unit of weight)

θ : Carbon tax rate (currency symbol/amount of carbon emissions)

Ec : Carbon emissions cost (currency symbol)

\widehat{K} : Amount of carbon emissions produced during the setup process (unit of weight/unit of time)

\widehat{h} : Amount of carbon emissions caused by holding items in the warehouse (unit of weight/unit of time)

\widehat{c} : Amount of carbon emissions generated during the feeding period (unit of weight/unit of time)

\widehat{p} : Amount of carbon emissions made during the purchasing activity (unit of weight/unit of time)

\widehat{z} : Amount of carbon emissions created during the inspection process (unit of weight/unit of time)

r : Inspection rate (unit of weight/unit of time)

α : Asymptotic weight of each item (unit of weight)

β : Integration constant (numeric value)

λ : Growth rate (numeric value/unit of time)

x : Percentage of slaughtered items that are of imperfect quality ($0 \leq x \leq 1$)

$E[x]$: Expected value of the percentage of imperfect items ($0 \leq E[x] \leq 1$)

$1 - E[x]$: Expected value of the percentage of perfect items ($0 \leq 1 - E[x] \leq 1$)

w_0 : Weight of a newborn item (unit of weight)

w_1 : Target weight of a grown item (unit of weight)

w_t : Weight of an item at time t (unit of weight)

t_1 : Growing period (unit of time)

t_2 : Inspection period for the backordering quantity (B) (unit of time)

t_3 : Inspection period for $yw_1 - B$ units of weight (unit of time)

t_4 : Consumption period of perfect items after inspection time (unit of time)

t_5 : Shortages accumulation period (unit of time)

Decision variables:

y : Order quantity of newborn items (units)

B : Backordering quantity (unit of weight)

s : Selling price of perfect items (currency symbol/unit of weight)

Decision-dependent variables:

T : Cycle time (unit of time)

Q_0 : Total weight at the beginning of the growing period, $Q_0 = yw_0$ (unit of weight)

Q_{t_1} : Total weight at the end of the growing period t_1 , $Q_{t_1} = yw_1$ (unit of weight)

Functions:

$D(s)$: Price-dependent demand function (unit of weight/unit of time)

$w_t(t)$: Growth function

$g(x)$: Probability density function of the percentage of imperfect items

TPU(y, B, s): Total profit (currency symbol/unit of time)

3. Assumptions

The inventory model is based on the following assumptions:

- (1) The planning horizon is infinite and a single kind of items is purchased, and these are capable of growing before the slaughter process.
- (2) The shortages are permitted, and these are completely backordered.

- (3) The items are slaughtered and are immediately inspected in order to sell them to consumers. Firstly, the backordering quantity is inspected in order to cover the shortages of the previous cycle.
- (4) There exists an inspection process that is 100% effective.
- (5) A random percentage of the slaughtered items is of imperfect quality.
- (6) Imperfect-quality items are not reworked or replaced.
- (7) All imperfect-quality items are salvaged and sold as a single lot at the end of the inspection process.
- (8) The feeding cost for growing the items is directly related to weight gained by these.
- (9) The holding cost to keep a weight unit of the slaughtered item in storage is incurred during both the inspection process and the consumption period.
- (10) The demand rate $D(s)$ is a polynomial function of selling price of the perfect-quality items. It is as follows: $D(s) = \pi - \rho s^n$.
- (11) The selling price of perfect-quality items is optimized, and it must be greater than that of the imperfect-quality items.
- (12) Carbon emissions are taken into account, and these occur in all operations of the inventory system, except in the shortage period.

4. Inventory Model Development

The inventory model for growing items with imperfect quality, carbon emissions, and shortages is depicted in Figure 1. Consider a situation where a company orders y newborn growing items from an outside supplier at the beginning of the growing period t_1 . Each newborn growing item has an initial weight of w_0 . In this moment, the total initial weight of the inventory is $Q_0 = yw_0$. The growing items are fed, and eventually, they grow through time until an objective weight of w_1 is attained. Then, the growing items are slaughtered at the end of the growing time t_1 . At this point, the final weight of the inventory is $Q_{t_1} = yw_1$, and this total weight contains a percentage x of imperfect items. The portion of imperfect items is a random variable with a known probability density distribution $f(x)$, and its expected value is $E[x]$. The shortages are permitted, and these are fully backordered. Therefore, immediately, the inspection process starts to screen the items to complete the backordering quantity (B) at a rate of r units of weight per unit of time during the period t_2 in order to satisfy immediate shortages from the previous cycle. So, at the end of the inspection period t_2 , the inventory model diminishes vertically by B units of weight. It is important to remark that the items continue to be inspected at the same rate r during the period t_3 till the total weight is screened. The length of the inspection time is $t_2 + t_3$. It is worth mentioning that, for the duration of t_3 , the on-hand inventory declines by both removing the imperfect items and current demand rate. At

the end of period t_3 , the imperfect items are salvaged and sold as a single lot with a less price. Consequently, the on-hand inventory drops vertically by $x y w_1$ units of weight. On the other hand, in the course of t_4 , the on-hand inventory decreases due to the current demand rate. The inventory level continues gradually consuming until it reaches zero at the end of period t_4 and the shortages period starts. As a final point, during t_5 , the shortages are accumulated at the current demand rate which are eventually satisfied in the next cycle. Without loss of generality and with the purpose to make the mathematical expressions more tractable, the cycle time is determined as follows: $T = t_3 + t_4 + t_5$.

Throughout t_1 , the items are growing according to a logistic growth function which relates the weight of items with time using three input parameters. These input parameters are the asymptotic weight of the items, the integration constant, and the growth rate, which are represented by α , β , and λ , respectively. Thus, the logistic growth function of the items is mathematically expressed by

$$w_t(t) = \frac{\alpha}{1 + \beta e^{-\lambda t}}. \quad (1)$$

As it was mentioned above, the growing items are slaughtered when their weight attains the objective weight of w_1 which occurs at the end of the growth period t_1 . Hence,

$$w_1 = w_t(t = t_1) = \frac{\alpha}{1 + \beta e^{-\lambda t_1}}. \quad (2)$$

The duration of the growth period (t_1) is calculated by solving equation (2) for t_1 . Thus,

$$t_1 = -\frac{\ln[(1/\beta)((\alpha/w_1) - 1)]}{\lambda}. \quad (3)$$

The inventory must be inspected before being sold in order to avoid to vend imperfect quality as good ones; therefore, firstly, the backordering quantity (B) of the previous cycle must be screened at an inspection rate r . Therefore, the duration of the inspection period (t_2) is computed as follows:

$$t_2 = \frac{B}{r}. \quad (4)$$

After the inspection of the backordering quantity is conducted, the screening process continues until the total weight is screened due to the fact that there are pending $yw_1 - B$ units of weight to be inspected. This is performed during the inspection time t_3 which is given as follows:

$$t_3 = \frac{yw_1 - B}{r}. \quad (5)$$

After the inspection time t_3 , the imperfect-quality items are withdrawn from the storage and sold. Now, the on-hand inventory contains only perfect-quality items, and these are consumed during t_4 . The time t_4 is determined as follows:

$$E[t_4] = \frac{yw_1 - B - Dt_3 - E[x]yw_1}{D(s)}. \quad (6)$$

The shortages accumulation period (t_5) is obtained with

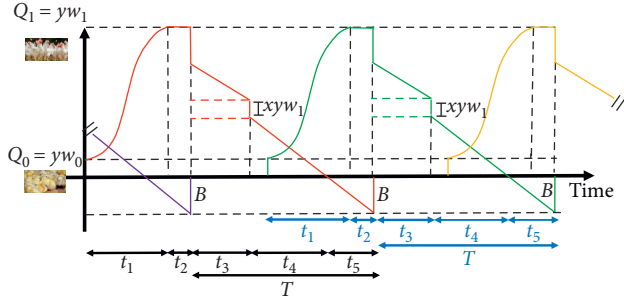


FIGURE 1: Inventory behavior for growing items with imperfect quality and shortages.

$$t_5 = \frac{B}{D(s)}. \quad (7)$$

With the aim to have more manageable equations and without loss of generality, it is defined that the expected cycle time T is computed by the sum of t_3 , t_4 , and t_5 . From equations (5)–(7), the expected cycle time $E[T]$ is calculated as

$$E[T] = t_3 + E[t_4] + t_5 = \frac{yw_1(1 - E[x])}{D(s)}. \quad (8)$$

The main goal of the inventory model for growing items with imperfect quality, carbon emissions, and shortages is to determine the optimal values for the selling price, order quantity, and backordering quantity that maximize the total profit of the company. The total profit is obtained by the difference between the total revenue and the total cost of the inventory system. The details of how to calculate each component of the total revenue and the total cost are given below.

4.1. Expected Revenue per Period. Due to the fact that the company vends both perfect- and imperfect-quality items, for that reason, the expected total revenue $E[TR]$ is calculated by the sum of total sales of imperfect and perfect items. The perfect-quality items are sold at a price of s per unit of weight, while at the end of the inspection period t_3 , the imperfect-quality items are vended as a single batch with a less price of v per unit of weight. As a result, the expected total revenue $E[TR]$ per period is given by

$$E[TR] = syw_1(1 - E[x]) + vyw_1E[x]. \quad (9)$$

4.2. Total Cost per Period. The expected total cost $E[TC]$ per period includes the purchasing cost, setup cost, feeding cost, expected holding cost, inspection cost, backordering cost, and expected carbon emissions cost. One has

$$E[TC] = Pc + Sc + Fc + E[Hc] + Zc + Bc + E[Ec]. \quad (10)$$

The objective of this inventory model is to maximize the expected total profit $E[TP]$ which is obtained by subtracting the expected total cost per period $E[TC]$ from the expected total revenue $E[TR]$ per period. Then,

$$E[TP] = E[TR] - Pc - Sc - Fc - E[Hc] - Zc - Bc - E[Ec]. \quad (11)$$

The following section provides a detailed discussion of the calculations of the aforesaid costs.

4.3. Purchasing Cost per Period. At the beginning of each period, the company buys y newborn items at a cost of p per unit of weight; each one with a weight of w_0 . Thus, the purchasing cost per period is obtained with

$$Pc = pyw_0. \quad (12)$$

4.4. Setup Cost per Period. At the commencement of each period, a setup cost of K is carried out. So, the setup cost per period is determined as

$$Sc = K. \quad (13)$$

4.5. Feeding Cost per Period. The growing items are fed during t_1 , and a feeding cost is incurred by the company at c per unit of weight. As the growing items become older and bigger, they need more food. For that reason, the quantity of food consumed by the items depends on the age (weight) of the items according to the growth function $w_t(t)$. The feeding cost per period is calculated with

$$\begin{aligned} Fc &= cy \int_0^{t_1} w_{t_1}(t) dt = cy \int_0^{t_1} \left(\frac{\alpha}{1 + \beta e^{-\lambda t}} \right) dt \\ &= cy \left(\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right). \end{aligned} \quad (14)$$

4.6. Expected Holding Cost per Period. The company incurs in a holding cost for maintaining in good conditions the items in storage during the cycle time T . The expected holding cost is determined by multiplying the holding cost (h) per unit of weight per unit of time by the inventory carried out during $t_2 + t_3 + t_4$. The whole inventory held is calculated as the sum of the areas $A_1 + A_2 + A_3 + A_4 + A_5$ (see Figure 5 in Appendix A). So, the expected holding cost is given by

$$\begin{aligned} E[Hc] &= h \left[\frac{y^2 w_1^2 E[(1-x)^2]}{2D(s)} - \frac{yw_1(1-E[x])B}{D(s)} \right. \\ &\quad \left. + \frac{B^2}{2D(s)} + \frac{y^2 w_1^2 E[x]}{r} - \frac{yw_1 E[x]B}{r} + \frac{yw_1 B}{r} \right]. \end{aligned} \quad (15)$$

The detailed derivation of the expected holding cost ($E[Hc]$) is given in Appendix A.

4.7. Inspection Cost per Period. Throughout $t_2 + t_3$, a 100% inspection process is performed at a rate of r with the aim of

splitting the perfect-quality items from imperfect-quality items. The company incurs in an inspection cost of z per unit of weight. In consequence, the inspection cost per period is expressed as

$$Zc = zyw_1. \quad (16)$$

4.8. Shortage Cost per Period. During the course of t_5 , the shortages are accumulated till these reach the backordering quantity of B units of weight. The company has a backordering cost for the management of the accumulation of shortages. The backordering cost is computed by multiplying shortage cost (b) per unit of weight per unit of time by area A_6 (see Figure 5 in Appendix A). Therefore, the backordering cost is calculated as

$$Bc = \frac{bB^2}{2D(s)}. \quad (17)$$

The detailed derivation of the backordering cost (Bc) is specified in Appendix B.

There are several research works that suppose that the carbon emissions generated by the companies are due to the transportation and warehousing activities. But, in fact, there are a lot of operations that emit carbon emissions. For example, the operations involved in the following process also cause carbon emissions: purchasing, setup, feeding, holding inventory, and inspection, among others. It is important to mention that some growing items per se generate carbon emissions (cows, goats, pigs, etc.). The carbon emissions caused by these processes are determined as follows:

4.9. Carbon Emissions Caused by the Purchasing Action.

$$\widehat{P}c = \widehat{p}yw_0. \quad (18)$$

4.10. Carbon Emissions Produced by the Setup Activity.

$$\widehat{S}c = \widehat{K}. \quad (19)$$

4.11. Carbon Emissions Generated during the Feeding Process.

$$\begin{aligned} \widehat{F}c &= \widehat{c}y \int_0^{t_1} w_{t_1}(t)dt = \widehat{c}y \int_0^{t_1} \left(\frac{\alpha}{1 + \beta e^{-\lambda t}} \right) dt \\ &= \widehat{c}y \left(\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right). \end{aligned} \quad (20)$$

4.12. Carbon Emissions Created in Holding Inventory's Operations.

$$\begin{aligned} E[\widehat{H}c] &= \widehat{h} \left[\frac{y^2 w_1^2 E[(1-x)^2]}{2D(s)} - \frac{yw_1(1-E[x])B}{D(s)} \right. \\ &\quad \left. + \frac{B^2}{2D(s)} + \frac{y^2 w_1^2 E[x]}{r} - \frac{yw_1 E[x]B}{r} + \frac{yw_1 B}{r} \right]. \end{aligned} \quad (21)$$

4.13. Carbon Emissions Made by the Inspection Process.

$$\widehat{Z}c = \widehat{z}yw_1. \quad (22)$$

Carbon tax (θ) is one of the well-known mechanisms imposed by government regulations as a penalty. This means that the companies need to pay a tax on the amount of carbon emissions. As a result, the carbon emissions cost per period that the company must pay is

$$\begin{aligned} Ec &= \theta \left[\widehat{p}yw_0 + \widehat{K} + \widehat{z}yw_1 + \widehat{c}y \left(\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right) \right. \\ &\quad \left. + \widehat{h} \left[\frac{y^2 w_1^2 E[(1-x)^2]}{2D(s)} - \frac{yw_1(1-E[x])B}{D(s)} + \frac{B^2}{2D(s)} + \frac{y^2 w_1^2 E[x]}{r} - \frac{yw_1 E[x]B}{r} + \frac{yw_1 B}{r} \right] \right]. \end{aligned} \quad (23)$$

4.14. Expected Total Profit Function. The expected total profit $E[TP]$ per period is calculated in the following manner: the expected total revenue $E[TR]$ per period minus the expected total cost $E[TC]$ per period. Basically, the expected total

profit $E[TP]$ is formulated by substituting equations (9) and (12) to (17) and equation (23) into equation (11). Then, the expected total profit $E[TPU]$ per unit time per period is determined as $E[TPU] = (E[TP]/E[T])$. Thus,

$$\begin{aligned}
 E[\text{TPU}(y, B, s)] = & sD(s) + \frac{vD(s)E[x]}{(1-E[x])} - \frac{pD(s)w_0}{w_1(1-E[x])} - \frac{KD(s)}{yw_1(1-E[x])} - \frac{zD(s)}{(1-E[x])} \\
 & - \frac{cD(s)}{w_1(1-E[x])} \left[\alpha t_1 + \frac{\alpha}{\lambda} \left[\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta) \right] \right] \\
 & - h \left[\frac{yw_1 E[(1-x)^2]}{2(1-E[x])} - B + \frac{B^2}{2yw_1(1-E[x])} + \frac{D(s)yw_1 E[x]}{r(1-E[x])} + \frac{D(s)B}{r} \right] \\
 & - \frac{bB^2}{2yw_1(1-E[x])} - \theta \left[\frac{\widehat{p}D(s)w_0}{w_1(1-E[x])} + \frac{\widehat{K}D(s)}{yw_1(1-E[x])} + \frac{\widehat{z}D(s)}{(1-E[x])} \right. \\
 & \left. + \frac{\widehat{c}D(s)}{w_1(1-E[x])} \left[\alpha t_1 + \frac{\alpha}{\lambda} \left[\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta) \right] \right] \right. \\
 & \left. + \widehat{h} \left[\frac{yw_1 E[(1-x)^2]}{2(1-E[x])} - B + \frac{B^2}{2yw_1(1-E[x])} + \frac{D(s)yw_1 E[x]}{r(1-E[x])} + \frac{D(s)B}{r} \right] \right].
 \end{aligned} \tag{24}$$

Considering that the demand rate depends on selling price with a polynomial function $D(s) = \pi - \rho s^n$, the expected total profit $E[\text{TPU}(y, B, s)]$ per unit of time becomes

$$\begin{aligned}
 E[\text{TPU}(y, B, s)] = & s(\pi - \rho s^n) + \frac{v(\pi - \rho s^n)E[x]}{(1-E[x])} - \frac{p(\pi - \rho s^n)w_0}{w_1(1-E[x])} - \frac{K(\pi - \rho s^n)}{yw_1(1-E[x])} \\
 & - \frac{z(\pi - \rho s^n)}{(1-E[x])} - \frac{c(\pi - \rho s^n)}{w_1(1-E[x])} \left[\alpha t_1 + \frac{\alpha}{\lambda} \left[\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta) \right] \right] \\
 & - h \left[\frac{yw_1 E[(1-x)^2]}{2(1-E[x])} - B + \frac{B^2}{2yw_1(1-E[x])} + \frac{(\pi - \rho s^n)yw_1 E[x]}{r(1-E[x])} + \frac{(\pi - \rho s^n)B}{r} \right] \\
 & - \frac{bB^2}{2yw_1(1-E[x])} - \theta \left[\frac{\widehat{p}(\pi - \rho s^n)w_0}{w_1(1-E[x])} + \frac{\widehat{K}(\pi - \rho s^n)}{yw_1(1-E[x])} + \frac{\widehat{z}(\pi - \rho s^n)}{(1-E[x])} \right. \\
 & \left. + \frac{\widehat{c}(\pi - \rho s^n)}{w_1(1-E[x])} \left[\alpha t_1 + \frac{\alpha}{\lambda} \left[\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta) \right] \right] \right. \\
 & \left. + \widehat{h} \left[\frac{yw_1 E[(1-x)^2]}{2(1-E[x])} - B + \frac{B^2}{2yw_1(1-E[x])} + \frac{(\pi - \rho s^n)yw_1 E[x]}{r(1-E[x])} + \frac{(\pi - \rho s^n)B}{r} \right] \right].
 \end{aligned} \tag{25}$$

Then, the objective is to maximize the expected total profit $E[\text{TPU}(y, B, s)]$ per unit of time. Therefore, the optimization problem is formulated as follows:

$$\text{Max}_{(y, B, s) \in \Omega} E[\text{TPU}(y, B, s)]$$

$$\text{where } \Omega = \left\{ (y, B, s): y > 0, 0 \leq B \leq yw_1 \text{ and } p \leq s \leq \left(\frac{\pi}{\rho} \right)^{(1/n)} \right\}. \tag{26}$$

It is important to remark that the abovementioned maximization formulation is a nonlinear optimization problem.

5. Solution Procedure

Firstly, the unconstrained optimization problem is considered in order to obtain some theoretical results. Secondly, an algorithm for finding the optimal solution to the constrained problem given by equation (26) is developed.

5.1. Theoretical Results. The aim is to find the optimal values for order quantity (y), backordering quantity (B), and selling price (s) that maximize the expected total profit per unit of time. As the expected total profit per unit of time function $E[\text{TPU}(y, B, s)]$ is continuous and twice differentiable with respect to the three decision variables (y, B, s) on the interval $[0, \infty]$, there exists a global maximum on that interval.

The necessary conditions which must be satisfied for the optimality of solution that maximizes the expected total profit per unit of time function are $((\partial E[\text{TPU}(y, B, s)]/\partial y) = 0$, $(\partial E[\text{TPU}(y, B, s)]/\partial B) = 0$, and $((\partial E[\text{TPU}(y, B, s)]/\partial s) = 0$.

The first partial derivative of $E[\text{TPU}(y, B, s)]$ given in equation (25) with respect to y is expressed as follows:

$$\begin{aligned} \frac{\partial E[\text{TPU}(y, B, s)]}{\partial y} &= \frac{K(\pi - \rho s^n)}{y^2 w_1 (1 - E[x])} \\ &- h \left[\frac{w_1 E[(1-x)^2]}{2(1-E[x])} - \frac{B^2}{2y^2 w_1 (1-E[x])} + \frac{(\pi - \rho s^n) w_1 E[x]}{r(1-E[x])} \right] + \frac{bB^2}{2y^2 w_1 (1-E[x])} \\ &- \theta \left[\frac{\widehat{K}(\pi - \rho s^n)}{y^2 w_1 (1-E[x])} + \widehat{h} \left[\frac{w_1 E[(1-x)^2]}{2(1-E[x])} - \frac{B^2}{2y^2 w_1 (1-E[x])} + \frac{(\pi - \rho s^n) w_1 E[x]}{r(1-E[x])} \right] \right] = 0. \end{aligned} \quad (27)$$

The first partial derivative of $E[\text{TPU}(y, B, s)]$ with respect to B is given by

$$\begin{aligned} \frac{\partial [\text{TPU}(y, B, s)]}{\partial B} &= -h \left[-1 + \frac{B}{y w_1 (1-E[x])} + \frac{(\pi - \rho s^n)}{r} \right] - \frac{bB}{y w_1 (1-E[x])} \\ &- \theta \widehat{h} \left[-1 + \frac{B}{y w_1 (1-E[x])} + \frac{(\pi - \rho s^n)}{r} \right] = 0. \end{aligned} \quad (28)$$

The first partial derivative of $E[\text{TPU}(y, B, s)]$ with respect to s is

$$\begin{aligned} \frac{\partial E[\text{TPU}(y, B, s)]}{\partial s} &= \pi - (n+1)\rho s^n - \frac{\rho n s^{n-1}}{w_1 (1-E[x])} \left[v E[x] w_1 - (p + \theta \widehat{p}) w_0 - \frac{(K + \theta \widehat{K})}{y} \right. \\ &- (z + \theta \widehat{z}) w_1 - (c + \theta \widehat{c}) \left[\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right] \\ &- \left. \frac{(h + \theta \widehat{h})}{r} [y w_1^2 E[x] + B w_1 (1 - E[x])] \right] = 0. \end{aligned} \quad (29)$$

By solving equations (27) and (28) simultaneously for the decision variables y and B , one has

$$y = \sqrt{\frac{2(\pi - \rho s^n) r (K + \theta \widehat{K})}{w_1^2 (h + \theta \widehat{h}) [r E[(1-x)^2] + 2(\pi - \rho s^n) E[x]] - \left([(h + \theta \widehat{h}) w_1 (1 - E[x]) [r - (\pi - \rho s^n)]]^2 / ((h + b + \theta \widehat{h}) r) \right)},} \quad (30)$$

$$B = \frac{(h + \theta \widehat{h}) y w_1 (1 - E[x]) [r - (\pi - \rho s^n)]}{(h + b + \theta \widehat{h}) r}. \quad (31)$$

The sufficient conditions for the optimality of the solution that maximize $E[TPU(y, B, s)]$ are given in Appendix C.

5.2. *Algorithm for Finding the Optimal Solution.* The following Algorithm 1 is proposed taking into account the theoretical results and constrains.

5.3. *Special Cases.* The proposed inventory model developed in this research work is a general inventory model because it

contains the following inventory models as special cases: Sebatjane and Adetunji's [40] improved inventory model, Maddah and Jaber [11], Chang and Ho [13], Shih [61], Silver [62], and the traditional EOQ inventory models with and without shortages.

- (i) When $n = 1$, it means that the product has a linear price-dependent demand $(\pi - \rho s)$. It is expressed as follows:

$$y = \sqrt{\frac{2(\pi - \rho s)r(K + \theta\widehat{K})}{w_1^2(h + \theta\widehat{h})[rE[(1 - x)^2] + 2(\pi - \rho s)E[x]] - \left(\left[(h + \theta\widehat{h})w_1(1 - E[x])[r - (\pi - \rho s)] \right]^2 / ((h + b + \theta\widehat{h})r) \right)}, \quad (32)$$

$$B = \frac{(h + \theta\widehat{h})yw_1(1 - E[x])[r - (\pi - \rho s)]}{(h + b + \theta\widehat{h})r}, \quad (33)$$

$$\begin{aligned} \pi - 2\rho s - \frac{\rho}{w_1(1 - E[x])} \left[vE[x]w_1 - (p + \theta\widehat{p})w_0 - \frac{(K + \theta\widehat{K})}{y} - (z + \theta\widehat{z})w_1 \right. \\ \left. - (c + \theta\widehat{c}) \left[\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right] - \frac{(h + \theta\widehat{h})}{r} [yw_1^2E[x] + Bw_1(1 - E[x])] \right] = 0. \end{aligned} \quad (34)$$

- (ii) When shortages and carbon tax rate are not considered and the demand is not dependent on price (i.e., $b \rightarrow \infty, \theta \rightarrow 0$ and $\rho \rightarrow 0$), then an improved version of Sebatjane and Adetunji's [40] inventory model is obtained. It is shown as follows:

$$y = \sqrt{\frac{2\pi K}{hw_1^2[E[(1 - x)^2] + ((2\pi E[x])/r)]}}. \quad (35)$$

- (iii) When shortages are permitted, carbon tax rate is not involved, the demand is not dependent on price, and the products are not of the type of growing items, and thus, the feeding process is not incorporated (i.e., $b > 0, \theta \rightarrow 0, \rho \rightarrow 0$, and $c \rightarrow 0$), and then, the inventory model of Chang and Ho [13] is determined. It is given by

$$y = \sqrt{\frac{2\pi K}{h[E[(1 - x)^2] + ((2\pi E[x])/r)] - ([h(1 - E[x])]^2 / (h + b))}}, \quad (36)$$

$$B = \frac{hy(1 - E[x])}{(h + b)}. \quad (37)$$

- (iv) When shortages are not allowed, carbon tax rate is not taken account, the demand is not dependent on price, and the products are not of the type of growing items ($b \rightarrow \infty, \theta \rightarrow 0, \rho \rightarrow 0$, and $c \rightarrow 0$); then, Maddah and Jaber's [11] inventory model is derived. It is presented as follows:

$$y = \sqrt{\frac{2\pi K}{h[E[(1 - x)^2] + ((2\pi E[x])/r)]}}. \quad (38)$$

- (v) When shortages are not tolerable, carbon tax rate is not considered, the demand does not depend on price, the goods are not of the type of growing

Step 1. Provide the input parameters of the inventory system.
 Step 2. Compute the selling price (s'), the order quantity (y'), and the backordering quantity (B') by solving simultaneously equations (29)–(31).
 Step 3. If the optimality conditions are satisfied, then go to Step 4. Else, go to Step 8.
 Step 4. If $p \leq s \leq (\pi/\rho)^{(1/m)}$, then go to Step 7. Else, go to Step 5.
 Step 5. If $s > (\pi/\rho)^{(1/m)}$, then set $s' = (\pi/\rho)^{(1/m)}$, determine the order quantity (y') with equation (30) and the backordering quantity (B') with equation (31), and go to Step 6. Else, set $s' = p$ and calculate the order quantity (y') with equation (30) and the backordering quantity (B') with equation (31) and go to Step 6.
 Step 6. Calculate the expected total profit per unit of time $E[\text{TPU}(y', B', s')]$ with equation (25).
 Step 7. Report the solution: $(y^*, B^*, s^*) = (y', B', s')$ and $E[\text{TPU}^*(y^*, B^*, s^*)] = E[\text{TPU}(y', B', s')]$.
 Step 8. Stop.

ALGORITHM 1: Algorithm to find the optimal solution.

items, and inspection rate is sufficiently large (i.e., $b \rightarrow \infty, \theta \rightarrow 0, \rho \rightarrow 0, c \rightarrow 0$ and $r \rightarrow \infty$); then, it is converted to the inventory model of Shih [61] and Silver [62]. It is expressed as

$$y = \sqrt{\frac{2\pi K}{h[E[(1-x)^2]]}} \quad (39)$$

- (vi) When shortages are permissible, carbon tax rate is not incorporated, the demand is not dependent on price, the products are not of the type of growing items, and the products are of perfect quality (i.e., $b > 0, \theta \rightarrow 0, \rho \rightarrow 0, c \rightarrow 0$, and $x = 0$); then, the traditional inventory model with shortages and with full backordering is found. It is written as follows:

$$y = \sqrt{\frac{2\pi K}{h}} \sqrt{\frac{h+b}{b}}, \quad (40)$$

$$B = \frac{hy}{(h+b)}. \quad (41)$$

- (vii) When shortages are not allowable, carbon tax rate is not taken into consideration, the demand does not depend on price, the goods are not of type of growing items, and these are of good quality (i.e., $b \rightarrow \infty, \theta \rightarrow 0, \rho \rightarrow 0, c \rightarrow 0$, and $x = 0$); then, it is transformed to the traditional inventory model without shortages proposed by Harris [1]. It is

$$y = \sqrt{\frac{2\pi K}{h}}. \quad (42)$$

It is important to remark that the inventory model developed in this research does not reduce to Salameh and Jaber's [8] and Wee et al.'s [10] inventory models because

those inventory models contain a flaw that was identified and corrected by Maddah and Jaber [11] and Chang and Ho [13], respectively. Basically, Maddah and Jaber [11] and Chang and Ho [13] proposed improved versions for Salameh and Jaber's [8] and Wee et al.'s [10] inventory models, respectively. For this reason, the proposed inventory model developed in this research work, in fact, converges to Maddah and Jaber's [11] and Chang and Ho's [13] inventory models as it was mentioned above.

6. Numerical Examples

This section presents and solves some numerical examples in order to illustrate the applicability of the proposed inventory model.

Example 1. This example considers the input parameters of Sebatjane and Adetunji [40]. To solve this inventory problem, it is stated $b \rightarrow \infty, \theta \rightarrow 0$, and $\rho \rightarrow 0$. The values for the data are $\pi = 1000000$, $\rho = 0$, $\theta = 0$, $s = 0.05$ ZAR/g, $v = 0.02$ ZAR/g, $K = 1000$ ZAR/cycle, $h = 0.04$ ZAR/g/year, $c = 0.2$ ZAR/g, $p = 0.025$ ZAR/g, $z = 0.00025$ ZAR/g, $r = 5256000$ g/year, $\alpha = 6870$ g, $\beta = 120\lambda = 40$ /year, $w_0 = 57$ g, and $w_1 = 1500$ g, and the percentage of imperfect growing items follows a uniform distribution ($x \sim U[\gamma, \delta]$) with the probability density function $g(x)$ which is given as follows:

$$x \sim g(x) = \begin{cases} \frac{1}{\delta - \gamma}, & \gamma \leq x \leq \delta, \\ 0, & \text{otherwise.} \end{cases} \quad (43)$$

Considering $x \sim U[0, 0.04]$,

$$x \sim g(x) = \begin{cases} 25, & 0 \leq x \leq 0.04, \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

Then, $E[x]$ and $E[(1-x)^2]$ are computed as follows:

$$E[x] = \int_{\gamma}^{\delta} x f(x) dx = \frac{\gamma + \delta}{2} = \frac{0 + 0.04}{2} = 0.02,$$

$$E[(1-x)^2] = \int_{\gamma}^{\delta} (1-x)^2 f(x) dx = \frac{\gamma^2 + \gamma\delta + \delta^2}{3} + 1 - \gamma - \delta, \tag{45}$$

$$E[(1-x)^2] = \frac{0^2 + 0(0.04) + (0.04)^2}{3} + 1 - 0 - 0.04 = 0.960533333.$$

As it was mentioned before, the proposed inventory model generates an improved version of Sebatjane and Adetunji's [40] inventory model. Additionally, the proposed inventory model also can be used for optimizing the order quantity (y) and backordering quantity (B) when the selling price (s) is given. Obviously, it also optimizes the three decision variables (y, B, s). Therefore, the numerical example is solved for different values of δ for the four inventory models: (I) Sebatjane and Adetunji [40] (original version), (II) Sebatjane and Adetunji [40] (improved version), (III) the proposed inventory model when selling price (s) is given, and (IV) the proposed inventory model when the selling price (s) is optimized. Table 2 shows the comparison of the optimal solutions among the four inventory models.

From Table 2, it is observed that the expected total profit of both the proposed inventory models (see columns III and IV) is greater than that of Sebatjane and Adetunji's [40] inventory models (see columns I and II).

Example 2. This example uses the input parameters of Wee et al. [10] and Chang and Ho [13]. To solve this inventory system, the following is established: $b > 0, \theta \rightarrow 0, \rho \rightarrow 0$, and $c \rightarrow 0$. The data are as follows: $\pi = 50000, \rho = 0, \theta = 0, c = 0, s = 50$ \$/unit, $v = 20$ \$/unit, $K = 100$ \$/cycle, $h = 5$ \$/unit/year, $b = 10$ \$/unit/year, $p = 25$ \$/unit, $z = 0.5$ \$/unit, $r = 175200$ units/year, and $x \sim U[0, 0.04]$. The optimal solution is as follows: $E[TPU^*(y^*, B^*)] = 1213562$ \$/year, $y^* = 1751.671$ units, and $B^* = 572.2127$ units. This solution is the same as in the work of Chang and Ho [13].

Example 3. Now, let us consider some dataset of Sebatjane and Adetunji [40] which is $v = 0.02$ ZAR/g, $K = 1000$ ZAR/cycle, $c = 0.2$ ZAR/g, $p = 0.025$ ZAR/g, $z = 0.00025$ ZAR/g, $r = 5256000$ g/year, $\alpha = 6870$ g, $\beta = 120$, $\lambda = 40$ /year, $w_0 = 57$ g, $w_1 = 1500$ g, and $x \sim U[0, 0.04]$. Here, the holding cost is $h = 0.2$ ZAR/g/year. For the implementation of the proposed inventory model, additional information is required. These data are related to the type of the demand function dependent on the price, the backordering cost, and carbon emissions. Therefore, it is necessary to state the following data: Assume that the demand rate of the growing items follows a polynomial function given by $D(s) = \pi - \rho s^n$ with values of $\pi = 135000$, $\rho = 1050$, and $n = 2$. The backordering cost is $b = 0.1$ ZAR/g/year. The relevant input parameters related to carbon

emissions are $\theta = 0.0045$ ZAR/tons, $\widehat{K} = 2000$ tons/year, $h = 0.2$ tons/year, $\widehat{c} = 0.65$ tons/year, $\widehat{p} = 0.375$ tons/year, and $\widehat{z} = 0.005$ tons/year. By applying the proposed algorithm, the optimal solution is obtained: $E[TPU^*(y^*, B^*, s^*)] = 584997.4$ ZAR/year, $y^* = 34.26474$ units of newborn growing items, $B^* = 33054.63$ g, and $s^* = 6.555838$ ZAR/g. Figures 2–4 demonstrate graphically the concavity property of the expected total profit with respect to pairs of decision variables y and B ; y and s ; and B and s , respectively.

7. Sensitivity Analysis

This section provides a sensitivity analysis in order to investigate the effects of changing the input parameters on the expected total profit per unit of time ($E[TPU^*]$), order quantity (y^*), backordering quantity (B^*), and selling price (s^*) of the growing items. The sensitivity analysis is performed taking into consideration the data of Example 3 when only one parameter changes at a time and other parameters are kept at their original values. Specifically, the sensitivity analysis is performed to study the effect of scale, sensitivity and power index demand parameters, selling price and expected value of a percent of poorer-quality growing items, setup cost, holding cost, backordering cost, feeding cost, purchasing cost, inspection cost, carbon tax rate, and the amount of carbon emissions caused by the operations of the processes in ordering, holding, feeding, purchasing, and screening. The results of the sensitivity analysis are given in Tables 3–5. Based on the behavioral changes as reflected in Tables 3–5, the following observations are made:

- (1) The value of $E[TPU^*]$ is highly sensitive to the demand parameters π, ρ, n and less sensitive to other parameters. On the one hand, the higher the value of the scale parameter of demand π , the higher the value of $E[TPU^*]$ due to the fact that demand increases; therefore, the sales increase, and this leads to high profits. For this reason, it is suggested for the manager to apply advertising actions in order to boost the demand. On the other hand, when the sensitivity parameter of the demand ρ grows, the $E[TPU^*]$ declines because this parameter has a negative impact on the demand, making it drop. For this reason, it is advisable for the decision maker to

TABLE 2: A comparison of the optimal solutions of the four inventory models.

			I		II		III			IV			
			Sebatjane and Adetunji [40] (original), $b \rightarrow \infty, \theta = 0, \rho = 0$		Sebatjane and Adetunji [40] (improved inventory model), $b \rightarrow \infty, \theta = 0, \rho = 0$		The proposed inventory model, $\rho = 0, s = 0.05$			The proposed inventory model, $\rho = 700000$			
γ	δ	$E[x]$	y^*	$E[TPU^*]$	y^*	$E[TPU^*]$	y^*	B^*	$E[TPU^*]$	y^*	B^*	s^*	$E[TPU^*]$
0	0.04	0.02	151.5143	34641.73	151.5039	34641.11	167.9070	57103.70	35518.41	138.9623	51818.67	0.6940517	449132.6
0	0.10	0.05	155.2887	35013.46	155.2186	35009.38	171.7566	56624.78	35880.02	142.5000	51509.93	0.6939366	449374.6
0	0.20	0.10	161.8760	35680.21	161.5588	35662.24	178.2198	55663.16	36519.49	148.5423	50865.87	0.6937328	449801.7
0	0.30	0.15	168.8356	36413.78	168.0293	36369.20	184.6550	54469.01	37209.73	154.7091	50032.07	0.6935122	450261.3
0	0.40	0.20	176.1594	37225.05	174.5420	37137.38	190.9380	53009.26	37957.34	160.9085	48973.44	0.6932723	450757.0
0	0.50	0.25	183.8244	38127.53	180.9801	37975.53	196.9174	51252.47	38770.55	167.0178	47653.14	0.6930098	451293.2

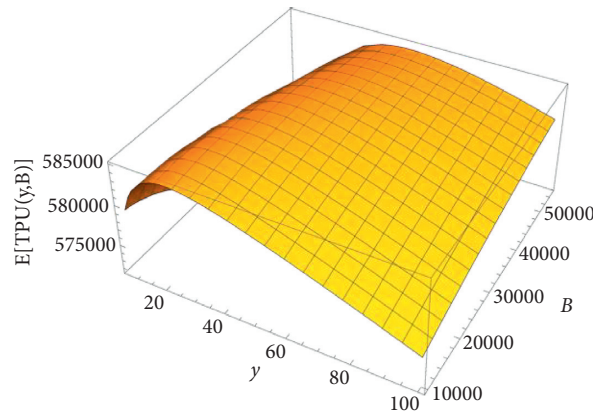


FIGURE 2: Concavity of $E[TPU(y, B)]$ w.r.t. y, B

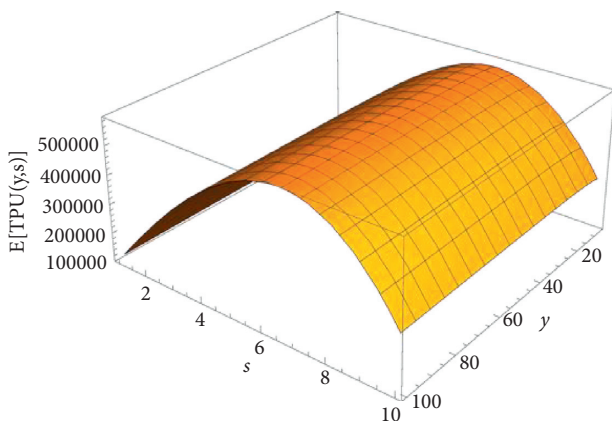


FIGURE 3: Concavity of $E[TPU(y, s)]$ w.r.t. y, s

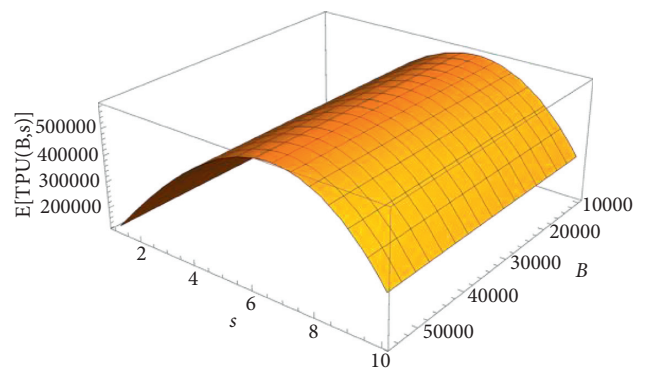


FIGURE 4: Concavity of $E[TPU(B, s)]$ w.r.t. B, s

reduce the selling price. However, a higher value of K induces higher values of s but a lower value for $E[TPU^*]$. This means that as the setup cost increases, the company must raise selling price s .

- (2) The value of y is more sensitive to the parameters π, n, K, h, b and less sensitive to other parameters. The higher the value of the parameter b , the smaller the value of y .
- (3) The value of B is more sensitive to the parameters π, n, K, h, b and less sensitive to other parameters. The higher the value of b , the smaller the value of B .

TABLE 3: Effects of the demand parameters (π, ρ, n), the price of imperfect growing items (v), and the percent of the imperfect growing items ($E[x]$) on the optimal solution.

Parameter	Value	y^*	B^*	s^*	$E[TPU^*]$
π	85000	27.51468	26714.18	5.205486	291168.5
	135000	34.26474	33054.63	6.555838	584997.4
	185000	39.64369	37996.92	7.672019	940097.9
	235000	44.17254	42062.76	8.645232	1347288
	285000	48.10619	45509.16	9.519438	1800598
ρ	650	34.26961	33059.16	8.329802	744653.9
	750	34.26828	33057.92	7.755266	692945.8
	850	34.26703	33056.76	7.285369	650655.1
	950	34.26586	33055.67	6.891773	615231.4
	1050	34.26474	33054.63	6.555838	584997.4
n	2	34.26474	33054.63	6.555838	584997.4
	5	37.98523	36485.12	1.850223	202864.7
	7	38.84940	37274.30	1.490172	170704.7
	10	39.54134	37903.96	1.281047	151875.7
	14	40.02588	38343.67	1.167557	141757.1
v	0.02	34.26474	33054.63	6.555838	584997.4
	0.06	34.26541	33055.26	6.555565	585070.7
	0.10	34.26608	33055.88	6.555293	585144.1
	0.14	34.26675	33056.50	6.555020	585217.5
	0.18	34.26742	33057.12	6.554748	585290.8
$E[x]$	0.01	33.94088	33076.32	6.555872	584987.5
	0.02	34.26474	33054.63	6.555838	584997.4
	0.03	34.59095	33028.81	6.555804	585007.0
	0.04	34.91934	32998.64	6.555770	585016.4
	0.05	35.24975	32963.88	6.555736	585025.6

TABLE 4: Impacts of the costs of the inventory system (K, h, b, c, p, z, θ) on the optimal solution.

Parameter	Value	y^*	B^*	s^*	$E[TPU^*]$
K	1000	34.26474	33054.63	6.555838	584997.4
	1200	37.50542	36180.93	6.556512	584656.6
	1400	40.48714	39057.41	6.557133	584343.1
	1600	43.26350	41735.80	6.557711	584051.2
	1800	45.87182	44252.08	6.558254	583776.9
h	0.12	38.34897	30325.67	6.554933	585380.6
	0.16	35.88691	31977.51	6.555439	585160.0
	0.2	34.26474	33054.63	6.555838	584997.4
	0.24	33.09099	33786.80	6.556172	584869.7
	0.28	32.18620	34295.59	6.556464	584765.0
b	0.1	34.26474	33054.63	6.555838	584997.4
	0.15	30.54445	25267.19	6.556552	584559.0
	0.2	28.43147	20585.99	6.557049	584259.0
	0.25	27.05971	17420.14	6.557416	584039.2
	0.3	26.09415	15121.73	6.557700	583870.6
c	0.05	34.26816	33057.81	6.554445	585372.3
	0.1	34.26702	33056.75	6.554909	585247.3
	0.15	34.26588	33055.69	6.555374	585122.3
	0.2	34.26474	33054.63	6.555838	584997.4
	0.25	34.26360	33053.58	6.556302	584872.4
p	0.025	34.26474	33054.63	6.555838	584997.4
	0.05	34.26395	33053.90	6.556162	584910.2
	0.075	34.26315	33053.16	6.556485	584823.1
	0.1	34.26236	33052.42	6.556809	584736.0
	0.125	34.26156	33051.68	6.557133	584648.9
z	0.00025	34.26474	33054.63	6.555838	584997.4
	0.0005	34.26453	33054.44	6.555923	584974.4
	0.00075	34.26432	33054.25	6.556008	584951.5
	0.001	34.26412	33054.05	6.556094	584928.6
	0.00125	34.26391	33053.86	6.556179	584905.6

TABLE 4: Continued.

Parameter	Value	y^*	B^*	s^*	$E[TPU^*]$
θ	0.0045	34.26474	33054.63	6.555838	584997.4
	0.01	34.41339	33258.23	6.555956	584955.2
	0.0155	34.56096	33460.86	6.556074	584913.2
	0.021	34.70746	33662.55	6.556192	584871.2
	0.0265	34.85293	33863.29	6.556309	584829.3

TABLE 5: Impacts of the carbon emission parameters ($\widehat{K}, \widehat{h}, \widehat{c}, \widehat{p}, \widehat{z}$) on the optimal solution.

Parameter	Value	y^*	B^*	s^*	$E[TPU^*]$
\widehat{K}	2000	34.26474	33054.63	6.555838	584997.4
	2500	34.30291	33091.45	6.555846	584993.3
	3000	34.34103	33128.23	6.555854	584989.3
	3500	34.37910	33164.96	6.555862	584985.3
	4000	34.41714	33201.65	6.555870	584981.3
\widehat{h}	0.2	34.26474	33054.63	6.555838	584997.4
	0.4	34.23432	33074.31	6.555846	584994.2
	0.6	34.20411	33093.81	6.555854	584991.0
	0.8	34.17412	33113.15	6.555862	584987.8
	1	34.14434	33132.32	6.555870	584984.7
\widehat{c}	0.65	34.26474	33054.63	6.555838	584997.4
	1.3	34.26468	33054.57	6.555865	584990.0
	1.95	34.26461	33054.51	6.555892	584982.7
	2.6	34.26454	33054.45	6.555919	584975.4
	3.25	34.26448	33054.39	6.555947	584968.1
\widehat{p}	0.375	34.26474	33054.63	6.555838	584997.4
	0.75	34.26469	33054.58	6.555860	584991.5
	1.125	34.26464	33054.53	6.555882	584985.6
	1.5	34.26458	33054.48	6.555904	584979.7
	1.875	34.26453	33054.44	6.555925	584973.8
\widehat{z}	0.005	34.26474	33054.63	6.555838	584997.4
	0.015	34.26470	33054.60	6.555853	584993.2
	0.025	34.26467	33054.56	6.555869	584989.1
	0.035	34.26463	33054.53	6.555884	584985.0
	0.045	34.26459	33054.49	6.555899	584980.8

TABLE 6: Sensitivity analysis of the key parameters of the inventory system.

Parameter	y^*	B^*	s^*	$E[TPU^*]$
$\pi \uparrow$	\uparrow	\uparrow	\uparrow	\uparrow
$\rho \uparrow$	\downarrow	\downarrow	\downarrow	\downarrow
$n \uparrow$	\uparrow	\uparrow	\downarrow	\downarrow
$v \uparrow$	\uparrow	\uparrow	\downarrow	\uparrow
$K \uparrow$	\uparrow	\uparrow	\uparrow	\downarrow
$h \uparrow$	\downarrow	\uparrow	\uparrow	\downarrow
$b \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$c \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$p \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$z \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$\theta \uparrow$	\uparrow	\uparrow	\uparrow	\downarrow
$\widehat{K} \uparrow$	\uparrow	\uparrow	\uparrow	\downarrow
$\widehat{h} \uparrow$	\downarrow	\uparrow	\uparrow	\downarrow
$\widehat{c} \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$\widehat{p} \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$\widehat{z} \uparrow$	\downarrow	\downarrow	\uparrow	\downarrow
$E[x] \uparrow$	\uparrow	\downarrow	\downarrow	\uparrow

- (4) The value of s is more sensitive to the parameters π, ρ, n and less sensitive to other parameters. The higher the value of the parameter π , the higher the value of s . This means that as demand scale parameter π increases, the company must raise selling price which will also directly impact positively the expected total profit. The higher the value of the parameters ρ, n , the smaller the value of s .
- (5) When the expected value of the defective growing items $E[x]$ decreases, the expected total profit also decreases $E[TPU^*]$.
- (6) Changes on carbon emission parameters have a regular influence on the expected total profit $E[TPU^*]$.

With the information of Tables 3–5 is constructed Table 6. Table 6 summarizes the sensitivity analysis study in a visual manner which is more helpful for the decision makers due to that these can more easily observe how changes in parameters affect to the decisions variables and the expected total profit per unit of time in order to take the best decisions based on numerical facts.

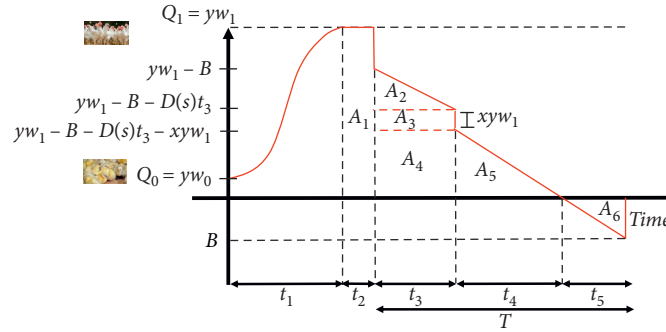


FIGURE 5: Inventory behavior through time.

8. Conclusions and Future Research

This paper develops an inventory model for growing items with imperfect quality and carbon emissions in which the demand rate is price sensitive according to a polynomial function. Furthermore, the shortages are permissible and these are fully backordered. To optimize the expected total profit per unit of time, some theoretical results are derived, and with these results, an effective and efficient solution procedure is developed. It is important to state that the algorithm generates the optimal solution to order quantity, backordering quantity, and selling price which maximizes the expected total profit per unit of time. The effects of carbon tax and the amount of carbon emissions are studied. Some important managerial insights are obtained from numerical examples and sensitivity analysis. The results show that carbon tax has a regular role on the reduction of carbon emissions. It is important to pay attention to quality and ensure that the percentage of defective items is kept to a minimum. Furthermore, if the setup cost increases, the company must raise selling price s , and when the demand elasticity parameter increases, the company must raise selling price which also directly impacts the total profit. Furthermore, the proposed inventory model is a generalized inventory model due the fact that several previously published inventory models are particular cases. It was found that the policy that permits shortages with full backordering is more economical than the one that avoids shortages. Moreover, it was also found that the proposed inventory model outperforms the inventory model of Sebatjane and Adetunji [40].

There are several possible extensions of the proposed inventory model that can be explored. These extensions constitute future research endeavors in the inventory

management of imperfect growing items. For instance, consider to investigate the effect that only a percentage of defective items can be reworked, and the others must be eliminated immediately. Other research studies that can be conducted are to include new aspects such as stock-dependent demand, nonlinear holding cost, vendor-managed inventory (VMI) with consignment stock (CS), inflation, volume discounts, deterioration, trade credit, supply chain environment, and a vendor-buyer inventory model with multiple shipments, partial backordering, advertising, and multiple products subject to constraints such as space, budget, and time. These are, among others, some interesting and challenging subjects of ongoing future investigation that academicians and researchers would like to study in the future.

Appendix

A. Determination of the Expected Holding Cost ($E[Hc]$)

The holding cost is calculated by the multiplication of the unit holding cost (h) and the total inventory accumulated during $t_2 + t_3 + t_4$, and this is the area above level zero in Figure 5. This area is computed as the sum of the areas $A_1 + A_2 + A_3 + A_4 + A_5$.

Considering Figure 5, the five areas are defined as follows.

The area A_1 is obtained as follows:

$$A_1 = (t_2)(yw_1) = \left(\frac{B}{r}\right)(yw_1) = \frac{yw_1B}{r}. \tag{A.1}$$

The area A_2 is found as follows:

$$A_2 = \frac{(t_3)(D(s))(t_3)}{2} = \frac{(D(s))(t_3^2)}{2} = \frac{(D(s))((yw_1 - B)/r)^2}{2} = \frac{(D(s))(yw_1 - B)^2}{2r^2}. \tag{A.2}$$

The area A_3 is determined in the following manner:

$$A_3 = (t_3)(xyw_1) = \left(\frac{yw_1 - B}{r}\right)(xyw_1) = \frac{y^2w_1^2x}{r} - \frac{yw_1xB}{r}. \tag{A.3}$$

The area A_4 is defined as follows:

$$\begin{aligned} A_4 &= (t_3)(yw_1 - B - D(s)t_3 - xyw_1) \\ &= (t_3)(yw_1(1 - x) - B - D(s)t_3), \end{aligned} \tag{A.4}$$

$$A_4 = (yw_1(1 - x) - B)t_3 - D(s)t_3^2.$$

The area A_5 is given by

$$\begin{aligned}
 A_5 &= \frac{(t_4)(yw_1 - B - D(s)t_3 - xyw_1)}{2} = \left(\frac{yw_1 - B - D(s)t_3 - xyw_1}{D(s)} \right) \left(\frac{yw_1(1-x) - B - D(s)t_3}{2} \right), \\
 A_5 &= \left(\frac{yw_1(1-x) - B - D(s)t_3}{D(s)} \right) \left(\frac{yw_1(1-x) - B - D(s)t_3}{2} \right) = \frac{(yw_1(1-x) - B - D(s)t_3)^2}{2D(s)}, \\
 A_5 &= \frac{(yw_1(1-x) - B)^2 - 2(yw_1(1-x) - B)D(s)t_3 + (D(s)t_3)^2}{2D(s)}, \\
 A_5 &= \frac{(yw_1(1-x) - B)^2}{2D(s)} - (yw_1(1-x) - B)t_3 + \frac{D(s)t_3^2}{2}.
 \end{aligned} \tag{A.5}$$

The total inventory accumulated (TIA) during $t_2 + t_3 + t_4$ is computed as follows:

$$TIA = A_1 + A_2 + A_3 + A_4 + A_5. \tag{A.6}$$

Substituting the corresponding areas,

$$\begin{aligned}
 TIA &= \frac{yw_1B}{r} + \frac{(D(s))(t_3^2)}{2} + \frac{y^2w_1^2x}{r} - \frac{yw_1xB}{r} + (yw_1(1-x) - B)t_3 - D(s)t_3^2 \\
 &+ \frac{(yw_1(1-x) - B)^2}{2D(s)} - (yw_1(1-x) - B)t_3 + \frac{D(s)t_3^2}{2}.
 \end{aligned} \tag{A.7}$$

Simplifying,

$$\begin{aligned}
 TIA &= \frac{yw_1B(1-x)}{r} + \frac{y^2w_1^2x}{r} + \frac{(yw_1(1-x) - B)^2}{2D(s)}, \\
 TIA &= \frac{yw_1B(1-x)}{r} + \frac{y^2w_1^2x}{r} + \frac{y^2w_1^2(1-x)^2 - 2yw_1(1-x)B + B^2}{2D(s)}, \\
 TIA &= \frac{yw_1B(1-x)}{r} + \frac{y^2w_1^2x}{r} + \frac{y^2w_1^2(1-x)^2 - 2yw_1(1-x)B + B^2}{2D(s)}.
 \end{aligned} \tag{A.8}$$

Taking the expected value of the total inventory accumulated (TIA), the following expression is obtained:

$$E[TIA] = \frac{yw_1B(1 - E[x])}{r} + \frac{y^2w_1^2E[x]}{r} + \frac{y^2w_1^2E[(1-x)^2] - 2yw_1(1 - E[x])B + B^2}{2D(s)}. \tag{A.9}$$

Rearranging terms,

$$E[TIA] = \frac{y^2w_1^2E[(1-x)^2]}{2D(s)} - \frac{yw_1(1 - E[x])B}{D(s)} + \frac{B^2}{2D(s)} + \frac{y^2w_1^2E[x]}{r} - \frac{yw_1E[x]B}{r} + \frac{yw_1B}{r}. \tag{A.10}$$

Finally, the expected holding cost ($E[Hc]$) is given by

$$E[Hc] = h \left[\frac{y^2 w_1^2 E[(1-x)^2]}{2D(s)} - \frac{y w_1 (1-E[x])B}{D(s)} + \frac{B^2}{2D(s)} + \frac{y^2 w_1^2 E[x]}{r} - \frac{y w_1 E[x]B}{r} + \frac{y w_1 B}{r} \right]. \quad (A.11)$$

B. Determination of the Backordering Cost (B_c)

The backordering cost is determined by the multiplication of the unit backordering cost (b) and the total shortages accumulated during t_5 , and this is the area A_6 shown below level zero in Figure 5. This area (A_6) is obtained as follows:

$$A_6 = \frac{(t_5)(B)}{2} = \frac{(B/D(s))(B)}{2} = \frac{B^2}{2D(s)}. \quad (B.1)$$

Thus, the backordering cost (B_c) is given by

$$B_c = \frac{bB^2}{2D(s)}. \quad (B.2)$$

C. Sufficient Conditions for the Optimality

For the sake of brevity, only the final equations are provided for the direct and cross second-order partial derivatives.

The direct second-order partial derivatives are given as follows.

The second-order partial derivative of equation (25) with respect to y is as follows:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} = \frac{B^2 [h + b + \theta \hat{h}] + 2(\pi - \rho s^n) [K + \theta \hat{K}]}{y^3 w_1 (1 - E[x])} < 0. \quad (C.1)$$

The second-order partial derivative of equation (25) with respect to B is expressed as

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2} = \frac{h + b + \theta \hat{h}}{y w_1 (1 - E[x])} < 0. \quad (C.2)$$

The second-order partial derivative of equation (25) with respect to s is given as follows:

$$\begin{aligned} \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s^2} = & -(n+1)\rho n s^{n-1} - \frac{\rho n(n-1)s^{n-2}}{w_1(1-E[x])} \left[vE[x]w_1 - (p + \theta \hat{p})w_0 - \frac{(K + \theta \hat{K})}{y} \right. \\ & \left. - (z + \theta \hat{z})w_1 - (c + \theta \hat{c}) \left[\alpha t_1 + \frac{\alpha}{\lambda} [\ln(1 + \beta e^{-\lambda t_1}) - \ln(1 + \beta)] \right] \right] \\ & - \frac{(h + \theta \hat{h})}{r} [y w_1^2 E[x] + B w_1 (1 - E[x])] < 0. \end{aligned} \quad (C.3)$$

The second-order cross partial derivatives are given as follows.

The second-order cross partial derivative ($\partial^2/\partial y \partial B$) of equation (25) is presented as follows:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial B} = \frac{B[h + b + \theta \hat{h}]}{y^2 w_1 (1 - E[x])}. \quad (C.4)$$

The second-order cross partial derivative ($\partial^2/\partial y \partial s$) of equation (25) is written in the following manner:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial s} = \frac{\rho n s^{n-1}}{y^2 w_1 (1 - E[x])} (K + \theta \hat{K}) + \frac{w_1 E[x] \rho n s^{n-1}}{r(1 - E[x])} (h + \theta \hat{h}). \quad (C.5)$$

The second-order cross partial derivative ($\partial^2/\partial B \partial y$) of equation (25) is shown in the following way:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial y} = \frac{B[h + b + \theta \hat{h}]}{y^2 w_1 (1 - E[x])}. \quad (C.6)$$

The second-order cross partial derivative ($\partial^2/\partial B \partial s$) of equation (25) is given as follows:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial s} = \frac{\rho n s^{n-1} [h + \theta \hat{h}]}{r}. \quad (C.7)$$

The second-order cross partial derivative ($\partial^2/\partial s \partial y$) of equation (25) is given as follows:

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s \partial y} = \frac{\rho n s^{n-1}}{y^2 w_1 (1 - E[x])} (K + \theta \hat{K}) + \frac{w_1 E[x] \rho n s^{n-1}}{r(1 - E[x])} (h + \theta \hat{h}). \quad (C.8)$$

The second-order cross partial derivative ($\partial^2/\partial s\partial B$) of equation (25) is

$$\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s\partial B} = \frac{\rho n s^{n-1} [h + \theta \hat{h}]}{r} \quad (\text{C.9})$$

Optimality is given as follows.

To prove the optimality of three decision variables, it is required to construct the Hessian matrix H , which is given as follows:

$$H = \begin{bmatrix} \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y\partial B} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y\partial s} \\ \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B\partial y} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B\partial s} \\ \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s\partial y} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s\partial B} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s^2} \end{bmatrix}. \quad (\text{C.10})$$

The sufficient conditions for optimality through the Hessian matrix H are as follows:

$\text{Det}(H_1) < 0$, $\text{Det}(H_2) > 0$, and $\text{Det}(H_3) < 0$, where H_1 , H_2 , and H_3 are the following matrixes:

$$H_1 = \left[\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} \right],$$

$$H_2 = \begin{bmatrix} \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y\partial B} \\ \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B\partial y} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2} \end{bmatrix},$$

$$H_3 = \begin{bmatrix} \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y\partial B} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y\partial s} \\ \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B\partial y} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B\partial s} \\ \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s\partial y} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s\partial B} & \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s^2} \end{bmatrix}. \quad (\text{C.11})$$

Thus,

$$\begin{aligned}
 \text{Det}(H_1) &= \frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2} = \frac{B^2[h + b + \theta\hat{h}] + 2(\pi - \rho s^n)[K + \theta\hat{K}]}{y^3 w_1(1 - E[x])} < 0, \\
 \text{Det}(H_2) &= \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2}\right) - \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial B}\right)^2 > 0, \\
 \text{Det}(H_2) &= \left(\frac{B^2[h + b + \theta\hat{h}] + 2(\pi - \rho s^n)[K + \theta\hat{K}]}{y^3 w_1(1 - E[x])}\right)\left(\frac{h + b + \theta\hat{h}}{y w_1(1 - E[x])}\right) \\
 &\quad - \left(\frac{B[h + b + \theta\hat{h}]}{y^2 w_1(1 - E[x])}\right)^2 > 0, \\
 \text{Det}(H_2) &= \left(\frac{2(\pi - \rho s^n)[K + \theta\hat{K}]}{y^4 w_1^2(1 - E[x])^2}\right)(h + b + \theta\hat{h}) > 0, \\
 \text{Det}(H_3) &= \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s^2}\right) \\
 &\quad + \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial y}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s \partial B}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial s}\right) \\
 &\quad + \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s \partial y}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial B}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial s}\right) \\
 &\quad - \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial s}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B^2}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s \partial y}\right) \\
 &\quad - \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial s}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s \partial B}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y^2}\right) \\
 &\quad - \left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial s^2}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial y \partial B}\right)\left(\frac{\partial^2 E[\text{TPU}(y, B, s)]}{\partial B \partial y}\right) < 0.
 \end{aligned} \tag{C.12}$$

The abovementioned analysis proves that the function $E[\text{TPU}(y, B, s)]$ is strictly concave and shows that the Hessian is negative-definite. Therefore, the optimal solution for the decision variables (y^*, B^*, s^*) exists and maximizes the expected total profit.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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