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Research Article

Stability Analysis of Time-Varying Delay Systems with an Improved Type of LKF

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This paper further investigates the problem of stability for a general linear system with time-varying delays. Firstly an improved type of Lyapunov–Krasovskii functional is introduced with integral and nonintegral terms and time-correlation terms. Referring a few existing papers, some valid inequalities mathematical analysis techniques are used in this paper in order to reduce the conservatism of the system. Finally, two examples are presented to demonstrate the advantages of the proposed tactics in this paper.

1. Introduction

It is widely acknowledged that, in practical systems such as switched systems [1], T-S fuzzy systems [2], and neural networks [3], time delays inevitably occur and affect the system performance [4, 5]. Therefore, in recent decades, researchers have paid so much attention to analyzing the stability and accomplishing the stabilization of time-delayed systems. The Lyapunov–Krasovskii functional (LKF), recognized as one of the most efficient tools for these questions, offers both the methods as well as the challenges for dealing with such problems. On one hand, like in a previous work [6–9], the way to construct LKF with augmented vector is commonly accepted. Meanwhile, adding more multi-integral seems also will be helpful to reduce the conservatism in some cases [10]. On the other hand, many innovative integral inequalities have been proposed to deal with the derivative of LKF, for example, Jensen inequality [11, 12], Wirtinger-based inequality [13], Bessel–Legendre inequality [14], free-matrix-based integral inequality [15, 16], and many new refined types of these integral inequalities such as shown in [17, 18]. Combining the recent studies of LMIs discussed above [13, 19] and improved quadratic integral inequalities such as in [20, 21], a general type of LKF may be employed for stability analysis of time-delayed systems [22, 23], with both the construction of LKF and the way to cope with the derivative of LKF been considered [24, 25].

Motivated by the contents above, an attempt to construct the LKF has been made for the time-varying delay systems in this paper. Consulting the method of LMIs in previous articles with an augmented quadratic term, analysis of a time-varying delay system with less conservatism is revealed.

This paper consists of the following parts: Section 2 states the description and definition of the system, as well as the lemmas used later. Then, the achievements of this paper are proved in Theorem 1. Two numerical examples are provided to confirm the results at the end of this paper.

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $P > 0$ means that P is a real symmetric and positive definite matrix. The superscripts "-1" and "T" indicate the inverse and the transpose of a matrix, respectively. $\mathrm{Sym}X = X + X^T$, $\mathrm{col}\ldots = [x_1^T, x_2^T, \ldots, x_n^T]^T$, $\mathrm{diag}\{\ldots\}$ represents the block diagonal matrix.

2. Problem Statement

Firstly, we take the following system model with a time-varying delay into account:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)) \\ x(t) = \phi(t) \quad t \in [-d, 0] \end{cases}, \tag{1}$$

where $x(t) \in R^n$ represents the state vector. $\phi(t)$ is the initial condition with $t \in [-d, 0]$, $d > 0$.A, and A_d are system matrices with compatible dimensions. The time delay function $d(t)$ is continuous and defined as

$$0 \le d(t) \le d, \quad \mu_1 \le \dot{d}(t) \le \mu_2. \tag{2}$$

The scalar μ_1 and μ_2, respectively, represent the lower and upper bounds of the time-varying delay.

In order to deduce the results of this paper, some lemmas are introduced before the main results.

Lemma 1 (See [26]). *Let $N \in \mathbb{N}, \zeta \in \mathbb{R}^m$ and x be a continuous and differentiable function: $[\alpha, \beta] \longrightarrow R^n$. Matrix $Z(\in \mathbb{R}^{n\times n}) > 0, M(\in \mathbb{R}^{(N+1)n\times m})$. The following inequality holds:*

$$-\int_\alpha^\beta \dot{x}^T(u) Z \dot{x}(u) \mathrm{d}u \le 2\zeta_N^T \Gamma_N^T M \xi + (\beta - \alpha)\xi^T M^T \widetilde{Z} M \xi, \tag{3}$$

with

$$\begin{aligned}
\Gamma_N &= \left[\pi_N^T(0), \ \pi_N^T(1) \ \ldots \ \pi_N^T(N) \right], \\
\widetilde{Z} &= \left\{ \frac{1}{Z}, \frac{1}{3Z} \ \cdots \ \frac{1}{(2N+1)Z} \right\}, \\
\zeta_N &= \begin{cases} \left[x^T(\beta) \ \ x^T(\alpha) \right] & \text{if } N = 0 \\ x^T(\beta) \ \ x^T(\alpha) \ \ \frac{1}{\beta-\alpha}\mathbf{\Theta}_0^T \ \cdots \ \frac{1}{\beta-\alpha}\mathbf{\Theta}_{N-1}^T & \text{if } N > 0 \end{cases}, \\
\pi_N(k) &= \begin{cases} \left[I \ \ -I \right] & \text{if } N = 0 \\ \left[I \ \ (-1)^{k+1} I \ \ \theta_{Nk}^0 I \ \ldots \ \theta_{Nk}^{N-1} I \right] & \text{if } N > 0 \end{cases}, \\
\theta_{Nk}^j &= \begin{cases} (2j+1)\left((-1)^{k+j} - 1 \right) & \text{if } j \le k \\ 0 & \text{if } j > k \end{cases}, \\
F_k(u) &= (-1)^k \sum_{i=0}^{k} \left[(-1)^i \binom{k}{i} \binom{k+i}{i} \right] \left(\frac{u-\alpha}{\beta-\alpha} \right)^i, \\
\mathbf{\Theta}_k &= \int_\alpha^\beta F_k(u) x(u) \mathrm{d}u.
\end{aligned} \tag{4}$$

Lemma 2 (See [27]). *Let $f(d(t)) = a_2 d^2 + a_1 d + a_0$, where a_2, a_1 and $a_0 \in \mathbb{R}$,*

$$f(d) \le 0, \forall d \in [0, d], \tag{5}$$

if the following conditions hold:

(i) $f(0) \le 0$

(ii) $f(d) \le 0$

(iii) $f(0) - d_a^2 \le 0$

Lemma 3 (See [28]). *For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$, and vector function*

ω: $[0,\ \gamma] \longrightarrow \mathbb{R}$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^{\gamma}\omega(s)\mathrm{d}s\right)^T M\left(\int_0^{\gamma}\omega(s)\mathrm{d}s\right)\leq\gamma\left(\int_0^{\gamma}\omega^T(s)M\omega(s)\mathrm{d}s\right). \tag{6}$$

Lemma 4 (See [29]). *Given positive integers m,n and a scalar $\beta\in(0,1)$, given matrices $R_1>0, R_2>0$ and $H_1, H_2\in\mathbb{R}^{n\times m}$. For all vecotrs $\xi\in\mathbb{R}^m$, the function $\Theta(\beta,R_1,R_2)$ is given by*

$$\Theta(\beta,R_1,R_2)=\frac{1}{\beta}\xi^T H_1^T R_1 H_1\xi_+\frac{1}{1-\beta}\xi^T H_2^T R_2 H_2\xi. \tag{7}$$

Then, if there exists a matrix $X\in\mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R_1 & X \\ X^T & R_2\end{bmatrix}>0$, the following equality holds:

$$\underbrace{\min\Theta(\beta,R_1,R_2)}_{\beta\in(0,1)}\geq\begin{bmatrix} H_1\xi \\ H_2\xi\end{bmatrix}^T\begin{bmatrix} R_1 & X \\ X^T & R_2\end{bmatrix}\begin{bmatrix} H_1\xi \\ H_2\xi\end{bmatrix}. \tag{8}$$

3. Main Result

In this section, the stability of the system is analyzed. For brevity, the following nomenclature is used to simplify vector and matrix representations:

$$\begin{aligned}
\eta_1(t)&=\mathrm{col}\Bigg\{x(t),x(t-d(t)),x(t-d),\int_{t-d(t)}^{t}x(s)\mathrm{d}s,\int_{t-d}^{t-d(t)}x(s)\mathrm{d}s\\
&\qquad\frac{1}{d(t)}\int_{-d(t)}^{0}\int_{t+\theta}^{t}x(s)\mathrm{d}s\mathrm{d}\theta,\frac{1}{d-d(t)}\int_{-d}^{-d(t)}\int_{t+\theta}^{t-\tau(t)}x(s)\mathrm{d}s\mathrm{d}\theta\Bigg\},\\
\eta_2(s)&=\mathrm{col}\left\{\dot{x}(s),x(s),x(t-d(t)),\int_s^t\dot{x}(\theta)d(\theta),\int_{t-d}^{s}\dot{x}(\theta)d(\theta),\int_{t-d(t)}^{s}\dot{x}(\theta)d(\theta)\right\},\\
\eta_3(t)&=\mathrm{col}\left\{x(t),x(\ t-d(t),\int_{t-d(t)}^{t}x(s)\mathrm{d}s,\frac{1}{d(t)}\int_{-d(t)}^{0}\int_{t+\theta}^{t}x(s)\mathrm{d}s\mathrm{d}\theta\right\},\\
\eta_4(t)&=\mathrm{col}\left\{x(t-d(t)\ ,x(t-d),\int_{t-d}^{t-d(t)}x(s)\mathrm{d}s,\frac{1}{d-d(t)}\int_{-d}^{-d(t)}\int_{t+\theta}^{t-\tau(t)}x(s)\mathrm{d}s\mathrm{d}\theta\right\},\\
\xi(t)&=\mathrm{col}\{\xi_1(t),\xi_2(t),\xi_3(t)\},\xi_1(t)=\mathrm{col}\{x(t),x(t-d(t)),x(t-d),\dot{x}(t-d(t)),\dot{x}(t-d)\},\\
\xi_2(t)&=\mathrm{col}\left\{\frac{1}{d(t)}\int_{t-d(t)}^{t}x(s)\mathrm{d}s,\frac{1}{d-d(t)}\int_{t-d}^{t-d(t)}x(s)\mathrm{d}s\right\},\\
\xi_3(t)&=\mathrm{col}\left\{\frac{1}{d^2(t)}\int_{-d(t)}^{0}\int_{t+\theta}^{t}x(s)\mathrm{d}s\mathrm{d}\theta,\frac{1}{(d-d(t))^2}\int_{-d}^{-d(t)}\int_{t+\theta}^{t-d(t)}x(s)\mathrm{d}s\mathrm{d}\theta\right\},\\
e_i&=\left[0_{n\times(i-1)}\ \ I_n\ \ 0_{n\times(9-i)}\right], i=1,2,\ldots,9.
\end{aligned} \tag{9}$$

Theorem 1. *In this section, the stability of system (1) is analyzed under the condition of the new LKF. With known scalars $\mu_1,\mu_2<1$ and $d>0$, system (1) is asymptotically stable if there exist appropriately dimensional symmetrical matrices $P>0, Q_1, Q_2>0, Z_1, \mathbb{Z}=Z_2+Z_3>0, U_1, U_2>0,$ and $V_1, V_2>0$ satisfying the following LMIS:*

$$\begin{bmatrix} \Psi & \sqrt{d}N_1 \\ * & -\widetilde{Z_3} \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \Psi & \sqrt{d}N_2 \\ * & -\widetilde{Z_3} \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \Psi & \sqrt{d}N_2 \\ * & -\widetilde{Z_3} \end{bmatrix} - d^2\Omega d^2(t) < 0, \tag{12}$$

$$\begin{aligned}
\Phi_1 &= \mathrm{Sym}\left\{\Pi_1^T P \Pi_2\right\},\\
\Phi_2 &= \Pi_3^T Q_1 \Pi_3 + (1-\dot{d}(t))\Pi_4^T(Q_2 - Q_1)\Pi_4 - \Pi_5^T Q_2 \Pi_5^T + \mathrm{Sym}\left\{\Pi_6^T Q_1 \Pi_8 + \Pi_7^T Q_2 \Pi_8\right\},\\
\Phi_3 &= e_1^T(U_1 + \mathrm{d}Z_1)e_1 + e_2^T(U_2 - U_1)e_2 - e_3^T U_2 e_3 + \mathrm{d}e_0^T Z_2 e_0 - \frac{1}{d}\Pi_{13}^T \Upsilon \Pi_{13},\\
\Phi_4 &= \mathrm{d}e_0 Z_3 e_0 + \mathrm{Sym}\{N_1\Pi_{14} + N_2\Pi_{15}\},\\
\Phi_5 &= \dot{d}(t)\Pi_9^T V_1 \Pi_9 + d(t)\mathrm{Sym}\left\{\Pi_9^T V_1 \Pi_{10}\right\} - \dot{d}(t)\Pi_{11}^T V_2 \Pi_{11} + (d - \mathrm{d}(t))\mathrm{Sym}\left\{\Pi_{11}^T V_1 \Pi_{12}\right\},\\
\Phi_6 &= (d - d(t))\left(N_1 \widetilde{Z_3}^{-1} N_1^T\right) + d(t) N_2 \widetilde{Z_3}^{-1} N_2^T,\\
\Psi &= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 - \Phi_6,\\
\Pi_1 &= \mathrm{col}\{e_1, e_2, e_3, d(t)e_6, (d-d(t))e_7, d(t)e_8, (d-d(t))_9\},\\
\Pi_2 &= \mathrm{col}\left\{e_d, (1-\dot{d}(t))e_4, e_5, e_1 - (1-\dot{d}(t))e_2, (1-\dot{d}(t))e_2 - e_3, e_1 - (1-\dot{d}(t))e_6 - \dot{d}(t)e_8, (1-\dot{d}(t))e_2 - e_7 + \dot{d}(t)e_9\right\},\\
\Pi_3 &= \mathrm{col}\{e_d, e_1, e_2, 0, e_1 - e_3, e_1 - e_2\},\\
\Pi_4 &= \mathrm{col}\{e_4, e_2, e_2, e_1 - e_2, e_2 - e_3, 0\},\\
\Pi_5 &= \mathrm{col}\{e_5, e_3, e_2, e_1 - e_3, 0, e_3 - e_2\},\\
\Pi_6 &= \mathrm{col}\{e_1 - e_2, d(t)e_6, d(t)e_2, d(t)(e_1 - e_6), d(t)(e_6 - e_3), d(t)(e_6 - e_2)\},\\
\Pi_7 &= \mathrm{col}\{e_2 - e_3, (d-d(t))e_7, (d-d(t))e_2, (d-d(t))e_1 - e_7), (d-d(t))(e_7 - e_3), (d-d(t))(e_7 - e_2)\},\\
\Pi_8 &= \mathrm{col}\left\{0, 0, (1-\dot{d}(t))e_4, e_d, -e_5, -(1-\dot{d}(t))e_4\right\},\\
\Pi_9 &= \mathrm{col}\{e_1, e_2, d(t)e_6, d(t)e_8\} \quad \Pi_{9a} = \mathrm{col}\{e_1, e_2, 0, 0\} \quad \Pi_{9b} = \mathrm{col}\{0, 0, e_6, e_8\},\\
\Pi_{10} &= \mathrm{col}\left\{e_d, (1-\dot{d}(t))e_4, e_1 - (1-\dot{d}(t))e_2, -\dot{d}(t)e_8 + e_1 - (1-\dot{d}(t))e_6\right\},\\
\Pi_{11} &= \mathrm{col}\{e_2, e_3, (d-d(t))e_7, (d-d(t))e_9\} \quad \Pi_{11a} = \mathrm{col}\{e_2, e_3, 0, 0\} \quad \Pi_{11b} = \mathrm{col}\{0, 0, e_7, e_9\},\\
\Pi_{12} &= \mathrm{col}\left\{(1-\dot{d}(t))e_4, e_5, (1-\dot{d}(t))e_2 - e_3, \dot{d}(t)e_9 + (1-\dot{d}(t))e_2 - e_7\right\},\\
\Pi_{13} &= \mathrm{col}\{d(t)e_6, e_1 - e_2, (d-d(t))e_7, e_2 - e_3\},\\
\Pi_{14} &= \mathrm{col}\{e_1 - e_2, e_1 + e_2 - 2e_6, e_1 - e_2 + 6e_6 - 12e_8\},\\
\Pi_{15} &= \mathrm{col}\{e_2 - e_3, e_2 + e_3 - 2e_7, e_2 - e_3 + 6e_7 - 12e_9\},\\
e_d &= Ae_1 + A_d e_2,\\
\overline{U}_1 &= \begin{bmatrix} Z_1 & U_1 \\ * & Z_2 \end{bmatrix},\\
\overline{U}_2 &= \begin{bmatrix} Z_1 & U_2 \\ * & Z_2 \end{bmatrix},\\
H &= \begin{bmatrix} H_1 & H_2 \\ * & H_3 \end{bmatrix}.
\end{aligned} \tag{13}$$

Proof. We design the functional below for system

$$W(t) = \sum_{i=1}^{3} V_i(t) + w(t), \tag{14}$$

with each part of $W(t)$ as

$$\begin{aligned} V_1(t) &= \eta_1^T(t)P\eta_1(t), \\ V_2(t) &= \int_{t-d(t)}^{t} \eta_2^T(s)Q_1\eta_2(s)\mathrm{d}s + \int_{t-d}^{t-d(t)} \eta_2^T(s)Q_2\eta_2(s)\mathrm{d}s, \\ V_3(t) &= \int_{-d}^{0}\int_{t+\theta}^{t} x^T(s)Z_1x(s)\mathrm{d}s\mathrm{d}\theta + \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\mathbb{Z}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta, \\ w(t) &= d(t)\eta_3^T(t)V_1\eta_3(t) + (d-d(t))\eta_4^T(t)V_2\eta_4(t). \end{aligned} \tag{15}$$

The positive of $W(t)$ can be proved with matrices $P, Q_1, Q_2, Z_1, \mathbb{Z}, V_1, V_2 > 0$, respectively. The derivation of each part in $W(t)$ is

$$\dot{V}_1(t) = 2\eta_1^T(t)P\dot{\eta}_1(t) = \xi^T(t)\mathrm{Sym}\{\Pi_1^T P\Pi_2\}\xi(t), \tag{16}$$

$$\begin{aligned} \dot{V}_2(t) &= \eta_2^T(t)Q_1\eta_2(t) + (1-\dot{d}(t))\eta_2^T(t-d(t))(Q_2-Q_1)\eta_2(t-d(t)) - \eta_2^T(t-d)Q_2\eta_2(t-d) \\ &\quad + 2\int_{t-d(t)}^{t} \eta_2^T(s)Q_1\dot{\eta}_2(s)\mathrm{d}s + 2\int_{t-d}^{t-d(t)} \eta_2^T(s)Q_2\dot{\eta}_2(s)\mathrm{d}s, \\ &= \xi^T(t)\left(\Pi_3^TQ_1\Pi_3 + (1-\dot{d}(t))\Pi_4^T(Q_2-Q_1)\Pi_4 - \Pi_5Q_2\Pi_5 + \mathrm{Sym}\{\Pi_6^TQ_1\Pi_8 + \Pi_7^TQ_2\Pi_8\}\right)\xi(t), \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{w}(t) &= \dot{d}(t)\eta_3^T(t)V_1\eta_3(t) - \dot{d}(t)\eta_4^T(t)V_2\eta_4(t) + 2\, d(t)\eta_3^T\big)V_1\dot{\eta}_3(t) + 2(d-d(t))\eta_4^TV_2\dot{\eta}_4(t), \\ &= \xi^T(t)\left(\dot{d}(t)\Pi_9^TV_1\Pi_9 + d(t)\mathrm{Sym}\{\Pi_9^TV_1\Pi_{10}\} - \dot{d}(t)\Pi_{11}^TV_2\Pi_{11} + (d-d(t))\mathrm{Sym}\{\Pi_{11}^TV_1\Pi_{12}\}\right), \end{aligned} \tag{18}$$

and $\dot{V}_3(t)$ could be divided into two parts:

$$\dot{V}_{3a}(t) = \xi^T(t)\left(de_1^TZ_1e_1 + de_0^TZ_2e_0\right)\xi(t) - \int_{t-d}^{t} x^T(s)Z_1x(s)\mathrm{d}s - \int_{t-d}^{t} \dot{x}^T(s)Z_2\dot{x}(s)\mathrm{d}s, \tag{19}$$

$$\dot{V}_{3b}(t) = \xi^T(t)\left(de_0^TZ_3e_0\right)\xi(t) - \int_{t-d}^{t} \dot{x}^T(s)Z_3\dot{x}(s)\mathrm{d}s. \tag{20}$$

Based on the formula of integration by parts, the following zero equalities hold for symmetric matrices U_1 and U_2 with appropriate dimension.

$$\begin{aligned} 0 &= x^T(t)U_1x(t) - x^T(t-d(t))U_1x(t-d(t)) - 2\int_{t-d(t)}^{t} x^T(s)U_1\dot{x}(s)\mathrm{d}s, \\ &= \xi^T(t)\left(e_1^TU_1e_1 - e_2^TU_1e_2\right)\xi(t) - 2\int_{t-d(t)}^{t} x^T(s)U_1\dot{x}(s)\mathrm{d}s, \end{aligned} \tag{21}$$

$$
\begin{aligned}
0 &= x^T(t-d(t))U_2x(t-d(t)) - x^T(t-d)U_2x(t-d) - 2\int_{t-d}^{t-d(t)} x^T(s)U_2\dot{x}(s)\mathrm{d}s,\\
&=\xi^T(t)e_2^TU_2e_2 - e_3^TU_2e_3\big)\xi(t) - 2\int_{t-d(t)}^{t} x^T(s)U_1\dot{x}(s)\mathrm{d}s.
\end{aligned}
\tag{22}
$$

Adding equations (21) and (22) to (20), one has

$$
\begin{aligned}
\dot{V}_{3a}(t) &= \xi^T(t)\left[e_1^T(U_1+\mathrm{d}Z_1)e_1 + e_2^T(U_2-U_1)e_2 - e_3^TU_2e_3 + \mathrm{d}e_0^TZ_2e_0\right]\xi(t)\\
&\quad - \int_{t-d(t)}^{t}\left(\begin{bmatrix} x(s)\\ \dot{x}(s)\end{bmatrix}^T \overline{U_1}\begin{bmatrix} x(s)\\ \dot{x}(s)\end{bmatrix}\right) - \int_{t-d}^{t-d(t)}\left(\begin{bmatrix} x(s)\\ \dot{x}(s)\end{bmatrix}^T \overline{U_2}\begin{bmatrix} x(s)\\ \dot{x}(s)\end{bmatrix}\right).
\end{aligned}
\tag{23}
$$

Then, through applying Lemmas 3 and 4, equation (23) would yield

$$
\begin{aligned}
\dot{V}_{3a}(t) &\le \xi^T(t)\left[e_1^T(U_1+\mathrm{d}Z_1)e_1 + e_2^T(U_2-U_1)e_2 - e_3^TU_2e_3 + \mathrm{d}e_0^TZ_2e_0\right]\xi(t)\\
&\quad -\frac{1}{d(t)}\xi^T(t)\begin{bmatrix} d(t)e_6\\ e_1-e_2\end{bmatrix}^T\overline{U_1}\begin{bmatrix} d(t)e_6\\ e_1-e_2\end{bmatrix}\xi(t) - \frac{1}{d-d(t)}\xi\left(t^T\right)\begin{bmatrix} (d-d(t))e_7\\ e_2-e_3\end{bmatrix}^T\overline{U_2}\begin{bmatrix} (d-d(t))e_7\\ e_2-e_3\end{bmatrix}\xi(t)\\
&\le \xi(t)^T\Bigg\{e_1^T(U_1+\mathrm{d}Z_1)e_1 + e_2^T(U_2-U_1)e_2 - e_3^TU_2e_3 + \mathrm{d}e_0^TZ_2e_0\\
&\quad -\frac{1}{d}\begin{bmatrix} d(t)e_6\\ e_1-e_2\\ (d-d(t))e_7\\ e_2-e_3\end{bmatrix}^T\begin{bmatrix}\overline{U_1} & H\\ * & \overline{U_2}\end{bmatrix}\begin{bmatrix} d(t)e_6\\ e_1-e_2\\ (d-d(t))e_7\\ e_2-e_3\end{bmatrix}\Bigg\}\xi(t)\\
&\le \xi(t)^T\left\{e_1^T(U_1+\mathrm{d}Z_1)e_1 + e_2^T(U_2-U_1)e_2 - e_3^TU_2e_3 + \mathrm{d}e_0^TZ_2e_0 - \frac{1}{d}\Pi_{13}^T\Upsilon\Pi_{13}\right\}.
\end{aligned}
\tag{24}
$$

According to Lemma 1, with $N=2$, the integration in $\dot{V}_{3b}(t)$ is expressed as

$$
\begin{aligned}
&-\int_{t-d}^{t}\dot{x}^T(s)Z_3\dot{x}(s)\mathrm{d}s = -\int_{t-d}^{t-d(t)}\dot{x}^T(s)Z_3\dot{x}(s)\mathrm{d}s - \int_{t-d(t)}^{t}\dot{x}^T(s)Z_3\dot{x}(s)\mathrm{d}s\\
&\le -(d-d(t))\xi(t)^T\left(N_1Z_3^{-1}N_1^T + \frac{1}{3}N_1Z_3^{-1}N_1^T + \frac{1}{5}N_1Z_3^{-1}N_1^T\right)\xi(t) + 2\xi^T(t)N_1\Pi_{14}\\
&\quad - d(t)\xi^T(t)\left(N_2Z_3^{-1}N_2^T + \frac{1}{3}N_2Z_3^{-1}N_2^T + \frac{1}{5}N_2Z_3^{-1}N_2^T\right)\xi(t) + 2\xi^T(t)N_2\Pi_{15},\\
&= \xi^T(t)\Big\{\mathrm{Sym}\{N_1\Pi_{14}+N_2\Pi_{15}\} - (d-d(t))\left(N_1Z_3^{-1}N_1^T + \frac{1}{3}N_1Z_3^{-1}N_1^T + \frac{1}{5}N_1Z_3^{-1}N_1^T\right)\\
&\quad + d(t)\left(N_2Z_3^{-1}N_2^T + \frac{1}{3}N_2Z_3^{-1}N_2^T + \frac{1}{5}N_2Z_3^{-1}N_2^T\right\}\xi(t),\\
&= \xi^T(t)\{\mathrm{Sym}\{N_1\Pi_{14}+N_2\Pi_{15}\} - (d-d(t))\left(N_1\widetilde{Z_3}^{-1}N_1^T\right) + d(t)\left(N_2\widetilde{Z_3}^{-1}N_2^T\right\}\xi(t).
\end{aligned}
\tag{25}
$$

For equation (19), the expressions of $\eta_3(t)$ and $\eta_4(t)$ are divided into two parts, respectively, i.e.,

$$\eta_3(t) = \xi^T(t)\left(\mathrm{col}\{e_1, e_2, 0, 0\} + \mathrm{col}\{0, 0, d(t)e_6, d(t)e_8\}\right)\xi(t) = \xi^T(t)\left(\Pi_{9a} + d(t)\Pi_{9b}\right)\xi(t), \tag{26}$$

$$\eta_4(t) = \xi^T(t)\left(\mathrm{col}\{e_2, e_3, 0, 0\} + \mathrm{col}\{0, 0, (d - d(t))e_7, (d - d(t))e_9\}\right)\xi(t) = \xi^T(t)\left(\Pi_{11a} + (d - d(t))\Pi_{11b}\right)\xi(t). \tag{27}$$

Therefore, all the quadratic terms of $\dot{W}(t)$ exist in equations (24), (26), and (27), and the coefficient $\Omega_{d^2(t)}$ of the quadratic terms is

$$\begin{aligned}\Omega_{d^2(t)} = {} & \dot{d}(t)\Pi_{9a}^T U_1 \Pi_{9a} + \mathrm{Sym}\{\Pi_{9a} U_1 \Pi_{10}\} - \dot{d}(t)\Pi_{11a}^T V_2 \Pi_{11a} \\ & + \mathrm{Sym}\{\Pi_{11a} V_2 \Pi_{12}\} - \frac{1}{d}\Big\{e_6^T(Z_1 + U_1 + H_1 + H_2)e_6 \\ & - \mathrm{Sym}\left\{e_6^T(Z_1 + U_1 + H_1 + H_2)e_7\right\} + e_7^T(Z_1 + U_2 + H_1 + H_2)e_7\Big\}.\end{aligned} \tag{28}$$

Combining equations (16)–(18) and (24)–(28) and referring Lemma 4 simultaneously, $\dot{W}(t)$ would be negative if LMIs (10)–(12) hold through utilizing Schur complement. Based on Lyapunov stable analysis of dynamic system, the system discussed in this paper is stable. □

Remark 1. To consider both the conservatism and computational complexity of the system, a nonintegral term of $w(t)$ is introduced into this with flexibal coefficients $d(t)$ and $d - d(t)$. In addition, the vector $\eta_i(t)$ is not fixed as well, which makes the LKF more general for some conditions of practical system. If the upper bound is much more high than the lower bound, the part $d(t)\xi_3^T(t)V_1\xi_3(t)$ could be ignored and so the same is in reverse.

Remark 2. Depending on the definition of Lemma 1, it is not difficult to get tighter bounds of the system through adjusting the number of order. With the increase of the order N, the double integral state vectors in $\xi_1(t)$ are required to translate into triple or multiple integrals in order to reduce the conservatism. Thus, these state vectors may satisfy the demand of a complex system.

4. Numerical Example

In this section, two numerical examples are provided while comparing other results existed in previous research to demonstrate the proposed methods.

We define the system matrices A and A_d as

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{29}$$

The example is aimed at searching the maximum value for the upper bounds of the delay (MVUBD) with the derivative of known delay. We set $\mu_1 = -\mu_2 = \mu$ and assign a value to μ, and the results of the MVUBD computed by Theorem 1 are shown in Table 1. Some other results from former research are also listed in Table 1 for the sake of comparison. Figures 1–3 show the state trajectories with different μ and MVUBD. From the simulation results, it is concluded that the achieved improvements in this paper are less conservative than other results.

In this example, the state matrices A and A_d have been set as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{30}$$

In this example, the value of μ_1 is still set as μ_2, and the results of the MVUBD which is computed by Theorem 1 are shown in Table 2. Figures 4–6 also show the state trajectory with different μ and MVUBD. The results of MVUBD with different μ in Table 2 are much better, which means the method proposed in not only valid but also advanced. It is

Table 1: The MVUBD under the condition $\mu_1 = -\mu_2$ with a certain μ.

μ	0.1	0.2	0.8
[16]	4.788	4.060	2.615
[4]	4.831	4.141	2.713
[30]	4.908	4.199	2.735
[31]	4.910	4.216	2.789
[26]	4.921	4.218	2.792
[32]	4.930	4.235	2.807
Theorem 1	4.940	4.264	2.868

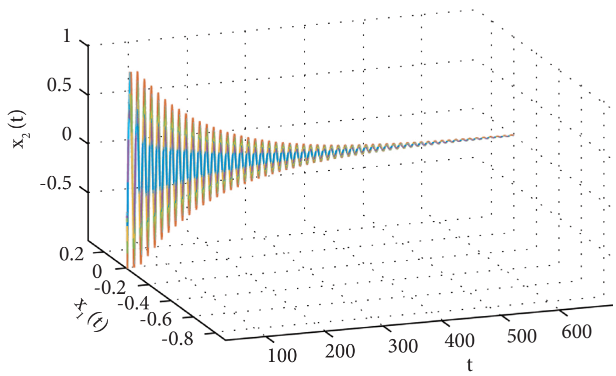

Figure 1: The state trajectory with $\mu = 0.1$ and MVUBD 4.940.

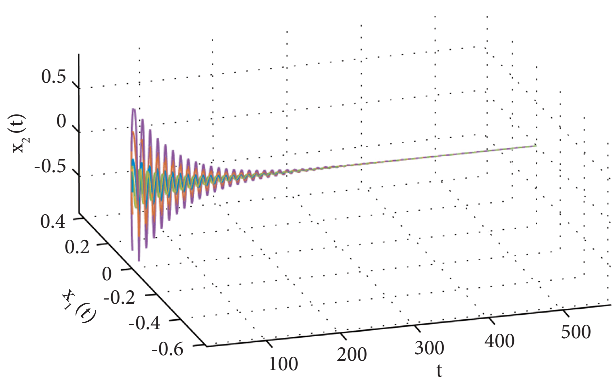

Figure 2: The state trajectory with $\mu = 0.2$ and MVUBD 4.264.

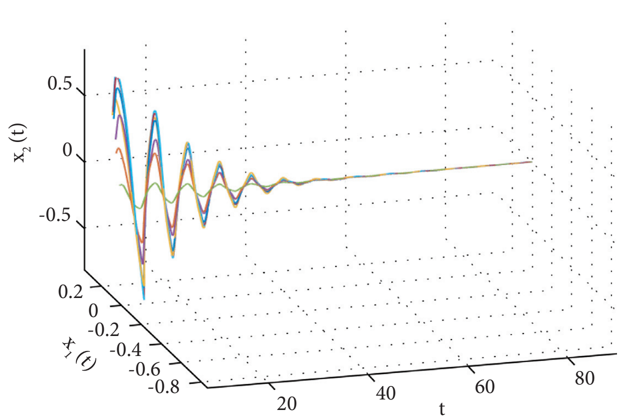

Figure 3: The state trajectory with $\mu = 0.8$ and MVUBD 2.868.

Table 2: The MVUBD under the condition $\mu_1 = -\mu_2$ and a certain μ.

μ	0.1	0.2	0.5
[13]	6.590	3.672	1.411
[16]	7.148	4.466	2.352
[4]	7.167	4.517	2.415
[32]	7.176	4.543	2.496
[31]	7.230	4.556	2.509
[26]	7.308	4.670	2.664
Theorem 1	7.348	4.759	2.897

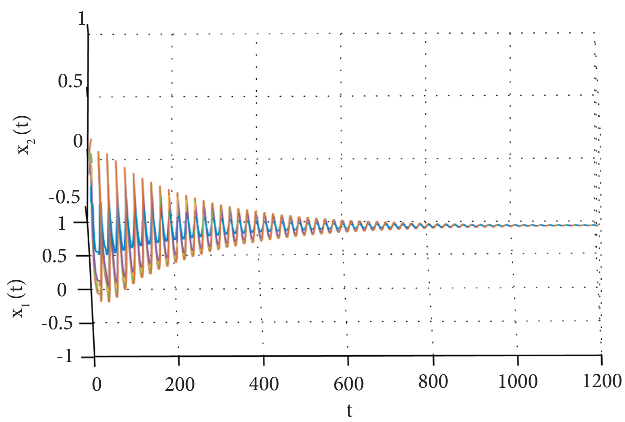

Figure 4: The state trajectory with $\mu = 0.1$ and MVUBD 7.348.

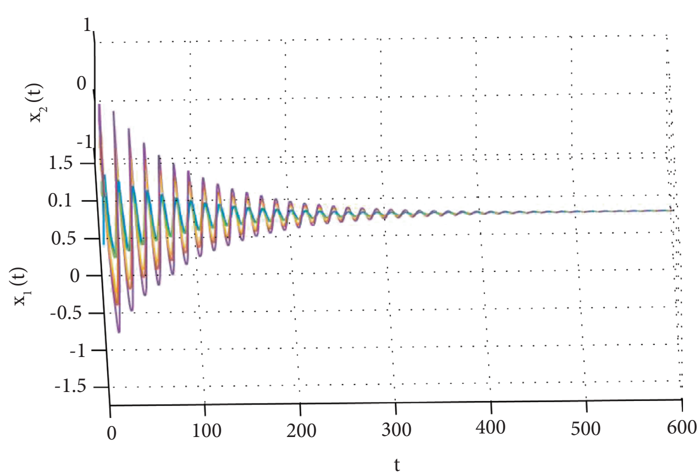

Figure 5: The state trajectory with $\mu = 0.2$ and MVUBD 4.759.

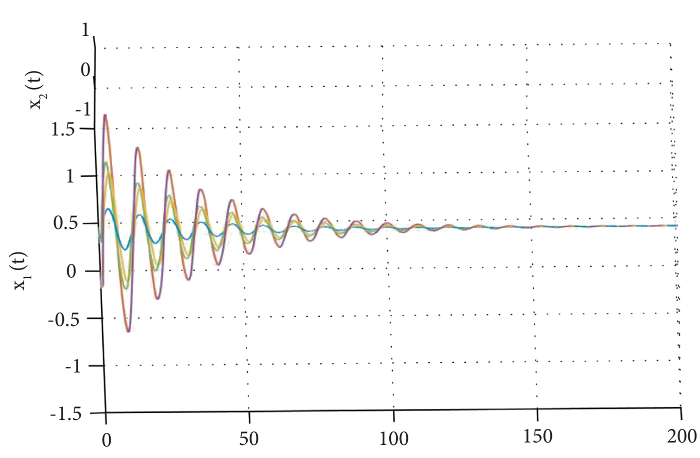

Figure 6: The state trajectory with $\mu = 0.5$ and MVUBD 2.897.

worth noticing that the MVUBD reaches almost 18.97% than the data published lately when μ is 0.5.

5. Conclusions

A further investigation of the stability for system with time-varying delays has been figured out in this paper. With different methods of solving inequalities, results of less-conservative stability criteria have been achieved. At the end of the paper, two examples, compared with results from other papers, are represented in order to show the availability of the such methods. Based on the construction of LKF in this paper, some other problem such as sampled data control may also be considered in the same way. In the future work, the improved type of LKF proposed in this paper will be extended to a chaotic Lurie system, T-S fuzzy systems, Markov jump systems, quaternion-valued or memristor-based neural networks, and complex dynamical networks.

Data Availability

The raw/processed data required to reproduce these findings cannot be shared at this time as the data also form part of an ongoing study.

Conflicts of Interest

The authors declare no conflicts of interest.

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