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Research Article

Uniformly Most Reliable Three-Terminal Graph of Dense Graphs

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A graph G with k specified target vertices in vertex set is a k-terminal graph. The k-terminal reliability is the connection probability of the fixed k target vertices in a k-terminal graph when every edge of this graph survives independently with probability p. For the class of two-terminal graphs with a large number of edges, Betrand, Goff, Graves, and Sun constructed a locally most reliable two-terminal graph for p close to 1 and illustrated by a counterexample that this locally most reliable graph is not the uniformly most reliable two-terminal graph. At the same time, they also determined that there is a uniformly most reliable two-terminal graph in the class obtained by deleting an edge from the complete graph with two target vertices. This article focuses on the uniformly most reliable three-terminal graph of dense graphs with n vertices and m edges. First, we give the locally most reliable three-terminal graphs of n and m in certain ranges for p close to 0 and 1. Then, it is proved that there is no uniformly most reliable three-terminal graph with specific n and m, where $n \geq 7$ and $\binom{n}{2} - \lfloor (n-3)/2 \rfloor \leq m \leq \binom{n}{2} - 2$. Finally, some uniformly most reliable graphs are given for n vertices and m edges, where $4 \leq n \leq 6$ and $m = \binom{n}{2} - 2$ or $n \geq 5$ and $m = \binom{n}{2} - 1$.

1. Introduction

Network reliability is a hot topic which has been generally investigated using graph theoretic models. Network with n vertices and m edges can be modeled as a graph G with the same number of vertices, edges, and interconnections as the network. For all-terminal reliability (connection probability of all vertices of a graph), lots of authors investigate the existence of a uniformly most reliable (all-terminal) graph for various values of n and m [1–8]. However, the research on k-terminal reliability (connection probability of k target vertices in a graph, where $2 \leq k < n$) is mainly about the algorithm of computing the k-terminal reliability polynomial [9–13], but only a few results on the construction of the uniformly most reliable k-terminal graph.

In [8], Kelmans has shown that for $\binom{n}{2} - \lfloor n/2 \rfloor \leq m \leq \binom{n}{2} - 2$, the uniformly most reliable graph is a complete graph with a matching removed (the matching of a graph is a set of edges with no common vertices between each other). In fact, the design of the real network often only needs to ensure the connectivity of $k\,(2 \leq k < n)$ critical vertices (target vertices). Therefore, the construction of the most reliable k-terminal graph has high application value. There are a few studies on the construction of the most reliable k-terminal structure. In [14], Betrand et al. proved that there is no uniformly most reliable two-terminal graph when $\binom{n}{2} - \lfloor (n-2)/2 \rfloor \leq m \leq \binom{n}{2} - 2$. At the same time, they also proved that when $m = \binom{n}{2} - 1$, the uniformly most reliable two-terminal graph is a complete graph with removing an edge between nontarget vertices. It is natural to consider the following problems.

Problems: for the three-terminal graphs with a large number of edges, is there a uniformly most reliable graph? If it exists, what is its construction? If it does not exist, can we construct the locally most reliable three-terminal graph?

With these questions, we further study the existence of uniformly most reliable three-terminal graphs for large m. It is difficult to find the exact cases that three target vertices are connected, which is NP-complete [15]. In this paper, we just consider the uniformly most reliable three-terminal dense graphs. In Section 2, some related basic definitions and notations are given. In Section 3, the locally most reliable three-terminal graphs for given m are determined. We show that there is no uniformly most reliable three-terminal graph for n vertices and m edges where $n \geq 7$ and $\binom{n}{2} - \lfloor (n-3)/2 \rfloor \leq m \leq \binom{n}{2} - 2$ and give the uniformly most reliable graphs for $4 \leq n \leq 6$ and $m = \binom{n}{2} - 2$. In Section 4, a uniformly most reliable three-terminal graph with $n \geq 5$ vertices and $m = \binom{n}{2} - 1$ edges is determined. The results are summarized in Section 5.

2. Basic Concepts and Notations

For notations and terminologies not defined here, we refer to [16]. Let $\beta_G(H)$ denote the number of subgraphs which is isomorphic to H in G. The complement of G, denoted by $\overline{G}$, is the graph obtained by deleting the edges of G and adding edges between all nonadjacent vertices in G. If $u, v \in V(G)$, then $G \cup \{uv\}$ denotes the addition of the edge uv to G, and $G - \{uv\}$ denotes the deletion of the edge uv from G. The union of simple graphs G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let P_n be the path with n vertices, K_n be the complete graph with n vertices, and S_n be a star with n vertices and $n-1$ edges.

A graph G with three specified target vertices $r, s,$ and t in $V(G)$ is a three-terminal graph. Using $\mathscr{G}_{n,m}$ denotes the set of all simple three-terminal graphs with n vertices and m edges. The connectivity probability of the three specified target vertices r, s, t in graph $G \in \mathscr{G}_{n,m}$ when each edge of G survives independently with a fixed probability p is called the three-terminal reliability of G (or the three-terminal reliability polynomial of G), denoted by $R_3(G; p)$. A $v_1, v_2, \ldots, v_j$-subgraph is a subgraph of G in which vertices $v_1, v_2, \ldots, v_j$ are connected in the subgraph. In particular, if the $v_1, v_2, \ldots, v_j$-subgraph with i edges does not contain any $v_1, v_2, \ldots, v_j$-subgraph with less than i edges, then it is minimal; otherwise, it is nonminimal. Clearly, rst-subgraph is a subgraph of G in which three target vertices $r,$ $s,$ and t are connected in the subgraph. Similar to the definition of two-terminal reliability [14], the three-terminal reliability polynomial of the graph $G \in \mathscr{G}_{n,m}$ can be written as

$$R_3(G; p) = \sum_{i=2}^{m} N_i(G) p^i (1-p)^{m-i}, \tag{1}$$

where $N_i(G)$ (or simply N_i) is the number of rst-subgraphs of graph G with i edges.

Similar as the definitions of the uniformly most reliable two-terminal graph [14] and the locally most reliable all-terminal graph [3], we defined the uniformly most reliable graph and the locally most reliable graph for three-terminal graphs.

Definition 1. A graph G is the uniformly most reliable graph in $\mathscr{G}_{n,m}$, if $R_3(G; p) \geq R_3(H; p)$ for all $H \in \mathscr{G}_{n,m}$ and all $0 \leq p \leq 1$. In particular, for $p_0 = 0$ (or 1), if there is an $\varepsilon > 0$ such that $R_3(G; p) \geq R_3(H; p)$ for all $H \in \mathscr{G}_{n,m}$ and for all $p \in [0, 1] \cap (p_0 - \varepsilon, p_0 + \varepsilon)$, then G is the locally most reliable graph in $\mathscr{G}_{n,m}$ for p close to 0(or for p close to 1).

Example 1. Figure 1 shows all types of simple three-terminal graph in $\mathscr{G}_{4,4}$ with three target vertices $r,$ $s,$ and t. Each edge of these graphs survives independently with probability p.

In G_1, the rst-subgraphs with 2 edges are $\{rs, rt\}$, $\{rs, st\}$, and $\{rt, st\}$, and the rst-subgraphs with 3 edges are $\{rs, rt, st\}$, $\{rs, rt, sv_4\}$, $\{rs, st, sv_4\}$, and $\{rt, st, sv_4\}$, while the rst-subgraph with 4 edges is $\{rs, rt, st, sv_4\}$. Obviously, $N_2(G_1) = 3$, $N_3(G_1) = 4$, and $N_4(G_1) = 1$. Similarly, we can calculate $N_i(G_j)$, $2 \leq i, j \leq 4$, which are $N_2(G_2) = 1$, $N_3(G_2) = 3$, and $N_4(G_2) = 1$; $N_2(G_3) = 1$, $N_3(G_3) = 4$, and $N_4(G_3) = 1$; and $N_2(G_4) = 0$, $N_3(G_4) = 3$, and $N_4(G_4) = 1$. Figure 2 shows a visualization of reliability polynomials for all graphs in $\mathscr{G}_{4,4}$. Clearly, for all $0 < p < 1$, $R_3(G_1; p) > R_3(G_3; p) > R_3(G_2; p) > R_3(G_4; p)$, so G_1 is the uniformly most reliable graph in $\mathscr{G}_{4,4}$.

Example 2. Figure 3 shows two special simple three-terminal graphs in $\mathscr{G}_{8,26}$ with three target vertices $r,$ $s,$ and t. Each edge of these graphs survives independently with probability p. By calculation, we give a plot of $R_3(H_1; p) - R_3(H_2; p)$ as shown in Figure 4. Clearly, $R_3(H_1; p) > R_3(H_2; p)$ for p close to 1, and $R_3(H_1; p) < R_3(H_2; p)$ for p close to 0. This article later proves that H_1 is the locally most reliable graph for p close to 1 and H_2 is the locally most reliable graph for p close to 0 in $\mathscr{G}_{8,26}$.

Many studies focus on determining a uniformly most reliable graph for given number of vertices n and edges m, as shown in [1, 2, 14]. If there is no uniformly most reliable graph, researchers usually focus on determining the locally most reliable graph for p close to 0 or 1, as shown in [3, 17]. At present, there are few studies on determining whether there is a uniformly most reliable three-terminal graph. Therefore, in this paper, we study the uniformly most reliable graph and the locally most reliable graph of dense three-terminal graph. One can see that some dense graphs have uniformly most reliable three-terminal graphs as Example 1, and some does not have the uniformly most reliable graph but have locally most reliable three-terminal graphs as Example 2.

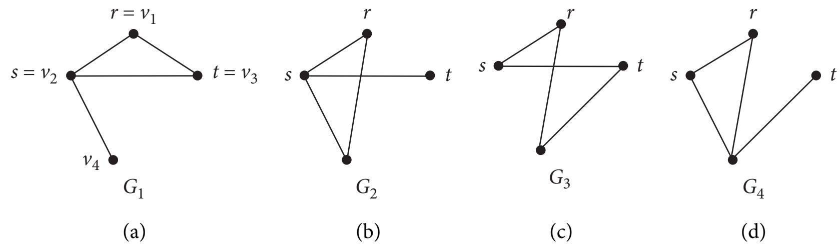

FIGURE 1: All simple three-terminal graphs in $\mathscr{G}_{4,4}$ with three target vertices r, s, and t.

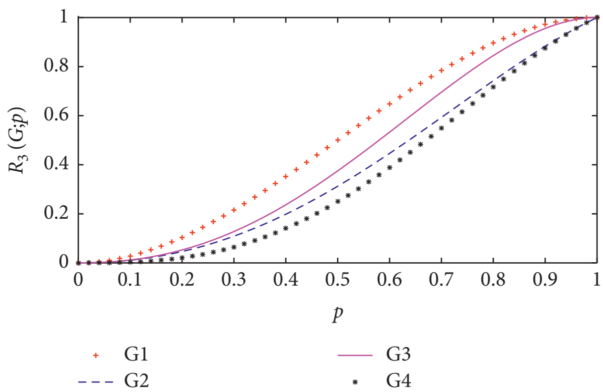

FIGURE 2: A visualization of reliability polynomials for graphs in $\mathscr{G}_{4,4}$.

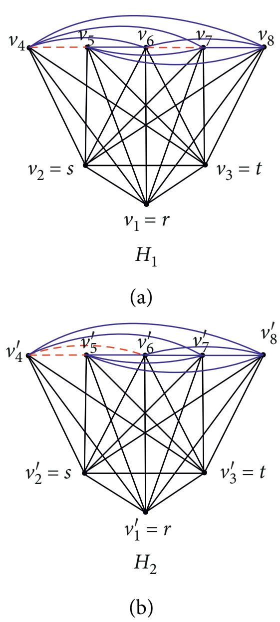

FIGURE 3: Two special three-terminal graphs in $\mathscr{G}_{8,26}$ with three target vertices r, s, and t. The red dotted lines indicate the deleted edges.

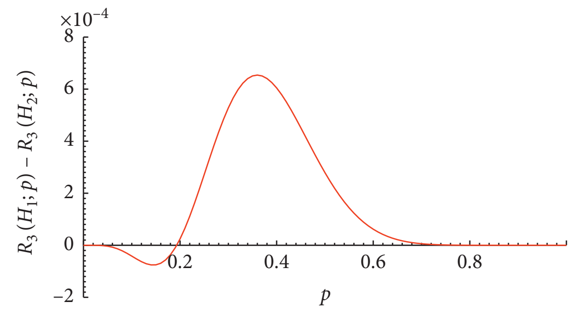

FIGURE 4: A plot of $R_3(H_1; p) - R_3(H_2; p)$.

3. Some Locally Most Reliable Three-Terminal Graphs

In this section, the locally most reliable three-terminal graph with $n \geq 7$ and $m \in \left[\binom{n}{2} - (n-4), \binom{n}{2} - 2\right]$ for p close to 0 is determined and the locally most reliable three-terminal graph with $n \geq 7$ and $m \in \left[\binom{n}{2} - \lfloor (n-3)/2 \rfloor, \binom{n}{2} - 2\right]$ for p close to 1 is also determined. Then, it is shown that for $n \geq 7$ and $m \in \left[\binom{n}{2} - \lfloor (n-3)/2 \rfloor, \binom{n}{2} - 2\right]$, there is no uniformly most reliable graph in $\mathscr{G}_{n,m}$. For $n = 4, 5, 6$ and $m = \binom{n}{2} - 2$, there is a uniformly most reliable three-terminal graph. To prove these results, we first introduce some related lemmas.

Lemma 1 (see [17]). *Let $n \geq 1$ and $0 \leq m \leq n-1$ be positive integers. If $m \neq 3$, then the unique simple graph on n vertices and m edges with the maximum number of P_3 is $S_{m+1} \cup \overline{K_{n-m-1}}$. If $m = 3$, then there are two simple graphs with the maximum number of P_3: $K_3 \cup \overline{K_{n-3}}$ and $S_4 \cup \overline{K_{n-4}}$.*

In general, it is difficult to calculate the three-terminal reliability polynomial of graph. Therefore, we study the locally most reliable graph by the following Lemma 2, which is extracted from [14].

Lemma 2. *Let the three-terminal reliable polynomials of $G, H \in \mathscr{G}_{n,m}$ be $R_3(G; p) = \sum_{i=2}^{m} N_i(G) p^i (1-p)^{m-i}$ and $R_3(H; p) = \sum_{i=2}^{m} N_i(H) p^i (1-p)^{m-i}$.*

Let $N_i(G) = N_i(H)$ for $1 \leq i < k$ and for $l < i \leq m$, where k and l are integers. Then,

(1) for p close to 0, if $N_k(G) > N_k(H)$, then $R_3(G; p) > R_3(H; p)$;

(2) for p close to 1, if $N_l(G) > N_l(H)$, then $R_3(G;p) > R_3(H;p)$.

By Lemma 2, we get the following conclusions:

(1) If $G \in \mathcal{G}_{n,m}$ is the locally most reliable graph for p close to 0, then it must contain the triangle rst and the value of $N_3(G)$ is the maximum among graphs containing the triangle rst in $\mathcal{G}_{n,m}$.

(2) An rst-cutset of G is a set of edges whose deletion results in the disconnection of vertices r, s, and t in G and the number of edges is its size. The edge connectivity of r, s, and t in G is the smallest size of an rst-cutset of G, denoted by $\lambda_{rst}(G)$ or simply $\lambda(G)$. If $G \in \mathcal{G}_{n,m}$ is the locally most reliable graph for p close to 1, then it must have the maximum $\lambda_{rst}(G)$. Since $N_i = \binom{m}{i}$ $(m-\lambda+1 \le i \le m)$ and $N_{m-\lambda} = \binom{m}{\lambda} - a$, where a is the number of the rst-cutsets with size λ, G must have the minimum a among the graphs with the largest edge connectivity of r, s, and t in $\mathcal{G}_{n,m}$.

Now, we demonstrate the locally most reliable graph for three-terminal graphs for p close to 0 or 1. We first introduce two important graphs for this section, as follows:

Let $n \ge 7$ and $2 \le l \le n-4$ be positive integers. Using $A_{n,l}$ denotes the three-terminal graph on n vertices and $\binom{n}{2} - l$ edges with vertex set $V(A_{n,l}) = \{r = v_1, s = v_2, t = v_3, v_4, \ldots, v_n\}$ and edge set $E(A_{n,l}) = \{v_iv_j | 1 \le i < j \le n\} - \{v_4v_{i+3} | 2 \le i \le l+1\}$.

Let $n \ge 7$ and $2 \le l \le \lfloor (n-3)/2 \rfloor$ be positive integers. Using $A'_{n,l}$ denotes the three-terminal graph on n vertices and $\binom{n}{2} - l$ edges with vertex set $V(A'_{n,l}) = \{r = v_1, s = v_2, t = v_3, v_4, \ldots, v_n\}$ and edge set $E(A'_{n,l}) = \{v_iv_j | 1 \le i < j \le n\} - \{v_{2i}v_{2i+1} | 2 \le i \le l+1\}$.

Figure 5 depicts these two three-terminal graphs with 11 vertices and 51 edges.

Theorem 1. *Let $n \ge 7$, $2 \le l \le n-4$ and $m = \binom{n}{2} - l$ be positive integers. Then,*

(1) if $l = 3$, then the graph $A^*_{n,3} = A_{n,3} \cup \{v_4v_7\} - \{v_5v_6\}$ is the unique locally most reliable graph in $\mathcal{G}_{n,m}$ for p close to 0;

(2) if $l \ne 3$, then the graph $A_{n,l}$ is the unique locally most reliable graph in $\mathcal{G}_{n,m}$ for p close to 0.

Proof. Suppose that G is the locally most reliable graph in $\mathcal{G}_{n,m}$ for p close to 0. Then, by Lemma 2, G must contain the triangle rst and N_3 (the number of rst-subgraphs with 3 edges in G) must be maximum among graphs containing the triangle rst in $\mathcal{G}_{n,m}$.

For the sake of calculating N_3, using a, b, and c denotes the number of rst-subgraphs with 3 edges containing 2,1,0 edges in triangle rst, respectively. Then, a is the number of sets $\{rs, st, v_iv_j\}$, $\{rs, rt, v_iv_j\}$, and $\{rt, st, v_iv_j\}$ $(1 \le i, j \le n)$, b is the number of sets $\{rs, rv_i, tv_i\}$, $\{rs, sv_i, tv_i\}$, $\{rt, rv_i, sv_i\}$, $\{rt, tv_i, sv_i\}$, $\{st, rv_i, sv_i\}$, and $\{st, rv_i, tv_i\}$ $(4 \le i \le n)$, and c is the number of set $\{v_ir, v_is, v_it\}$ $(4 \le i \le n)$. It is no hard to see that $N_3 = a + b + c$. Clearly, for graphs containing the triangle rst in $\mathcal{G}_{n,m}$, a is a constant which equals $3(m-3)+1$, and N_3 takes the maximum value if and only if b and c attains the maximum value. Note that if c takes the maximum value $n-3$, then the value of b also reaches its maximum; that is, $E(G)$ contains the edges v_ir, v_is, v_it for all $4 \le i \le n$. Now, consider the remaining $\binom{n}{2} - l - 3n + 6$ edges among nontarget vertices in G that has not been described. Since G is a dense graph, it is often easier to consider the deleted l edges among nontarget vertices.

By Lemma 2, we continue to calculate the coefficients $N_i = b_i + c_i$ $(i > 3)$, where b_i and c_i are the number of minimal rst-subgraphs and nonminimal rst-subgraphs with i edges, respectively. We now compute N_4. Clearly, b_4 is the sum of the numbers of sets $\{sv_i, v_it, sv_j, v_jr\}$, $\{sv_i, v_iv_j, v_jt, sr\}$, and $\{sv_i, v_iv_j, v_jt, v_jr\}$ $(4 \le i, j \le n)$. The nonminimal rst-subgraph with 4 edges includes two cases: one is containing the minimal rst-subgraph with 2 edges and the other is containing the minimal rst-subgraph with 3 edges but no minimal rst-subgraph with 2 edges. By calculation, $b_4 = 6\binom{n-3}{2} + 12(m-3n+6) + 6(m-3n+6)$ and $c_4 = 3\binom{m-3}{2} + (m-3) + (n-3)(m-3) + 6(n-3)(m-6)$ which are constants. So, N_4 is a constant.

Furthermore, we need to calculate N_5. Clearly, b_5 is the sum of the numbers of sets $\{sv_i, v_iv_j, v_jr, rv_k, v_kt\}$, $\{sv_i, v_iv_j, v_jv_k, v_kt, rt\}$, $\{sv_i, v_iv_j, v_jr, v_jv_k, v_kt\}$, and $\{rv_i, sv_i, v_iv_j, v_jv_k, v_kt\}$ $(4 \le i, j, k \le n)$. The nonminimal rst-subgraph with 5 edges includes three cases: containing the minimal rst-subgraph with 2 edges, containing the minimal rst-subgraph with 3 edges but no minimal rst-subgraph with 2 edges, and containing the minimal rst-subgraph with 4 edges but no minimal rst-subgraph with less than 4 edges. By calculation, $c_5 = 3\binom{m-3}{3} + \binom{m-3}{2} + 7(n-3)\binom{m-6}{2} + 3(n-3)(m-6) - 12\binom{n-3}{2} + 18(m-3n+6)(m-10) + 6(m-3n+6) + 6(m-9)\binom{n-3}{2}$, which is a constant. And

$$
\begin{aligned}
b_5 &= 12(m-3n+6)(n-5) + 24\sum_{i=4}^{n}\binom{d(v_i)-3}{2} \\
&= 12\sum_{i=4}^{n}(d(v_i)-3)^2 + 12(m-3n+6)(n-7) \\
&= 24\sum_{i=4}^{n}\binom{d_i}{2} + 24l + 12(n-3)(n-4)^2 \\
&\quad - 48(n-4)l + 12(m-3n+6)(n-7),
\end{aligned}
\tag{2}
$$

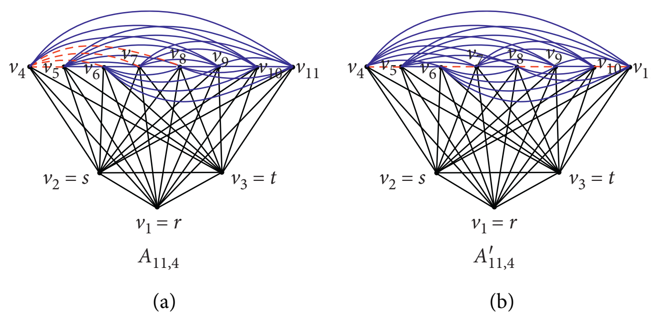

Figure 5: Graph $A_{11,4}$ (a) and Graph $A'_{11,4}$ (b). The red dotted lines indicate the deleted edges.

where $d(v_i)$ is the degree of v_i in G, and d_i is the number of edges deleted on the nontarget vertex $v_i\,(4 \le i \le n)$, which is the degree of v_i in $\overline{G}$. Note that the number of subgraphs as P_3 in $\overline{G}$ is $\sum_{i=4}^{n}\binom{d_i}{2}$. Obviously, N_5 of $R_3(G;p)$ is related to $\beta_{\overline{G}}(P_3)$, the number of subgraphs as P_3 in $\overline{G}$. There are at least three isolate vertices in $\overline{G}$, which are target vertices in G. $\beta_{\overline{G}}(P_3)$ is corresponding with $n-3$ nontarget vertices.

(1) By Lemma 1, if $l = 3$, then the number of P_3 in a simple graph with $n-3$ vertices and l edges reaches the maximum if the graph is either $K_3 \cup \overline{K_{n-6}}$ or $S_4 \cup \overline{K_{n-7}}$. Thus, $\overline{G}$ is either $K_3 \cup \overline{K_{n-6}} \cup \{r,s,t\}$ or $S_4 \cup \overline{K_{n-7}} \cup \{r,s,t\}$, which means that G is either $A_{n,3}$ or $A^*_{n,3}$, where $N_5(A_{n,3}) = N_5(A^*_{n,3})$. So, we need to compare $N_6(A_{n,3})$ and $N_6(A^*_{n,3})$, which can be calculated as the same analysis of N_5. For convenient, if $H \in \mathscr{G}(n,m)$, then let $\widehat{H}$ be the graph $H-\{s,t\}$. Then, $N_6(A_{n,3}) = \binom{m-3}{3} + 3\binom{m-3}{4} + 3(n-3)\binom{m-6}{2} + 7(n-3)\binom{m-6}{3} + \left[6\binom{m-9}{2} - 12(m-9) - 13\right] \quad \binom{n-3}{2} - 30\binom{n-3}{3} + [18\binom{m-10}{2} + 6(m-10) - 36(n-5) + 12(n-5) \quad (m-13) + 12(m-3n+7)] \quad (m-3n+6) + (12n+24m-459)\beta_{\widehat{A_{n,3}}}(P_3) - 66\beta_{\widehat{A_{n,3}}}(K_3) + 30\beta_{\widehat{A_{n,3}}}(P_4) + 6\beta_{\widehat{A_{n,3}}}(S_4)$ and $N_6(A^*_{n,3}) = \binom{m-3}{3} + 3\binom{m-3}{4} + 3(n-3)\binom{m-6}{2} + 7(n-3)\binom{m-6}{3} + \left[6\binom{m-9}{2} - 12(m-9) - 13\right] \quad \binom{n-3}{2} - 30\binom{n-3}{3} + \left[18\binom{m-10}{2} + 6(m-10) - 36(n-5) + 12(n-5) \quad (m-13) + 12(m-3n+7)\right] (m-3n+6) + (12n+24m-459)\beta_{\widehat{A_{n,}}}3^*(P_3) - 66\beta_{\widehat{A^*_{n,3}}}(K_3) + 30\beta_{\widehat{A^*_{n,3}}}(P_4) + 6\beta_{\widehat{A^*_{n,3}}}(S_4)$. By calculation, $\beta_{\widehat{A_{n,3}}}(P_4) = \beta_{\widehat{A^*_{n,3}}}(P_4)$, $\beta_{\widehat{A_{n,3}}}(S_4) - \beta_{\widehat{A^*_{n,3}}}(S_4) = -1$ and $\beta_{\widehat{A_{n,3}}}(K_3) - \beta_{\widehat{A^*_{n,3}}}(K_3) = 1$. So,

$$\begin{aligned} N_6(A_{n,3}) - N_6(A^*_{n,3}) = {} & 30\left[\beta_{\widehat{A_{n,3}}}(P_4) - \beta_{\widehat{A^*_{n,3}}}(P_4)\right] \\ & + 6\left[\beta_{\widehat{A_{n,3}}}(S_4) - \beta_{\widehat{A^*_{n,3}}}(S_4)\right] \\ & - 66\left[\beta_{\widehat{A_{n,3}}}(K_3) - \beta_{\widehat{A^*_{n,3}}}(K_3)\right] = -72 < 0. \end{aligned} \tag{3}$$

Therefore, if $l = 3$, the graph $A^*_{n,3}$ is the unique locally most reliable graph in $\mathscr{G}_{n,m}$ for p close to 0.

(2) By Lemma 1, if $l \ne 3$, then the number of P_3 in a simple graph with $n-3$ vertices and l edges is maximized only if the graph is $S_{l+1} \cup \overline{K_{n-4-l}}$. Thus, $\overline{G}$ is $S_{l+1} \cup \overline{K_{n-4-l}} \cup \{r,s,t\}$, which means that G is $A_{n,l}$. Therefore, if $l \ne 3$, then the graph $A_{n,l}$ is the unique locally most reliable graph in $\mathscr{G}_{n,m}$ for p close to 0.

Theorem 2. *Let $n \ge 7$, $2 \le l \le \lfloor (n-3)/2 \rfloor$, and $m = \binom{n}{2} - l$ be positive integers. Then, $A'_{n,l}$ is the unique locally most reliable graph in $\mathscr{G}_{n,m}$ for p close to 1.*

Proof. Let $G \in \mathscr{G}_{n,m}$ be the most reliable graph for p close to 1. By Lemma 2, G must have the largest edge connectivity of r, s, and t, which is the size of the smallest rst-cutset of G as large as possible. For convenient, let $\mathscr{H}_{n,m}$ be the set of three-terminal graph with the largest edge connectivity of r, s, and t. Obviously, $G \in \mathscr{H}_{n,m}$.

For any graph $H \in \mathscr{H}_{n,m}$, let C be the minimal rst-cutset of H, then there must exist a component containing just one target vertex and $k\,(0 \le k \le n-3)$ nontarget vertices $u_i\,(1 \le i \le k)$ in $H - C$, without loss of generality, setting this target vertex as r. Clearly, $\lambda(H) \le \min\{d(r), d(s), d(t)\} \le n-1$. If $d(r) = d(s) = d(t) = n-1$, then $|C| \ge d(r) - k + 2k = d(r) + k \ge n-1$. Hence, λ can arrive at the maximum value $n-1$ if and only if $d(r) = d(s) = d(t) = n-1$. Thus, three target vertices in H are adjacent to every vertex in $V(H)$. However, the connections among nontarget vertices are various. In order to get the precise construction of G, by Lemma 2, we need to consider the coefficient $N_{m-i}\,(i \geq \lambda)$. Obviously, $N_{m-i} = \binom{m}{m-i} - m_i$, where m_i is the number of the rst-cutsets of size i. For given i, $\binom{m}{m-i}$ is a constant and N_{m-i} is corresponding with m_i. Let δ be the minimum degree of H. Since $n \geq 7$ and $\binom{n}{2} - \lfloor (n-3)/2 \rfloor \leq m \leq \binom{n}{2} - 2$, we have $\delta \geq n - 1 - \lfloor (n-3)/2 \rfloor \geq 4$.

Continue to calculate the minimal rst-cutset of H, $|C| = \lambda + \sum_{i=1}^{k} d\,(u_i) - 2k - 2m' \geq \lambda + \delta k - 2k - 2\binom{k}{2} = \lambda - k^2 + (\delta - 1)k$, where m' is the edge number among nontarget vertices $u_1, u_2, \ldots, u_k$. If $k > \delta - 2$, then $|C| \geq d(r) + k > \lambda + \delta - 2$. If $1 \leq k \leq \delta - 2$, then $|C| \geq \lambda + \delta - 2$, where the equation holds if the components containing r in $H - C$ are following two cases: one is composed of target vertex r and one nontarget vertex with degree δ; the other is composed of target vertex r and $\delta - 2$ nontarget vertices with degree δ and the induced subgraph by vertices in this component is $K_{\delta-1}$. So, the number of C with size $\lambda + 1, \lambda + 2, \ldots, \lambda + \delta - 3$ is 0. Meanwhile, the number of C with size λ is 3. By calculation, we see that $m_j(H) = 3\binom{m-\lambda}{j-\lambda}$ for $\lambda \leq j \leq \lambda + \delta - 3$ and $m_{\lambda+\delta-2}(H) = 3\binom{m-\lambda}{\delta-2} + 3n_1 + 3n_2$, where n_1 is the number of nontarget vertices with degree δ and n_2 is the number of $K_{\delta-2}$ in the induced subgraph of nontarget vertices with degree δ.

Similarly, as the calculation of H, we have $m_j(G) = 3\binom{m-\lambda}{j-\lambda}$ for $\lambda \leq j \leq \lambda + \delta(G) - 3$ and $m_{\lambda+\delta(G)-2}(G) = 3\binom{m-\lambda}{\delta(G)-2} + 3n_1' + 3n_2'$, where n_1' is the number of nontarget vertices with degree $\delta(G)$ and n_2' is the number of $K_{\delta(G)-2}$ in the induced subgraph of nontarget vertices with degree δ in G. Obviously, $n_1' > 0$, $n_2' \geq 0$ and if $\delta(G) < \delta$, then $m_j(G) = m_j(H)$ for $\lambda \leq j \leq \lambda + \delta(G) - 3$, and $m_{\lambda+\delta(G)-2}(G) > 3\binom{m-\lambda}{\delta(G)-2}$ but $m_{\lambda+\delta(G)-2}(H) = 3\binom{m-\lambda}{\delta(G)-2}$. So, $N_{m-j}(G) = N_{m-j}(H)$ for $\lambda \leq j \leq \lambda + \delta(G) - 3$ and $N_{m-(\lambda+\delta(G)-2)}(G) < N_{m-(\lambda+\delta(G)-2)}(H)$, which contradicts the assumption that G is the most reliable graph for p close to 1. Then, we have $\delta(G) \geq \delta$.

Clearly, $\delta(G) = \lfloor 2(m - 3n + 6)/(n-3) \rfloor + 3$. Since $\binom{n}{2} - \lfloor (n-3)/2 \rfloor \leq m \leq \binom{n}{2} - 2$, the minimum degree of graph G is larger than $n - 2$, which implies that for each $v \in V(G)$, $d(v) \geq n - 2$.

Therefore, $A_{n,l}'$ is the unique locally most reliable graph in $\mathcal{G}_{n,m}$ for p close to 1.

As a straightforward consequence of Theorems 1 and 2, we obtain the following theorem.

Theorem 3. *Let $n \geq 7$ and $2 \leq l \leq \lfloor (n-3)/2 \rfloor$ be positive integers. If $m = \binom{n}{2} - l$, then there is no uniformly most reliable three-terminal graph in $\mathcal{G}_{n,m}$.*

Three specific classes of graphs with order n and size $m = \binom{n}{2} - 2$ are uniformly most reliable graphs.

Remark 1. For $n = 4$ and $m = 4$, there is a uniformly most reliable three-terminal graph in $\mathcal{G}_{4,4}$ (see Example 1). For $n = 5$ and $m = 8$, there is a uniformly most reliable three-terminal graph is G_1 in $\mathcal{G}_{5,8}$ (graphs are shown in Figure 6, and the comparison of the reliability polynomial is shown in Table 1). For n = 6 and m = 13, there is a uniformly most reliable three-terminal graph is G_1 in $\mathcal{G}_{6,13}$ (graphs are shown in Figure 7, and the comparison of the reliability polynomial is shown in Table 2).

4. The Uniformly Most Reliable Three-Terminal Graph

For the three-terminal graph with $\binom{n}{2}$ edges, it is complete graph, which is also the uniformly most reliable graph. In this section, we determine the uniformly most reliable graph in $\mathcal{G}_{n,m}$ with $m = \binom{n}{2} - 1$.

When we remove one edge from K_n with three target vertices, there are only three cases: the edge between target vertices; the edge between a target vertex and a nontarget vertex; and the edge between nontarget vertices. Let $n \geq 5$ and $m = \binom{n}{2} - 1$ be positive integers.

(1) Using X_n denotes the three-terminal graph on n vertices and m edges with vertex set $V(X_n) = \{r = x_1, s = x_2, t = x_3, x_4, \ldots, x_n\}$ and edge set $E(X_n) = \{x_i x_j \mid 1 \leq i < j \leq n\} - \{rs\}$.

(2) Using Y_n denotes the three-terminal graph on n vertices and m edges with vertex set $V(Y_n) = \{r = y_1, s = y_2, t = y_3, y_4, \ldots, y_n\}$ and edge set $E(Y_n) = \{y_i y_j \mid 1 \leq i < j \leq n\} - \{r y_4\}$.

(3) Using Z_n denotes the three-terminal graph on n vertices and m edges with vertex set $V(Z_n) = \{r = z_1, s = z_2, t = z_3, z_4, \ldots, z_n\}$ and edge set $E(Z_n) = \{z_i z_j \mid 1 \leq i < j \leq n\} - \{z_4 z_5\}$.

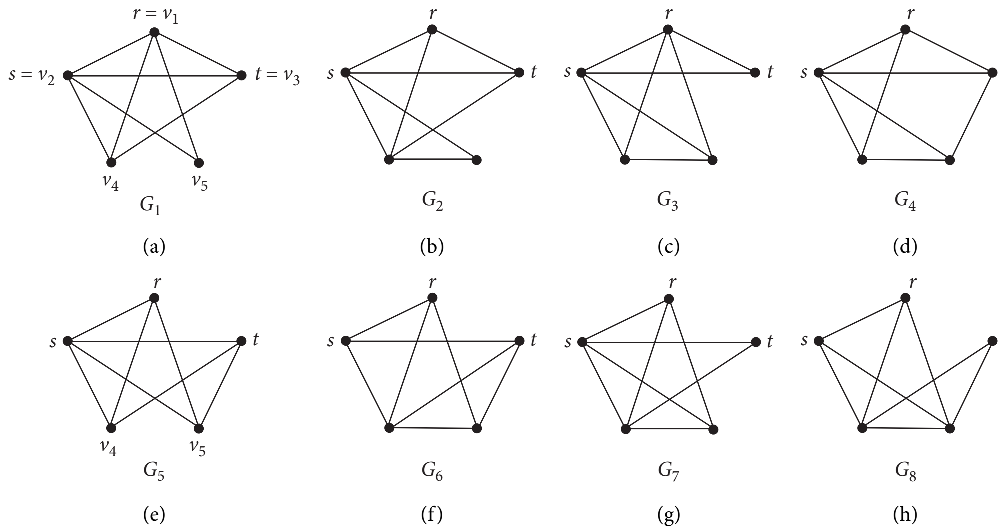

FIGURE 6: All simple three-terminal graphs in $\mathcal{G}_{5,8}$ with three target vertices r, s, and t.

TABLE 1: Reliability polynomials of graphs for $R_3(G; p)$.

$R_3(G; p)$	N_2	N_3	N_4	N_5	N_6	N_7	N_8
$R_3(G_1; p)$	3	25	60	55	28	8	1
$R_3(G_2; p)$	3	23	57	54	28	8	1
$R_3(G_3; p)$	3	20	51	50	27	8	1
$R_3(G_4; p)$	3	20	56	54	28	8	1
$R_3(G_5; p)$	1	16	55	54	28	8	1
$R_3(G_6; p)$	1	13	51	53	28	8	1
$R_3(G_7; p)$	1	12	46	49	27	8	1
$R_3(G_8; p)$	0	6	42	48	27	8	1

Theorem 4. *Let $n \geq 5$ and $m = \binom{n}{2} - 1$ be positive integers. Then, Z_n is the unique uniformly most reliable graph in $\mathcal{G}_{n,m}$.*

Proof. By the definition of three-terminal reliability polynomial, if we can prove that there are more rst-subgraphs with i edges in Z_n than in X_n and Y_n for each $2 \leq i \leq \binom{n}{2} - 1$, then Z_n is the unique uniformly most reliable graph in $\mathcal{G}_{n,m}$.

We complete this proof by constructing two injective maps f_X and f_Y, from the rst-subgraphs with i edges in X_n and Y_n to the rst-subgraphs with i edges in Z_n, respectively.

Construct the map f_X:

Let S be an rst-subgraph with i edges in X_n, where $2 \leq i \leq \binom{n}{2} - 1$. According to whether $x_4x_5 \in S$, there are two cases we need to consider.

Case 1: if S does not contain the edge x_4x_5, then $f_X(S) = \{z_iz_j | x_ix_j \in S\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. And this image does not contain the edge rs.

Case 2: assume that S contains the edge x_4x_5.

Case 2.1: if $S - \{x_4x_5\}$ is still an rst-subgraph, then $f_X(S) = \{z_iz_j | x_ix_j \in S\} \cup \{rs\} - \{z_4z_5\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. Since this image contains the edge rs, it is distinct from Case 1. And $f_X(S) - \{rs\}$ is still an rst-subgraph.

Case 2.2: if $S - \{x_4x_5\}$ is not an rst-subgraph, but an rt-subgraph or an st-subgraph, then $f_X(S) = \{z_iz_j | x_ix_j \in S\} \cup \{rs\} - \{z_4z_5\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. Since the image contains rs and $f_X(S) - \{rs\}$ is not an rst-subgraph, it is distinct from the above cases. It is clear to see that in Case 2.2, $f_X(S)$ contains either the edge st and an edge rz_i for some $4 \leq i \leq n$, or the edge rt and an edge sz_j for some $4 \leq j \leq n$, or an edge rz_i and an edge sz_j for some $4 \leq i, j \leq n$.

Case 2.3: assume that $S - \{x_4x_5\}$ is not an rst-subgraph, or rt-subgraph, or st-subgraph. In this case, all rst-subgraphs, rt-subgraphs, and st-subgraphs in S contain the edge x_4x_5. Thus, the image of the map defined by the above cases is not an rst-subgraph of Z_n. Let S' be a minimal rst-subgraph in S. In $S' - \{x_4x_5\}$, there are two components containing x_4 and x_5, respectively. Targets r and s are in the same component; otherwise, $S - \{x_4x_5\}$ is either an rt-subgraph or an st-subgraph. Without loss of generality, let r, s be in the component containing x_4 in $S' - \{x_4x_5\}$ and let t be in the same component as x_5 in $S' - \{x_4x_5\}$.

Case 2.3.1: if S' is consisted of an edge sx_j, a minimal x_4x_j-subgraph, the edge rx_4, the edge x_4x_5, and a minimal x_5t-subgraph, then $f_X(S) = \{z_iz_j | x_ix_j \in S\} \cup \{z_5z_j | sx_j \in S\} \cup \{rs\} - \{z_4z_5\}$. S does not have both edges x_5x_j and sx_j; otherwise, an edge x_5x_j, an edge sx_j, and a x_5t-subgraph will get a st-subgraph that does not contain the edge x_4x_5 which contradicts the condition of Case 2.3. Therefore, $f_X(S)$ has the same number of edges as S. In $f_X(S)$, we have an rst-subgraph of Z_n which is consisted of an edge z_5z_j, a

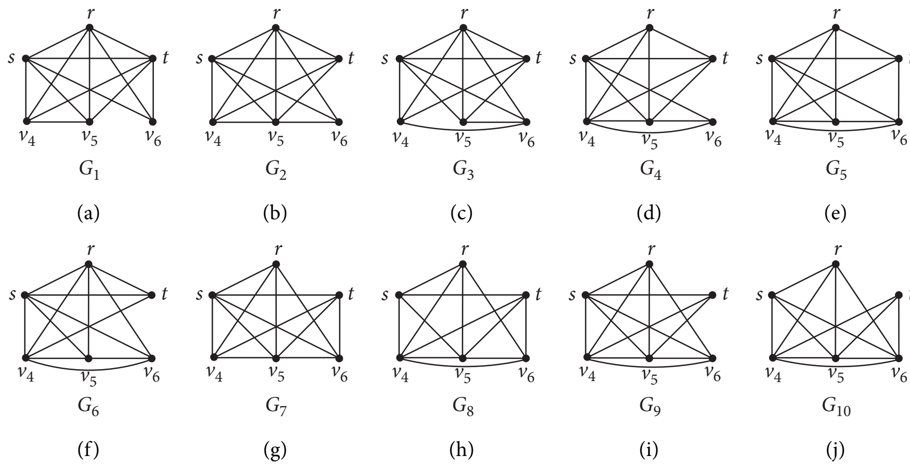

FIGURE 7: All simple three-terminal graphs in $\mathscr{G}_{6,13}$ with three target vertices r, s, and t.

TABLE 2: Reliability polynomials of graphs for $R_3(G;p)$.

$R_3(G;p)$	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}	N_{13}
$R_3(G_1;p)$	3	52	337	1017	1605	1689	1284	715	286	78	13	1
$R_3(G_2;p)$	3	47	304	955	1550	1661	1276	714	286	78	13	1
$R_3(G_3;p)$	3	47	297	953	1552	1662	1276	714	286	78	13	1
$R_3(G_4;p)$	3	45	283	907	1501	1634	1268	713	286	78	13	1
$R_3(G_5;p)$	3	42	264	889	1494	1633	1268	713	286	78	13	1
$R_3(G_6;p)$	3	42	259	849	1428	1577	1240	705	285	78	13	1
$R_3(G_7;p)$	1	26	227	863	1486	1632	1268	713	286	78	13	1
$R_3(G_8;p)$	1	23	199	805	1432	1604	1260	712	286	78	13	1
$R_3(G_9;p)$	1	22	188	761	1365	1548	1232	704	285	78	13	1
$R_3(G_{10};p)$	0	9	132	687	1308	1520	1224	703	285	78	13	1

z_4z_j-subgraph, the edge rz_4, a z_5t-subgraph, and the edge rs. Since $f_X(S)$ contains the edge rs but does not contain any edge $sz_j\,(3 \le j \le n)$ and $f_X(S) - \{rs\}$ is not an rst-subgraph, $f_X(S)$ is distinct from the above cases. The image $f_X(S)$ in this case contains the edge rz_4.

Case 2.3.2: if S' is consisted of an edge rx_j, a minimal sx_4x_j-subgraph, the edge x_4x_5, and a minimal x_5t-subgraph, then $f_X(S) = \{z_iz_j \mid x_ix_j \in S\} \cup \{z_5z_j \mid rx_j \in S\} \cup \{rs\} - \{z_4z_5\}$. Similarly, as Case 2.3.1, S does not have both edges x_5x_j and rx_j. Therefore, $f_X(S)$ has the same number of edges as S. In $f_X(S)$, we have an rst-subgraph of Z_n which is consisted of an edge z_5z_j, a sz_4z_j-subgraph, a z_5t-subgraph, and the edge rs. Since $f_X(S)$ contains edges rs and sz_j for some $4 \le j \le n$ but does not contain any edge $rz_i\,(3 \le i \le n)$ and $f_X(S) - \{rs\}$ is not an rst-subgraph, $f_X(S)$ is distinct from the above cases.

Case 2.3.3: if S' is consisted of the edge rx_4, the edge sx_4, the edge x_4x_5 and a minimal x_5t-subgraph, then $f_X(S) = \{z_iz_j \mid x_ix_j \in S\} \cup \{rz_5 \mid sx_4 \in S\} \cup \{z_5z_j \mid sx_j \in S, j \ne 4\} \cup \{rs\} - \{z_4z_5\}$. Similarly, as Case 2.3.1, S does not have both edges x_5x_j and sx_j. Therefore, $f_X(S)$ has the same number of edges as S. In $f_X(S)$, we have an rst-subgraph of Z_n is consisted of the edge rz_4, the edge rz_5, a z_5t-subgraph, and the edge rs. Since $f_X(S)$ contains edges rs, rz_4, and rz_5 but does not contain any edge $sz_j\,(3 \le i \le n)$ and $f_X(S) - \{rs\}$ is not an rst-subgraph, $f_X(S)$ is distinct from the above cases.

Since the map f_X defined on each of these cases yields rst-subgraph of Z_n as disjoint images, the map is injective.

Because there are at least as many rst-subgraphs with i edges in Z_n as in X_n for $2 \le i \le \binom{n}{2} - 1$, Z_n is more reliable than X_n for all $p\,(0 \le p \le 1)$.

Construct the map f_Y:

Let S be an rst-subgraph with i edges in Y_n, where $2 \le i \le \binom{n}{2} - 1$. According to whether $y_4y_5 \in S$, there are two cases we need to consider.

Case 1: if S does not contain the edge y_4y_5, then $f_Y(S) = \{z_iz_j \mid y_iy_j \in S\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. And this image does not contain the edge rz_4.

Case 2: assume that S contains the edge y_4y_5.

Case 2.1: if $S - \{y_4y_5\}$ is still an rst-subgraph, then $f_Y(S) = \{z_iz_j \mid y_iy_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. Since the image contains the edge rz_4, it is distinct from Case 1. And $f_Y(S) - \{rz_4\}$ is still an rst-subgraph.

Case 2.2: if $S - \{y_4y_5\}$ is not an rst-subgraph but is an rsy_5-subgraph and a y_4t-subgraph, or an ry_5-subgraph and an sty_4-subgraph, or an rty_5-subgraph and an sy_4-subgraph, then $f_Y(S) = \{z_iz_j | y_iy_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$. The image is an rst-subgraph of Z_n with the same number of edges as S. Since the image contains rz_4 and $f_Y(S) - \{rz_4\}$ is not an rst-subgraph, it is distinct from the above cases. Since S contains an edge ry_j for some $2 \le j \le n$ and $j \ne 4$, the image also contains an edge rz_j for some $2 \le j \le n$ and $j \ne 4$.

Case 2.3: assume that $S - \{y_4y_5\}$ does not contain an rst-subgraph, or an rsy_5-subgraph and a y_4t-subgraph, or an ry_5-subgraph and an sty_4-subgraph, or an rty_5-subgraph and an sy_4-subgraph. In this case, all rst-subgraphs contain the edge y_4y_5. And $S - \{y_4y_5\}$ is either an rsy_4-subgraph and a y_5t-subgraph, or an rty_4-subgraph and an sy_5-subgraph, or an ry_4-subgraph and an sty_5-subgraph. Therefore, the image of the map defined for the above cases is not an rst-subgraph of Z_n. Let S' be a minimal rst-subgraph in S.

Case 2.3.1: if S' is consisted of an edge ry_j, a minimal sy_4y_j-subgraph, the edge y_4y_5, and a minimal y_5t-subgraph, then $f_Y(S) = \{z_iz_j | y_i \quad y_j \in S\} \cup \{z_5z_j | ry_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$. If S contains both edges y_5y_j and ry_j, then the edge y_5y_j, the edge ry_j, an sy_4y_j-subgraph, and a y_5t-subgraph will yield an rst-subgraph without the edge y_4y_5, which contradicts the condition of Case 2.3. Consequently, S does not have both edges y_5y_j and ry_j and $f_Y(S)$ has the same number of edges as S. In $f_Y(S)$, we have an rst-subgraph of Z_n which is consisted of an edge z_5z_j, an sz_4z_j-subgraph, a z_5t-subgraph, and the edge rz_4. Since $f_Y(S)$ contains the edge rz_4 but does not contain any edge $rz_j\,(2 \le j \le n, j \ne 4)$ and $f_Y(S) - \{rz_4\}$ is not an rst-subgraph, $f_Y(S)$ is distinct from the above cases. In this case, each rs-subgraph in image $f_Y(S)$ does not contain z_5.

Case 2.3.2: if S' is consisted of an edge ry_j, a minimal ty_4y_j-subgraph, the edge y_4y_5, and a minimal sy_5-subgraph, then $f_Y(S) = \{z_iz_j \quad | y_iy_j \in S\} \cup \{z_5z_j \,| ry_j \quad \in S\} \cup \{rz_4\} - \{z_4z_5\}$. Similarly, as Case 2.3.1, S does not have both edges y_5y_j and ty_j. Therefore, $f_Y(S)$ has the same number of edges as S. In $f_Y(S)$, we have an rst-subgraph of Z_n which is consisted of an edge z_5z_j, a tz_4z_j-subgraph, a sz_5-subgraph, and the edge rz_4. Since $f_Y(S)$ contains the edge rz_4 but does not contain any edge $rz_j\,(2 \le j \le n, j \ne 4)$ and $f_Y(S) - \{rz_4\}$ is not an rst-subgraph and all rs-subgraph in $f_Y(S)$ contain z_5, $f_Y(S)$ is distinct from the above cases. In $f_Y(S)$, each rt-subgraph does not contain z_5.

Case 2.3.3: if S' is consisted of an edge ry_j, a minimal y_4y_j-subgraph, the edge y_4y_5, and a minimal sty_5-subgraph, then $f_Y(S) = \{z_iz_j | y_iy \quad _j \in S\} \cup \{z_5 \, z_j | ry_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$. Similarly, as Case 2.3.1, S does not have both edges y_5y_j and ry_j. Therefore, $f_Y(S)$ has the same number of edges as S. In $f_Y(S)$, we have an rst-subgraph of Z_n which is consisted of an edge z_5z_j, a z_4z_j-subgraph, a stz_5-subgraph, and the edge rz_4. Since $f_Y(S)$ contains the edge rz_4 but does not contain any edge $rz_j\,(2 \le j \le n, j \ne 4)$ and $f_Y(S) - \{rz_4\}$ is not an rst-subgraph and all rs-subgraphs and rt-subgraphs in $f_Y(S)$ contain z_5, $f_Y(S)$ is distinct from the above cases.

Since the map f_Y defined on each of these cases yields rst-subgraph of Z_n as disjoint images, the map is injective.

Because there are at least as many rst-subgraphs with i edges in Z_n as in Y_n for $2 \le i \le \binom{n}{2} - 1$, Z_n is more reliable than Y_n for all $p\,(0 \le p \le 1)$.

From the above argument, we conclude that Z_n is the unique most reliable graph in $\mathcal{G}_{n,m}$ for all $p\,(0 \le p \le 1)$.

5. Conclusion

The reliability of three-terminal graphs with large number of edges are investigated in this article, which fills in the blank of this research.

When the number of vertices n is 4, 5, or 6 and the number of edges m is $\binom{n}{2} - 2$, the uniformly most reliable graph is determined with comparisons in Example 1 and Remark 1. When $n \ge 7$ and $\binom{n}{2} - \lfloor (n-3)/2 \rfloor \le m \le \binom{n}{2} - 2$, there is no uniformly most reliable graph. However, the locally most reliable graph in $\mathcal{G}_{n,m}$ for p close to 0 or 1 is justified by Theorems 1 and 2, respectively. It is worth considering whether there is a uniformly most reliable graph in the class of three-terminal graphs by deleting more edges from K_n with three target vertices.

The uniformly most reliable graph in $\mathcal{G}_{n,\binom{n}{2}-1}$ is determined, which is a graph by removing an edge between nontarget vertices from K_n with three target vertices. This conclusion is significant in comparison with the conclusion given by Betrand et al. [14], which states that the uniformly most reliable graph with $\binom{n}{2} - 1$ edges for two-terminal graphs is a graph by deleting an edge between nontarget vertices from K_n with two target vertices. For $m = \binom{n}{2} - 1$, we conjecture that the uniformly most reliable graph for k-terminal graphs is a graph by removing an edge between nontarget vertices from K_n with k target vertices.

Based on these results, we found that when $m \ge \binom{n}{2} - \lfloor (n-3)/2 \rfloor$, there are no uniformly most reliable three-terminal graphs in most three-terminal graph classes, and then it is interesting to study the uniformly most reliable three-terminal graphs for sparse graphs or other general graphs. In addition, it is significant to extend the results of this paper and [14] to k-terminal graphs.

Data Availability

No data were used to support this study.

Disclosure

This paper is already published in the preprint given in the following link: "https://arxiv.org/abs/1912.11361."

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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