

Research Article

Refinements and Generalizations of Some Fractional Integral Inequalities via Strongly Convex Functions

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In this article, we have established the Hadamard inequalities for strongly convex functions using generalized Riemann–Liouville fractional integrals. The findings of this paper provide refinements of some fractional integral inequalities. Furthermore, the error bounds of these inequalities are given by using two generalized integral identities.

1. Introduction

Let $f: I \rightarrow \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$ and $x, y \in I$, where $x < y$. Then, the following inequality holds:

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(v)dv \leq \frac{f(x) + f(y)}{2}. \quad (1)$$

The above inequality is well-known as the Hadamard inequality. This inequality provides lower and upper estimates for integral average of a convex function. Since the appearance of this result in literature, it has drawn attention of many mathematicians of recent age and it is one of the most extensively studied results for convex functions. In [1, 2], Sarikaya et al. have studied it via Riemann–Liouville fractional integrals of convex functions. After these versions of Hadamard inequality, many researchers were motivated and elegantly produced fractional inequalities using different types of fractional integrals. Also, many new classes of functions have been introduced in the establishment of fractional Hadamard inequalities; for details, we refer the readers to [3–11].

Fractional calculus studies the integrals and derivatives of any arbitrary order, real or complex. Its history begins at the end of seventeenth century, when G. W. Leibniz and Marquis de l'Hospital in 1695 introduced it for first time by discussing the differentiation of functions of order 1/2. However, it experienced a rapid growth over the short span of time. For example, Lagrange, Laplace, Lacroix, Fourier, Abel, Liouville, Riemann, Green, Holmgren, Grunwald, Letnikov, Sonin, Laurent, Nekrassov, Krug, and Weyl made their major contributions to establish a solid foundation of fractional calculus (see [12–14] and references there in). Fractional integral and derivative operators are the key factors in the development of fractional calculus. Recently, the generalizations [15–17], extensions [18–20], and applications [21–23] for fractional operators have been made by many researchers in mathematics, fluid mechanics [24–26], biological population models [27], and numerical methods [28].

Our aim in this paper is to utilize generalized Riemann–Liouville fractional integrals with monotonically increasing function. The Hadamard inequality is studied for these integral operators of strongly convex functions, and

also, by using some integral identities, error bounds are established. Next, we give the definition of strongly convex function introduced by Polyak [29] (see also [30]).

Definition 1. Let D be a convex subset of \mathbb{X} , $(\mathbb{X}, \|\cdot\|)$ be a normed space. A function $f: D \subset \mathbb{X} \rightarrow \mathbb{R}$ will be called strongly convex function with modulus $C \geq 0$ if

$$f(xt + (1-t)y) \leq tf(x) + (1-t)f(y) - Ct(1-t)\|y-x\|^2, \quad (2)$$

holds $\forall x, y \in D \subseteq \mathbb{X}, t \in [0, 1]$. For $C = 0$, (2) gives the definition of convex function.

In the following, we give the definition of Riemann–Liouville fractional integrals.

Definition 2. Let $f \in L_1[a, b]$. Then, left-sided and right-sided Riemann–Liouville fractional integrals of a function f of order μ where $\Re(\mu) > 0$ are defined as follows:

$$I_{a^+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a, \quad (3)$$

$$I_{b^-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b. \quad (4)$$

The fractional versions of Hadamard inequality by Riemann–Liouville fractional integrals are given in the following theorems.

Theorem 1 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following fractional integral inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}, \quad (5)$$

with $\alpha > 0$.

Theorem 2 (see [2]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following fractional integral inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{(a+b/2)^+}^\alpha f(b) + I_{(a+b/2)^-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned} \quad (6)$$

with $\alpha > 0$.

Theorem 3 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following fractional integral inequality holds:

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \right| \\ &\leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha}\right) [|f'(a)| + |f'(b)|]. \end{aligned} \quad (7)$$

In the following, refinements of Theorem 1–3 are given.

Theorem 4 (see [31]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is strongly convex function on $[a, b]$ with modulus $C \geq 0$, then the following fractional integral inequalities hold:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)} \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2}{(\alpha+1)(\alpha+2)}, \end{aligned} \quad (8)$$

with α .

Theorem 5 (see [32]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is strongly convex function on $[a, b]$ with modulus $C \geq 0$, then the following fractional integral inequalities hold:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2}{2(\alpha+1)(\alpha+2)} \\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} [I_{(a+b/2)^+}^\alpha f(b) + I_{(a+b/2)^-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}, \end{aligned} \quad (9)$$

with α .

Theorem 6 (see [32]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $f \in L_1[a, b]$. If f is strongly convex function on $[a, b]$ with modulus $C \geq 0$, then the following fractional integral inequalities hold:

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \right| \\ &\leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha}\right) [|f'(a)| + |f'(b)|] \\ &\quad - \frac{C(b-a)^3}{(\alpha+2)(\alpha+3)} \left(1 - \frac{\alpha+4}{2^{\alpha+2}}\right), \end{aligned} \quad (10)$$

with α .

In [33], k -fractional Riemann–Liouville integrals are defined as follows.

Definition 3. Let $f \in L_1[a, b]$. Then, k -fractional Riemann–Liouville integrals of order μ , where $\Re(\mu) > 0$, $k > 0$, are defined by

$${}_k I_{a^+}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{(\mu/k)-1} f(t) dt, \quad x > a, \quad (11)$$

$${}_k I_{b^-}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{(\mu/k)-1} f(t) dt, \quad x < b, \quad (12)$$

where $\Gamma_k(\cdot)$ is defined as follows [34]:

$$\Gamma_k(\mu) = \int_0^\infty t^{\alpha-1} e^{-(t^{k/\mu})} dt, \quad \Re(\mu) > 0. \quad (13)$$

If $k = 1$, (11) and (12) coincide with (3) and (4).

Farid et al. [35, 36] proved the following k -fractional Hadamard inequalities.

Theorem 7 (see [35]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a, b]$, then the following inequalities for k -fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{a^+}^\alpha f(b) + {}_k I_{b^-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}. \quad (14)$$

Theorem 8 (see [36]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a, b]$, then the following inequalities for k -fractional integrals hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{(a+b/2)^+}^\alpha f(b) + {}_k I_{(a+b/2)^-}^\alpha f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (15)$$

Theorem 9 (see [35]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for k -fractional integrals hold:

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{a^+}^\alpha f(b) + {}_k I_{b^-}^\alpha f(a) \right] \right| \\ &\leq \frac{b-a}{2((\alpha/k)+1)} \left(1 - \frac{1}{2^{(\alpha/k)}} \right) [|f'(a)| + |f'(b)|]. \end{aligned} \quad (16)$$

In the following, we give the definition of generalized Riemann–Liouville fractional integrals by a monotonically increasing function:

Definition 4 (see [37]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on (a, b) , having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by

$${}_k I_{a^+}^{\mu,\psi} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} f(t) dt, \quad x > a, \quad (17)$$

$${}_k I_{b^-}^{\mu,\psi} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\mu-1} f(t) dt, \quad x < b. \quad (18)$$

If ψ is identity function, then (17) and (18) coincide with (3) and (4).

The k -analogue of generalized Riemann–Liouville fractional integrals are defined as follows:

Definition 5 (see [38]). Let $f: [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also, let ψ be an increasing and positive function on (a, b) , having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ where $\Re(\mu) > 0$, $k > 0$, are defined by

$$\begin{aligned} {}_k I_{a^+}^{\mu,\psi} f(x) &= \frac{1}{k\Gamma_k(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\mu/k)-1} f(t) dt, \quad x > a, \\ {}_k I_{b^-}^{\mu,\psi} f(x) &= \frac{1}{k\Gamma_k(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{(\mu/k)-1} f(t) dt, \quad x < b. \end{aligned} \quad (19)$$

$${}_k I_{a^+}^{\mu,\psi} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\mu/k)-1} f(t) dt, \quad x > a. \quad (20)$$

If $k = 1$, (19) and (20) coincide with (17) and (18). If ψ is taken as identity function, (19) and (20) coincide with (11) and (12). If ψ is taken as identity function along with $k = 1$, (19) and (20) coincide with (3) and (4). For more details of above defined fractional integrals, one can see [13, 39].

In Section 2, we establish Hadamard inequalities for generalized Riemann–Liouville fractional integrals of strongly convex functions. The particular cases are given as consequences of these inequalities which are connected with already published results. In Section 3, by using two integral identities for generalized fractional integrals, the error bounds of fractional Hadamard inequalities are established. The findings of this paper are connected with results that are explicitly proved in [1, 2, 31, 35, 36, 40–44].

2. Main Results

Theorem 10. Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, suppose that f is strongly convex function on $[a, b]$ with modulus $C \geq 0$, ψ is an increasing and positive monotone function on (a, b) , having a

continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following k -fractional integral inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2(\alpha^2 - k\alpha + 2k^2)}{4(\alpha+k)(\alpha+2k)} \\ \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ \leq \frac{f(a) + f(b)}{2} - \frac{Ck\alpha(b-a)^2}{(\alpha+k)(\alpha+2k)}, \end{aligned} \quad (21)$$

with $\alpha > 0$.

Proof. Since the function f is strongly convex function, so for $x, y \in [a, b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \frac{C}{4}|x-y|^2. \quad (22)$$

Let $x = at + (1-t)b$, $y = (1-t)a + tb$ for $t \in [0, 1]$ in (22) and multiplying the resulting inequality with $t^{(\alpha/k)-1}$ on both sides, we get

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)t^{(\alpha/k)-1} &\leq f(at + (1-t)b)t^{(\alpha/k)-1} \\ &\quad + f((1-t)a + bt)t^{(\alpha/k)-1} \\ &\quad - \frac{C}{2}(b-a)^2(1-2t)^2t^{(\alpha/k)-1}. \end{aligned} \quad (23)$$

Integrating (23) over the interval $[0, 1]$, we get

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) \int_0^1 t^{(\alpha/k)-1} dt &\leq \int_0^1 f(at + (1-t)b)t^{(\alpha/k)-1} dt \\ &\quad + \int_0^1 f((1-t)a + bt)t^{(\alpha/k)-1} dt \\ &\quad - \frac{C}{2}(b-a)^2 \int_0^1 (1-2t)^2t^{(\alpha/k)-1} dt, \\ \frac{2k}{\alpha} f\left(\frac{a+b}{2}\right) &\leq \int_0^1 f(at + (1-t)b)t^{(\alpha/k)-1} dt \\ &\quad + \int_0^1 f((1-t)a + bt)t^{(\alpha/k)-1} dt \\ &\quad - \frac{Ck(b-a)^2(\alpha^2 - k\alpha + 2k^2)}{2\alpha(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (24)$$

Multiplying (24) by $\alpha/2k$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\alpha}{2k} \left(\int_0^1 f(at + (1-t)b)t^{(\alpha/k)-1} dt \right. \\ &\quad \left. + \int_0^1 f((1-t)a + bt)t^{(\alpha/k)-1} dt \right) - \frac{C(b-a)^2(\alpha^2 - k\alpha + 2k^2)}{4(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (25)$$

Taking $u \in [a, b]$ so that $\psi(u) = at + b(1-t)$, that is, $t = (b-\psi(u))/(b-a)$, and $v \in [a, b]$ so that $\psi(v) = a(1-t) + bt$, that is, $t = (\psi(v)-a)/(b-a)$, in (25), then by applying Definition 5, we get the following inequality:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2(\alpha^2 - k\alpha + 2k^2)}{4(\alpha+k)(\alpha+2k)} \\ \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right]. \end{aligned} \quad (26)$$

Since f is strongly convex function, for $t \in [0, 1]$, we also have following inequality:

$$\begin{aligned} f(ta + (1-t)b) + f((1-t)a + tb) &\leq f(a) + f(b) \\ &\quad - 2Ct(1-t)(b-a)^2. \end{aligned} \quad (27)$$

Multiplying (27) with $t^{(\alpha/k)-1}$ and then integrating over the interval $[0, 1]$, we get

$$\begin{aligned} \int_0^1 t^{(\alpha/k)-1} f(ta + (1-t)b) dt + \int_0^1 t^{(\alpha/k)-1} f((1-t)a + tb) dt \\ \leq (f(a) + f(b)) \int_0^1 t^{(\alpha/k)-1} dt - 2C(b-a)^2 \int_0^1 t^{(\alpha/k)} (1-t) dt, \\ \int_0^1 t^{(\alpha/k)-1} f(ta + (1-t)b) dt + \int_0^1 t^{(\alpha/k)-1} f((1-t)a + tb) dt \\ \leq \frac{k(f(a) + f(b))}{\alpha} - \frac{2Ck^2(b-a)^2}{(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (28)$$

Multiplying (28) by $\alpha/2k$, we get

$$\begin{aligned} \frac{\alpha}{2k} \left(\int_0^1 t^{(\alpha/k)-1} f(ta + (1-t)b) dt + \int_0^1 t^{(\alpha/k)-1} f((1-t)a + tb) dt \right) \\ \leq \frac{f(a) + f(b)}{2} - \frac{Ck\alpha(b-a)^2}{(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (29)$$

Again taking $\psi(u) = at + b(1-t)$, that is, $t = (b-\psi(u))/(b-a)$, and $\psi(v) = a(1-t) + bt$, that is, $t = (\psi(v)-a)/(b-a)$, in (29), then by applying Definition 5, we get the following inequality:

$$\begin{aligned} \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ \leq \frac{f(a) + f(b)}{2} - \frac{Ck\alpha(b-a)^2}{(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (30)$$

Combining (26) and (30), we get (21). \square

Remark 1. Under the assumption of Theorem 10, one can get the following results:

- (i) If $C = 0$, $k = 1$, and ψ is identity function in (21), then Theorem 1 is obtained.
- (ii) If $C = 0$ and ψ is identity function in (21), then Theorem 7 is obtained.
- (iii) If $k = 1$ and ψ is identity function in (21), then Theorem 4 is obtained.
- (iv) If $\alpha = 1$, $k = 1$, $C = 0$, and ψ is identity function in (21), then Hadamard inequality is obtained.
- (v) If $C = 0$ in (21), then the inequality (Theorem 1) stated in [41] is obtained.
- (vi) If $C = 0$ and $k = 1$ in (21), then the inequality (Theorem 2.1) stated in [40] is obtained.
- (vii) If $k = 1$, $\alpha = 1$, and ψ is identity function in (21), then the inequality (Theorem 6) stated in [44] is obtained.

Corollary 1. Under the assumption of Theorem 10 with $k = 1$ in (21), the following inequality holds:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)} \\ & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2}{(\alpha+1)(\alpha+2)}. \end{aligned} \quad (31)$$

Corollary 2. Under the assumption of Theorem 10 with ψ as identity function in (21), the following inequality holds:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2(\alpha^2 - k\alpha + 2k^2)}{4(\alpha+k)(\alpha+2k)} \\ & \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_kI_{a^+}^\alpha f(b) + {}_kI_{b^-}^\alpha f(a) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{Ck\alpha(b-a)^2}{(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (32)$$

Theorem 11. Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, suppose that f is strongly convex function on $[a, b]$ with modulus $C \geq 0$ and ψ is an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following k -fractional integral inequalities hold:

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) + \frac{k^2 C(b-a)^2}{2(\alpha+k)(\alpha+2k)} \\ & \leq \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3k)}{4(\alpha+k)(\alpha+2k)}, \end{aligned} \quad (33)$$

with $\alpha > 0$.

Proof. Let $x = (at/2) + (2-t/2)b$ and $y = (2-t/2)a + (bt/2)$ in (22), and multiplying the resulting inequality with $t^{(\alpha/k)-1}$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) t^{(\alpha/k)-1} \leq \frac{1}{2} \left[f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) t^{(\alpha/k)-1} + f\left(\left(\frac{2-t}{2}\right)a + \frac{bt}{2}\right) t^{(\alpha/k)-1} \right] \\ & \quad - \frac{C}{4} (b-a)^2 (1-t)^2 t^{(\alpha/k)-1}. \end{aligned} \quad (34)$$

Integrating (34) over $[0, 1]$, we get

$$\frac{2k}{\alpha} f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) t^{(\alpha/k)-1} dt + \int_0^1 f\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\alpha/k)-1} dt - \frac{k^3 C(b-a)^2}{\alpha(\alpha+k)(\alpha+2k)}. \quad (35)$$

Multiplying above inequality with $\alpha/2k$, we get

$$f\left(\frac{a+b}{2}\right) + \frac{Ck^2(b-a)^2}{2(\alpha+k)(\alpha+2k)} \leq \frac{\alpha}{2k} \left(\int_0^1 f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) t^{(\alpha/k)-1} dt + \int_0^1 f\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\alpha/k)-1} dt \right). \quad (36)$$

Taking $u \in [a, b]$ so that $\psi(u) = (at/2) + b(2-t/2)$, that is, $t = (2(b-\psi(u))/b-a)$, and $v \in [a, b]$ so that $\psi(v) = a(2-t/2) + (bt/2)$, that is, $t = (2(\psi(v)-a)/b-a)$,

in (36), then by applying Definition 5, we get following inequality:

$$f\left(\frac{a+b}{2}\right) + \frac{Ck^2(b-a)^2}{2(\alpha+k)(\alpha+2k)} \leq \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right]. \quad (37)$$

Since f is strongly convex function on $[a, b]$, for $t \in [0, 1]$, we have the following inequality:

$$f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) + f\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \leq f(a) + f(b) - \frac{Ct(2-t)(b-a)^2}{2}. \quad (38)$$

Multiplying (38) with $t^{(\alpha/k)-1}$ on both sides and integrating over $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) t^{(\alpha/k)-1} dt + \int_0^1 f\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\alpha/k)-1} dt \\ & \leq (f(a) + f(b)) \int_0^1 t^{(\alpha/k)-1} dt - \frac{C(b-a)^2}{2} \int_0^1 (2-t)t^{(\alpha/k)} dt. \end{aligned} \quad (39)$$

Multiplying (39) with $\alpha/2k$ on both sides, we get

$$\begin{aligned} & \frac{\alpha}{2k} \left(\int_0^1 f\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) t^{(\alpha/k)-1} dt + \int_0^1 f\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) t^{(\alpha/k)-1} dt \right) \\ & \leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3k)}{4(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (40)$$

Again taking $\psi(u) = (at/2) + b(2-t/2)$, that is, $t = (2(b-\psi(v))/b-a)$, and $\psi(v) = a(2-t/2) + (bt/2)$, that is, $t = (2(\psi(v)-a)/b-a)$, in (40), then by applying Definition 5, we get the following inequality:

$$\begin{aligned} & \frac{2^{\alpha/k} \Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3k)}{4(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (41)$$

Combining (37) and (41), (33) is obtained. \square

Remark 2. Under the assumption of Theorem 11, one can get the following results:

- (i) If $k = 1$, $C = 0$, and ψ is identity function in (33), then Theorem 2 is obtained.
- (ii) If $k = 1$ and ψ is identity function in (33), then Theorem 5 is obtained.
- (iii) If $C = 0$ and ψ is identity function in (33), then inequality (Theorem 2.1) stated in [36] is obtained.

(iv) If $k = 1$, $C = 0$, $\alpha = 1$, and ψ is identity function in (33), then Hadamard inequality is obtained.

(v) If $k = 1$, $\alpha = 1$, and ψ is identity function in (33), then the inequality (Theorem 6) stated in [44] is obtained.

Corollary 3. Under the assumption of Theorem 11 with $C = 0$ in (33), the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (42)$$

Corollary 4. Under the assumption of Theorem 11 with $k = 1$ in (33), the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{C(b-a)^2}{2(\alpha+1)(\alpha+2)} &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ &\leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}. \end{aligned} \quad (43)$$

Corollary 5. Under the assumption of Theorem 11 with ψ as identity function in (33), the following inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{Ck^2(b-a)^2}{2(\alpha+k)(\alpha+2k)} \\ &\leq \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{(a+b/2)^+}^\alpha f(b) + {}_kI_{(a+b/2)^-}^\alpha f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2} - \frac{C\alpha(b-a)^2(\alpha+3k)}{4(\alpha+k)(\alpha+2k)}. \end{aligned} \quad (44)$$

3. Error Bounds of Hadamard Inequalities for Strongly Convex Functions

In this section, we provide the error bounds of fractional Hadamard inequalities using generalized Riemann–Liouville fractional integrals via strongly convex functions. Estimations here are further refined as compared to those already established for convex functions. The following lemma is useful to prove the next result.

Lemma 1 (see [41]). Let $a < b$ and $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . Also, suppose that $f' \in L[a, b]$, ψ is an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) , and $\alpha \in (0, 1)$. Then, for $k > 0$, the following identity holds:

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_kI_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\ = \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha/k} - t^{\alpha/k}] f'(ta + (1-t)b) dt. \end{aligned} \quad (45)$$

Theorem 12. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. Also, suppose that $|f'|$ is strongly convex function on $[a, b]$ with modulus $C \geq 0$ and $\psi(x)$ is an

increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following k -fractional integral inequalities hold:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2((\alpha/k)+1)} \left(1 - \frac{1}{2^{(\alpha/k)}} \right) [|f'(a)| + |f'(b)|] - \frac{C(b-a)^3}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right), \end{aligned} \quad (46)$$

with $\alpha > 0$.

Proof. From Lemma 1 and strongly convexity of $|f'|$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) \right] \right| \\ & \leq \frac{b-a}{2} \int_0^1 |(1-t)^{(\alpha/k)} - t^{(\alpha/k)}| |f'(ta + (1-t))| dt \\ & \leq \frac{b-a}{2} \int_0^1 |(1-t)^{(\alpha/k)} - t^{(\alpha/k)}| (t|f'(a)| + (1-t)|f'(b)| - Ct(1-t)|b-a|^2) dt \\ & = \frac{b-a}{2} \int_0^{1/2} ((1-t)^{(\alpha/k)} - t^{(\alpha/k)}) (t|f'(a)| + (1-t)|f'(b)| - Ct(1-t)|b-a|^2) dt \\ & \quad + \int_{1/2}^1 ((1-t)^{(\alpha/k)} - t^{(\alpha/k)}) (t|f'(a)| + (1-t)|f'(b)| - Ct(1-t)|b-a|^2) dt. \end{aligned} \quad (47)$$

It can be noted that

$$\begin{aligned} & \int_0^{1/2} ((1-t)^{(\alpha/k)} - t^{(\alpha/k)}) (t|f'(a)| + (1-t)|f'(b)| - Ct(1-t)|b-a|^2) dt \\ & = |f'(a)| \int_0^{1/2} (t(1-t)^{(\alpha/k)} - t^{(\alpha/k)+1}) dt + |f'(b)| \int_0^{1/2} ((1-t)^{(\alpha/k)+1} - (1-t)t^{(\alpha/k)}) dt \\ & \quad - C(b-a)^2 \left(\int_0^{1/2} t(1-t)^{(\alpha/k)+1} dt - \int_0^{1/2} (1-t)t^{(\alpha/k)+1} dt \right) \\ & = |f'(a)| \left(\frac{1}{((\alpha/k)+1)((\alpha/k)+2)} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) + |f'(b)| \left(\frac{1}{(\alpha/k)+2} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) \\ & \quad - \frac{C(b-a)^2}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right). \end{aligned} \quad (48)$$

By similar evaluation, one can have

$$\begin{aligned} & \int_{1/2}^1 \left(t^{(\alpha/k)} - (1-t)^{(\alpha/k)} \right) \left(t|f'(a)| + (1-t)|f'(b)| - Ct(1-t)|b-a|^2 \right) dt \\ &= |f'(a)| \left(\frac{1}{(\alpha/k)+2} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) + |f'(b)| \left(\frac{1}{((\alpha/k)+1)((\alpha/k)+2)} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) \\ &\quad - \frac{C(b-a)^2}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right). \end{aligned} \quad (49)$$

Therefore, (47) implies

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) \right] \right| \\ & \leq \frac{b-a}{2} \left[|f'(a)| \left(\frac{1}{((\alpha/k)+1)((\alpha/k)+2)} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) + |f'(b)| \left(\frac{1}{(\alpha/k)+2} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) \right. \\ &\quad \left. - \frac{C(b-a)^2}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right) + |f'(a)| \left(\frac{1}{(\alpha/k)+2} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) \right. \\ &\quad \left. + |f'(b)| \left(\frac{1}{((\alpha/k)+1)((\alpha/k)+2)} - \frac{(1/2)^{(\alpha/k)+1}}{(\alpha/k)+1} \right) - \frac{C(b-a)^2}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right) \right]. \end{aligned} \quad (50)$$

From which after a little computation, one can get (46). \square

Remark 3. Under the assumption of Theorem 12, one can get the following results:

- (i) If $k = 1$ and ψ is identity function in (46), then Theorem 6 is obtained.
- (ii) If $C = 0$ and ψ is identity function in (46), then Theorem 9 is obtained.

(iii) If $k = 1$, $C = 0$, and ψ is identity function in (46), then Theorem 3 is obtained.

(iv) If $k = 1$, $C = 0$, $\alpha = 1$, and ψ is identity function in (46), then Theorem 2.2 in [42] is obtained.

(v) If $k = 1$ and ψ is identity function in (46), then Theorem 6 is obtained.

Corollary 6. Under the assumption of Theorem 12 with $C = 0$ in (46), the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2((\alpha/k)+1)} \left(1 - \frac{1}{2^{(\alpha/k)}} \right) [|f'(a)| + |f'(b)|]. \end{aligned} \quad (51)$$

Corollary 7. Under the assumption of Theorem 12 with $k = 1$ in (46), the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{\psi^{-1}(a)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) [|f'(a)| + |f'(b)|] - \frac{C(b-a)^3}{(\alpha+2)(\alpha+3)} \left(1 - \frac{\alpha+4}{2^{\alpha+2}} \right). \end{aligned} \quad (52)$$

Corollary 8. Under the assumption of Theorem 12 with ψ as identity function in (46), the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\alpha/k}} [{}_kI_{a^+}^\alpha f(b) + {}_kI_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{b-a}{2((\alpha/k)+1)} \left(1 - \frac{1}{2^{\alpha/k}} \right) [|f'(a)| + |f'(b)|] - \frac{C(b-a)^3}{((\alpha/k)+2)((\alpha/k)+3)} \left(1 - \frac{(\alpha/k)+4}{2^{(\alpha/k)+2}} \right). \end{aligned} \quad (53)$$

We now derive a new fractional integral identity for fractional integrals (19) and (20).

Lemma 2. Let $a < b$ and $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . Also, suppose that $f' \in L[a, b]$ and ψ is an

increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following identity holds:

$$\begin{aligned} & \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_kI_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \\ & = \frac{b-a}{4} \left[\int_0^1 t^{(\alpha/k)} f'\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) dt \right. \\ & \quad \left. - \int_0^1 t^{(\alpha/k)} f'\left(\left(\frac{2-t}{2}\right)a + \frac{bt}{2}\right) dt \right], \end{aligned} \quad (54)$$

with $\alpha > 0$.

Then, we have

Proof. Let

$$\begin{aligned} I_1 &= \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) \right], \\ I_2 &= \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right]. \end{aligned} \quad (55)$$

$$\begin{aligned} I_1 &= \frac{2^{(\alpha/k)-1}\alpha}{k(b-a)^{(\alpha/k)}} \left[\int_{\psi^{-1}(a+b/2)}^{\psi^{-1}(b)} \psi'(\nu) (b - \psi(\nu))^{(\alpha/k)-1} (f \circ \psi)(\nu) d\nu \right] \\ &= \frac{-2^{(\alpha/k)-1}}{(b-a)^{(\alpha/k)}} \left[\int_{\psi^{-1}(a+b/2)}^{\psi^{-1}(b)} f(\psi(\nu)) d(b - \psi(\nu))^{(\alpha/k)} \right] \\ &= \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{1}{2} \int_{\psi^{-1}(a+b/2)}^{\psi^{-1}(b)} f'(\psi(\nu)) \left(\frac{2(b - \psi(\nu))}{b-a} \right)^{(\alpha/k)} \psi'(\nu) d\nu. \end{aligned} \quad (56)$$

By substituting $t = (2(b - \psi(\nu))/b - a)$, we will get

$$I_1 = \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{b-a}{4} \int_0^1 t^{\alpha/k} f'\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) dt. \quad (57)$$

We also have

$$\begin{aligned} I_2 &= \frac{2^{(\alpha/k)-1} \alpha}{k(b-a)^{(\alpha/k)}} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(a+b/2)} \psi'(\nu) (\psi(\nu) - a)^{(\alpha/k)-1} f \circ \psi(\nu) d\nu \right] \\ &= \frac{2^{(\alpha/k)-1}}{(b-a)^{(\alpha/k)}} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(a+b/2)} d(\psi(\nu) - a)^{(\alpha/k)} (f(\psi(\nu))) \right] \\ &= \frac{1}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{2} \int_{\psi^{-1}(a)}^{\psi^{-1}(a+b/2)} f'(\psi(\nu)) \left(\frac{2(\psi(\nu)) - a}{b-a} \right)^{(\alpha/k)} \psi'(\nu) d\nu. \end{aligned} \quad (58)$$

By substituting $s = (2(\psi(\nu) - a)/b - a)$, we will get

$$I_2 = \frac{1}{2} f\left(\frac{a+b}{2}\right) - \frac{b-a}{4} \int_0^1 f'\left(a\left(\frac{2-s}{2}\right) + \frac{bs}{2}\right) s^{\alpha/k} ds. \quad (59)$$

By summing (57) and (59), we get (54). \square

Remark 4. Under the assumption of Lemma 2, one can get the following results:

- (i) If $k = 1$ and ψ is identity function in (54), then the identity (Lemma 3) stated in [2] is obtained.
- (ii) If $k = 1$, $\alpha = 1$, and ψ is identity function in (54), then the identity (Corollary 1) stated in [2] is obtained.
- (iii) If ψ is identity function in (54), then the identity (Lemma 3.1) stated in [36] is obtained.

Corollary 9. Under the assumption of Lemma 2 with $k = 1$ in (54), the following identity holds:

$$\begin{aligned} &\frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) - I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{4} \left[\int_0^1 t^\alpha f'\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) dt + \int_0^1 t^\alpha f'\left(\left(\frac{2-t}{2}\right)a + \frac{bt}{2}\right) dt \right]. \end{aligned} \quad (60)$$

Using above lemma, we give the following error bounds of the k -fractional Hadamard inequality.

Theorem 13. Let $f: I \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. Also, suppose that $|f'|^q$ is strongly convex

function on $[a, b]$ with modulus $C \geq 0$ for $q \geq 1$, and ψ is an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following k -fractional integral inequalities hold:

$$\begin{aligned} &\left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{b-a}{4((\alpha/k)+1)} \left(\frac{1}{2((\alpha/k)+2)} \right)^{1/q} \left[\left(\left(\frac{\alpha}{k} + 1 \right) |f'(a)|^q + \left(\frac{\alpha}{k} + 3 \right) |f'(b)|^q \right) \right. \\ &\quad \left. - \frac{C(b-a)^2 ((\alpha/k)+1)((\alpha/k)+4)}{2((\alpha/k)+3)} \right)^{1/q} + \left(\left(\frac{\alpha}{k} + 3 \right) |f'(a)|^q + \left(\frac{\alpha}{k} + 1 \right) |f'(b)|^q \right. \\ &\quad \left. - \frac{C(b-a)^2 ((\alpha/k)+1)((\alpha/k)+4)}{2((\alpha/k)+3)} \right)^{1/q} \right], \end{aligned} \quad (61)$$

with $\alpha > 0$.

Proof. From Lemma 2 and strongly convexity of $|f'|$, let $q = 1$, we have

$$\begin{aligned}
 & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
 & \leq \frac{b-a}{4} \left[\int_0^1 t^{(\alpha/k)} \left| f'\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) \right| dt + \int_0^1 t^{(\alpha/k)} \left| f'\left(a\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \right| dt \right] \\
 & \leq \frac{b-a}{4} \left[(|f'(a)| + |f'(b)|) \int_0^1 t^{(\alpha/k)} dt - \frac{C}{2}(b-a)^2 \int_0^1 t^{(\alpha/k)+1} (2-t) dt \right] \\
 & \leq \frac{b-a}{4((\alpha/k)+1)} \left[(|f'(a)| + |f'(b)|) - \frac{C(b-a)^2 ((\alpha/k)+4)((\alpha/k)+1)}{((\alpha/k)+2)((\alpha/k)+k)} \right].
 \end{aligned} \tag{62}$$

Now, for $q > 1$, we proceed as follows.

From Lemma 2 and using power mean inequality, we get

$$\begin{aligned}
 & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{\alpha/k}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
 & \leq \frac{b-a}{4} \left(\int_0^1 t^{(\alpha/k)} dt \right)^{1-(1/q)} \left[\left(\int_0^1 t^{(\alpha/k)} \left| f'\left(\frac{at}{2} + \left(\frac{2-t}{2}\right)b\right) \right|^q dt \right)^{(1/q)} \right. \\
 & \quad \left. + \left(\int_0^1 t^{(\alpha/k)} \left| f'\left(\left(\frac{2-t}{2}\right)a + \frac{bt}{2}\right) \right|^q dt \right)^{(1/q)} \right].
 \end{aligned} \tag{63}$$

Strongly convexity of $|f'|^q$ gives

$$\begin{aligned}
 & \left| \frac{2^{(\alpha/k)-1} \Gamma(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
 & \leq \frac{b-a}{4((\alpha/k)+1)^{1/p}} \left[\left(|f'(a)|^q \int_0^1 \frac{t^{(\alpha/k)+1}}{2} dt + |f'(b)|^q \int_0^1 \left(\frac{2t^{(\alpha/k)} - t^{(\alpha/k)+1}}{2} \right) dt \right. \right. \\
 & \quad \left. \left. - \frac{C(b-a)^2}{4} \int_0^1 t^{(\alpha/k)+1} (2-t) dt \right)^{1/q} + \left(|f'(a)|^q \int_0^1 \left(\frac{2t^{(\alpha/k)} - t^{(\alpha/k)+1}}{2} \right) dt \right. \right. \\
 & \quad \left. \left. + |f'(b)|^q \int_0^1 \frac{t^{(\alpha/k)+1}}{2} dt - \frac{C(b-a)^2}{4} \int_0^1 t^{(\alpha/k)+1} (2-t) dt \right)^{1/q} \right] \\
 & \leq \frac{b-a}{4((\alpha/k)+1)^{1/p}} \left[\left(\frac{|f'(a)|^q}{2((\alpha/k)+2)} + \frac{|f'(b)|^q ((\alpha/k)+3)}{2((\alpha/k)+1)((\alpha/k)+2)} - \frac{C(b-a)^2 ((\alpha/k)+4)}{4((\alpha/k)+2)((\alpha/k)+3)} \right)^{1/q} \right. \\
 & \quad \left. + \left(\frac{|f'(a)|^q ((\alpha/k)+3)}{2((\alpha/k)+1)((\alpha/k)+2)} + \frac{|f'(b)|^q}{2((\alpha/k)+2)} - \frac{C(b-a)^2 ((\alpha/k)+4)}{4((\alpha/k)+2)((\alpha/k)+3)} \right)^{1/q} \right],
 \end{aligned} \tag{64}$$

which after a little computation gives the required result. \square

Remark 5. Under the assumption of Theorem 13, one can get the following results:

- (i) If $C = 0$ and ψ is identity function in (61), then the inequality (Theorem 3.1) stated in [36] is obtained.

(ii) If $C = 0$, $k = 1$, and ψ is identity function in (61), then the inequality (Theorem 5) stated in [2] is obtained.

(iii) If $q = 1$, $C = 0$, $k = 1$, $\alpha = 1$, and ψ is identity function in (61), then the inequality (Theorem 2.2) stated in [43] is obtained.

Corollary 10. *Under the assumption of Theorem 13 with $C = 0$ in (61), the following inequality holds:*

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4((\alpha/k)+1)} \left(\frac{1}{2((\alpha/k)+2)} \right)^{1/q} \left[\left(\frac{\alpha}{k} + 1 \right) |f'(a)|^q \right. \\ & \quad \left. + \left(\frac{\alpha}{k} + 3 \right) |f'(b)|^q + \left(\frac{\alpha}{k} + 3 \right) |f'(a)|^q + \left(\frac{\alpha}{k} + 1 \right) |f'(b)|^q \right]^{1/q}. \end{aligned} \quad (65)$$

Corollary 11. *Under the assumption of Theorem 13 with $k = 1$ in (61), the following inequality holds:*

$$\begin{aligned} & \left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[{}_I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4(\alpha+1)} \left(\frac{1}{2(\alpha+2)} \right)^{1/q} \left[((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q \right. \\ & \quad \left. - \frac{C(b-a)^2((\alpha/k)+1)(\alpha+4)}{2(\alpha+3)} \right]^{1/q} + \left[((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q \right. \\ & \quad \left. - \frac{C(b-a)^2(\alpha+1)(\alpha+4)}{2(\alpha+3)} \right]^{1/q}. \end{aligned} \quad (66)$$

Corollary 12. *Under the assumption of Theorem 13 with ψ is identity function in (61), the following inequality holds:*

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{(a+b/2)^+}^\alpha f(b) - {}_k I_{(a+b/2)^-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4((\alpha/k)+1)} \left(\frac{1}{2((\alpha/k)+2)} \right)^{1/q} \left[\left(\frac{\alpha}{k} + 1 \right) |f'(a)|^q + \left(\frac{\alpha}{k} + 3 \right) |f'(b)|^q \right. \\ & \quad \left. - \frac{C(b-a)^2((\alpha/k)+1)((\alpha/k)+4)}{2((\alpha/k)+3)} \right]^{1/q} + \left[\left(\frac{\alpha}{k} + 3 \right) |f'(a)|^q + \left(\frac{\alpha}{k} + 1 \right) |f'(b)|^q - \frac{C(b-a)^2((\alpha/k)+1)((\alpha/k)+4)}{2((\alpha/k)+3)} \right]^{1/q}. \end{aligned} \quad (67)$$

Corollary 13. Under the assumption of Theorem 13 with $k = 1$, $q = 1$, $\alpha = 1$, and ψ as identity function in (61), the following inequality holds:

$$\left| \frac{1}{b-a} \int_a^b f(v)dv - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)| - \frac{5C(b-a)^2}{12} \right]. \quad (68)$$

Theorem 14. Let $f: I \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. Also, suppose that $|f'|^q$ is strongly convex function on $[a, b]$ for $q > 1$, and ψ is an increasing and positive

monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b) . Then, for $k > 0$, the following k -fractional integral inequalities hold:

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{(\alpha p/k) + 1} \right)^{1/p} \left[\left((|f'(a)| + 3^{1/q} |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\ & \quad \left. + \left((3^{1/q} |f'(a)| + |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right], \end{aligned} \quad (69)$$

with $\alpha > 0$.

Proof. From Lemma 2 and using the property of modulus, we get

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f(\psi^{-1}(b))) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{4} \left[\int_0^1 \left| t^{(\alpha/k)} f'\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) \right| dt + \int_0^1 \left| t^{(\alpha/k)} f'\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) \right| dt \right]. \end{aligned} \quad (70)$$

Now applying Hölder's inequality for integrals, we get

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{g^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{4} \left[\left(\int_0^1 t^{\alpha p/k} dt \right)^{1/p} \left[\int_0^1 \left| f'\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 t^{\alpha p/k} dt \right)^{1/p} \left[\int_0^1 \left| f'\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) \right|^q dt \right]^{1/q} \right]. \end{aligned} \quad (71)$$

Using strongly convexity of $|f'|^q$, we get

$$\begin{aligned}
& \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[k I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + k I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
& \leq \frac{b-a}{4} \left(\frac{1}{(\alpha p/k)+1} \right)^{1/p} \left[\left(|f'(a)|^q \int_0^1 \frac{t}{2} dt + |f'(b)|^q \int_0^1 \frac{2-t}{2} dt \right. \right. \\
& \quad \left. \left. - \frac{C(b-a)^2}{4} \int_0^1 t(2-t) dt \right)^{1/q} + \left(|f'(a)|^q \int_0^1 \frac{2-t}{2} dt + |f'(b)|^q \int_0^1 \frac{t}{2} dt - \frac{C(b-a)^2}{4} \int_0^1 t(2-t) dt \right)^{1/q} \right] \\
& = \frac{b-a}{4} \left(\frac{1}{(\alpha p/k)+1} \right)^{1/p} \left[\left(|f'(a)|^q + 3|f'(b)|^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\
& \quad \left. + \left(3|f'(a)|^q + |f'(b)|^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right] \\
& \leq \frac{b-a}{4} \left(\frac{1}{(\alpha p/k)+1} \right)^{1/p} \left[\left((|f'(a)| + 3^{1/q}|f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\
& \quad \left. + \left((3^{1/q}|f'(a)| + |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right]. \tag{72}
\end{aligned}$$

Here, we have used the fact $a^q + b^q \leq (a+b)^q$, for $q > 1$, $a, b \geq 0$. This completes the proof. \square

Remark 6. Under the assumption of Theorem 14, one can get the following results:

- (i) If $C = 0$ and ψ is identity function in (69), then the inequality (Theorem 3.2) stated in [36] is obtained.

(ii) If $k = 1$, $C = 0$, $\alpha = 1$ and ψ is identity function in (69), then the inequality (Theorem 2.3) stated in [43] is obtained.

Corollary 14. Under the assumption of Theorem 14 with $C = 0$ in (69), the following inequality holds:

$$\begin{aligned}
& \left| \frac{2^{(\alpha/k)-1} \Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
& \leq \frac{b-a}{4} \left(\frac{4}{(\alpha p/k)+1} \right)^{1/p} [|f'(a)| + |f'(b)|]. \tag{73}
\end{aligned}$$

Corollary 15. Under the assumption of Theorem 14 with $k = 1$ in (69), the following inequality holds:

$$\begin{aligned}
& \left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\
& \leq \frac{b-a}{16} \left(\frac{4}{\alpha p+1} \right)^{1/p} \left[\left((|f'(a)| + 3^{1/q}|f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\
& \quad \left. + \left((3^{1/q}|f'(a)| + |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right]. \tag{74}
\end{aligned}$$

Corollary 16. Under the assumption of Theorem 14 with $C = 0$ and $k = 1$ in (69), the following inequality holds:

$$\begin{aligned} & \left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[I_{\psi^{-1}(a+b/2)^+}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(a+b/2)^-}^{\alpha,\psi} (f \circ \psi)(\psi^{-1}(a)) - f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{4} \left(\frac{4}{\alpha p+1} \right)^{1/p} [|f'(a)| + |f'(b)|]. \end{aligned} \quad (75)$$

Corollary 17. Under the assumption of Theorem 14 with ψ is identity function in (69), the following inequality holds:

$$\begin{aligned} & \left| \frac{2^{(\alpha/k)-1}\Gamma_k(\alpha+k)}{(b-a)^{(\alpha/k)}} \left[{}_kI_{(a+b/2)^+}^\alpha f(b) + {}_kI_{(a+b/2)^-}^\alpha f(a) - f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{(\alpha p/k)+1} \right)^{1/p} \left[\left((|f'(a)| + 3^{1/q}|f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\ & \quad \left. + \left((3^{1/q}|f'(a)| + |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right]. \end{aligned} \quad (76)$$

Corollary 18. Under the assumption of Theorem 14 with $k = 1$, $\alpha = 1$, and ψ is identity function in (69), the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \left[\left((|f'(a)| + 3^{1/q}|f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right. \\ & \quad \left. + \left((3^{1/q}|f'(a)| + |f'(b)|)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \right]. \end{aligned} \quad (77)$$

4. Conclusion

In this paper, we have studied Hadamard inequalities and their error estimations using generalized Riemann–Liouville fractional integrals of strongly convex functions. The Hadamard inequalities obtained in this work are refinements as well as generalizations of many well-known inequalities. The error estimations of the Hadamard inequalities for differentiable strongly convex functions are better as compared to those which are obtained for convex functions. The authors are analyzing other well-known fractional integral operators for several kinds of functions in their future work.

Data Availability

No additional data are required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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