Research Article

Jacobian Consistency of a Smoothing Function for the Weighted Second-Order Cone Complementarity Problem

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In this paper, a weighted second-order cone (SOC) complementarity function and its smoothing function are presented. Then, we derive the computable formula for the Jacobian of the smoothing function and show its Jacobian consistency. Also, we estimate the distance between the subgradient of the weighted SOC complementarity function and the gradient of its smoothing function. These results will be critical to achieve the rapid convergence of smoothing methods for weighted SOC complementarity problems.

1. Introduction

The weighted second-order cone complementarity problem (WSOCCP) is, for a given weight vector $w \in \mathcal{K}$ and a continuously differentiable function $F: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m}$, to find vectors $(x, s, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\begin{cases} x \circ s = w, \\ F(x, s, y) = 0, \\ x \in \mathcal{K}, \\ s \in \mathcal{K}, \end{cases} \quad (1)$$

where $\circ$ represents the Jordan product and $\mathcal{K}$ is the Cartesian product of second-order cone, that is, $\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_r}$ with $\sum_{i=1}^r n_i = n, i = 1, \dots, r$. The set $\mathcal{K}^{n_i} (i = 1, \dots, r)$ is the second-order cone (SOC) of dimension n_i defined by

$$\mathcal{K}^{n_i} := \{x_i = (x_{i0}, x_{i1}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : x_{i0} - \|x_{i1}\| \geq 0\}, \quad (2)$$

and the interior of the SOC $\mathcal{K}^{n_i}$ is the set

$$\text{int}\mathcal{K}^{n_i} = \{x_i = (x_{i0}, x_{i1}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} : x_{i0} - \|x_{i1}\| > 0\}. \quad (3)$$

Here $\|\cdot\|$ is the Euclidean norm, and

$$\text{int}\mathcal{K} = \text{int}\mathcal{K}^{n_1} \times \text{int}\mathcal{K}^{n_2} \times \cdots \times \text{int}\mathcal{K}^{n_r}. \quad (4)$$

Obviously, if $w = 0$, WSOCCP (1) reduces to second-order cone complementarity problem (SOCCP). In this article, we may assume that $r = 1$ and $\mathcal{K} = \mathcal{K}^n$ in the following analysis, since it can easily be extended to the general case.

In order to reformulate several equilibrium problems in economics and study highly efficient algorithms to solve these problems, Potra [1] introduced the notion of a weighted complementarity problem (WCP). He showed that the Fisher market equilibrium problem can be modeled as a monotone linear WCP. Moreover, the linear programming and weighted centering (LPWC) problem, which was introduced by Anstreicher [2], can also be formulated as a monotone linear WCP. And Potra [1] analyzed two interior-point methods for solving the monotone linear WCP over the nonnegative orthant. Since then, many scholars are dedicated to investigating the theories and solution methods

of WCP. Tang [3] gave a new nonmonotone smoothing-type algorithm to solve the linear WCP. Chi et al. [4] studied the existence and uniqueness of the solution for a class of WCPs.

As is well known, smoothing methods have superior theoretical and numerical performances. For solving the SOCCP by smoothing methods, we usually reformulate the SOCCP as a system of equations based on parametric smoothing functions of SOC complementarity functions [5, 6]. The smoothing parameter involved in smoothing functions may be treated as a variable [7] or a parameter with an appropriate parameter control [8]. In the latter case, the Jacobian consistency is important to achieve a rapid convergence of Newton methods or Newton-like methods. Hayashi et al. [8] proposed a combined smoothing and regularized method for monotone SOCCP, and based on the Jacobian consistency of the smoothing natural residual function, they proved that the method has global and quadratic convergence. Krejić and Rapajić [9] gave a non-monotone Jacobian smoothing inexact Newton method for nonlinear complementarity problem and proved the global and local superlinear convergence of the method. Chen et al. [10] presented a modified Jacobian smoothing method for the nonsmooth complementarity problem and established the global and fast local convergence for the method.

In this paper, we consider the function $\varphi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ for WSOCCP

$$\varphi(x, s, w) := x + s - \sqrt{x^2 + s^2 + x \circ s + w}, \quad (5)$$

with a given vector $w \in \mathcal{K}^n$. If $w = 0$, φ (5) reduces to the SOC complementarity function [6] with $\tau = 3$:

$$\varphi(x, s, 0) := x + s - \sqrt{x^2 + s^2 + x \circ s}. \quad (6)$$

Since φ is nonsmooth, we define the following smoothing function φ_μ :

$$\varphi_\mu(x, s, w) := x + s - \sqrt{x^2 + s^2 + x \circ s + w + \mu^2 e}, \quad (7)$$

where $\mu \in \mathbb{R}$ is a smoothing parameter.

The main contribution of this paper is to show the Jacobian consistency of the smoothing function (7) and estimate the distance between the subgradient of the weighted SOC complementarity function (5) and the gradient of its smoothing function (7). These properties will be critical to solve weighted SOC complementarity problems by smoothing methods.

The paper is organized as follows. In Section 2, we review some concepts and properties. In Section 3, we derive the computable formula for the Jacobian of the smoothing function in WSOCCP. In Section 4, we show the Jacobian consistency of the smoothing function and estimate the distance between the gradient of smoothing function and the subgradient of the weighted SOC complementarity function. Some conclusions are reported in Section 5.

Throughout this paper, $\mathbb{R}_+$ denotes the set of non-negative numbers. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the space of n -dimensional real column vectors and the space of matrices, respectively. We use $\|\cdot\|$ to denote the Euclidean

norm and define $\|x\| := \sqrt{x^T x}$ for a vector x or the corresponding induced matrix norm. For simplicity, we often use $x = (x_0; x_1)$ instead of the column vector $x = (x_0, x_1^T)^T$. $\text{int}\mathcal{K}^n$ and $\text{bd}\mathcal{K}^n$ mean the topological interior and the boundary of the SOC $\mathcal{K}^n$, respectively. For a given set $S \subset \mathbb{R}^{m \times n}$, $\text{conv}S$ denotes the convex hull of S in $\mathbb{R}^{m \times n}$, and for any matrix $X \in \mathbb{R}^{m \times n}$, $\text{dist}(X, S)$ denotes $\inf\{\|X - Y\|: Y \in S\}$.

2. Preliminaries

In this section, we briefly recall some definitions and results about the Euclidean Jordan algebra [11] associated with the SOC $\mathcal{K}^n$ and subdifferentials [12].

For any $x, s \in \mathbb{R}^n$, their Jordan product is defined as $x \circ s = (x^T s; x_0 s_1 + s_0 x_1)$, and $e = (1, 0, \dots, 0) \in \mathbb{R}^n$ is unit element of this algebra. Given an element $x = (x_0; x_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define the symmetric matrix

$$L(x) = \begin{pmatrix} x_0 & x_1^T \\ x_1 & x_0 I \end{pmatrix}, \quad (8)$$

where I represents the $(n-1) \times (n-1)$ identity matrix. It is easy to verify that $x \circ s = L(x)s$ for any $s \in \mathbb{R}^n$. Moreover, $L(x)$ is positive definite (and hence invertible) if and only if $x \in \text{int}\mathcal{K}^n$.

For each $x = (x_0; x_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, let λ_1, λ_2 and $u^{(1)}, u^{(2)}$ be the spectral values and the associated spectral vectors of x , given by

$$\begin{aligned} \lambda_i &= x_0 + (-1)^i \|x_1\|, \\ u^{(i)} &= \begin{cases} \frac{1}{2} \begin{pmatrix} 1; (-1)^i \frac{x_1}{\|x_1\|} \end{pmatrix}, & \text{if } x_1 \neq 0, \\ \frac{1}{2} \begin{pmatrix} 1; (-1)^i \bar{x}_1 \end{pmatrix}, & \text{otherwise,} \end{cases} \end{aligned} \quad (9)$$

for $i = 1, 2$, with any $\bar{x}_1 \in \mathbb{R}^{n-1}$ such that $\|\bar{x}_1\| = 1$. Then, x admits a spectral factorization associated with SOC $\mathcal{K}^n$ in the form of

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}. \quad (10)$$

For any $x = (x_0; x_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, let $x' = (x_0; -x_1)$ [13]. Then, $x'' = x$, $(x + s)' = x' + s'$, and $(cx)' = cx'$ for any $c \in \mathbb{R}$. Moreover, $x \circ x' = x_0^2 - \|x_1\|^2 = 0$ if $x \in \text{bd}\mathcal{K}^n$.

Suppose that $G: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a locally Lipschitzian function; then, from Rademacher's theorem [14], G is differentiable almost everywhere. The Bouligand (B-) subdifferential and the Clarke subdifferential of G at z are defined by

$$\begin{aligned} \partial_B G(z) &:= \left\{ \lim_{\hat{z} \rightarrow z} G'(\hat{z}): \hat{z} \in D_G \right\} \text{ and } \partial G(z) \\ &= \text{conv} \partial_B G(z), \end{aligned} \quad (11)$$

where D_G denotes the set of points at which G is differentiable. Obviously, $\partial G(z) = \{G'(z)\}$ if G is continuously differentiable at z .

Definition 1 (see [12]). Let $G: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a locally Lipschitzian function and $G_\mu: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a continuously differentiable function for any $\mu > 0$, and for any $z \in \mathbb{R}^m$, we have $\lim_{\mu \rightarrow 0} G_\mu(z) = G(z)$. Then, G_μ satisfies the Jacobian consistency property if for any $z \in \mathbb{R}^m$, $\lim_{\mu \rightarrow 0} \text{dist}(G'_\mu(z), \partial G(z)) = 0$.

3. Smoothing Function

In this section, we study the properties of the smoothing function (7).

Definition 2 (see [8]). For a nondifferentiable function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$, we consider a function $f_\mu: \mathbb{R}^m \rightarrow \mathbb{R}^n$ with a parameter $\mu > 0$ that has the following properties:

- (i) f_μ is differentiable for any $\mu > 0$
- (ii) $\lim_{\mu \rightarrow 0} f_\mu(x) = f(x)$ for any $x \in \mathbb{R}^m$

Such a function f_μ is called a smoothing function of f .

Lemma 1. For any $w \in \mathcal{K}^n$ and $\mu \in \mathbb{R}$, one has

$$\varphi_\mu(x, s, w) = 0 \Leftrightarrow x \circ s = w + \mu^2 e, \quad x \in \mathcal{K}^n, s \in \mathcal{K}^n. \quad (12)$$

Proof. We first suppose that $x \circ s = w + \mu^2 e$, $x \in \mathcal{K}^n$, $s \in \mathcal{K}^n$. Then,

$$\begin{aligned} 0 &= x \circ s - w - \mu^2 e \\ &= (x + s)^2 - (x^2 + s^2 + x \circ s + w + \mu^2 e), \end{aligned} \quad (13)$$

and hence

$$x + s = \sqrt{x^2 + s^2 + x \circ s + w + \mu^2 e}. \quad (14)$$

That is, $\varphi_\mu(x, s, w) = 0$.

Conversely, suppose that $\varphi_\mu(x, s, w) = 0$; then, it follows from (7) that

$$x + s = \sqrt{x^2 + s^2 + x \circ s + w + \mu^2 e} \in \mathcal{K}^n. \quad (15)$$

Upon squaring both sides of it, we obtain

$$x \circ s = w + \mu^2 e \in \mathcal{K}^n. \quad (16)$$

Let

$$\omega := x + s = \sqrt{x^2 + s^2 + x \circ s + w + \mu^2 e} \in \mathcal{K}^n, \quad (17)$$

which implies

$$\begin{aligned} \omega &\in \mathcal{K}^n, \\ \omega^2 &= x^2 + s^2 + x \circ s + w + \mu^2 e \in \mathcal{K}^n. \end{aligned} \quad (18)$$

Therefore,

$$\begin{aligned} \omega^2 - s^2 &= x^2 + x \circ s + w + \mu^2 e \in \mathcal{K}^n, \\ \omega^2 - x^2 &= s^2 + x \circ s + w + \mu^2 e \in \mathcal{K}^n. \end{aligned} \quad (19)$$

Further, it follows from Proposition 3.4 [15] that

$$\begin{aligned} x &= \omega - s \in \mathcal{K}^n, \\ s &= \omega - x \in \mathcal{K}^n. \end{aligned} \quad (20)$$

□

Let $w = (w_0; w_1) \in \mathcal{K}^n$, $\mu \in \mathbb{R}$, $x = (x_0; x_1)$, $s = (s_0; s_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and the mapping $v^\mu: \mathbb{R}^{2n} \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}$ be defined by

$$v^\mu = (v_0^\mu; v_1^\mu) = v^\mu(x, s, w) := x^2 + s^2 + x \circ s + w + \mu^2 e, \quad (21)$$

For simplicity, we use v to denote v^μ when $\mu = 0$, that is,

$$v = (v_0; v_1) = v(x, s, w) := x^2 + s^2 + x \circ s + w. \quad (22)$$

By direct calculations, we have

$$\begin{aligned} v_0^\mu &= \|x\|^2 + \|s\|^2 + x^T s + w_0 + \mu^2 = v_0 + \mu^2, \\ v_1^\mu &= 2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + w_1 = v_1. \end{aligned} \quad (23)$$

Therefore, $v^\mu = (v_0^\mu; v_1)$. From the definition of spectral factorization, v^μ can be decomposed as

$$v^\mu = \lambda_1(v^\mu) u_1(v) + \lambda_2(v^\mu) u_2(v), \quad (24)$$

where $\lambda_1(v^\mu)$, $\lambda_2(v^\mu)$, and $u_1(v)$, $u_2(v)$ are the spectral values and the associated spectral vectors of v^μ given by

$$\begin{aligned} \lambda_i(v^\mu) &= \|x\|^2 + \|s\|^2 + x^T s + w_0 + \mu^2 \\ &\quad + (-1)^i \|2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + w_1\|, \end{aligned} \quad (25)$$

and

$$u_i(v) = \frac{1}{2} \left(1; (-1)^i \bar{v}_1 \right), \quad (26)$$

for $i = 1, 2$, where

$$\bar{v}_1 := \frac{v_1}{\|v_1\|} = \frac{2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + w_1}{\|2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + w_1\|}, \quad (27)$$

if $v_1 \neq 0$; otherwise, $\bar{v}_1$ is any vector in $\mathbb{R}^{n-1}$ such that $\|\bar{v}_1\| = 1$. For any given $w = (w_0; w_1) \in \mathcal{K}^n$ and any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$, it can be verified that

$$\begin{aligned} v^\mu &= x^2 + s^2 + x \circ s + w + \mu^2 e \\ &= \left(x + \frac{s}{2} \right)^2 + \frac{3}{4} s^2 + w + \mu^2 e \\ &= \left(s + \frac{x}{2} \right)^2 + \frac{3}{4} x^2 + w + \mu^2 e \in \text{int} \mathcal{K}^n, \end{aligned} \quad (28)$$

for any $\mu > 0$, and

$$\begin{aligned} v &= x^2 + s^2 + x \circ s + w \\ &= \left(x + \frac{s}{2} \right)^2 + \frac{3}{4} s^2 + w \\ &= \left(s + \frac{x}{2} \right)^2 + \frac{3}{4} x^2 + w \in \mathcal{K}^n. \end{aligned} \quad (29)$$

Given $\mu \in \mathbb{R}$ and $x = (x_0; x_1), s = (s_0; s_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define

$$\omega^\mu = (\omega_0^\mu; \omega_1^\mu) = \omega^\mu(x, s, w) := \sqrt{x^2 + s^2 + x \circ s + w + \mu^2 e}, \quad (30)$$

and when $\mu = 0$,

$$\omega = (\omega_0; \omega_1) = \omega(x, s, w) := \sqrt{x^2 + s^2 + x \circ s + w}. \quad (31)$$

The spectral factorization of ω^μ and ω is as follows:

$$\begin{aligned} \omega^\mu &= \sqrt{\lambda_1(v^\mu)} u_1(v) + \sqrt{\lambda_2(v^\mu)} u_2(v), \\ \omega &= \sqrt{\lambda_1(v)} u_1(v) + \sqrt{\lambda_2(v)} u_2(v). \end{aligned} \quad (32)$$

By (29), we can partition $\mathbb{R}^{2n}$ as $\mathbb{R}^{2n} = \mathcal{O} \cup \mathcal{I} \cup \mathcal{B}$, where

$$\begin{aligned} \mathcal{O} &:= \{(x, s) \in \mathbb{R}^{2n}: v \in \{0\}\} \\ &= \{(x, s) \in \mathbb{R}^{2n}: \lambda_2(v) = \lambda_1(v) = 0\}, \\ \mathcal{I} &:= \{(x, s) \in \mathbb{R}^{2n}: v \in \text{int} \mathcal{K}^n\} \\ &= \{(x, s) \in \mathbb{R}^{2n}: \lambda_2(v) \geq \lambda_1(v) > 0\}, \\ \mathcal{B} &:= \{(x, s) \in \mathbb{R}^{2n}: v \in b d \mathcal{K}^n \setminus \{0\}\} \\ &= \{(x, s) \in \mathbb{R}^{2n}: 2v_0 = \lambda_2(v) > \lambda_1(v) = 0\}. \end{aligned} \quad (33)$$

Lemma 2. For any given $w \in \mathcal{K}^n$ and any $(\mu, x, s) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, let φ and φ_μ be defined as (5) and (7), respectively. Then, we have

(i) The function φ_μ is continuously differentiable everywhere with any $\mu > 0$, and its Jacobian is given by

$$\varphi'_\mu(x, s, w) = \begin{pmatrix} I - L\left(x + \frac{s}{2}\right)L^{-1}(\omega^\mu) \\ I - L\left(s + \frac{x}{2}\right)L^{-1}(\omega^\mu) \end{pmatrix}. \quad (34)$$

Here $L^{-1}(\omega^\mu) = (1/\sqrt{v_0^\mu})I$ if $v_1 = 0$; otherwise,

$$\begin{aligned} L^{-1}(\omega^\mu) &= L_1(v^\mu) + L_2(v^\mu) \\ &= \begin{pmatrix} b_\mu & c_\mu \bar{v}_1^T \\ c_\mu \bar{v}_1 & a_\mu I + (b_\mu - a_\mu) \bar{v}_1 \bar{v}_1^T \end{pmatrix}, \end{aligned} \quad (35)$$

with

$$L_1(v^\mu) = \frac{1}{2\sqrt{\lambda_1(v^\mu)}} \begin{pmatrix} 1 & -\bar{v}_1^T \\ -\bar{v}_1 & \bar{v}_1 \bar{v}_1^T \end{pmatrix}, \quad (36)$$

$$L_2(v^\mu) = \frac{1}{2\sqrt{\lambda_2(v^\mu)}} \begin{pmatrix} 1 & \bar{v}_1^T \\ \bar{v}_1 & \bar{v}_1 \bar{v}_1^T \end{pmatrix} + a_\mu \begin{pmatrix} 0 & 0^T \\ 0 & I - \bar{v}_1 \bar{v}_1^T \end{pmatrix}, \quad (37)$$

where

$$\begin{aligned} a_\mu &= \frac{2}{\sqrt{\lambda_1(v^\mu)} + \sqrt{\lambda_2(v^\mu)}}, \\ b_\mu &= \frac{1}{2} \left(\frac{1}{\sqrt{\lambda_1(v^\mu)}} + \frac{1}{\sqrt{\lambda_2(v^\mu)}} \right), \\ c_\mu &= \frac{1}{2} \left(\frac{1}{\sqrt{\lambda_2(v^\mu)}} - \frac{1}{\sqrt{\lambda_1(v^\mu)}} \right). \end{aligned} \quad (38)$$

(ii) For any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$, we have $\lim_{\mu \rightarrow 0} \varphi_\mu(x, s, w) = \varphi(x, s, w)$. Thus, φ_μ is a smoothing function of φ .

(iii) For any $\mu, \nu \in \mathbb{R}_+$,

$$\|\varphi_\mu(x, s, w) - \varphi_\nu(x, s, w)\| \leq \sqrt{r} |\mu - \nu|. \quad (39)$$

Proof

(i) For any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $\mu > 0$, according to Corollary 5.4 [15] and (28), formula (34) holds. By Proposition 5.2 and its proof [15], we get formula (35).

(ii) Given any $x = (x_0; x_1), s = (s_0; s_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For any $\mu > 0$, we obtain from the spectral factorization of v^μ and v that

$$\begin{aligned} \varphi_\mu(x, s, w) &= x + s - \left(\sqrt{\lambda_1(v^\mu)} u_1(v) + \sqrt{\lambda_2(v^\mu)} u_2(v) \right), \\ \varphi(x, s, w) &= x + s - \left(\sqrt{\lambda_1(v)} u_1(v) + \sqrt{\lambda_2(v)} u_2(v) \right), \end{aligned} \quad (40)$$

where

$$\begin{aligned} \lambda_i(v) &= \|x\|^2 + \|s\|^2 + x^T s + w_0 + (-1)^i \|2x_0 x_1 + 2s_0 s_1 \\ &\quad + x_0 s_1 + s_0 x_1 + w_1\|, \end{aligned} \quad (41)$$

and $\lambda_i(v^\mu)$ and $u_i(v)$ are, respectively, given by (25) and (26) for $i = 1, 2$. It is obvious that

$$\lambda_i(v^\mu) = \lambda_i(v) + \mu^2, \quad (42)$$

for $i = 1, 2$. Then,

$$\begin{aligned} &\lim_{\mu \rightarrow 0} \left(\sqrt{\lambda_1(v^\mu)} u_1(v) + \sqrt{\lambda_2(v^\mu)} u_2(v) \right) \\ &= \lim_{\mu \rightarrow 0} \left(\sqrt{\lambda_1(v) + \mu^2} u_1(v) + \sqrt{\lambda_2(v) + \mu^2} u_2(v) \right) \\ &= \sqrt{\lambda_1(v)} u_1(v) + \sqrt{\lambda_2(v)} u_2(v), \end{aligned} \quad (43)$$

and $\lim_{\mu \rightarrow 0} \varphi_\mu(x, s, w) = \varphi(x, s, w)$. Thus, by (i) and Definition 2, φ_μ is a smoothing function of φ .
 (iii) By following the proof of Proposition 5.1 [15], we obtain the desired result. $\square$

Next, we study some properties of φ , which will be used in the subsequent analysis.

Lemma 3. For any $x = (x_0; x_1), s = (s_0; s_1), \tilde{w} = (\tilde{w}_0; \tilde{w}_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, let $x^2 + s^2 + \tilde{w}^2 \in \text{bd}\mathcal{K}^n$. Then, we have

$$\begin{aligned} x_0^2 &= \|x_1\|^2, \\ s_0^2 &= \|s_1\|^2, \\ \tilde{w}_0^2 &= \|\tilde{w}_1\|^2, \\ x_0 s_0 &= x_1^T s_1, \\ x_0 \tilde{w}_0 &= x_1^T \tilde{w}_1, \\ s_0 \tilde{w}_0 &= s_1^T \tilde{w}_1, \\ x_0 s_1 &= s_0 x_1, \\ x_0 \tilde{w}_1 &= \tilde{w}_0 x_1, \\ s_0 \tilde{w}_1 &= \tilde{w}_0^T s_1. \end{aligned} \quad (44)$$

Proof. We can obtain the desired result by following the proof of Lemma 2 [16]. $\square$

Lemma 4. For any $x = (x_0; x_1), s = (s_0; s_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, let $v = (v_0; v_1) = x^2 + s^2 + x \circ s + w \in \text{bd}\mathcal{K}^n$. Then, one has

$$x \circ x' = 0, \quad (45)$$

$$s \circ s' = 0,$$

$$\begin{aligned} x \circ s' &= 0, \\ x \circ \tilde{w}' &= 0, \end{aligned} \quad (46)$$

$$\begin{aligned} s \circ \tilde{w}' &= 0, \\ \tilde{w} \circ \tilde{w}' &= 0, \end{aligned} \quad (47)$$

$$\begin{aligned} x \circ v' &= 0, \\ s \circ v' &= 0, \end{aligned} \quad (48)$$

$$\begin{aligned} x_0^2 + s_0^2 + x_0 s_0 + \frac{w_0}{2} &= \left\| x_0 x_1 + s_0 s_1 + x_0 s_1 + \frac{w_1}{2} \right\| \\ &= \|x_1\|^2 + \|s_1\|^2 + x_1^T s_1 + \frac{w_1}{2}, \end{aligned} \quad (49)$$

where $\tilde{w} := \sqrt{w}$. Moreover, the following equivalence holds:

$$\begin{aligned} v_0 = 0 &\Leftrightarrow v_1 = 0 \Leftrightarrow v = 0 \\ \Leftrightarrow x_0 = s_0 = w_0 = 0 &\Leftrightarrow x_1 = s_1 = w_1 = 0 \Leftrightarrow (x, s, w) = (0, 0, 0). \end{aligned} \quad (50)$$

Proof. Since

$$\begin{aligned} v &= x^2 + s^2 + x \circ s + w \\ &= \left(x + \frac{s}{2}\right)^2 + \frac{3}{4}s^2 + \tilde{w}^2 \\ &= \left(s + \frac{x}{2}\right)^2 + \frac{3}{4}x^2 + \tilde{w}^2 \in \text{bd}\mathcal{K}^n, \end{aligned} \quad (51)$$

from Lemma 3, we have

$$\begin{aligned} \left(x + \frac{s}{2}\right) \circ \left(x + \frac{s}{2}\right)' &= 0, \quad s \circ s' = 0, \\ \left(x + \frac{s}{2}\right) \circ s' &= 0, \quad s \circ \tilde{w}' = 0, \\ \left(x + \frac{s}{2}\right) \circ \tilde{w}' &= 0, \quad \tilde{w} \circ \tilde{w}' = 0, \\ \left(s + \frac{x}{2}\right) \circ \left(s + \frac{x}{2}\right)' &= 0, \quad x \circ x' = 0, \\ \left(s + \frac{x}{2}\right) \circ x' &= 0, \quad x \circ \tilde{w}' = 0, \\ \left(s + \frac{x}{2}\right) \circ \tilde{w}' &= 0. \end{aligned} \quad (52)$$

It follows from these equalities that the results in (45)–(47) hold. Since $v \in \text{bd}\mathcal{K}^n$, we have $\lambda_1(v) = 0$, i.e.,

$$\|x\|^2 + \|s\|^2 + x^T s + w_0 = \|2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + w_1\|. \quad (53)$$

By the last relation and (45)–(47), we obtain that (49) holds. To prove (48), we only need to verify $x_0 v_1 = v_0 x_1$ and $x_1^T v_1 = x_0 v_0$ by the symmetry of x and s in v . From (45)–(47) and (49),

$$\begin{aligned} x_0 v_1 &= x_0 (2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + 2\tilde{w}_0 \tilde{w}_1) \\ &= 2(x_0^2 + s_0^2 + x_0 s_0 + \tilde{w}_0^2) x_1 \\ &= 2\left(\|x_1\|^2 + \|s_1\|^2 + x_1^T s_1 + \frac{w_0}{2}\right) x_1 \\ &= v_0 x_1, \\ x_1^T v_1 &= x_1^T (2x_0 x_1 + 2s_0 s_1 + x_0 s_1 + s_0 x_1 + 2\tilde{w}_0 \tilde{w}_1) \\ &= 2x_0 \left(\|x_1\|^2 + s_0^2 + x_1^T s_1 + \tilde{w}_0^2\right) \\ &= 2x_0 \left(x_0^2 + s_0^2 + x_0 s_0 + \frac{w_0}{2}\right) \\ &= x_0 v_0. \end{aligned} \quad (54)$$

From (51), the equivalence is also true. $\square$

4. Jacobian Consistency

In this section, we will show the Jacobian consistency property and estimate the distance between the gradient of the smoothing function (7) and the subgradient of the WSOCCP complementarity function (5). For any $\mu \in \mathbb{R}$, $w \in \mathcal{K}^n$, let $z := (x, s, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Based on smoothing function (7), we define $\Phi_\mu: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{2n+m}$ by

$$\Phi(z) := \begin{pmatrix} F(x, s, y) \\ \varphi(x, s, w) \end{pmatrix}, \quad (55)$$

$$\Phi_\mu(z) := \begin{pmatrix} F(x, s, y) \\ \varphi_\mu(x, s, w) \end{pmatrix}. \quad (56)$$

From (1) and (56) and Lemma 1,

$$\Phi_\mu(z) = 0 \Leftrightarrow z = (x, s, y) \text{ solves WSOCCP (1)}. \quad (57)$$

Since the function $\Phi(z)$ is typically nonsmooth, Newton's method cannot be applied to the system $\Phi(z) = 0$ directly. Thus, we can approximately solve the smooth system $\Phi_\mu(z) = 0$ at each iteration and make $\|\Phi_\mu(z)\|$ decrease gradually by reducing μ to zero. First, we show that the function $\Phi_\mu(z)$ satisfies the Jacobian consistency.

Lemma 5. For any arbitrary but fixed vector $w \in \mathcal{K}^n$, we have for any $(\mu, x, s) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$,

$$J_\varphi^0(x, s) := \lim_{\mu \rightarrow 0} \varphi'_\mu(x, s, w) = \begin{pmatrix} I - L\left(x + \frac{s}{2}\right)J \\ I - L\left(s + \frac{x}{2}\right)J \end{pmatrix}, \quad (58)$$

where

$$J := \begin{cases} L^{-1}(\omega), & \text{if } (x, s) \in \mathcal{J}, \\ \frac{1}{2\sqrt{2}v_0} \begin{pmatrix} 1 & \bar{v}_1^T \bar{v}_4 I - 3\bar{v}_1 \bar{v}_1^T \end{pmatrix}, & \text{if } (x, s) \in \mathcal{B}, \\ 0, & \text{if } (x, s) \in \mathcal{O}. \end{cases} \quad (59)$$

Proof. By (34) and the symmetry of x and s , it suffices to prove

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{s}{2}\right)L^{-1}(\omega^\mu) = L\left(x + \frac{s}{2}\right)J. \quad (60) \quad \square$$

Case 1. If $(x, s) \in \mathcal{J}$, it follows from (25) that

$$\begin{aligned} \lim_{\mu \rightarrow 0} \omega^\mu &= \lim_{\mu \rightarrow 0} \left[\sqrt{\lambda_1(v^\mu)} u_1(v) + \sqrt{\lambda_2(v^\mu)} u_2(v) \right] \\ &= \lim_{\mu \rightarrow 0} \left[\sqrt{\lambda_1(v) + \mu^2} u_1(v) + \sqrt{\lambda_2(v) + \mu^2} u_2(v) \right] \\ &= \sqrt{\lambda_1(v)} u_1(v) + \sqrt{\lambda_2(v)} u_2(v) \\ &= \omega \in \text{int} \mathcal{K}^n. \end{aligned} \quad (61)$$

Therefore,

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{s}{2}\right)L^{-1}(\omega^\mu) = L\left(x + \frac{s}{2}\right)L^{-1}(\omega). \quad (62)$$

Case 2. If $(x, s) \in \mathcal{B}$, it is easy to prove (51), and

$$\begin{aligned} 2v_0 &= \lambda_2(v) > \lambda_1(v) = 0, \\ \|v_1\| &= v_0 = \left\| x + \frac{s}{2} \right\|^2 + \frac{3}{4} \|s\|^2 + w_0 > 0. \end{aligned} \quad (63)$$

Thus, we obtain the following from (25):

$$\lambda_1(v^\mu) = \lambda_1(v) + \mu^2 = \mu^2 > 0, \quad (64)$$

$$\lambda_2(v^\mu) = \lambda_2(v) + \mu^2 = 2v_0 + \mu^2 > 0. \quad (65)$$

For any $\mu \neq 0$, we may get from (35) that $L^{-1}(\omega^\mu) = L_1(v^\mu) + L_2(v^\mu)$. We first prove for any $\mu \neq 0$,

$$L\left(x + \frac{s}{2}\right)L_1(v^\mu) = 0. \quad (66)$$

Let

$$\vartheta := (1; \bar{v}_1) = \frac{1}{\|v_1\|} (v_0; v_1) = \frac{v}{v_0}. \quad (67)$$

Based on (36), (48), and (64), we have

$$\begin{aligned} L\left(x + \frac{s}{2}\right)L_1(v^\mu) &= \frac{1}{2\sqrt{\lambda_1(v^\mu)}} L\left(x + \frac{s}{2}\right) \vartheta \vartheta^T \\ &= \frac{1}{2|\mu|} \left(x + \frac{s}{2}\right)^\circ \vartheta \vartheta^T \\ &= \frac{1}{2|\mu|v_0^2} \left(x + \frac{s}{2}\right)^\circ v v^T \\ &= 0. \end{aligned} \quad (68)$$

Next, we prove $\lim_{\mu \rightarrow 0} L_2(v^\mu) = J$. From (37), (64), and (65), we have

$$\begin{aligned}
\lim_{\mu \rightarrow 0} L_2(v^\mu) &= \lim_{\mu \rightarrow 0} \frac{1}{2\sqrt{2v_0 + \mu^2}} \begin{pmatrix} 1 & \bar{v}_1^T \\ \bar{v}_1 & \bar{v}_1 \bar{v}_1^T \end{pmatrix} \\
&\quad + \lim_{\mu \rightarrow 0} \frac{2}{\sqrt{\mu^2} + \sqrt{2v_0 + \mu^2}} \begin{pmatrix} 0 & 0^T \\ 0 & I - \bar{v}_1 \bar{v}_1^T \end{pmatrix} \\
&= \frac{1}{2\sqrt{2v_0}} \begin{pmatrix} 1 & \bar{v}_1^T \\ \bar{v}_1 & 4I - 3\bar{v}_1 \bar{v}_1^T \end{pmatrix} = J.
\end{aligned} \tag{69}$$

Combining (68) and (69) yields

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{s}{2}\right) L^{-1}(\omega^\mu) = \lim_{\mu \rightarrow 0} L\left(x + \frac{s}{2}\right) L_2(v^\mu) = L\left(x + \frac{s}{2}\right) J. \tag{70}$$

Case 3. If $(x, s) \in \mathcal{O}$, it follows from Lemma 4 that $(x, s, w) = (0, 0, 0)$ and

$$\omega^\mu = \sqrt{v^\mu} = |\mu|e \in \text{int}\mathcal{K}^n,$$

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{s}{2}\right) L^{-1}(\omega^\mu) = \lim_{\mu \rightarrow 0} O \cdot \frac{1}{|\mu|} e = O = L\left(x + \frac{s}{2}\right) J. \tag{71}$$

Lemma 6. For any arbitrary but fixed vector $w \in \mathcal{K}^n$, we have for any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{pmatrix} I - U_x \\ I - U_s \end{pmatrix} \in \partial_B \varphi(x, s, w), \tag{72}$$

where

$$\begin{aligned}
U_x &= \pm \frac{1}{2} Z + L\left(x + \frac{s}{2}\right) J, \\
U_s &= \pm Z + L\left(s + \frac{x}{2}\right) J, \\
Z &= \begin{cases} O, & \text{if } (x, s) \in \mathcal{J}, \\ \frac{1}{2} \begin{pmatrix} 1 & -\bar{v}_1^T - \bar{v}_1 \bar{v}_1^T \end{pmatrix}, & \text{if } (x, s) \in \mathcal{B}, \\ I, & \text{if } (x, s) \in \mathcal{O}, \end{cases}
\end{aligned} \tag{73}$$

and J is defined by (59).

Proof. By Proposition 5.2 [15] and the chain rule for differentiation, the complementarity function φ is continuously differentiable at any $(x, s) \in \mathcal{J}$ with

$$\varphi'(x, s, w) = \begin{pmatrix} I - L\left(x + \frac{s}{2}\right) L^{-1}(\omega) \\ I - L\left(s + \frac{x}{2}\right) L^{-1}(\omega) \end{pmatrix} \in \partial_B \varphi(x, s, w). \tag{74}$$

Thus, it suffices to consider the two cases: $(x, s) \in \mathcal{B}$ and $(x, s) \in \mathcal{O}$.

For any $(x, s) \in \mathcal{B}$ or $(x, s) \in \mathcal{O}$, let $(x, \hat{s}) = (x, s + \mu e)$ with sufficiently small $\mu \neq 0$, and define

$$\hat{v} = (\hat{v}_0; \hat{v}_1) := x^2 + \hat{s}^2 + x \circ \hat{s} + w,$$

$$\hat{\omega} = (\hat{\omega}_0; \hat{\omega}_1) := \sqrt{\hat{v}},$$

$$\hat{\vartheta}_1 := \frac{\hat{v}_1}{\|\hat{v}_1\|}, \tag{75}$$

$$\hat{\lambda}_i = \lambda_i(\hat{v}) := \hat{v}_0 + (-1)^i \|\hat{v}_1\|, \quad i = 1, 2.$$

Then, we have

$$\begin{aligned}
\hat{v} &= x^2 + (s + \mu e)^2 + x \circ (s + \mu e) + w \\
&= v + \mu x + 2\mu s + \mu^2 e, \\
\hat{v}_0 &= v_0 + \mu x_0 + 2\mu s_0 + \mu^2, \\
\hat{v}_1 &= v_1 + \mu x_1 + 2\mu s_1,
\end{aligned} \tag{76}$$

$$\hat{\lambda}_i = v_0 + \mu x_0 + 2\mu s_0 + \mu^2 + (-1)^i \|v_1 + \mu x_1 + 2\mu s_1\|, \quad i = 1, 2. \tag{77}$$

Obviously, when $\mu \rightarrow 0$, we have $(x, \hat{s}) \rightarrow (x, s)$, $\hat{v} \rightarrow v$, $\hat{\omega} \rightarrow \omega$ and $\hat{\lambda}_i \rightarrow \lambda_i(v)$ for $i = 1, 2$. Then by (7), it suffices to show

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{\hat{s}}{2}\right) L^{-1}(\hat{\omega}) = U_x, \tag{78}$$

$$\lim_{\mu \rightarrow 0} L\left(\hat{s} + \frac{x}{2}\right) L^{-1}(\hat{\omega}) = U_s,$$

if φ is differentiable at $(x, \hat{s})$. $\square$

Case 4. If $(x, s) \in \mathcal{B}$, we obtain $v \in (\text{bd}\mathcal{K}^n / \{0\})$, and from (45), (46), and (48),

$$\begin{aligned}
\|\hat{v}_1\|^2 &= \|v_1 + \mu x_1 + 2\mu s_1\|^2 \\
&= \|v_1\|^2 + \mu^2 \|x_1\|^2 + 4\mu^2 \|s_1\|^2 + 4\mu v_1^T s_1 \\
&\quad + 2\mu v_1^T x_1 + 4\mu^2 x_1^T s_1 \\
&= (v_0 + \mu x_0 + 2\mu s_0)^2.
\end{aligned} \tag{79}$$

The last relation together with $v_0 > 0$ implies that for sufficiently small μ , we have

$$\|\hat{v}_1\| = v_0 + \mu x_0 + 2\mu s_0 > 0. \quad (80)$$

For sufficiently small $\mu \neq 0$, we obtain from (77) and (80),

$$\hat{\lambda}_1 = v_0 + \mu x_0 + 2\mu s_0 + \mu^2 - \|\hat{v}_1\| = \mu^2 > 0, \quad (81)$$

$$\hat{\lambda}_2 = v_0 + \mu x_0 + 2\mu s_0 + \mu^2 + \|\hat{v}_1\| = 2(v_0 + \mu x_0 + 2\mu s_0) + \mu^2 > 0. \quad (82)$$

It follows from (81) and (82) that $\hat{v} \in \text{int}\mathcal{K}^n$, and hence φ is differentiable at $(x, \hat{s})$.

Now we will prove

$$\lim_{\mu \rightarrow 0} L\left(x + \frac{\hat{s}}{2}\right) L^{-1}(\hat{\omega}) = U_x, \quad (83)$$

where $L^{-1}(\hat{\omega}) = L_1(\hat{v}) + L_2(\hat{v})$, in which $L_1(\hat{v})$ and $L_2(\hat{v})$ are given by (36) and (37) with $\hat{v}$ and $\hat{\vartheta}_1$ replacing v^μ and $\bar{v}_1$, respectively. By the expression of $\hat{v}_1$ and (80),

$$\begin{aligned} \hat{\vartheta} &:= (1; \hat{\vartheta}_1) = \frac{1}{\|\hat{v}_1\|} (\|\hat{v}_1\|; \hat{v}_1) \\ &= \frac{1}{\|\hat{v}_1\|} (v_0 + \mu x_0 + 2\mu s_0; v_1 + \mu x_1 + 2\mu s_1) \\ &= \frac{1}{\|\hat{v}_1\|} (v + \mu x + 2\mu s). \end{aligned} \quad (84)$$

By (45), (46), (48), and (84), we have

$$\begin{aligned} \left(x + \frac{s}{2}\right)^\circ \hat{\vartheta}' &= \frac{1}{\|\hat{v}_1\|} \left(x + \frac{s}{2}\right)^\circ (v + \mu x + 2\mu s)' \\ &= \frac{1}{\|\hat{v}_1\|} \left[\left(x + \frac{s}{2}\right)^\circ v' + 2\mu \left(x + \frac{s}{2}\right)^\circ \left(\frac{x}{2} + s\right)' \right] \\ &= 0. \end{aligned} \quad (85)$$

Thus, from (36) and (81),

$$\begin{aligned} L\left(x + \frac{\hat{s}}{2}\right) L_1(\hat{v}) &= \frac{1}{2\sqrt{\hat{\lambda}_1}} \left(x + \frac{s}{2} + \frac{\mu e}{2}\right)^\circ \hat{\vartheta}' \hat{\vartheta}'^T \\ &= \frac{1}{2|\mu|} \left[\left(x + \frac{s}{2}\right)^\circ \hat{\vartheta}' \hat{\vartheta}'^T + \frac{\mu}{2} \hat{\vartheta}' \hat{\vartheta}'^T \right] \\ &= \frac{\text{sgn}(\mu)}{4} \hat{\vartheta}' \hat{\vartheta}'^T. \end{aligned} \quad (86)$$

It follows from (73)–(84) that as $\mu \rightarrow 0$,

$$\hat{\lambda}_1 \rightarrow \lambda_1(v) = 0,$$

$$\hat{\lambda}_2 \rightarrow \lambda_2(v) = 2v_0,$$

$$\hat{\vartheta}_1 \rightarrow \bar{v}_1, \quad (87)$$

$$\frac{1}{2} \hat{\vartheta}' \hat{\vartheta}'^T = \frac{1}{2} \begin{pmatrix} 1 & -\hat{\vartheta}_1^T \\ -\hat{\vartheta}_1 & -\hat{\vartheta}_1 \hat{\vartheta}_1^T \end{pmatrix} \rightarrow Z.$$

Then, by following the proof of Case 5 in Lemma 5, we have

$$\lim_{\mu \rightarrow 0} L_2(\hat{v}) = \frac{1}{2\sqrt{2v_0}} \begin{pmatrix} 1 & \bar{v}_1^T \\ \bar{v}_1 & 4I - 3\bar{v}_1 \bar{v}_1^T \end{pmatrix} = J. \quad (88)$$

Therefore, we obtain from (86) and (88) that

$$\begin{aligned} \lim_{\mu \rightarrow \pm 0} L\left(x + \frac{\hat{s}}{2}\right) L^{-1}(\hat{\omega}) &= \lim_{\mu \rightarrow \pm 0} L\left(x + \frac{\hat{s}}{2}\right) L_1(\hat{v}) \\ &\quad + \lim_{\mu \rightarrow \pm 0} L\left(x + \frac{\hat{s}}{2}\right) L_2(\hat{v}) \\ &= \lim_{\mu \rightarrow \pm 0} \frac{\text{sgn}(\mu)}{4} \hat{\vartheta}' \hat{\vartheta}'^T + L\left(x + \frac{s}{2}\right) J \\ &= \pm \frac{1}{2} Z + L\left(x + \frac{s}{2}\right) J \\ &= U_x. \end{aligned} \quad (89)$$

Next we will prove

$$\lim_{\mu \rightarrow 0} L\left(\hat{s} + \frac{x}{2}\right) L^{-1}(\hat{\omega}) = U_s. \quad (90)$$

By (45), (46), (48), (81), and (84), we have

$$\begin{aligned} \left(s + \frac{x}{2}\right)^\circ \hat{\vartheta}' &= \frac{1}{\|\hat{v}_1\|} \left(s + \frac{x}{2}\right)^\circ (v + \mu x + 2\mu s)' \\ &= \frac{1}{\|\hat{v}_1\|} \left[\left(s + \frac{x}{2}\right)^\circ v' + 2\mu \left(s + \frac{x}{2}\right)^\circ \left(\frac{x}{2} + s\right)' \right] \\ &= 0, \end{aligned} \quad (91)$$

and then

$$\begin{aligned}
L\left(\hat{s} + \frac{x}{2}\right)L_1(\hat{v}) &= \frac{1}{2\sqrt{\lambda_1}}\left(s + \mu e + \frac{x}{2}\right) \circ \hat{\vartheta}_I \hat{\vartheta}'^T \\
&= \frac{1}{2|\mu|} \left[\left(s + \frac{x}{2}\right) \circ \hat{\vartheta}_I \hat{\vartheta}'^T + \mu \hat{\vartheta}_I \hat{\vartheta}'^T \right] \quad (92) \\
&= \frac{\text{sgn}(\mu)}{2} \hat{\vartheta}_I \hat{\vartheta}'^T.
\end{aligned}$$

Therefore, we obtain from (88) and (92) that

$$\begin{aligned}
\lim_{\mu \rightarrow \pm 0} L\left(\hat{s} + \frac{x}{2}\right)L^{-1}(\hat{w}) &= \lim_{\mu \rightarrow \pm 0} L\left(\hat{s} + \frac{x}{2}\right)L_1(\hat{v}) \\
&\quad + \lim_{\mu \rightarrow \pm 0} L\left(\hat{s} + \frac{x}{2}\right)L_2(\hat{v}) \\
&= \lim_{\mu \rightarrow \pm 0} \frac{\text{sgn}(\mu)}{2} \hat{\vartheta}_I \hat{\vartheta}'^T + L\left(s + \frac{x}{2}\right)J \\
&= \pm Z + L\left(s + \frac{x}{2}\right)J \\
&= U_s. \quad (93)
\end{aligned}$$

Case 5. If $(x, s) \in \mathcal{O}$, it follows from Lemma 4 that $(x, s, w) = (0, 0, 0)$. Thus, $\hat{v} = \mu^2 e \in \text{int}\mathcal{K}^n$, $\hat{w} = |\mu|e$, and

$$\begin{aligned}
\lim_{\mu \rightarrow \pm 0} L\left(x + \frac{\hat{s}}{2}\right)L^{-1}(\hat{w}) &= \lim_{\mu \rightarrow \pm 0} \frac{\mu}{2} I \cdot \frac{1}{|\mu|} I = \lim_{\mu \rightarrow \pm 0} \frac{\text{sgn}(\mu)}{2} I \\
&= \pm \frac{1}{2} I = U_x, \\
\lim_{\mu \rightarrow \pm 0} L\left(\hat{s} + \frac{x}{2}\right)L^{-1}(\hat{w}) &= \lim_{\mu \rightarrow \pm 0} \mu I \cdot \frac{1}{|\mu|} I = \lim_{\mu \rightarrow \pm 0} \text{sgn}(\mu) I \\
&= \pm I = U_s. \quad (94)
\end{aligned}$$

□

Now we show the Jacobian consistency of the function Φ_μ (56) and then estimate an upper bound of the parameter $\mu > 0$ for the predicted accuracy of the distance between the gradient of Φ_μ (56) and the subgradient of Φ (55).

Theorem 1. *The following results hold. (i) The function Φ_μ defined by (56) with $\mu > 0$ satisfies the Jacobian consistency. (ii) For given $\tau > 0$ and any point $z := (x, s, y) \in \mathbb{R}^{2n+m}$, let $\rho(x, s)$ be any function such that*

$$\rho(x, s) \geq \left\| \begin{array}{c} L\left(x + \frac{s}{2}\right)J \\ L\left(s + \frac{x}{2}\right)J \end{array} \right\|, \quad (95)$$

and let $\bar{\mu}: \mathbb{R}^{2n} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ be defined by

$$\bar{\mu}(x, s, \tau) := \begin{cases} \frac{\lambda_1(v)\tau}{\sqrt{\rho^2(x, s) - \lambda_1(v)\tau^2}}, & \text{if } (x, s) \in \mathcal{J} \text{ and } \tau < (\rho(x, s)/\sqrt{\lambda_1(v)}), \\ \frac{v_0\tau}{\sqrt{2\rho(x, s)(2\rho(x, s) - \tau\sqrt{2v_0})}}, & \text{if } (x, s) \in \mathcal{B} \text{ and } \tau < 2\rho(x, s)/\sqrt{2v_0}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (96)$$

Then, for any $\mu \in \mathbb{R}$ such that $0 < |\mu| \leq \bar{\mu}(x, s, \tau)$, we have

$$\text{dist}\left(\Phi_{\mu_I}(z), \partial\Phi(z)\right) < \tau. \quad (97)$$

Proof. By (56), it suffices to show the Jacobian consistency of φ_μ with $\mu > 0$. Define

$$V^i := \begin{pmatrix} I - U_x^i \\ I - U_s^i \end{pmatrix}, \quad (98)$$

where

$$\begin{aligned}
U_x^i &= (-1)^i \frac{1}{2} Z + L\left(x + \frac{s}{2}\right)J, \\
U_s^i &= (-1)^i Z + L\left(s + \frac{x}{2}\right)J, \quad (99)
\end{aligned}$$

for $i = 1, 2$, J and Z are defined by (59) and (73). Let

$$V := \frac{1}{2}(V^1 + V^2) = \begin{pmatrix} I - L\left(x + \frac{s}{2}\right)J \\ I - L\left(s + \frac{x}{2}\right)J \end{pmatrix}. \quad (100)$$

It follows from Lemma 5 and Lemma 6 that

$$V = J_\varphi^0(x, s) = \lim_{\mu \rightarrow 0} \varphi_\mu(x, s, w), \quad (101)$$

and $V^1, V^2 \in \partial_B \varphi(x, s, w)$. Hence,

$$V = \frac{1}{2}(V^1 + V^2) \in \partial \varphi(x, s, w), \quad (102)$$

which together with Definition 1 and Lemma 2 implies the Jacobian consistency of φ_μ with $\mu > 0$. (ii) For any $z := (x, s, y) \in \mathbb{R}^{2n+m}$, it follows from the proof of Theorem 1(i) that

$$\begin{aligned} J_\varphi^0(x, s) &= V \in \partial \varphi(x, s, w), \\ J_\Phi^0(z) &:= \begin{pmatrix} J_\varphi^0(x, s) & O \\ F_{x,s}'(x, s, y) & F_y'(x, s, y) \end{pmatrix} \in \partial \Phi(x, s, y). \end{aligned} \quad (103)$$

Thus, we obtain from (34) and (100) that

$$\begin{aligned} \text{dist}(\Phi_\mu'(z), \partial \Phi(z)) &\leq \|\Phi_\mu'(z) - J_\Phi^0(z)\| \\ &= \|\varphi_\mu'(z) - J_\varphi^0(z)\| \\ &= \left\| \begin{pmatrix} L\left(x + \frac{s}{2}\right)(L^{-1}(\hat{w}) - J) \\ L\left(s + \frac{x}{2}\right)(L^{-1}(\hat{w}) - J) \end{pmatrix} \right\|. \end{aligned} \quad (104)$$

Then, similar to the proof of Proposition 4.1 [13], we have

$$\text{dist}(\Phi_\mu'(z), \partial \Phi(z)) \leq |g_0(x, s) - g_\mu(x, s)| \cdot \left\| \begin{pmatrix} L\left(x + \frac{s}{2}\right)J \\ L\left(s + \frac{x}{2}\right)J \end{pmatrix} \right\|, \quad (105)$$

where $g_\mu: \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$ is given by

$$g_\mu(x, s) := \begin{cases} \frac{1}{\sqrt{\lambda_1(v) + \mu^2}}, & \text{if } (x, s) \in \mathcal{J}, \\ \frac{2}{\sqrt{2v_0 + \mu^2} + |\mu|} \frac{1}{\sqrt{\lambda_1(v) + \mu^2}}, & \text{if } (x, s) \in \mathcal{B}, \\ 0, & \text{if } (x, s) \in \mathcal{O}. \end{cases} \quad (106)$$

Hence, by following the proof of Theorem 4.1 [13], the result holds. $\square$

5. Conclusions

In this paper, we show the Jacobian consistency of the smoothing function φ_μ for WSOCCP, which will play a key role in analyzing the rapid convergence of smoothing methods. Moreover, in order to adjust a parameter appropriately in smoothing methods, we estimate the distance between the gradient of the smoothing function φ_μ and the subgradient of the weighted SOC complementarity function φ .

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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