



Research Article

Study of Fractional Integral Operators Containing Mittag-Leffler Functions via Strongly (α, m) -Convex Functions

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The main aim of this paper is to give refinement of bounds of fractional integral operators involving extended generalized Mittag-Leffler functions. A new definition, namely, strongly (α, m) -convex function is introduced to obtain improvements of bounds of fractional integral operators for convex, m -convex, and (α, m) -convex functions. The results of this paper will provide simultaneous generalizations as well as refinements of various published results.

1. Introduction

Convexity is one of the most important and key concept in mathematics, and many researchers have extended, generalized, and refined it in different ways. Numerous generalizations and extensions have been produced in recent past, for example, in generalization and extension point of views, m -convexity, (α, m) -convexity, s -convexity, (s, m) -convexity, h -convexity, and (h, m) -convexity are remarkable, and in refinement point of view, the strongly convexity is the tremendous notion. In this paper, we have introduced the notion of strongly (α, m) -convex function. By utilizing this refined form of convex function, we obtain refinements of the bounds of fractional integral operators involving Mittag-Leffler functions in their kernels. Therefore, the results of this paper are refinements of all the results proved in [1]. First, we give definitions of convex, strongly convex, and (α, m) -convex functions.

Definition 1 (see [2]). Let I be an interval on real line. A function $f: I \rightarrow \mathbb{R}$ is said to be convex function if the following inequality holds:

$$f(tu_1 + (1-t)u_2) \leq tf(u_1) + (1-t)f(u_2), \quad (1)$$

for all $u_1, u_2 \in I$ and $t \in [0, 1]$.

Definition 2 (see [3]). Let I be an interval on real line. A real-valued function f is said to be strongly convex with modulus $\lambda \geq 0$ on I if, for each $u_1, u_2 \in I$ and $t \in [0, 1]$, we have

$$f(tu_1 + (1-t)u_2) \leq tf(u_1) + (1-t)f(u_2) - \lambda t(1-t)|u_2 - u_1|^2. \quad (2)$$

Definition 3 (see [4]). A function $f: [0, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be (α, m) -convex function, where $(\alpha, m) \in [0, 1]^2$ and $b > 0$ if, for every $u_1, u_2 \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tu_1 + m(1-t)u_2) \leq t^\alpha f(u_1) + m(1-t^\alpha)f(u_2). \quad (3)$$

The well-known Mittag-Leffler function is denoted by $E_\xi(\cdot)$ and defined as follows (see [5]):

$$E_\xi(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\xi n + 1)}, \quad (4)$$

where $t, \xi \in \mathbb{C}$, $\Re(\xi) > 0$, and $\Gamma(\cdot)$ is the gamma function. It is a natural extension of exponential, hyperbolic, and trigonometric functions. This function and its extensions appear

as solution of fractional integral equations and fractional differential equations. For a detailed study of Mittag-Leffler function and its extensions, see [6–10]. The following extended generalized Mittag-Leffler function is introduced by Andrić et al.

Definition 4 (see [11]). Let $\mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then, the Mittag-Leffler function $E_{\mu, \xi, l}^{\gamma, \delta, k, c}(t; p)$ is defined by

$$E_{\mu, \xi, l}^{\gamma, \delta, k, c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \quad (5)$$

where β_p is defined by

$$\beta_p(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-p/t(1-t)} dt, \quad (6)$$

and $(c)_{nk} = \Gamma(c + nk)/\Gamma(c)$.

A derivative formula of the extended generalized Mittag-Leffler function is given in the following lemma.

Lemma 1 (see [11]). If $m \in \mathbb{N}, \omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k < \delta + \Re(\mu)$, then

$$\begin{aligned} & \left(\frac{d}{dt}\right)^m \left[t^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c}(\omega t^\mu; p) \right] \\ &= t^{\xi-m-1} E_{\mu, \xi-m, l}^{\gamma, \delta, k, c}(\omega t^\mu; p), \quad \Re(\xi) > m. \end{aligned} \quad (7)$$

Remark 1. The extended generalized Mittag-Leffler function (5) produces the related functions defined in [8–10, 12, 13], see Remark 1.3 in [14].

Next, we have the definition of the generalized fractional integral operator containing the extended generalized Mittag-Leffler function (5).

Definition 5 (see [11]). Let $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then, the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$\left(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} f \right)(x; p) = \int_a^x (x-t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c}(\omega(x-t)^\mu; p) f(t) dt \quad (8)$$

and

$$\left(\epsilon_{\mu, \xi, l, \omega, b^-}^{\gamma, \delta, k, c} f \right)(x; p) = \int_x^b (t-x)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c}(\omega(t-x)^\mu; p) f(t) dt. \quad (9)$$

Remark 2. The operators defined in (8) and (9) produce several kinds of known fractional integral operators, see Remark 1.4 in [14].

The classical Riemann–Liouville fractional integral operator is defined as follows.

Definition 6 (see [13]). Let $f \in L_1[a, b]$. Then, Riemann–Liouville fractional integral operators of order $\xi > 0$ are defined by

$$I_{a^+}^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_x^b (x-t)^{\xi-1} f(t) dt, \quad x > a, \quad (10)$$

$$I_{b^-}^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_a^x (t-x)^{\xi-1} f(t) dt, \quad x < b. \quad (11)$$

It can be noted that $(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} f)(x; 0) = I_{a^+}^\xi f(x)$ and $(\epsilon_{\mu, \xi, l, \omega, b^-}^{\gamma, \delta, k, c} f)(x; 0) = I_{b^-}^\xi f(x)$. From fractional integral operators (8) and (9), we can have

$$\begin{aligned} J_{\xi, a^+}(x; p) &:= \left(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} 1 \right)(x; p) \\ &= (x-a)^\xi E_{\mu, \xi+1, l}^{\gamma, \delta, k, c}(\omega(x-a)^\mu; p), \end{aligned} \quad (12)$$

$$\begin{aligned} J_{\eta, b^-}(x; p) &:= \left(\epsilon_{\mu, \eta, l, \omega, b^-}^{\gamma, \delta, k, c} 1 \right)(x; p) \\ &= (b-x)^\eta E_{\mu, \eta+1, l}^{\gamma, \delta, k, c}(\omega(b-x)^\mu; p). \end{aligned} \quad (13)$$

In view of wide applications of Riemann–Liouville fractional integrals and derivatives, the problems which involve this integral operator are studied extensively by many authors. The aim of this paper is to provide fractional integral inequalities which are generalizations of Riemann–Liouville fractional integral inequalities. These inequalities also give associated inequalities for fractional integral operators containing Mittag-Leffler functions with different parameters.

The bounds of fractional integrals (8) and (9) for (α, m) -convex functions are given in the following theorems.

Theorem 1 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is positive (α, m) -convex, then for $\xi, \eta \geq 1$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \eta, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq \left(\frac{f(a) + m\alpha f(x_0/m)}{\alpha + 1} \right) (x_0 - a) J_{\xi-1, a^+} (x_0; p) \\ & \quad + \left(\frac{f(b) + m\alpha f(x_0/m)}{\alpha + 1} \right) (b - x_0) J_{\eta-1, b^-} (x_0; p), \quad x_0 \in [a, b]. \end{aligned} \tag{14}$$

Theorem 2 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is (α, m) -convex, then,

for $\xi, \eta \geq 1$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned} & \left| \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) - \left(J_{\xi-1, a^+} (x_0; p) f(a) + J_{\eta-1, b^-} (x_0; p) f(b) \right) \right| \\ & \leq \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha + 1} \right) (x_0 - a) J_{\xi-1, a^+} (x_0; p) + \left(\frac{|f'(b)| + m\alpha |f'(x_0/m)|}{\alpha + 1} \right) (b - x_0) J_{\eta-1, b^-} (x_0; p), \quad x_0 \in [a, b]. \end{aligned} \tag{15}$$

Theorem 3 (see [1]). Let $f: [a, b] \rightarrow \mathbb{R}$, $a > b$, be a real-valued function. If f is positive, (α, m) -convex, and $f(a + mb - x_0/m) = f(x_0)$, then, for $\xi, \eta > 0$, the following

fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned} & \frac{1}{1/2^\alpha + m(1 - 1/2^\alpha)} \left(f\left(\frac{a + mb}{2}\right) \left(J_{\eta+1, b^-} (a; p) + J_{\xi+1, a^+} (b; p) \right) \right) \\ & \leq \left(\epsilon_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \\ & \leq \left(J_{\eta-1, b^-} (a; p) + J_{\xi-1, a^+} (b; p) \right) (b - a)^2 \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} \right). \end{aligned} \tag{16}$$

In Section 2, by using definition of strongly (α, m) -convex function, we establish new refinements of the bounds of generalized fractional integral operators. Also, the refinements of bounds of these operators are presented in the form of Hadamard-like inequality by using strongly (α, m) -convex functions. The results of this paper are connected with several well-known inequalities.

2. Main Results

First, we define strongly (α, m) -convex function.

Definition 7. A function $f: [0, +\infty) \rightarrow \mathbb{R}$ is said to be strongly (α, m) -convex function, with modulus $\lambda \geq 0$, for $(\alpha, m) \in [0, 1]^2$, if

$$\begin{aligned} f(tu_1 + m(1-t)u_2) & \leq t^\alpha f(u_1) + m(1-t^\alpha) f(u_2) \\ & \quad - \lambda m t^\alpha (1-t^\alpha) |u_2 - u_1|^2 \end{aligned} \tag{17}$$

holds, for all $u_1, u_2 \in [0, +\infty)$ and $t \in [0, 1]$.

Remark 3

- (i) By setting $\alpha = 1$ in (17), strongly m -convex function can be obtained [15]
- (ii) By setting $\lambda = 0$ in (17), (α, m) -convex function can be obtained
- (iii) By setting $\alpha = m = 1$ and $\lambda = 0$ in (17), convex function can be obtained
- (iv) By setting $\alpha = 1$ and $\lambda = 0$ in (17), m -convex function can be obtained
- (v) By setting $\alpha = m = 1$ in (17), strongly convex function can be obtained

In the following, by using strongly (α, m) -convex functions the refinement of already proved results are given.

Theorem 4. Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is positive and strongly (α, m) -convex, then, for $\xi, \eta \geq 1$, the

following fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \eta, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq \left(\frac{f(a) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(x_0 - ma)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (x_0 - a) J_{\xi-1, a^+} (x_0; p) \\ & \quad + \left(\frac{f(b) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(mb - x_0)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (b - x_0) J_{\eta-1, b^-} (x_0; p), \\ & x_0 \in [a, b]. \end{aligned} \tag{18}$$

Proof. Let $x_0 \in [a, b]$. Then, for $t \in [a, x_0]$ and $\xi \geq 1$, one can have the following inequality:

$$\begin{aligned} (x_0 - t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - t)^\mu; p) & \leq (x_0 - a)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} \\ & \cdot (\omega(x_0 - a)^\mu; p). \end{aligned} \tag{19}$$

The function f is strongly (α, m) -convex; therefore, one can obtain

$$\begin{aligned} f(t) & \leq \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha f(a) + m \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) f\left(\frac{x_0}{m}\right) \\ & \quad - \lambda m \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left(\frac{x_0 - ma}{m} \right)^2. \end{aligned} \tag{20}$$

By multiplying (19) and (20) and then integrating over $[a, x_0]$, we obtain

$$\begin{aligned} & \int_a^{x_0} (x_0 - t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - t)^\mu; p) f(t) dt \\ & \leq (x_0 - a)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) \left(f(a) \int_a^{x_0} \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha dt \right. \\ & \quad \left. + m f\left(\frac{x_0}{m}\right) \int_a^{x_0} \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) dt \right. \\ & \quad \left. - \frac{\lambda}{m} (x_0 - ma)^2 \int_a^{x_0} \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) dt \right). \end{aligned} \tag{21}$$

From which we have that the left integral operator satisfies the following inequality:

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq (x_0 - a) J_{\xi-1, a^+} (x_0; p) \left(\frac{f(a) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(x_0 - ma)^2}{m(\alpha + 1)(2\alpha + 1)} \right). \end{aligned} \tag{22}$$

Now, on the contrary, for $t \in (x_0, b]$ and $\eta \geq 1$, one can have the following inequality:

$$\begin{aligned} (t - x_0)^{\eta-1} E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(t - x_0)^\mu; p) & \leq (b - x_0)^{\eta-1} E_{\mu, \eta, l}^{\gamma, \delta, k, c} \\ & \cdot (\omega(b - x_0)^\mu; p). \end{aligned} \tag{23}$$

Again from strongly (α, m) -convexity of f , we have

$$\begin{aligned} f(t) & \leq \left(\frac{t - x_0}{b - x_0} \right)^\alpha f(b) + m \left(1 - \left(\frac{t - x_0}{b - x_0} \right)^\alpha \right) f\left(\frac{x_0}{m}\right) \\ & \quad - \lambda m \left(\frac{t - x_0}{b - x_0} \right)^\alpha \left(1 - \left(\frac{t - x_0}{b - x_0} \right)^\alpha \right) \left(\frac{mb - x_0}{m} \right)^2. \end{aligned} \tag{24}$$

By multiplying (23) and (24) and then integrating over $[x_0, b]$, we have

$$\begin{aligned} & \int_{x_0}^b (t - x_0)^{\eta-1} E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(t - x_0)^\mu; p) f(t) dt \\ & \leq (b - x_0)^{\eta-1} E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(b - x_0)^\mu; p) \\ & \quad \times \left(f(a) \int_{x_0}^b \left(\frac{t - x_0}{b - x_0} \right)^\alpha dt + m f\left(\frac{x_0}{m}\right) \right. \\ & \quad \left. \cdot \int_{x_0}^b \left(1 - \left(\frac{t - x_0}{b - x_0} \right)^\alpha \right) dt \right. \\ & \quad \left. - \frac{\lambda}{m} (mb - x_0)^2 \int_{x_0}^b \left(\frac{t - x_0}{b - x_0} \right)^\alpha \left(1 - \left(\frac{t - x_0}{b - x_0} \right)^\alpha \right) dt \right). \end{aligned} \tag{25}$$

The right integral operator satisfies the following inequality:

$$\begin{aligned} & \left(\epsilon_{\mu, \eta, \lambda, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq (b - x_0) J_{\eta-1, b^-} (x_0; p) \left(\frac{f(b) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(mb - x_0)^2}{m(\alpha + 1)(2\alpha + 1)} \right). \end{aligned} \tag{26}$$

By adding (22) and (26), the required inequality (18) is established. \square

Remark 4

- (i) Inequality (18) provides the refinement of Theorem 2.1 in [1]
- (ii) If $\alpha = 1$ in (18), then result for strongly m -convex will be obtained

- (iii) If $\alpha = m = 1$ and $\lambda = 0$ in (18), then Corollary 1 [16] is obtained
- (iv) If $\alpha = m = 1$, $\lambda = 0$, and $\omega = p = 0$ in (18), then Theorem 1 in [17] is obtained
- (v) If $\alpha = 1$ and $t = t^s$ in (18), then Theorem 3 in [18] is obtained

Corollary 1. *If we set $\xi = \eta$ in (18), then the following inequality is obtained:*

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, \lambda, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \xi, \lambda, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq \left(\frac{f(a) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(x_0 - ma)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (x_0 - a) J_{\xi-1, a^+} (x_0; p) \\ & \quad + \left(\frac{f(b) + m\alpha f(x_0/m)}{\alpha + 1} - \frac{\lambda\alpha(mb - x_0)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (b - x_0) J_{\xi-1, b^-} (x_0; p), \end{aligned} \tag{27}$$

$$x_0 \in [a, b].$$

Remark 5. Inequality (27) provides the refinement of Corollary 2.1 in [1].

Corollary 2. *Along with assumptions of Theorem 1, if $f \in L_\infty[a, b]$, then the following inequality is obtained:*

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, \lambda, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \eta, \lambda, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq \frac{\|f\|_\infty (1 + m\alpha)}{\alpha + 1} \left[(x_0 - a) J_{\xi-1, a^+} (x_0; p) + (b - x_0) J_{\eta-1, b^-} (x_0; p) \right] \\ & \quad - \left[\frac{\lambda\alpha(x_0 - ma)^2 (x_0 - a)}{m(\alpha + 1)(2\alpha + 1)} J_{\xi-1, a^+} (x_0; p) + \frac{\lambda\alpha(mb - x_0)^2 (b - x_0)}{m(\alpha + 1)(2\alpha + 1)} J_{\eta-1, b^-} (x_0; p) \right]. \end{aligned} \tag{28}$$

Corollary 3. *For $\xi = \eta$ in (28), we get the following result:*

$$\begin{aligned} & \left(\epsilon_{\mu, \xi, \lambda, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(\epsilon_{\mu, \xi, \lambda, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) \\ & \leq \frac{\|f\|_\infty (1 + m\alpha)}{\alpha + 1} \left((x_0 - a) J_{\xi-1, a^+} (x_0; p) + (b - x_0) J_{\xi-1, b^-} (x_0; p) \right) \\ & \quad - \left(\frac{\lambda\alpha(x_0 - ma)^2 (x_0 - a)}{m(\alpha + 1)(2\alpha + 1)} J_{\xi-1, a^+} (x_0; p) + \frac{\lambda\alpha(mb - x_0)^2 (b - x_0)}{m(\alpha + 1)(2\alpha + 1)} J_{\xi-1, b^-} (x_0; p) \right). \end{aligned} \tag{29}$$

Theorem 5. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is strongly (α, m) -convex, then for*

$\xi, \eta \geq 1$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned}
& \left| \left(e_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(e_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) - \left(J_{\xi-1, a^+} (x_0; p) f(a) + J_{\eta-1, b^-} (x_0; p) f(b) \right) \right| \\
& \leq \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha + 1} - \frac{\lambda\alpha(x_0 - ma)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (x_0 - a) J_{\xi-1, a^+} (x_0; p) \\
& \quad + \left(\frac{|f'(b)| + m\alpha |f'(x_0/m)|}{\alpha + 1} - \frac{\lambda\alpha(mb - x_0)^2}{m(\alpha + 1)(2\alpha + 1)} \right) (b - x_0) J_{\eta-1, b^-} (x_0; p),
\end{aligned} \tag{30}$$

$$x_0 \in [a, b].$$

Proof. As $x_0 \in [a, b]$ and $t \in [a, x_0)$, by using strongly (α, m) -convexity of $|f'|$, we have

$$\begin{aligned}
|f'(t)| & \leq \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha |f'(a)| + m \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left| f' \left(\frac{x_0}{m} \right) \right| \\
& \quad - \lambda m \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left(\frac{x_0 - ma}{m} \right)^2.
\end{aligned} \tag{31}$$

From (31), one can have

$$\begin{aligned}
& \int_a^{x_0} (x_0 - t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - t)^\mu; p) f'(t) dt \\
& \leq (x_0 - a)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) \\
& \quad \left(\left(\frac{x_0 - t}{x_0 - a} \right)^\alpha |f'(a)| + m \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left| f' \left(\frac{x_0}{m} \right) \right| - \lambda m \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left(\frac{x_0 - ma}{m} \right)^2 \right).
\end{aligned} \tag{32}$$

After integrating above inequality over $[a, x_0]$, we obtain

$$\begin{aligned}
& \int_a^{x_0} (x_0 - t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - t)^\mu; p) f'(t) dt \\
& \leq (x_0 - a)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) \left(|f'(a)| \int_a^{x_0} \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha dt \right. \\
& \quad \left. + m \left| f' \left(\frac{x_0}{m} \right) \right| \int_a^{x_0} \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) dt \right. \\
& \quad \left. - \frac{\lambda}{m} (x_0 - ma)^2 \int_a^{x_0} \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) dt \right) \\
& = \frac{(x_0 - a)^\xi E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p)}{\alpha + 1} \left(|f'(a)| + m\alpha \left| f' \left(\frac{x_0}{m} \right) \right| - \frac{\lambda\alpha(x_0 - ma)^2}{m(2\alpha + 1)} \right).
\end{aligned} \tag{34}$$

The left-hand side of (34) is calculated as follows:

$$\int_a^{x_0} (x_0 - t)^{\xi-1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(x_0 - t)^\mu; p) f'(t) dt. \tag{35}$$

$$\begin{aligned}
& \left| f'(t) \leq \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha |f'(a)| + m \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left| f' \left(\frac{x_0}{m} \right) \right| \right. \\
& \quad \left. - \lambda m \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \left(1 - \left(\frac{x_0 - t}{x_0 - a} \right)^\alpha \right) \left(\frac{x_0 - ma}{m} \right)^2 \right|.
\end{aligned} \tag{32}$$

The product of (19) and (32) gives the following inequality:

Put $x_0 - t = z$, that is, $t = x_0 - z$; also using the derivative property (7) of Mittag-Leffler function, we have

$$\int_0^{x_0-a} z^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f'(x_0-z) dz$$

$$= (x_0-a)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x_0-a)^\mu; p) f(a) - \int_0^{x_0-a} z^{\xi-2} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f(x_0-z) dz. \tag{36}$$

Now, put $x_0 - z = t$ in the second term of the right-hand side of the above equation, and then using (8), we obtain

Therefore, (34) takes the following form:

$$\int_0^{x_0-a} z^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega z^\mu; p) f'(x_0-z) dz$$

$$= (x_0-a)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(x_0-a)^\mu; p) f(a) \tag{37}$$

$$- \left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} f \right) (x_0; p).$$

$$\left(J_{\xi-1,a^+}(x_0; p) \right) f(a) - \left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} f \right) (x_0; p)$$

$$\leq (x_0-a) J_{\xi-1,a^+}(x_0; p) \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)} \right). \tag{38}$$

Also, from (31), one can have

$$f'(t) \geq - \left(\left(\frac{x_0-t}{x_0-a} \right)^\alpha |f'(a)| + m \left(1 - \left(\frac{x_0-t}{x_0-a} \right)^\alpha \right) \left| f' \left(\frac{x_0}{m} \right) \right| \right)$$

$$- \lambda m \left(\frac{x_0-t}{x_0-a} \right)^\alpha \left(1 - \left(\frac{x_0-t}{x_0-a} \right)^\alpha \right) \left(\frac{x_0-ma}{m} \right)^2. \tag{39}$$

Following the same procedure as we did for (32), one can obtain

$$\left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} f \right) (x_0; p) - J_{\xi-1,a^+}(x_0; p) f(a)$$

$$\leq (x_0-a) J_{\xi-1,a^+}(x_0; p) \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)} \right). \tag{40}$$

From (38) and (40), we obtain

$$\left| \left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} f \right) (x_0; p) - J_{\xi-1,a^+}(x_0; p) f(a) \right|$$

$$\leq (x_0-a) J_{\xi-1,a^+}(x_0; p) \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha(x_0-ma)^2}{m(\alpha+1)(2\alpha+1)} \right). \tag{41}$$

Now, we let $x_0 \in [a, b]$ and $t \in (x_0, b]$. Then, by using strongly (α, m) -convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{t-x_0}{b-x_0}\right)^\alpha |f'(b)| + m \left(1 - \left(\frac{t-x_0}{b-x_0}\right)^\alpha\right) \left|f'\left(\frac{x_0}{m}\right)\right| - \lambda m \left(\frac{t-x_0}{b-x_0}\right)^\alpha \left(1 - \left(\frac{t-x_0}{b-x_0}\right)^\alpha\right) \left(\frac{mb-x_0}{m}\right)^2. \tag{42}$$

On the same lines as we have done for (19), (32), and (39), one can get, from (23) and (12), the following inequality:

$$\left| \left(e_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) - J_{\eta-1, b^-} (x_0; p) f(b) \right| \leq (b-x_0) J_{\eta-1, b^-} (x_0; p) \left(\frac{|f'(b)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha (mb-x_0)^2}{m(\alpha+1)(2\alpha+1)} \right). \tag{43}$$

From inequalities (41) and (43), (30) is obtained. \square

Remark 6

- (i) Inequality (30) provides the refinement of Theorem 2.2 [1]
- (ii) If $\alpha = 1$ in (30), then result for strongly m -convex will be obtained
- (iii) If $\alpha = m = 1$ and $\lambda = 0$ in (30), then Corollary 2 [16] is obtained

- (iv) If $\alpha = m = 1$, $\lambda = 0$, and $\omega = p = 0$ in (30), then Theorem 2 [17] is obtained
- (v) If $\alpha = 1$ and $t = t^s$ in (30), then Theorem 5 in [18] is obtained

Corollary 4. *If we put $\xi = \eta$ in (30), then the following inequality is obtained:*

$$\left| \left(e_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (x_0; p) + \left(e_{\mu, \xi+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (x_0; p) - \left(J_{\xi-1, a^+} (x_0; p) f(a) + J_{\xi-1, b^-} (x_0; p) f(b) \right) \right| \leq \left(\frac{|f'(a)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha (x_0-ma)^2}{m(\alpha+1)(2\alpha+1)} \right) (x_0-a) J_{\xi-1, a^+} (x_0; p) + \left(\frac{|f'(b)| + m\alpha |f'(x_0/m)|}{\alpha+1} - \frac{\lambda\alpha (mb-x_0)^2}{m(\alpha+1)(2\alpha+1)} \right) (b-x_0) J_{\xi-1, b^-} (x_0; p), \tag{44}$$

$$x_0 \in [a, b].$$

Remark 7. Inequality (44) provides the refinement of Corollary 2.2 in [8].

To prove our next result, we consider following lemma.

Lemma 2. *Let $f: [a, b] \rightarrow \mathbb{R}$, $a < mb$, be strongly (α, m) -convex function. If $f(a+mb-x_0/m) = f(x_0)$, $m \neq 0$ and $(\alpha, m) \in [0, 1]^2$; then, the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq f(x_0) \left(\frac{1}{2^\alpha} + m \left(1 - \frac{1}{2^\alpha}\right) \right) - \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}} \right) \times (a+mb-x_0-mx_0)^2. \tag{45}$$

Proof. As f is strongly (α, m) -convex function, we have

$$f\left(\frac{a+mb}{2}\right) \leq \frac{f((1-t)a+mtb)}{2^\alpha} + m \left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{ta+m(1-t)b}{m}\right) - \lambda m \left(\frac{2^\alpha - 1}{2^{2\alpha}}\right) \left(\frac{ta+m(1-t)b}{m} - ((1-t)a+mtb)\right)^2. \tag{46}$$

Let $x_0 = a(1 - t) + mtb$, and we have $a + mb - x_0 = ta + m(1 - t)b$.

$$f\left(\frac{a + mb}{2}\right) \leq \frac{f(x_0)}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right)f\left(\frac{a + mb - x_0}{m}\right) - \frac{\lambda}{m}\left(\frac{2^\alpha - 1}{2^{2\alpha}}\right)(a + mb - x_0 - mx_0)^2. \tag{47}$$

Hence, by using $f(a + mb - x_0/m) = f(x_0), m \neq 0$, inequality (45) can be obtained. \square

Theorem 6. Let $f: [a, b] \rightarrow \mathbb{R}, a > b$, be a real-valued function. If f is positive, strongly (α, m) -convex, and $f(a + mb - x_0/m) = f(x_0), m \neq 0$, then, for $\xi, \eta > 0$, the following fractional integral inequality for generalized integral operators (8) and (9) holds:

$$\begin{aligned} & \frac{1}{1/2^\alpha + m(1 - 1/2^\alpha)} \left(f\left(\frac{a + mb}{2}\right) (J_{\eta+1, b^-}(a; p) + J_{\xi+1, a^+}(b; p)) + \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}} \right) (M1 + M2) \right) \\ & \leq \left(\epsilon_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) + \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \\ & \leq (J_{\eta-1, b^-}(a; p) + J_{\xi-1, a^+}(b; p)) (b - a)^2 \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} - \frac{\lambda\alpha(a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right), \end{aligned} \tag{48}$$

where

$$\begin{aligned} M1 &= (b - a)^{\eta+2} J_{\eta+1, b^-}(a; p) + 2(1 + m)(b - a)^{\eta+1} J_{\eta+2, b^-}(a; p) + 2(1 + m)^2 J_{\eta+3, b^-}(a; p), \\ M2 &= (b - a)^{\xi+2} J_{\xi+1, a^+}(b; p) + 2(1 + m)(b - a)^{\xi+1} J_{\xi+2, a^+}(b; p) + 2(1 + m)^2 J_{\xi+3, a^+}(b; p). \end{aligned} \tag{49}$$

Proof. For $x_0 \in [a, b]$, we have

$$\begin{aligned} (x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c}(\omega(x_0 - a)^\mu; p) &\leq (b - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c}(\omega(b - a)^\mu; p) \\ &\cdot (\omega(b - a)^\mu; p), \eta > 0. \end{aligned} \tag{50}$$

As f is strongly (α, m) -convex, so, for $x_0 \in [a, b]$, we have

$$\begin{aligned} f(x_0) &\leq \left(\frac{x_0 - a}{b - a}\right)^\alpha f(b) + m\left(1 - \left(\frac{x_0 - a}{b - a}\right)^\alpha\right) f\left(\frac{a}{m}\right) \\ &- \lambda m \left(\frac{x_0 - a}{b - a}\right)^\alpha \left(1 - \left(\frac{x_0 - a}{b - a}\right)^\alpha\right) \left(\frac{a - mb}{m}\right)^2. \end{aligned} \tag{51}$$

By multiplying (50) and (51) and then integrating over $[a, b]$, we obtain

$$\begin{aligned} & \int_a^b (x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c}(\omega(x_0 - a)^\mu; p) f(x_0) dx_0 \\ & \leq (b - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c}(\omega(b - a)^\mu; p) \left(f(b) \int_a^b \left(\frac{x_0 - a}{b - a}\right)^\alpha dx_0 \right. \\ & \quad \left. + m f\left(\frac{a}{m}\right) \int_a^b \left(1 - \left(\frac{x_0 - a}{b - a}\right)^\alpha\right) dx_0 - \frac{\lambda}{m} (a - mb)^2 \right. \\ & \quad \left. \times \int_a^b \left(\frac{x_0 - a}{b - a}\right)^\alpha \left(1 - \left(\frac{x_0 - a}{b - a}\right)^\alpha\right) dx_0 \right). \end{aligned} \tag{52}$$

From which we have

$$\begin{aligned} & \left(\epsilon_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) \leq (b - a)^{\eta+1} E_{\mu, \eta, l}^{\gamma, \delta, k, c}(\omega(b - a)^\mu; p) \\ & \times \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} - \frac{\lambda\alpha(a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right), \end{aligned} \tag{53}$$

that is,

$$\left(\epsilon_{\mu, \eta+1, l, \omega, b^-}^{\gamma, \delta, k, c} f \right) (a; p) \leq (b-a)^2 J_{\eta-1, b^-} (a; p) \times \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} - \frac{\lambda\alpha(a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right). \tag{54}$$

$$(b-x_0)^\xi E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(b-x_0)^\mu; p) \leq (b-a)^\xi E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(b-a)^\mu; p), \xi > 0. \tag{55}$$

Now, on the contrary, for $x_0 \in [a, b]$, we have

By multiplying (51) and (55) and then integrating over $[a, b]$, we obtain

$$\begin{aligned} & \int_a^b (b-x_0)^\xi E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(b-x_0)^\mu; p) f(x_0) dx_0 \\ & \leq (b-a)^\xi E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(b-a)^\mu; p) \left(f(b) \int_a^b \left(\frac{x_0-a}{b-a} \right)^\alpha dx_0 \right. \\ & \quad \left. + mf \left(\frac{a}{m} \right) \int_a^b \left(1 - \left(\frac{x_0-a}{b-a} \right)^\alpha \right) dx_0 \right. \\ & \quad \left. - \frac{\lambda}{m} (a - mb)^2 \int_a^b \left(\frac{x_0-a}{b-a} \right)^\alpha \left(1 - \left(\frac{x_0-a}{b-a} \right)^\alpha \right) dx_0 \right). \end{aligned} \tag{56}$$

From which we have

$$\left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \leq (b-a)^{\xi+1} E_{\mu, \xi, l}^{\gamma, \delta, k, c} (\omega(b-a)^\mu; p) \times \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} - \frac{\lambda\alpha(a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right), \tag{57}$$

$$\left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f \right) (b; p) \leq (b-a)^2 J_{\xi-1, a^+} (b; p) \times \left(\frac{f(b) + m\alpha f(a/m)}{\alpha + 1} - \frac{\lambda\alpha(a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right). \tag{58}$$

Adding (54) and (58), we get the second inequality of (48). Multiplying (45) with $(x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p)$ and integrating over $[a, b]$, we obtain

that is,

$$\begin{aligned} & f \left(\frac{a + mb}{2} \right) \int_a^b (x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) dx_0 \leq \left(\frac{1}{2^\alpha} + m \left(1 - \frac{1}{2^\alpha} \right) \right) \\ & \int_a^b (x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) f(x_0) dx_0 - \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}} \right) \\ & \times \int_a^b (x_0 - a)^\eta E_{\mu, \eta, l}^{\gamma, \delta, k, c} (\omega(x_0 - a)^\mu; p) (a + mb - x_0 - mx_0)^2 dx_0. \end{aligned} \tag{59}$$

By using (9) and (12), we obtain

$$\begin{aligned}
 f\left(\frac{a+mb}{2}\right) J_{\eta+1,b^-}(a;p) &\leq \left(\frac{1}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right)\right) \\
 &\times \left(\epsilon_{\mu,\eta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(a;p) - \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}}\right) \\
 &\times \left(\epsilon_{\mu,\eta+1,l,\omega,b^-}^{\gamma,\delta,k,c} (a+mb - x_0 - mx_0)^2\right)(a;p).
 \end{aligned} \tag{60}$$

The integral operator appearing in the last term is computed as follows:

$$\begin{aligned}
 &\left(\epsilon_{\mu,\eta+1,l,\omega,b^-}^{\gamma,\delta,k,c} (a+mb - x_0 - mx_0)^2\right)(a;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \\
 &\times \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{1}{(l)_{n\delta}} \int_b^a (x_0 - a)^{\eta+\mu n} (a+mb - x_0 - mx_0)^2 dx_0 \\
 &= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{1}{(l)_{n\delta}} \left(\frac{(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)}\right. \\
 &\left. + \frac{2(1+m)(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)(\eta+\mu n+2)} + \frac{2(1+m)^2(b-a)^{\eta+\mu n+3}}{(\eta+\mu n+1)(\eta+\mu n+2)(\eta+\mu n+3)}\right) \\
 &= (b-a)^{\eta+2} J_{\eta+1,b^-}(a;p) + 2(1+m)(b-a)^{\eta+1} J_{\eta+2,b^-}(a;p) \\
 &\quad + 2(1+m)^2 J_{\eta+3,b^-}(a;p).
 \end{aligned} \tag{61}$$

Now, inequality (43) becomes

$$\begin{aligned}
 f\left(\frac{a+mb}{2}\right) J_{\eta+1,b^-}(a;p) &\leq \left(\frac{1}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right)\right) \\
 &\times \left(\epsilon_{\mu,\eta+1,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(a;p) - \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}}\right) \left((b-a)^{\eta+2} J_{\eta+1,b^-}(a;p)\right. \\
 &\left. + 2(1+m)(b-a)^{\eta+1} J_{\eta+2,b^-}(a;p) + 2(1+m)^2 J_{\eta+3,b^-}(a;p)\right).
 \end{aligned} \tag{62}$$

By multiplying (45) with $(b-x_0)^\xi E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(b-x_0)^\mu; p)$ and integrating over $[a, b]$, also using (8) and (12), we obtain

$$\begin{aligned}
 f\left(\frac{a+mb}{2}\right) J_{\xi+1,a^+}(b;p) &\leq \left(\frac{1}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right)\right) \left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(b;p) \\
 &- \frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2\alpha}}\right) \left(\epsilon_{\mu,\xi+1,l,\omega,a^+}^{\gamma,\delta,k,c} (a+mb - x_0 - mx_0)^2\right)(b;p).
 \end{aligned} \tag{63}$$

The integral operator appearing in the last term is computed as follows:

$$\begin{aligned} & \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} (a + mb - x_0 - mx_0)^2 \right) (b; p) \\ &= (b - a)^{\xi+2} J_{\xi+1, a^+} (b; p) + 2(1 + m)(b - a)^{\xi+1} J_{\xi+2, a^+} (b; p) \\ & \quad + 2(1 + m)^2 J_{\xi+3, a^+} (b; p). \end{aligned} \tag{64}$$

Now, inequality (63) becomes

$$\begin{aligned} & f\left(\frac{a + mb}{2}\right) J_{\xi+1, a^+} (b; p) \leq \left(\frac{1}{2^\alpha} + m\left(1 - \frac{1}{2^\alpha}\right)\right) \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f\right) (b; p) \\ & \quad - \frac{\lambda}{m} \left(\frac{2^\alpha}{2^{2^\alpha} - 1}\right) \left((b - a)^{\xi+2} J_{\xi+1, a^+} (b; p)\right. \\ & \quad \left. + 2(1 + m)(b - a)^{\xi+1} J_{\xi+2, a^+} (b; p) + 2(1 + m)^2 J_{\xi+3, a^+} (b; p)\right). \end{aligned} \tag{65}$$

By adding (62) and (65), first inequality of (48) can be obtained. \square

Remark 8

- (i) Inequality (48) provides the refinement of Theorem 2.3 in [1]
- (ii) If $\alpha = 1$ in (48), then result for strongly m -convex will be obtained

- (iii) If $\alpha = m = 1$ and $\lambda = 0$ in (48), then Corollary 2 in [16] is obtained
- (iv) If $\alpha = m = 1$, $\lambda = 0$, and $\omega = p = 0$ in (48), then Theorem 2 in [17] is obtained
- (v) If $\alpha = 1$ and $t = t^s$ in (48), then Theorem 7 in [18] is obtained

Corollary 5. *If we put $\xi = \eta$ in (48), then the following inequality is obtained:*

$$\begin{aligned} & \frac{1}{1/2^\alpha + m(1 - 1/2^\alpha)} \left(f\left(\frac{a + mb}{2}\right) (J_{\xi+1, b^-} (a; p) + J_{\xi+1, a^+} (b; p)) \right. \\ & \quad \left. + \left(\frac{\lambda}{m} \left(\frac{2^\alpha - 1}{2^{2^\alpha}}\right) (M1 + M2)\right) \right) \leq \left(\epsilon_{\mu, \xi+1, l, \omega, b^-}^{\gamma, \delta, k, c} f\right) (a; p) + \left(\epsilon_{\mu, \xi+1, l, \omega, a^+}^{\gamma, \delta, k, c} f\right) (b; p) \\ & \leq (J_{\xi-1, b^-} (a; p) + J_{\xi-1, a^+} (b; p)) (b - a)^2 \left(\frac{f(b) + \max f(a/m)}{\alpha + 1} - \frac{\lambda \alpha (a - mb)^2}{m(\alpha + 1)(2\alpha + 1)} \right). \end{aligned} \tag{66}$$

Remark 9. Inequality (66) provides the refinement of Corollary 2.3 in [1].

3. Concluding Remarks

The presented results are the refinements of the bounds of generalized fractional integral operators given in (8) and (9) for strongly (α, m) -convex functions. From the presented results, one can obtain already proved results for convex, m -convex, and (α, m) -convex functions. Moreover, the refinements of some known fractional versions of the Hadamard inequality are also given.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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