Research Article

Fractional Birkhoffian Mechanics Based on Quasi-Fractional Dynamics Models and Its Noether Symmetry

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This paper focuses on the exploration of fractional Birkhoffian mechanics and its fractional Noether theorems under quasi-fractional dynamics models. The quasi-fractional dynamics models under study are nonconservative dynamics models proposed by El-Nabulsi, including three cases: extended by Riemann–Liouville fractional integral (abbreviated as ERLFI), extended by exponential fractional integral (abbreviated as EEFI), and extended by periodic fractional integral (abbreviated as EPFI). First, the fractional Pfaff–Birkhoff principles based on quasi-fractional dynamics models are proposed, in which the Pfaff action contains the fractional-order derivative terms, and the corresponding fractional Birkhoff's equations are obtained. Second, the Noether symmetries and conservation laws of the systems are studied. Finally, three concrete examples are given to demonstrate the validity of the results.

1. Introduction


Fractional calculus is an important mathematical tool in science and engineering [25–28]. In recent decades, the research of fractional calculus has developed greatly, and its application fields have expanded to automatic control, quantum mechanics, and mechanical systems [29–35]. Riewe [36, 37] introduced the fractional variational problem for the first time in the study of nonconservative mechanics. In 2005, El-Nabulsi established a dynamical model of nonconservative systems under the framework of fractional calculus [38] based on the definition of Riemann–Liouville fractional integral (ERLFI). El-Nabulsi expanded the idea of
dynamics modeling and successively put forward the dynamical models of nonconservative systems, which are extended by exponentially fractional integral (EEFI) and extended by periodic laws fractional integral (EPFI) \[39, 40\], respectively. The equations obtained from quasi-fractional dynamics models are similar to dynamical equations of classical conservative systems, which contain the generalized fractional external forces corresponding to dissipative forces, but the term with the fractional derivative does not show up. Different from other models, the fractional time integration of quasi-fractional dynamics models only needs one parameter. In this way, it simplifies the calculation of complex fractional calculus and provides a modeling method for nonconservative systems. Therefore, the quasi-fractional dynamics models can be used to study complex dynamical systems more conveniently. Frederico and Torres \[41\] first presented fractional Noether’s theorems. Since then, studies on fractional Noether symmetry of quasi-fractional dynamical systems based on quasi-fractional dynamical models have been obtained \[52–55\]. However, most of the previous dynamical equations and Noether conservation laws have only integral-order derivative terms. Here, we will further illustrate the application of the methods and results in this text, three examples are given in Section 5. In Section 6, we come to the conclusions.

2. Fractional Birkhoff’s Equations and Variation of Fractional Pfaff Action under Quasi-Fractional Dynamics Models

For an introduction to fractional calculus and its basic theory, please refer to the monographs \[27, 28\].

2.1. Fractional Birkhoffian System Based on ERLFI. We consider a fractional Birkhoffian system determined by Birkhoff’s variables \(a^\mu (\tau, a^\mu), \beta\) of the Birkhoffian is \(B = B(\tau, a^\mu), \beta\) is the order of fractional derivative, and \(0 \leq \beta < 1\).

Under the model of ERLFI, we define the Pfaff action as

\[
S_R = \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a^\mu) \frac{d^\alpha}{dt^\alpha} - B(\tau, a^\mu) \right] (t - \tau)^{\alpha - 1} dt,
\]

where \(\frac{d^\alpha}{dt^\alpha} (\mu = 1, 2, \ldots, 2n)\) is the fractional derivative term.

The variational principle,

\[
\delta S_R = 0,
\]

with commutative relation,

\[
\delta_\alpha \frac{d^\beta}{dt^\beta} = a^\mu \frac{d^\beta}{dt^\beta} \delta a^\mu,
\]

and boundary conditions,

\[
a^\mu |_{t=a} = a^\mu_1,
\]

\[
a^\mu |_{t=b} = a^\mu_2
\]

is called the fractional Pfaff–Birkhoff principle based on ERLFI.

According to principle (2), we drive

\[
\left( \frac{\partial R_\mu}{\partial a^\mu} \frac{d^\beta}{dt^\beta} - \frac{\partial B}{\partial a^\mu} \right) (t - \tau)^{\alpha - 1} + \frac{d^\beta}{dt^\beta} \left[ R_\mu(\tau, a^\mu) \right] = 0,
\]

\(\mu = 1, 2, \ldots, 2n\).

\[
(5)
\]

Equation (5) is the fractional Birkhoff’s equations based on ERLFI.

If \(\beta \to 1\), equation (5) becomes Birkhoff’s equations based on ERLFI. If \(\beta \to 1\) and \(\alpha \to 1\), equation (5) becomes classical Birkhoff’s equations \[58\].

Take the infinitesimal transformations:
\[\begin{align*}
\mathcal{T} &= \tau + \Delta \tau, \\
\mathcal{T}'(\tau) &= a^\mu(\tau) + \Delta a^\mu, \quad (\mu = 1, 2, \ldots, 2n),
\end{align*}\]
and their first-order extensions
\[\begin{align*}
\bar{\mathcal{T}} &= \tau + \varepsilon_0 \xi_0^\sigma(\tau, a^\sigma), \\
\bar{\mathcal{T}}'(\tau) &= a^\mu(\tau) + \varepsilon_0 \xi_0^\sigma(\tau, a^\sigma), \quad (\mu = 1, 2, \ldots, 2n),
\end{align*}\]
where \(\varepsilon_0\) is the infinitesimal parameter and \(\xi_0^\sigma\) and \(\xi_0^\mu\) are the generating functions.

Under transformation (6), the Pfaff action (1) is transformed into
\[\begin{align*}
S_R(\bar{\mathcal{T}}) &= \frac{1}{\Gamma(\alpha)} \int_{\mathcal{T}_0}^b \left[ R_\mu(\bar{\mathcal{T}}, \bar{\mathcal{T}}') a^\mu \Delta a^\mu - B(\bar{\mathcal{T}}, \bar{\mathcal{T}}') \right] (t - \bar{t})^{(\alpha - 1)} d\bar{t} \\
&- \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a') a^\mu \Delta a^\mu - B(\tau, a') \right] (t - \tau)^{\alpha - 1} d\tau.
\end{align*}\]
And, we have
\[\begin{align*}
S_R(\bar{\mathcal{T}}) - S_R(\mathcal{T}) &= \frac{1}{\Gamma(\alpha)} \int_{\mathcal{T}_0}^b \left[ R_\mu(\bar{\mathcal{T}}, \bar{\mathcal{T}}') a^\mu \Delta a^\mu - B(\bar{\mathcal{T}}, \bar{\mathcal{T}}') \right] (t - \bar{t})^{\alpha - 1} d\bar{t} \\
&- \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a') a^\mu \Delta a^\mu - B(\tau, a') \right] (t - \tau)^{\alpha - 1} d\tau
\end{align*}\]
Let \(\Delta S_R\) be nonisochronous variation of \(S_R\), which is the main line part of \(S_R(\bar{\mathcal{T}}) - S_R(\mathcal{T})\) relative to \(\varepsilon\), and we obtain
\[\begin{align*}
\Delta S_R &= \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left\{ \left( \frac{\partial R_\mu}{\partial a^\mu} a^\mu a^\mu - \frac{\partial B}{\partial t} \right) (t - \tau)^{\alpha - 1} \Delta a^\mu \right. \\
&+ \left. \left( \frac{\partial R_\mu}{\partial \tau} a^\mu a^\mu - \frac{\partial B}{\partial \tau} \right) (t - \tau)^{\alpha - 1} \Delta \tau \right\} d\tau \\
&+ \left( R_\mu a D^\beta a^\mu - B(\tau) \right) (t - \tau)^{\alpha - 1} \left( \frac{d}{d\tau} - \frac{d}{d(t - \tau)} \right) \Delta \tau \\
&+ \frac{R_\mu}{\alpha - 1} \Delta \tau \\
&+ \frac{R_\mu}{\alpha - 1} \Delta \tau \\
&+ \frac{R_\mu}{\alpha - 1} \Delta \tau \\
&+ \frac{R_\mu}{\alpha - 1} \Delta \tau \\
&+ \frac{R_\mu}{\alpha - 1} \Delta \tau.
\end{align*}\]
Equations (10) and (13) are two mutually equivalent formulas derived from Pfaff action (1).

2.2. Fractional Birkhoffian System Based on EEFI. Under the model of EEFI, we define the Pfaff action as

\[
S_E = \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \mathcal{R}_{\mu}(\tau, \alpha') D_{\mu}^a \alpha - B(\tau, \alpha') \right] \left( \cosh t - \cosh \tau \right)^{a-1} d\tau.
\]  
(14)

The fractional Pfaff–Birkhoff principle is

\[
\delta S_E = 0,
\]  
(15)

under commutative relation,

\[
\delta_{\mu} D_{\mu} \alpha = D_{\mu} \delta \alpha,
\]  
(16)

and boundary conditions,

\[
\alpha(\tau = a) = a_1, \\
\dot{\alpha}(\tau = b) = a_2.
\]  
(17)

The fractional Birkhoff's equations are

\[
\frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \left( \cosh t - \cosh \tau \right)^{a-1} \\
+ \tau D_{\beta} \left[ R_{\mu}(\cosh t - \cosh \tau)^{a-1} \right] = 0, \quad (\mu = 1, 2, \ldots, 2n).
\]  
(18)

If \( \beta \to 1 \), equation (18) becomes Birkhoff's equations based on EEFI. If \( \beta \to 1 \) and \( \alpha \to 1 \), equation (18) becomes classical Birkhoff’s equations [58].

According to formula (6), action (14) is transformed into

\[
S_E(\tau) = \frac{1}{\Gamma(a)} \int_{a}^{b} \mathcal{R}_{\mu}(\tau, \alpha') D_{\mu} \alpha - B(\tau, \alpha') \left( \cosh t - \cosh \tau \right)^{a-1} d\tau,
\]  
(19)

and we have

\[
S_E(\tau) - S_E(\gamma) = \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \mathcal{R}_{\mu}(\tau, \alpha') D_{\mu} \alpha - B(\tau, \alpha') \right] \left( \cosh t - \cosh \tau \right)^{a-1} d\tau \\
- \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \mathcal{R}_{\mu}(\gamma, \alpha') D_{\mu} \alpha - B(\gamma, \alpha') \right] \left( \cosh t - \cosh \gamma \right)^{a-1} d\tau \\
= \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \left( D_{\mu} \alpha + \mu D_{\mu} \Delta t \alpha_{\mu} + D_{\mu} (\Delta \alpha_{\mu}) \right) - B(\tau + \Delta \tau, \alpha_{\mu} + \Delta \alpha_{\mu}) \right] \left( \cosh t - \cosh \tau \right)^{a-1} \left( 1 + \frac{d}{d\tau} \Delta \tau \right) \\
- \left[ \mathcal{R}_{\mu}(\tau, \alpha') D_{\mu} \alpha - B(\tau, \alpha') \right] \left( \cosh t - \cosh \tau \right)^{a-1} d\tau.
\]  
(20)

So, the nonisochronous variation \( \Delta S_E \) of action \( S_E \) is

\[
\Delta S_E = \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \cosh t - \cosh \tau \right]^{a-1} \Delta \alpha \\
+ \frac{\partial R_{\mu}}{\partial \tau} D_{\mu} \alpha - \frac{\partial B}{\partial \tau} \cosh t - \cosh \tau\right]^{a-1} \Delta \tau \\
+ R_{\mu} D_{\mu} \alpha - B \cosh t - \cosh \tau\right]^{a-1} \frac{d}{d\tau} \Delta \tau \\
+ R_{\mu} D_{\mu} \alpha - B \cosh t - \cosh \tau\right]^{a-1} \Delta t \\
+ \alpha - \sin \theta \cosh \tau - \cosh \tau\right]^{a-1} \Delta t \right] d\tau.
\]  
(21)

Equation (21) can also be written as

\[
\Delta S_E = \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \frac{d}{d\tau} \left[ R_{\mu} D_{\mu} \alpha - B \cosh t - \cosh \tau\right]^{a-1} \Delta \tau \\
+ \frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \cosh t - \cosh \tau\right]^{a-1} \Delta \alpha \\
- \frac{\partial R_{\mu}}{\partial \tau} D_{\mu} \alpha - \frac{\partial B}{\partial \tau} \cosh t - \cosh \tau\right]^{a-1} ds \\
+ \left[ \frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \cosh t - \cosh \tau\right]^{a-1} \Delta \alpha \\
+ r D_{\beta} \left[ R_{\mu} \cosh t - \cosh \tau\right]^{a-1} \Delta \alpha \right] d\tau.
\]  
(22)

By using formula (7), we have

\[
\Delta S_E = \frac{1}{\Gamma(a)} \int_{a}^{b} \left[ \frac{d}{d\tau} \left[ R_{\mu} D_{\mu} \alpha - B \cosh t - \cosh \tau\right]^{a-1} \xi_0 \\
+ \frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \cosh t - \cosh \tau\right]^{a-1} \xi_0 \\
- \xi_0 D_{\mu} \alpha - B \cosh t - \cosh \tau\right]^{a-1} ds \\
+ \left[ \frac{\partial R_{\mu}}{\partial \alpha^\mu} D_{\mu} \alpha - \frac{\partial B}{\partial \alpha^\mu} \cosh t - \cosh \tau\right]^{a-1} \xi_0 \\
+ r D_{\beta} \left[ R_{\mu} \cosh t - \cosh \tau\right]^{a-1} \xi_0 \right] d\tau.
\]  
(23)

Equations (21) and (23) are two mutually equivalent formulas derived from Pfaff action (14).
2.3. Fractional Birkhoffian System Based on EPFI. Under the model of EPFI, we define the Pfaff action as

\[ S_p = \frac{1}{\Gamma (\alpha) \lambda} \int_{a}^{b} \left[ R_{\mu} (\tau, a^{\alpha}) D^{\beta}_{\alpha} a^{\mu} - B (\tau, a^{\alpha}) \right] \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) d\tau, \]

(24)

The fractional Pfaff–Birkhoff principle is

\[ \delta S_p = 0, \]

(25)

under commutative relation,

\[ \delta_{a} D^{\beta}_{\alpha} a^{\mu} = a D^{\beta}_{\alpha} \delta a^{\mu}, \]

(26)

and boundary conditions,

\[ a^{\alpha} \big|_{\tau=a} = a^{\alpha}_{a}, \]
\[ a^{\alpha} \big|_{\tau=b} = a^{\alpha}_{b}. \]

(27)

The fractional Birkhoff's equations are

\[ \left( \frac{\partial R_{\mu}}{\partial a^{\alpha}} D^{\beta}_{\alpha} a^{\mu} - \frac{\partial B}{\partial a^{\mu}} \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \]
\[ + \int_{a}^{b} \left[ D^{\beta}_{\gamma} \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right] = 0, \quad (\mu = 1, 2, \ldots, 2n). \]

(28)

If \( \beta \rightarrow 1 \), equation (28) becomes Birkhoff's equations based on EPFI. If \( \beta \rightarrow 1 \) and \( \alpha \rightarrow 1 \), equation (28) becomes the classical Birkhoff's equations [58].

According to formula (6), action (24) is transformed into

\[ S_p (\psi) - S_p (\gamma) = \frac{1}{\Gamma (\alpha) \lambda} \int_{a}^{b} \left[ R_{\mu} (\tau, a^{\alpha}) D^{\beta}_{\alpha} a^{\mu} - B (\tau, a^{\alpha}) \right] \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) d\tau, \]

(29)

and we have

\[ \Delta S_p = \frac{1}{\Gamma (\alpha) \lambda} \int_{a}^{b} \left[ \left( \frac{\partial R_{\mu}}{\partial a^{\alpha}} D^{\beta}_{\alpha} a^{\mu} - \frac{\partial B}{\partial a^{\mu}} \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Delta a^{\tau} + \left( \frac{\partial R_{\mu}}{\partial \tau} D^{\beta}_{a} a^{\mu} - \frac{\partial B}{\partial \tau} \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Delta \tau \]
\[ + \left( R_{\mu a} D^{\beta}_{a} a^{\mu} - B \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) d\tau \]
\[ - (\alpha - 1) \cos \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right] d\tau. \]

(31)
Equation (31) can also be written as

\[
\Delta S_p = \frac{1}{\Gamma(\alpha)} \int_a^b \frac{d}{d\tau} \left[ (R_{\mu a} D_{\tau a}^\alpha a^\mu - B) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Delta \tau + \int_a^\tau \left( \frac{\partial R_{\mu a}}{\partial \alpha} D_{\tau a}^\alpha a^\mu - \frac{\partial B}{\partial a} \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right] d\tau
\]

By using formula (7), we have

\[
\Delta S_p = \frac{1}{\Gamma(\alpha)} \int_a^b \frac{d}{d\tau} \left[ (R_{\mu a} D_{\tau a}^\alpha a^\mu - B) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right] d\tau
\]

Equations (31) and (33) are two mutually equivalent formulas derived from Pfaff action (24).

3. Fractional Noether Symmetries under Quasi-Fractional Dynamics Models

Next, we will define the Noether symmetries of the system under three quasi-fractional dynamics models and establish their criteria.

3.1. Fractional Noether Symmetries Based on ERLFI

\[
\left( \frac{\partial R_{\mu a}}{\partial a} D_{\tau a}^\alpha a^\mu - \frac{\partial B}{\partial a} \right) \Delta a^\mu + \left( \frac{\partial R_{\mu a}}{\partial \alpha} D_{\tau a}^\alpha a^\mu - \frac{\partial B}{\partial a} \right) \Delta \alpha + \left( R_{\mu a} D_{\tau a}^\alpha a^\mu - B \right) \frac{d}{d\tau} \Delta \tau \\
+ R_{\mu} \left[ a D_{\tau}^\alpha (a^\mu \Delta \tau) - a D_{\tau}^\alpha (a^\mu \Delta \tau) \right] - \left( R_{\mu a} D_{\tau}^\alpha a^\mu - B \right) \frac{\alpha - 1}{t - \tau} \Delta \tau = 0,
\]

needs to be satisfied.

Equation (35) can be written as \( r \) equations:

\[
\left( \frac{\partial R_{\mu a}}{\partial a} D_{\tau a}^\alpha a^\mu - \frac{\partial B}{\partial a} \right) \varepsilon_\sigma^\alpha + \left( \frac{\partial R_{\mu a}}{\partial \alpha} D_{\tau a}^\alpha a^\mu - \frac{\partial B}{\partial a} \right) \varepsilon_\sigma^\alpha + \left( R_{\mu a} D_{\tau}^\alpha a^\mu - B \right) \varepsilon_\sigma^\alpha \]

\[
+ R_{\mu} \left[ a D_{\tau}^\alpha (a^\mu \varepsilon_\sigma^\alpha) - a D_{\tau}^\alpha (a^\mu \varepsilon_\sigma^\alpha) \right] - \left( R_{\mu a} D_{\tau}^\alpha a^\mu - B \right) \frac{\alpha - 1}{t - \tau} \varepsilon_\sigma^\alpha = 0, \quad (\sigma = 1, 2, \ldots, r).
\]
If \( r = 1 \), equation (36) gives the fractional Noether identity based on ERLFI.

\[
\frac{d}{dt} \left( \left( R_\mu D_\mu a^\mu - B \right) (t - \tau)^{\alpha - 1} \xi^\tau_0 \right) + \int_a^t \left[ R_\mu D_\mu \left( \xi^\tau - \dot{\xi}^\tau \xi^\tau_0 \right) (t - s)^{\alpha - 1} - \left( \xi^\tau_0 - \dot{\xi}^\tau \xi^\tau_0 \right) D_\mu \left[ R_\mu (t - s)^{\alpha - 1} \right] ds \right] \\
+ \left[ \frac{\partial R_\mu}{\partial \xi^\tau} D_\mu a^\mu - \frac{\partial B}{\partial a^\mu} \right] (t - \tau)^{\alpha - 1} + D_\mu \left( R_\mu (t - \tau)^{\alpha - 1} \right) \left( \dot{\xi}^\tau - \dot{\dot{\xi}}^\tau \xi^\tau_0 \right) = 0, \quad (\sigma = 1, 2, \ldots, r),
\]

need to be satisfied.

**Definition 2.** If the Pfaff action (1) satisfies the equality

\[
\Delta S_R = -\frac{1}{\Gamma(\alpha)} \int_a^b \frac{d}{dt} \left( \Delta \xi^\tau \right) d\tau,
\]

where \( \Delta = \epsilon_{\mu}G^\mu \) and \( G^\mu = G^\mu (\tau, a^\mu) \) is the gauge function, then transformation (6) is said to be Noether quasi-symmetric for system (5).

\[
\left( \frac{\partial R_\mu}{\partial \alpha^\mu} D_\mu a^\mu - \frac{\partial B}{\partial a^\mu} \right) \Delta a^\mu + \left( \frac{\partial R_\mu}{\partial \tau} - a D_\mu \left( a^\mu \right) \Delta \tau \right) \Delta a^\mu \\
\left[ a D_\mu \dot{a}^\mu + a D_\mu \dot{a}^\mu \Delta \tau \right] \\
\left[ a D_\mu \dot{a}^\mu + a D_\mu \dot{a}^\mu \Delta \tau \right]
\]

needs to be satisfied.

**Criterion 2.** If transformation (7) is Noether symmetric, then the following \( r \) equations,

**Criterion 3.** If transformation (6) is Noether quasi-symmetric, then the equation,

\[
\left( \frac{\partial R_\mu}{\partial \alpha^\mu} D_\mu a^\mu - \frac{\partial B}{\partial a^\mu} \right) \dot{\xi}^\tau_0 + \left( \frac{\partial R_\mu}{\partial \tau} - a D_\mu \left( a^\mu \right) \dot{\tau} \right) \dot{\xi}^\tau_0 \\
+ \left[ a D_\mu \dot{a}^\mu + a D_\mu \dot{a}^\mu \dot{\tau} \right] \\
- \left[ a D_\mu \dot{a}^\mu + a D_\mu \dot{a}^\mu \dot{\tau} \right]
\]

Equation (39) can be written as \( r \) equations:

If \( r = 1 \), equation (40) also gives the fractional Noether identity based on ERLFI.

**Criterion 4.** If transformation (7) is Noether quasi-symmetric, then the following \( r \) equations,

\[
\frac{d}{dt} \left( \left( R_\mu D_\mu a^\mu - b \right) (t - \tau)^{\alpha - 1} \xi^\tau_0 \right) + \int_a^t \left[ R_\mu D_\mu \left( \xi^\tau - \dot{\xi}^\tau \xi^\tau_0 \right) (t - s)^{\alpha - 1} - \left( \xi^\tau_0 - \dot{\xi}^\tau \xi^\tau_0 \right) D_\mu \left[ R_\mu (t - s)^{\alpha - 1} \right] ds \right] \\
+ \left[ \frac{\partial R_\mu}{\partial \xi^\tau} D_\mu a^\mu - \frac{\partial B}{\partial a^\mu} \right] (t - \tau)^{\alpha - 1} + D_\mu \left( R_\mu (t - \tau)^{\alpha - 1} \right) \left( \dot{\xi}^\tau - \dot{\dot{\xi}}^\tau \xi^\tau_0 \right) = 0, \quad (\sigma = 1, 2, \ldots, r),
\]

need to be satisfied.
3.2. Fractional Noether Symmetries Based on EEFI

**Definition 3.** If the Pfaff action (14) satisfies the equality
\[ \Delta S_E = 0, \]  
(42)
then transformation (6) is said to be Noether symmetric for system (18).

According to Definition 3, using formulas (21) and (23), we have

**Criterion 5.** If transformation (6) is Noether symmetric, then the equation,
\[ (\alpha - 1) \sinh \tau \cosh t - \cosh \tau \Delta \tau = 0, \]  
(43)
needs to be satisfied.

Equation (43) can be written as \( r \) equations:

\[ \left( \frac{\partial R_{\mu}}{\partial a^\nu} D_\tau^\mu a^\mu - \frac{\partial B}{\partial a^\nu} \right) \xi^\sigma + \left( \frac{\partial R_{\mu}}{\partial \tau} D_\tau^\mu a^\mu - \frac{\partial B}{\partial \tau} \right) \xi^\sigma + \left( R_{\mu a} D_\tau^\sigma a^\mu - B \right) \xi^\sigma = 0, \]  
(44)

If \( r = 1 \), equation (44) gives the fractional Noether identity based on EEFI.

According to Definition 4, using formulas (21) and (23), transformation (6) is said to be Noether quasi-symmetric for system (18).

**Criterion 6.** If transformation (7) is Noether symmetric, then the following \( r \) equations,
\[ \left( \frac{\partial R_{\mu}}{\partial a^\nu} D_\tau^\mu a^\mu - \frac{\partial B}{\partial a^\nu} \right) \xi^\sigma + \left( \frac{\partial R_{\mu}}{\partial \tau} D_\tau^\mu a^\mu - \frac{\partial B}{\partial \tau} \right) \xi^\sigma + \left( R_{\mu a} D_\tau^\sigma a^\mu - B \right) \xi^\sigma = 0, \]  
(45)
need to be satisfied.

where \( \Delta G = \varepsilon_G G^\sigma = G^\sigma (\tau, a^\nu) \) is the gauge function.
Criterion 7. If transformation (6) is Noether quasi-symmetric, then the equation,
\[
\left(\frac{\partial R_\mu}{\partial a^\alpha}D_\beta^\mu a^\alpha - \frac{\partial B}{\partial a^\alpha}\right)\Delta a^\alpha + \left(\frac{\partial R_\mu}{\partial \tau}d_\tau^\mu a^\alpha - \frac{\partial B}{\partial \tau}\right)\Delta \tau \\
+ (R_\mu D_\beta^\mu a^\alpha - B)\frac{d}{d\tau}\Delta \tau + R_\mu (D_\beta^\mu a^\alpha - a D_\beta^\mu (\alpha^\alpha \Delta \tau) + a D_\beta^\mu \alpha^\alpha \Delta \tau)
\]
needs to be satisfied.

Equation (47) can be written as \( r \) equations:
\[
\left(\frac{\partial R_\mu}{\partial a^\alpha}D_\beta^\mu a^\alpha - \frac{\partial B}{\partial a^\alpha}\right)\xi^\alpha_{\sigma} + \left(\frac{\partial R_\mu}{\partial \tau}d_\tau^\mu a^\alpha - \frac{\partial B}{\partial \tau}\right)\xi^\alpha_{\sigma} = G^\alpha (\cosh t - \cosh \tau)^{1-a}, \quad (\sigma = 1, 2, \ldots, r).
\]

If \( r = 1 \), equation (48) also gives the fractional Noether identity based on EEFI.

Criterion 8. If transformation (7) is Noether quasi-symmetric, then the following \( r \) equations,
\[
\frac{d}{d\tau} \left(\frac{R_\mu D_\beta^\mu a^\alpha - B}{(\cosh t - \cosh \tau)^{a-1}}\xi^\alpha_{\sigma} + \int_a (R_\mu D_\beta^\mu (\xi^\alpha_{\sigma} - \alpha^\alpha \xi^\alpha_{\sigma}) (\cosh t - \cosh s)^{a-1} \right.
\]
\[
- \left(\xi^\alpha_{\sigma} - \alpha^\alpha \xi^\alpha_{\sigma}, D_\beta^\mu [R_\mu (\cosh t - \cosh s)^{a-1}]\right) ds \biggr|_{a}^{b} + \left[\frac{\partial R_\mu}{\partial a^\alpha}D_\beta^\mu a^\alpha - \frac{\partial B}{\partial a^\alpha}\right] (\cosh t - \cosh \tau)^{a-1} + D_\beta^\mu \left(R_\mu (\cosh t - \cosh \tau)^{a-1}\right)(\xi^\alpha_{\sigma} - \alpha^\alpha \xi^\alpha_{\sigma}) = -G^\alpha, \quad (\sigma = 1, 2, \ldots, r),
\]
need to be satisfied.

3.3. Fractional Noether Symmetries Based on EPFI

Definition 5. If the Pfaff action 24 satisfies the equality
\[
\Delta S_D = 0, \quad (50)
\]

Criterion 9. If transformation (6) is Noether symmetric, then the equation,
\[
\left(\frac{\partial R_\mu}{\partial a^\alpha}D_\beta^\mu a^\alpha - \frac{\partial B}{\partial a^\alpha}\right)\Delta a^\alpha + \left(\frac{\partial R_\mu}{\partial \tau}d_\tau^\mu a^\alpha - \frac{\partial B}{\partial \tau}\right)\Delta \tau + (R_\mu D_\beta^\mu a^\alpha - B)\frac{d}{d\tau}\Delta \tau \\
+ R_\mu (a D_\beta^\mu \Delta a^\alpha - a D_\beta^\mu (\alpha^\alpha \Delta \tau) + a D_\beta^\mu \alpha^\alpha \Delta \tau) - (R_\mu a D_\beta^\mu a^\alpha - B) \frac{\alpha - 1}{\tan((\alpha - 1)(t - \tau) + (\pi/2))}\Delta \tau = 0,
\]
needs to be satisfied.
Equation (51) can be written as $r$ equations:

$$
\frac{\partial R_\mu}{\partial a} \Delta \tau \Delta a^\rho \Delta a^\sigma + \left( \frac{\partial R_\mu}{\partial a} D_\tau^a \Delta a^\rho - \frac{\partial B}{\partial a} \right) \Delta a^\sigma + \left( R_{\mu a} D_\tau^a \Delta a^\rho - B \right) \frac{d}{d\tau} \Delta \tau
$$

$$
+ R_\mu \left( a D_\tau^a \Delta a^\rho - a D_\tau^a (\Delta \tau) + a D_\tau^a \Delta a^\rho \Delta \tau \right) - \left( R_{\mu a} D_\tau^a \Delta a^\rho - B \right) \frac{\alpha - 1}{\tan((\alpha - 1)(t - \tau) + (\pi/2))} \xi_0^\sigma = 0, \quad (\sigma = 1, 2, \ldots, r).
$$

If $r = 1$, equation (52) gives the fractional Noether identity based on EPFI.

**Criterion 10.** If transformation (7) is Noether symmetric, then the following $r$ equations,

$$
d\left( R_{\mu a} D_\tau^a \Delta a^\rho - B \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \xi_0^\sigma + \int_0^\tau \left[ R_{\mu a} D_\tau^a (\xi_0^\rho - \Delta a^\rho \xi_0^\sigma) \sin\left( (\alpha - 1)(t - s) + \frac{\pi}{2} \right) \right] ds
$$

$$
- \left( \xi_0^\rho - \Delta a^\rho \xi_0^\sigma \right) D_\rho^a \left[ R_\rho \sin\left( (\alpha - 1)(t - s) + \frac{\pi}{2} \right) \right] ds
$$

$$
+ \left[ \left( \frac{\partial R_\rho}{\partial a} D_\tau^a \Delta a^\rho - \frac{\partial B}{\partial a} \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) + r D_\rho^a \left( R_\rho \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right) \right] \left( \xi_0^\rho - \Delta a^\rho \xi_0^\sigma \right) = 0, \quad (\sigma = 1, 2, \ldots, r).
$$

need to be satisfied.

**Definition 6.** If the Pfaff action (24) satisfies the equality

$$
\Delta S_p = -\frac{1}{\Gamma (\alpha)} \int_0^\tau \frac{d}{d\tau} (\Delta G) d\tau,
$$

where $\Delta G = \epsilon \alpha G^\rho$ and $G^\rho = G^\rho (\tau, a^\rho)$ is the gauge function, then transformation (6) is said to be Noether quasi-symmetric for system (28).

**Criterion 11.** If transformation (6) is Noether quasi-symmetric, then the equation,

$$
\Delta \tau = \frac{(d/d\tau) (\Delta G)}{\sin((\alpha - 1)(t - \tau) + (\pi/2))}
$$

needs to be satisfied.
Equation (55) can be written as \( r \) equations:

\[
\left( \frac{\partial R_\mu}{\partial a^\alpha} D_\xi^\alpha d^\alpha - \frac{\partial B}{\partial a^\alpha} \right) \xi^\alpha_{\xi_0} + \left( \frac{\partial R_\mu}{\partial \tau^\alpha} D_\xi^\alpha d^\alpha - \frac{\partial B}{\partial \tau^\alpha} \right) \xi^\alpha_{\xi_0} + (R_{\mu a} D_\xi^\alpha d^\alpha - B) \xi^\alpha_{\xi_0}
\]

\[+ R_{\mu a} D_\xi^\alpha d^\alpha \left( \alpha \xi^\alpha_{\xi_0} + \alpha D_\xi^\alpha d^\alpha - (R_{\mu a} D_\xi^\alpha d^\alpha - B) \right) \frac{\alpha - 1}{\tan((\alpha - 1)(t - \tau) + (\pi/2))} \tag{56}\]

If \( r = 1 \), equation (56) gives the fractional Noether identity based on EPFI.

\[
\frac{d}{d\tau} \left( R_{\mu a} D_\xi^\alpha d^\alpha - B \right) \sin((\alpha - 1)(t - \tau) + \frac{\pi}{2}) \xi^\alpha_{\xi_0} + \int_{s}^{\tau} \left[ R_{\mu a} D_\xi^\alpha (\xi^\alpha_{\xi_0} - \alpha d^\alpha \xi^\alpha_{\xi_0}) \sin((\alpha - 1)(t - s) + \frac{\pi}{2}) \right] ds
\]

\[- \left( \xi^\alpha_{\xi_0} - \alpha d^\alpha \xi^\alpha_{\xi_0} \right) D_\xi^\alpha \left[ R_\mu \sin((\alpha - 1)(t - s) + \frac{\pi}{2}) \right] + \left( \frac{\partial R_\mu}{\partial a^\alpha} D_\xi^\alpha d^\alpha - \frac{\partial B}{\partial a^\alpha} \right) \sin((\alpha - 1)(t - \tau) + \frac{\pi}{2}) + \alpha D_\xi^\alpha \left( R_\mu \sin((\alpha - 1)(t - \tau) + \frac{\pi}{2}) \right) \right] \tag{57}\]

\[
\left( \xi^\alpha_{\xi_0} - \alpha d^\alpha \xi^\alpha_{\xi_0} \right) = -G^\alpha, \quad (\alpha = 1, 2, \ldots, r),
\]

need to be satisfied.

4. Fractional Noether’s Theorems under Quasi-Fractional Dynamics Models

Now, we prove Noether’s theorems for fractional Birkhoffian systems under three quasi-fractional dynamics models.

\[
I^\alpha = \left( R_{\mu a} D_\xi^\alpha d^\alpha - B \right)(t - \tau)^{\alpha - 1} \xi^\alpha_{\tau_0} + \int_{s}^{\tau} \left[ R_{\mu a} D_\xi^\alpha (\tau^\alpha_{\tau_0} - \alpha d^\alpha \tau^\alpha_{\tau_0}) (t - s)^{\alpha - 1} - (\tau^\alpha_{\tau_0} - \alpha d^\alpha \tau^\alpha_{\tau_0}) D_\xi^\alpha \left( R_\mu (t - s)^{\alpha - 1} \right) \right] ds = c^\alpha, \quad (\alpha = 1, 2, \ldots, r), \tag{58}\]

are \( r \) linearly independent conserved quantities.

Proof. From Definition 1, we get \( \Delta S_R = 0 \), namely,

\[
\frac{1}{\Gamma(\alpha)} \int_{a}^{b} \xi^\alpha_{\tau_0} \left[ \frac{d}{d\tau} \left( R_{\mu a} D_\xi^\alpha d^\alpha - B \right)(t - \tau)^{\alpha - 1} \xi^\alpha_{\tau_0} + \int_{s}^{\tau} \left( R_{\mu a} D_\xi^\alpha (\tau^\alpha_{\tau_0} - \alpha d^\alpha \tau^\alpha_{\tau_0}) (t - s)^{\alpha - 1} - (\tau^\alpha_{\tau_0} - \alpha d^\alpha \tau^\alpha_{\tau_0}) D_\xi^\alpha \left( R_\mu (t - s)^{\alpha - 1} \right) \right) ds \right]
\]

\[+ \left( \frac{\partial R_\mu}{\partial a^\alpha} D_\xi^\alpha d^\alpha - \frac{\partial B}{\partial a^\alpha} \right) (t - \tau)^{\alpha - 1} + \alpha D_\xi^\alpha \left( R_\mu (t - \tau)^{\alpha - 1} \right) \right] \xi^\alpha_{\tau_0} = 0. \tag{59}\]
By substituting (5) into the above formula and considering the arbitrariness of the integral interval and the independence of $\epsilon_\sigma$, we obtain

\[
\frac{d}{dr} \left[ (R_{\mu_D} D^\mu_D a^\mu - B) (t - r)^{a-1} \xi_0^\sigma + \int_a^r \left( R_{\mu_D} D^\mu_D \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) (t - s)^{a-1} - \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) D_\sigma^\mu \left( R_\mu (t - s)^{a-1} \right) \right) ds \right] = 0.
\] (60)

So, Theorem 1 is proved.

\[ \square \]

**Theorem 2.** If transformation (7) is Noether quasi-symmetric of system (5) based on ERLFI, then

\[
I^\sigma = \left( R_{\mu_D} D^\mu_D a^\mu - B \right) (t - r)^{a-1} \xi_0^\sigma + \int_a^r \left[ R_{\mu_D} D^\mu_D \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) (t - s)^{a-1} - \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) D_\sigma^\mu \left( R_\mu (t - s)^{a-1} \right) \right] ds + G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r),
\] (61)

are $r$ linearly independent conserved quantities.

**Proof.** Combining Definition 2 and formula (13), using equation (5), and considering the arbitrariness of the integral interval and the independence of $\epsilon_\sigma$, the conclusion is obtained.

\[ \square \]

**Theorem 3.** If transformation (7) is Noether symmetric of system (18) based on EEFI, then

\[
I^\sigma = \left( R_{\mu_D} D^\mu_D a^\mu - B \right) \left( \cosh t - \cosh \tau \right)^{a-1} \xi_0^\sigma + \int_a^\tau \left[ R_{\mu_D} D^\mu_D \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) (\cosh t - \cosh s)^{a-1} - \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) D_\sigma^\mu \left( R_\mu (\cosh t - \cosh s)^{a-1} \right) \right] ds = c^\sigma, \quad (\sigma = 1, 2, \ldots, r),
\] (62)

are $r$ linearly independent conserved quantities.

**Theorem 4.** If transformation (7) is Noether quasi-symmetric of system (18) based on EEFI, then

\[
I^\sigma = \left( R_{\mu_D} D^\mu_D a^\mu - B \right) \left( \cosh t - \cosh \tau \right)^{a-1} \xi_0^\sigma + \int_a^\tau \left[ R_{\mu_D} D^\mu_D \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) (\cosh t - \cosh s)^{a-1} - \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) D_\sigma^\mu \left( R_\mu (\cosh t - \cosh s)^{a-1} \right) \right] ds + G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r),
\] (63)

are $r$ linearly independent conserved quantities.

**Theorem 5.** If transformation (7) is Noether symmetric of system (28) based on EPFI, then

\[
I^\sigma = \left( R_{\mu_D} D^\mu_D a^\mu - B \right) \left( \sin \left( \alpha - 1 \right) (t - \tau) + \frac{\alpha}{2} \right) \xi_0^\sigma + \int_a^\tau \left[ R_{\mu_D} D^\mu_D \left( \xi_\mu^\sigma - \dot{\alpha}^\sigma \xi_0^\sigma \right) \sin \left( \alpha - 1 \right) (t - s) + \frac{\alpha}{2} \right] ds + G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r),
\] (64)

are $r$ linearly independent conserved quantities.
Theorem 6. If transformation (7) is Noether quasi-symmetric of system (28) based on EPFI, then

\[
I^\sigma = \left( R_{\mu a} D^\beta_a \xi^\sigma - B \right) \sin \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) C_0 + \int_a^b R_{\mu a} D^\beta_a \xi^\sigma \left( \frac{\pi}{4} \right) \sin \left( (\alpha - 1)(t - s) + \frac{\pi}{2} \right) \left[ R_{\mu a} D^\beta_a \xi^\sigma \left( \frac{\pi}{4} \right) \right] ds + G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r),
\]

are \( r \) linearly independent conserved quantities.

Obviously, if \( \beta \to 1 \), then Theorems 1–6 give Noether’s theorems for quasi-fractional Birkhoffian systems. If \( \alpha \to 1 \) and \( \beta \to 1 \), Theorems 1–6 give Noether’s theorems for classical Birkhoffian systems [58].

5. Examples

5.1. Example 1. Consider a fractional Birkhoffian system based on ERLFI. The Pfaff action is

\[
S_R = \frac{1}{\Gamma(\alpha)} \int_a^b \left( a^2 D^\alpha_a a^1 + a^4 D^\alpha_a a^3 - a^2 a^3 \right) (t - \tau)^{\alpha - 1} dt,
\]

where the Birkhoffian is \( B = a^2 a^3 \), and Birkhoff’s functions are \( R_1 = a^2, R_2 = 0, R_3 = a^4, \) and \( R_4 = 0 \).

From equation (5), Birkhoff’s equations are

\[
\begin{align*}
D^\beta \left[ a^2 (t - \tau)^{\alpha - 1} \right] &= 0, \\
D^\beta a^1 - a^3 &= 0, \\
-a^2 (t - \tau)^{\alpha - 1} + D^\beta \left[ a^4 (t - \tau)^{\alpha - 1} \right] &= 0, \\
D^\beta a^3 &= 0.
\end{align*}
\]

According to (40), the Noether identity gives

\[
\begin{align*}
&\left( a^2 D^\beta_a a^1 - a^3 \right) \xi^\sigma - a^2 \xi^\sigma + a^4 D^\beta_a a^3 - a^2 a^3 \right) (t - \tau)^{\alpha - 1} dt, \\
&+ a^4 \left[ a^2 D^\beta_a \xi^\sigma - a^3 \xi^\sigma \right] + a^4 \left[ a^2 D^\beta_a a^1 + a^4 D^\beta_a a^3 - a^2 a^3 \right] \xi^\sigma - \frac{\alpha - 1}{t - \tau} \\
&\cdot \left( a^2 D^\beta_a a^1 + a^4 D^\beta_a a^3 - a^2 a^3 \right) \xi^\sigma = -G^\sigma (t - \tau)^{1 - \alpha}.
\end{align*}
\]

Let

\[
\begin{align*}
\xi^\sigma_0 &= 1, \\
\xi^\sigma_1 &= a^1, \\
\xi^\sigma_2 &= 1, \\
\xi^\sigma_3 &= a^3, \\
\xi^\sigma_4 &= 1, \\
G^\sigma &= 0.
\end{align*}
\]

By Theorem 2, we obtain

\[ I = \left( a^2 D^\beta_a a^1 + a^4 D^\beta_a a^3 - a^2 a^3 \right) (t - \tau)^{\alpha - 1} = \text{const.} \quad \text{(70)} \]

5.2. Example 2. Consider a fractional Birkhoffian system based on EEFI. The Pfaff action is
where \( B = a^2a^3 + (a^3)^2 \), \( R_1 = a^2 + a^3 \), \( R_2 = 0 \), \( R_3 = a^4 \), and \( R_4 = 0 \).

From equation (18), Birkhoff’s equations are

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \dot{a}} &= \text{const}, \\
\frac{\partial \mathcal{L}}{\partial \dot{a}^3} &= \text{const}, \\
\frac{\partial \mathcal{L}}{\partial \dot{a}^3} &= \text{const}. 
\end{align*}
\]

According to (48), the Noether identity is

\[
\left( aD_t^a a^1 - a^3 \right) \xi_2 + \left( aD_t^a a^1 - a^3 \right) \xi_3 + aD_t^a a^3 \xi_0 + \left( a^2 + a^3 \right) \left[ aD_t^a \xi_0 - aD_t^a a^1 \xi_0 + aD_t^a a^3 \xi_0 \right] \\
+ \left( a^2 + a^3 \right) \left[ aD_t^a a^1 + a^4D_t^a a^3 - a^2a^3 - (a^3)^2 \right] \xi_0 = -\mathcal{G}^2 \left( \cosh t - \cosh \tau \right)^{a-1}. 
\]

Let

\[
\begin{align*}
\xi_0 &= 1, \\
\xi_1 &= a^1, \\
\xi_2 &= 0, \\
\xi_3 &= a^3, \\
\xi_4 &= 0, \\
\mathcal{G}^2 &= 0. 
\end{align*}
\]

By Theorem 3, we obtain

\[
I = \left( a^2 + a^3 \right) a^1 + a^4D_t^a a^3 - a^2a^3 - (a^3)^2 \right] \\
\cdot \left( \cosh t - \cosh \tau \right)^{a-1} \right) = \text{const.} 
\]

The conserved quantity (77) corresponds to the Noether symmetry (76).

If \( \beta \to 1 \), then we obtain

\[
I = \left[ (a^2 + a^3) a^1 + a^4a^3 - a^2a^3 - (a^3)^2 \right] \\
\cdot \left( \cosh t - \cosh \tau \right)^{a-1} = \text{const.} 
\]

Formula (78) is the conserved quantity of Birkhoffian system based on EEFI.

If \( \beta \to 1 \) and \( \alpha \to 1 \), then we obtain

\[
I = \left( a^2 + a^3 \right) a^1 + a^4a^3 - a^2a^3 - (a^3)^2 = \text{const.} 
\]

Formula (79) is the classical conserved quantity.

5.3. Example 3. Consider a fractional Birkhoffian system based on EPFI. The Pfaff action is

\[
S_p = \frac{1}{\Gamma(a)} \int_a^b \left[ \left( a^2 a^3 a^4 D_t^a a^4 + a^4D_t^a a^3 - a^2a^3 - (a^3)^2 \right) - \frac{1}{2} \left( a^4 \right)^2 \right] \cdot \sin \left( (\alpha - 1) (t - \tau) + \frac{\pi}{2} \right) \; dt, 
\]

where \( B = (1/2)(a^3)^2 + (1/2)(a^4)^2 \), \( R_1 = a^3 \), \( R_2 = a^4 \), and \( R_3 = R_4 = 0 \).
From equation (28), Birkhoff’s equations are

\[
\begin{align*}
\mathcal{L}^a & = \left[ a^3 \sin \left( (\alpha - 1) (t - \tau) + \frac{\pi}{2} \right) \right] = 0, \\
\mathcal{L}^b & = \left[ a^4 \sin \left( (\alpha - 1) (t - \tau) + \frac{\pi}{2} \right) \right] = 0, \\
\frac{da^3}{dt} & = 0, \\
\frac{da^4}{dt} & = 0.
\end{align*}
\]

(81)

According to (56), the Noether identity is

\[
\begin{align*}
\left( a^3 \frac{d^\alpha}{dt^\alpha} (a^3 - a^4) \xi_3 \right) + \left( a^4 \frac{d^\alpha}{dt^\alpha} (a^2 - a^4) \xi_4 \right) + a^3 \left( a^3 \frac{d^\alpha}{dt^\alpha} (\xi_1^0 - a^4 \xi_0^0) + a^3 \frac{d^\alpha}{dt^\alpha} (\xi_0^0) \right) \\
+ a^4 \left( a^4 \frac{d^\alpha}{dt^\alpha} (\xi_1^0) + a^4 \frac{d^\alpha}{dt^\alpha} (\xi_0^0) \right) + \left( a^3 a^4 \frac{d^\alpha}{dt^\alpha} a^1 + a^4 a^4 \frac{d^\alpha}{dt^\alpha} a^2 \right) + \left( \frac{1}{2} (a^3)^2 + \frac{1}{2} (a^4)^2 \right) \right) \xi_0^0 \\
- \frac{\alpha - 1}{\tan((\alpha - 1)(t - \tau) + \pi/2))} \left( a^3 a^4 \frac{d^\alpha}{dt^\alpha} a^1 + a^4 a^4 \frac{d^\alpha}{dt^\alpha} a^2 - \left( \frac{1}{2} (a^3)^2 + \frac{1}{2} (a^4)^2 \right) \right) \xi_0^0 \\
= - \mathcal{G}^\sigma \left\{ \frac{1}{\sin((\alpha - 1)(t - \tau) + \pi/2))} \right\}
\end{align*}
\]

(82)

Let

\[
\begin{align*}
\xi_0 & = 0, \\
\xi_1 & = a^2, \\
\xi_2 & = -a^1, \\
\xi_3 & = \xi_4 = 0, \\
\mathcal{G}^\sigma & = 0.
\end{align*}
\]

(83)

By Theorem 5, we obtain

\[
\begin{align*}
I & = \int_a^b \left\{ a^3 \frac{d^\alpha}{dt^\alpha} a^3 \sin \left( (\alpha - 1) (t - s) + \frac{\pi}{2} \right) \\
- a^4 \frac{d^\alpha}{dt^\alpha} a^1 \sin \left( (\alpha - 1) (t - s) + \frac{\pi}{2} \right) \\
- a^4 \frac{d^\alpha}{dt^\alpha} \left[ a^4 \sin \left( (\alpha - 1) (t - s) + \frac{\pi}{2} \right) \right] \\
+ a^4 \frac{d^\alpha}{dt^\alpha} \left[ a^4 \sin \left( (\alpha - 1) (t - s) + \frac{\pi}{2} \right) \right] \right\} ds \\
= \text{const.}
\end{align*}
\]

The conserved quantity (84) corresponds to the Noether symmetry (83).

When \( \beta \to 1 \) and \( \alpha \to 1 \), formula (84) becomes

\[
I = \left( a^3 a^2 - a^1 a^4 \right) \sin \left( (\alpha - 1) (t - \tau) + \frac{\pi}{2} \right) = \text{const.}
\]

(85)

Formula (85) is the conserved quantity of Birkhoffian system based on EPFI.

When \( \beta \to 1 \) and \( \alpha \to 1 \), formula (84) becomes

\[
I = a^3 a^2 - a^4 a^1 = \text{const.}
\]

(86)

Formula (86) is the classical conserved quantity.

6. Conclusions

By introducing fractional calculus into the dynamic modeling of nonconservative systems, the dynamic behavior and physical process of complex systems can be described more accurately, which provides the possibility for the quantization of nonconservative problems. Compared with fractional models, the quasi-fractional model greatly simplifies the calculation of complex fractional-order calculus, so it can be used to study complex nonconservative dynamic systems more conveniently. The dynamics of Birkhoffian system is an extension of Hamiltonian mechanics, and the fractional Birkhoffian system is an extension of integer Birkhoffian system. Therefore, fractional Birkhoffian dynamics is a research field worthy of further study and full of vitality.

The main contributions of this paper are as follows. Firstly, based on three quasi-fractional dynamics models, the fractional Pfaff–Birkhoff principles and fractional Birkhoff’s equations are established, in which the Pfaff action contains fractional-order derivative terms. Secondly, the fractional Noether symmetry is explored, and its definitions and criteria are...
established. Thirdly, Noether’s theorems for fractional Birkhoffian systems under three quasi-fractional dynamics models are proved, and fractional conservation laws are obtained.

Obviously, the results of the following two systems are special cases of this paper: (1) the quasi-fractional Birkhoffian systems based on quasi-fractional dynamics models, in which the Pfaff action contains only integer-order derivative terms; (2) the classical Birkhoffian systems under integer-order models. Therefore, our study is of great significance.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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