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*Research Article*

# Fractional Birkhoffian Mechanics Based on Quasi-Fractional Dynamics Models and Its Noether Symmetry

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This paper focuses on the exploration of fractional Birkhoffian mechanics and its fractional Noether theorems under quasi-fractional dynamics models. The quasi-fractional dynamics models under study are nonconservative dynamics models proposed by El-Nabulsi, including three cases: extended by Riemann–Liouville fractional integral (abbreviated as ERLFI), extended by exponential fractional integral (abbreviated as EEFI), and extended by periodic fractional integral (abbreviated as EPFI). First, the fractional Pfaff–Birkhoff principles based on quasi-fractional dynamics models are proposed, in which the Pfaff action contains the fractional-order derivative terms, and the corresponding fractional Birkhoff's equations are obtained. Second, the Noether symmetries and conservation laws of the systems are studied. Finally, three concrete examples are given to demonstrate the validity of the results.

## 1. Introduction

Symmetry theory plays an important role in mathematics, physics, and mechanics, and the study of symmetry properties of dynamic systems has become a very effective method to solve some practical problems. The most important and common symmetries are mainly of two kinds, namely, Noether symmetry and Lie symmetry. Noether's symmetry theory originated in 1918 and was first put forward by the famous mathematician Emmy Noether [1]. In this method, the relationship between symmetry and conserved quantity was established by using the invariance of Hamilton action under the infinitesimal group transformation of time and generalized coordinates. Candotti [2] and Desloge [3] applied Noether's theorem to classical mechanics. Djukić [4] established Noether's theorem for nonconservative systems. Liu [5] generalized Noether's theorem to nonholonomic mechanical systems. In 1979, Lutzky [6] applied the Lie method [7] of invariance of differential equations under infinitesimal group transformations to differential equations of motion for dynamical systems and started the study of Lie symmetry and conserved quantity of mechanical systems. Ibragimov [8] and Bluman [9] elaborated the role of Lie algebra and Lie group in studying the invariance of differential equations. Zhao [10] extended Lie symmetry theory to nonconservative mechanical systems. Mei [11, 12] systematically studied Noether symmetry, Lie symmetry of constrained mechanical systems, and corresponding conserved quantities. Recently, some new progress has been made in the study of these two symmetries (cf. [13–24] and references therein).

Fractional calculus is an important mathematical tool in science and engineering [25–28]. In recent decades, the research of fractional calculus has developed greatly, and its application fields have expanded to automatic control, quantum mechanics, and mechanical systems [29–35]. Riewe [36, 37] introduced the fractional variational problem for the first time in the study of nonconservative mechanics. In 2005, El-Nabulsi established a dynamical model of nonconservative systems under the framework of fractional calculus [38] based on the definition of Riemann–Liouville fractional integral (ERLFI). El-Nabulsi expanded the idea of

dynamics modeling and successively put forward the dynamical models of nonconservative systems, which are extended by exponentially fractional integral (EEFI) and extended by periodic laws fractional integral (EPFI) [39, 40], respectively. The equations obtained from quasi-fractional dynamics models are similar to dynamical equations of classical conservative systems, which contain the generalized fractional external forces corresponding to dissipative forces, but the term with the fractional derivative does not show up. Different from other models, the fractional time integration of quasi-fractional dynamics models only needs one parameter. In this way, it simplifies the calculation of complex fractional calculus and provides a modeling method for nonconservative systems. Therefore, the quasi-fractional dynamics models can be used to study complex dynamical systems more conveniently. Frederico and Torres [41] first presented fractional Noether's theorems. Since then, studies on fractional Noether symmetry and conservation laws have been extensively developed [42–49]. In addition, Torres and Frederico studied Noether's theorems of fractional action-like variation problems [50, 51]. In recent years, nonconservative dynamical systems based on quasi-fractional dynamical models have been studied deeply, and the corresponding dynamical equations and Noether conservation laws have been obtained [52–55]. However, most of the previous studies on the variational problems of quasi-fractional dynamics models are confined to Lagrangian framework and Hamiltonian framework.

It is well known that Birkhoffian mechanics is a new stage in the development of Hamiltonian mechanics [56–58]. Under canonical transformation, Hamilton canonical equation remains unchanged, but under general noncanonical transformation, it becomes Birkhoff's equation. Santilli [57] and Mei [58] both pointed out that Birkhoffian mechanics is the most general possible mechanics, which can be applied to hadron physics, space mechanics, statistical mechanics, biophysics, engineering, and other fields. Zhang and Zhai [59] has proposed the fractional Pfaff–Birkhoff principle and fractional Birkhoff's equations and proved that the fractional Hamilton principle is the special case of the fractional Pfaff–Birkhoff principle, and the fractional Hamilton equations and the fractional Lagrange equations are the special cases of the fractional Birkhoff's equations. Zhang and Zhou [52] proposed the quasi-fractional Pfaff-Birkhoff principle and derived corresponding quasi-fractional Birkhoff's equations which is based on the quasi-fractional model given by [38]. Up to now, some results have been obtained on Noether symmetry of fractional or quasi-fractional Birkhoffian systems, such as [52, 59–66]. However, the results of these quasi-fractional Birkhoffian systems are limited to the Pfaff action containing only integral-order derivative terms. Here, we will further extend fractional Birkhoffian mechanics on the basis of three quasi-fractional dynamical models given in [38–40], where for the Pfaff actions, we consider contain fractional-order derivative terms. The quasi-fractional Lagrangian system and quasi-fractional Hamiltonian system are special cases of the results presented in this paper.

The text is organized as follows. In Section 2, the fractional Pfaff–Birkhoff principles under quasi-fractional dynamics models are presented and Birkhoff's equations are given, and nonisochronous variational formulae of the Pfaff action are driven. In Section 3, fractional Noether symmetries are well defined and their criteria are established. In Section 4, fractional Noether theorems are proved. For illustrating the application of the methods and results in this text, three examples are given in Section 5. In Section 6, we come to the conclusions.

## 2. Fractional Birkhoff's Equations and Variation of Fractional Pfaff Action under Quasi-Fractional Dynamics Models

For an introduction to fractional calculus and its basic theory, please refer to the monographs [27, 28].

*2.1. Fractional Birkhoffian System Based on ERLFI.* We consider a fractional Birkhoffian system determined by Birkhoff's variables $a^\mu$ $(\mu = 1, 2, \ldots, 2n)$, whose Birkhoff's functions are $R_\mu = R_\mu(\tau, a^\nu)$, the Birkhoffian is $B = B(\tau, a^\nu)$, $\beta$ is the order of fractional derivative, and $0 \le \beta < 1$.

Under the model of ERLFI, we define the Pfaff action as

$$S_R = \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a^\nu) {}_aD_\tau^\beta a^\mu - B(\tau, a^\nu) \right] (t-\tau)^{\alpha-1} \mathrm{d}\tau, \tag{1}$$

where ${}_aD_\tau^\beta a^\mu$ $(\mu = 1, 2, \ldots, 2n)$ is the fractional derivative term.

The variational principle,

$$\delta S_R = 0, \tag{2}$$

with commutative relation,

$$\delta_a D_\tau^\beta a^\mu = {}_aD_\tau^\beta \delta a^\mu, \tag{3}$$

and boundary conditions,

$$\begin{aligned} a^\mu|_{\tau=a} &= a_1^\mu, \\ a^\mu|_{\tau=b} &= a_2^\mu, \end{aligned} \tag{4}$$

is called the fractional Pfaff–Birkhoff principle based on ERLFI.

According to principle (2), we drive

$$\left( \frac{\partial R_\nu}{\partial a^\mu} {}_aD_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu} \right) (t-\tau)^{\alpha-1} + {}_\tau D_b^\beta \left[ R_\mu (t-\tau)^{\alpha-1} \right] = 0, \quad (\mu = 1, 2, \ldots, 2n). \tag{5}$$

Equation (5) is the fractional Birkhoff's equations based on ERLFI.

If $\beta \longrightarrow 1$, equation (5) becomes Birkhoff's equations based on ERLFI. If $\beta \longrightarrow 1$ and $\alpha \longrightarrow 1$, equation (5) becomes classical Birkhoff's equations [58].

Take the infinitesimal transformations:

$$\begin{aligned}\overline{\tau} &= \tau + \Delta\tau,\\ \overline{a}^{\mu}(\overline{\tau}) &= a^{\mu}(\tau) + \Delta a^{\mu}, \quad (\mu = 1, 2, \ldots, 2n),\end{aligned} \tag{6}$$

and their first-order extensions

$$\begin{aligned}\overline{\tau} &= \tau + \varepsilon_{\sigma}\xi_0^{\sigma}(\tau, a^{\nu}),\\ \overline{a}^{\mu}(\overline{\tau}) &= a^{\mu}(\tau) + \varepsilon_{\sigma}\xi_{\mu}^{\sigma}(\tau, a^{\nu}), \quad (\mu = 1, 2, \ldots, 2n),\end{aligned} \tag{7}$$

where $\varepsilon_{\sigma}$ is the infinitesimal parameter and $\xi_0^{\sigma}$ and $\xi_{\mu}^{\sigma}$ are the generating functions.

Under transformation (6), the Pfaff action (1) is transformed into

$$S_R(\overline{\gamma}) = \frac{1}{\Gamma(\alpha)}\int_{\overline{a}}^{\overline{b}}\left[R_{\mu}(\overline{\tau}, \overline{a}^{\nu})\,{}_{\overline{a}}D_{\overline{\tau}}^{\beta}\overline{a}^{\mu} - B(\overline{\tau}, \overline{a}^{\nu})\right](t - \overline{\tau})^{\alpha-1}\mathrm{d}\overline{\tau}. \tag{8}$$

And, we have

$$\begin{aligned}&S_R(\overline{\gamma}) - S_R(\gamma)\\ &= \frac{1}{\Gamma(\alpha)}\int_{\overline{a}}^{\overline{b}}\left[R_{\mu}(\overline{\tau}, \overline{a}^{\nu})\,{}_{\overline{a}}D_{\overline{\tau}}^{\beta}\overline{a}^{\mu} - B(\overline{\tau}, \overline{a}^{\nu})\right]\\ &(t - \overline{\tau})^{\alpha-1}\mathrm{d}\overline{\tau}\\ &- \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left[R_{\mu}(\tau, a^{\nu})\,{}_{a}D_{\tau}^{\beta}a^{\mu} - B(\tau, a^{\nu})\right]\\ &(t - \tau)^{\alpha-1}\mathrm{d}\tau\\ &= \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left\{\left[R_{\mu}(\tau + \Delta\tau, a^{\nu} + \Delta a^{\nu})\right]\right\}\\ &\left({}_{a}D_{\tau}^{\beta}a^{\mu} + {}_{a}D_{\tau}^{\beta}\Delta a^{\mu} - {}_{a}D_{\tau}^{\beta}(\dot{a}^{\mu}\Delta\tau) + \Delta\tau\,{}_{a}D_{\tau}^{\beta}\dot{a}^{\mu}\right)\\ &- B(\tau + \Delta\tau, a^{\nu} + \Delta a^{\nu})]\\ &(t - (\tau + \Delta\tau))^{\alpha-1}\left(1 + \frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\\ &- \left[R_{\mu}(\tau, a^{\nu})\,{}_{a}D_{\tau}^{\beta}a^{\mu} - B(\tau, a^{\nu})\right](t - \tau)^{\alpha-1}\Big\}\mathrm{d}\tau.\end{aligned} \tag{9}$$

Let $\Delta S_R$ be nonisochronous variation of $S_R$, which is the main line part of $S_R(\overline{\gamma}) - S_R(\gamma)$ relative to $\varepsilon$, and we obtain

$$\begin{aligned}\Delta S_R = \frac{1}{\Gamma(\alpha)}\int_a^b &\left\{\left(\frac{\partial R_{\mu}}{\partial a^{\nu}}\,{}_{a}D_{\tau}^{\beta}a^{\mu} - \frac{\partial B}{\partial a^{\nu}}\right)(t - \tau)^{\alpha-1}\Delta a^{\nu}\right\}\\ &+ \left(\frac{\partial R_{\mu}}{\partial \tau}\,{}_{a}D_{\tau}^{\beta}a^{\mu} - \frac{\partial B}{\partial \tau}\right)(t - \tau)^{\alpha-1}\Delta\tau\\ &+ \left(R_{\mu a}D_{\tau}^{\beta}a^{\mu} - B\right)(t - \tau)^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\\ &+ R_{\mu}\left({}_{a}D_{\tau}^{\beta}\Delta a^{\mu} - {}_{a}D_{\tau}^{\beta}(\dot{a}^{\mu}\Delta\tau) + \Delta\tau\,{}_{a}D_{\tau}^{\beta}\dot{a}^{\mu}\right)(t - \tau)^{\alpha-1}\\ &- \left(R_{\mu a}D_{\tau}^{\beta}a^{\mu} - B\right)(\alpha - 1)(t - \tau)^{\alpha-2}\Delta\tau\Big\}\mathrm{d}\tau.\end{aligned} \tag{10}$$

Since

$$\begin{aligned}\delta a^{\mu} &= \Delta a^{\mu} - \dot{a}^{\mu}\Delta\tau,\\ \Delta\dot{a}^{\mu} &= \frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta a^{\mu}) - \dot{a}^{\mu}\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta\tau),\end{aligned} \tag{11}$$

then we obtain

$$\begin{aligned}\Delta S_R = \frac{1}{\Gamma(\alpha)}\int_a^b &\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu} - B\right)(t - \tau)^{\alpha-1}\Delta\tau\right.\right.\\ &\left.+ \int_a^{\tau}\left(R_{\mu a}D_{\tau}^{\beta}\delta a^{\mu}(t - s)^{\alpha-1} - \delta a^{\mu}\,{}_{s}D_{b}^{\beta}\left(R_{\mu}(t - s)^{\alpha-1}\right)\right)\mathrm{d}s\right]\\ &+ \left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\,{}_{a}D_{\tau}^{\beta}a^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right)(t - \tau)^{\alpha-1} + {}_{\tau}D_{b}^{\beta}\left(R_{\mu}(t - \tau)^{\alpha-1}\right)\right]\delta a^{\mu}\Big\}\mathrm{d}\tau.\end{aligned} \tag{12}$$

By using formula (7), we obtain

$$\begin{aligned}\Delta S_R = \frac{1}{\Gamma(\alpha)}\int_a^b \varepsilon_{\sigma} &\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu} - B\right)(t - \tau)^{\alpha-1}\xi_0^{\sigma}\right.\right.\\ &\left.+ \int_a^{\tau}\left(R_{\mu a}D_{s}^{\beta}\left(\xi_{\mu}^{\sigma} - \dot{a}^{\mu}\xi_0^{\sigma}\right)(t - s)^{\alpha-1} - \left(\xi_{\mu}^{\sigma} - \dot{a}^{\mu}\xi_0^{\sigma}\right)_{s}D_{b}^{\beta}\left(R_{\mu}(t - s)^{\alpha-1}\right)\right)\mathrm{d}s\right]\\ &+ \left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\,{}_{a}D_{\tau}^{\beta}a^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right)(t - \tau)^{\alpha-1} + {}_{\tau}D_{b}^{\beta}\left(R_{\mu}(t - \tau)^{\alpha-1}\right)\right]\left(\xi_{\mu}^{\sigma} - \dot{a}^{\mu}\xi_0^{\sigma}\right)\Big\}\mathrm{d}\tau.\end{aligned} \tag{13}$$

Equations (10) and (13) are two mutually equivalent formulas derived from Pfaff action (1).

*2.2. Fractional Birkhoffian System Based on EEFI.* Under the model of EEFI, we define the Pfaff action as

$$S_E = \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] (\cosh t - \cosh \tau)^{\alpha-1} \mathrm{d}\tau. \tag{14}$$

The fractional Pfaff–Birkhoff principle is

$$\delta S_E = 0, \tag{15}$$

under commutative relation,

$$\delta_a D_\tau^\beta a^\mu = {}_a D_\tau^\beta \delta a^\mu, \tag{16}$$

and boundary conditions,

$$\begin{aligned} a^\mu|_{\tau=a} &= a_1^\mu, \\ a^\mu|_{\tau=b} &= a_2^\mu. \end{aligned} \tag{17}$$

The fractional Birkhoff's equations are

$$\begin{aligned} &\left( \frac{\partial R_\nu}{\partial a^\mu} {}_a D_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu} \right) (\cosh t - \cosh \tau)^{\alpha-1} \\ &+ {}_\tau D_b^\beta \left[ R_\mu (\cosh t - \cosh \tau)^{\alpha-1} \right] = 0, \quad (\mu = 1, 2, \ldots, 2n). \end{aligned} \tag{18}$$

If $\beta \longrightarrow 1$, equation (18) becomes Birkhoff's equations based on EEFI. If $\beta \longrightarrow 1$ and $\alpha \longrightarrow 1$, equation (18) becomes classical Birkhoff's equations [58].

According to formula (6), action (14) is transformed into

$$S_E(\overline{\gamma}) = \frac{1}{\Gamma(\alpha)} \int_{\overline{a}}^{\overline{b}} \left[ R_\mu(\overline{\tau}, \overline{a}^\nu)_{\overline{a}} D_{\overline{\tau}}^\beta \overline{a}^\mu - B(\overline{\tau}, \overline{a}^\nu) \right] (\cosh t - \cosh \overline{\tau})^{\alpha-1} \mathrm{d}\overline{\tau}, \tag{19}$$

and we have

$$\begin{aligned} &S_E(\overline{\gamma}) - S_E(\gamma) \\ &= \frac{1}{\Gamma(\alpha)} \int_{\overline{a}}^{\overline{b}} \left[ R_\mu(\overline{\tau}, \overline{a}^\nu)_{\overline{a}} D_{\overline{\tau}}^\beta \overline{a}^\mu - B(\overline{\tau}, \overline{a}^\nu) \right] \\ &(\cosh t - \cosh \overline{\tau})^{\alpha-1} \mathrm{d}\overline{\tau} \\ &- \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] \\ &(\cosh t - \cosh \tau)^{\alpha-1} \mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left[ R_\mu(\tau + \Delta\tau, a^\nu + \Delta a^\nu) \right] \right\} \\ &\left( {}_a D_\tau^\beta a^\mu + {}_a D_\tau^\beta \Delta a^\mu - {}_a D_\tau^\beta (\dot{a}^\mu \Delta\tau) + \Delta\tau {}_a D_\tau^\beta \dot{a}^\mu \right) \\ &- B(\tau + \Delta\tau, a^\nu + \Delta a^\nu) \Big] \\ &(\cosh t - \cosh(\tau + \Delta\tau))^{\alpha-1} \left( 1 + \frac{\mathrm{d}}{\mathrm{d}\tau} \Delta\tau \right) \\ &- \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] (\cosh t - \cosh \tau)^{\alpha-1} \Big\} \mathrm{d}\tau. \end{aligned} \tag{20}$$

So, the nonisochronous variation $\Delta S_E$ of action $S_E$ is

$$\begin{aligned} \Delta S_E = &\frac{1}{\Gamma\alpha} \\ &\cdot \int_a^b \Bigg\{ \frac{\partial R_\mu}{\partial a^\nu} {}_a D_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu} \cos ht - \cos h\tau^{\alpha-1} \Delta a^\nu \\ &+ \frac{\partial R_\mu}{\partial \tau} {}_a D_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau} \cos ht - \cos h\tau^{\alpha-1} \Delta\tau \\ &+ R_\mu {}_a D_\tau^\beta a^\mu - B \cos ht - \cos h\tau^{\alpha-1} \frac{\mathrm{d}}{\mathrm{d}\tau} \Delta\tau \\ &+ R_\mu {}_a D_\tau^\beta \Delta a^\mu - {}_a D_\tau^\beta \dot{a}^\mu \Delta\tau + \Delta\tau {}_a D_\tau^\beta \dot{a}^\mu \cos ht - \cos h\tau^{\alpha-1} \\ &+ \alpha - 1 \sin h\tau \cos ht - \cos h\tau^{\alpha-2} \Delta\tau \Bigg\} \mathrm{d}\tau. \end{aligned} \tag{21}$$

Equation (21) can also be written as

$$\begin{aligned} \Delta S_E = &\frac{1}{\Gamma\alpha} \\ &\cdot \int_a^b \Bigg\{ \frac{d}{d\tau} \Big[ R_\mu {}_a D_\tau^\beta a^\mu - B \cosh t - \cosh \tau^{\alpha-1} \Delta\tau \\ &+ \int_a^\tau R_\mu {}_a D_\tau^\beta \delta a^\mu \cosh\ t - \cosh s^{\alpha-1} \\ &- \delta a^\mu {}_s D_b^\beta R_\mu \cosh\ t - \cosh s^{\alpha-1} \mathrm{d}s \Big] \\ &+ \Big[ \frac{\partial R_\nu}{\partial a^\mu} {}_a D_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu} \cos\ ht - \cos\ h\tau^{\alpha-1} \\ &+ {}_\tau D_b^\beta R_\mu \cosh t - \cosh \tau^{\alpha-1} \Big] \delta a^\mu \Bigg\} \mathrm{d}\tau. \end{aligned} \tag{22}$$

By using formula (7), we have

$$\begin{aligned} \Delta S_E = &\frac{1}{\Gamma\alpha} \\ &\cdot \int_a^b \varepsilon_\sigma \Bigg\{ \frac{d}{d\tau} \Big[ R_\mu {}_a D_\tau^\beta a^\mu - B \cosh t - \cosh \tau^{\alpha-1} \xi_0^\sigma \\ &+ \int_a^\tau R_\mu {}_a D_s^\beta \xi_\mu^\sigma - \dot{a}^\mu \xi_0^\sigma \cosh t - \cosh s^{\alpha-1} \\ &- \xi_\mu^\sigma - \dot{a}^\mu \xi_0^\sigma {}_s D_b^\beta R_\mu \cosh t - \cosh s^{\alpha-1} \mathrm{d}s \Big] \\ &+ \Big[ \frac{\partial R_\nu}{\partial a^\mu} {}_a D_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu} \cosh t - \cosh \tau^{\alpha-1} \\ &+ {}_\tau D_b^\beta R_\mu \cosh t - \cosh \tau^{\alpha-1} \Big] \xi_\mu^\sigma - \dot{a}^\mu \xi_0^\sigma \Bigg\} \mathrm{d}\tau. \end{aligned} \tag{23}$$

Equations (21) and (23) are two mutually equivalent formulas derived from Pfaff action (14).

*2.3. Fractional Birkhoffian System Based on EPFI.* Under the model of EPFI, we define the Pfaff action as

$$S_P = \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] \sin \cdot \left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right\} \mathrm{d}\tau. \tag{24}$$

The fractional Pfaff–Birkhoff principle is

$$\delta S_P = 0, \tag{25}$$

under commutative relation,

$$\delta_a D_\tau^\beta a^\mu = {}_a D_\tau^\beta \delta a^\mu, \tag{26}$$

and boundary conditions,

$$\begin{aligned} a^\mu|_{\tau=a} &= a_1^\mu, \\ a^\mu|_{\tau=b} &= a_2^\mu. \end{aligned} \tag{27}$$

The fractional Birkhoff's equations are

$$\left( \frac{\partial R_\nu}{\partial a^\mu} {}_a D_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu} \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) + {}_\tau D_b^\beta \left[ R_\mu \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right] = 0, \quad (\mu = 1, 2, \ldots, 2n). \tag{28}$$

If $\beta \longrightarrow 1$, equation (28) becomes Birkhoff's equations based on EPFI. If $\beta \longrightarrow 1$ and $\alpha \longrightarrow 1$, equation (28) becomes the classical Birkhoff's equations [58].

According to formula (6), action (24) is transformed into

$$S_P(\overline{\gamma}) = \frac{1}{\Gamma(\alpha)} \int_{\overline{a}}^{\overline{b}} \left[ R_\mu(\overline{\tau}, \overline{a}^\nu)_{\overline{a}} D_{\overline{\tau}}^\beta \overline{a}^\mu - B(\overline{\tau}, \overline{a}^\nu) \right] \sin \cdot \left( (\alpha - 1)(t - \overline{\tau}) + \frac{\pi}{2} \right) d\overline{\tau}, \tag{29}$$

and we have

$$\begin{aligned} S_P(\overline{\gamma}) - S_P(\gamma) &= \frac{1}{\Gamma(\alpha)} \int_{\overline{a}}^{\overline{b}} \left[ R_\mu(\overline{\tau}, \overline{a}^\nu)_{\overline{a}} D_{\overline{\tau}}^\beta \overline{a}^\mu - B(\overline{\tau}, \overline{a}^\nu) \right] \sin\left( (\alpha - 1)(t - \overline{\tau}) + \frac{\pi}{2} \right) \mathrm{d}\overline{\tau} \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^b \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left[ R_\mu(\tau + \Delta\tau, a^\nu + \Delta a^\nu) \right] \right\} \left( {}_a D_\tau^\beta a^\mu + {}_a D_\tau^\beta \Delta a^\mu \right) \\ &\quad \left( -{}_a D_\tau^\beta (\dot{a}^\mu \Delta\tau) + \Delta\tau {}_a D_\tau^\beta \dot{a}^\mu \right) - B(\tau + \Delta\tau, a^\nu + \Delta a^\nu) \Big] \\ &\quad \sin\left( (\alpha - 1)(t - (\tau + \Delta\tau)) + \frac{\pi}{2} \right) \left( 1 + \frac{\mathrm{d}}{\mathrm{d}\tau} \Delta\tau \right) \\ &\quad - \left[ R_\mu(\tau, a^\nu)_a D_\tau^\beta a^\mu - B(\tau, a^\nu) \right] \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Big\} \mathrm{d}\tau. \end{aligned} \tag{30}$$

So, we have

$$\begin{aligned} \Delta S_P &= \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left( \frac{\partial R_\mu}{\partial a^\nu} {}_a D_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu} \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Delta a^\nu + \left( \frac{\partial R_\mu}{\partial \tau} {}_a D_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau} \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \Delta\tau \right. \\ &\quad + \left( R_\mu {}_a D_\tau^\beta a^\mu - B \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \frac{\mathrm{d}}{\mathrm{d}\tau} \Delta\tau + R_\mu \left( {}_a D_\tau^\beta \Delta a^\mu - {}_a D_\tau^\beta (\dot{a}^\mu \Delta\tau) + \Delta\tau {}_a D_\tau^\beta \dot{a}^\mu \right) \sin\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \\ &\quad \left. - (\alpha - 1) \cos\left( (\alpha - 1)(t - \tau) + \frac{\pi}{2} \right) \right\} \mathrm{d}\tau. \end{aligned} \tag{31}$$

Equation (31) can also be written as

$$\begin{aligned}\Delta S_P = \frac{1}{\Gamma(\alpha)}\int_a^b \Bigg\{ \frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg[\Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big)\Delta\tau + \int_a^\tau \Big(R_{\mu a}D_s^\beta \delta a^\mu \sin\Big((\alpha-1)(t-s)+\frac{\pi}{2}\Big) \\ - \delta a^\mu {}_sD_b^\beta\Big(R_\mu \sin\Big((\alpha-1)(t-s)+\frac{\pi}{2}\Big)\Big)\Big)\mathrm{d}s\Bigg] + \Bigg[\Big(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu}\Big)\sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big) \\ + {}_\tau D_b^\beta\Big(R_\mu \sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big)\Big)\Bigg]\delta a^\mu\Bigg\}\mathrm{d}\tau.\end{aligned} \tag{32}$$

By using formula (7), we have

$$\begin{aligned}\Delta S_P = \frac{1}{\Gamma(\alpha)}\int_a^b \varepsilon_\sigma \Bigg\{ \frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg[\Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big)\xi_0^\sigma \\ + \int_a^\tau \Big(R_{\mu a}D_s^\beta\big(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\big)\sin\Big((\alpha-1)(t-s)+\frac{\pi}{2}\Big) - \big(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\big){}_sD_b^\beta\Big(R_\mu\sin\Big((\alpha-1)(t-s)+\frac{\pi}{2}\Big)\Big)\Big)\Bigg]\mathrm{d}s \\ + \Bigg[\Big(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu}\Big)\sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big) + {}_\tau D_b^\beta\Big(R_\mu\sin\Big((\alpha-1)(t-\tau)+\frac{\pi}{2}\Big)\Big)\Bigg]\big(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\big)\Bigg\}\mathrm{d}\tau.\end{aligned} \tag{33}$$

Equations (31) and (33) are two mutually equivalent formulas derived from Pfaff action (24).

## 3. Fractional Noether Symmetries under Quasi-Fractional Dynamics Models

Next, we will define the Noether symmetries of the system under three quasi-fractional dynamics models and establish their criteria.

### 3.1. Fractional Noether Symmetries Based on ERLFI

*Definition 1.* If the Pfaff action (1) satisfies the equality

$$\Delta S_R = 0, \tag{34}$$

then transformation (6) is said to be Noether symmetric for system (5).

According to Definition 1, using formulas (10) and (13), we have the following.

*Criterion 1.* If transformation (6) is Noether symmetric, then the equation,

$$\begin{aligned}\Big(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu}\Big)\Delta a^\nu + \Big(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau}\Big)\Delta\tau + \Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau \\ + R_\mu\Big[{}_aD_\tau^\beta\Delta a^\mu - {}_aD_\tau^\beta\big(\dot{a}^\mu\Delta\tau\big) + {}_aD_\tau^\beta\dot{a}^\mu\Delta\tau\Big] - \Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\frac{\alpha-1}{t-\tau}\Delta\tau = 0,\end{aligned} \tag{35}$$

needs to be satisfied.

Equation (35) can be written as $r$ equations:

$$\begin{aligned}\Big(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu}\Big)\xi_\nu^\sigma + \Big(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau}\Big)\xi_0^\sigma + \Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\dot{\xi}_0^\sigma \\ + R_\mu\Big({}_aD_\tau^\beta\xi_\mu^\sigma - {}_aD_\tau^\beta\big(\dot{a}^\mu\xi_0^\sigma\big) + {}_aD_\tau^\beta\dot{a}^\mu\xi_0^\sigma\Big) - \Big(R_{\mu a}D_\tau^\beta a^\mu - B\Big)\frac{\alpha-1}{t-\tau}\xi_0^\sigma = 0, \quad (\sigma = 1, 2, \ldots, r).\end{aligned} \tag{36}$$

If $r = 1$, equation (36) gives the fractional Noether identity based on ERLFI.

*Criterion 2.* If transformation (7) is Noether symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)(t-\tau)^{\alpha-1}\xi_0^{\sigma}+\int_a^{\tau}\left[R_{\mu a}D_s^{\beta}\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right)(t-s)^{\alpha-1}-\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right){}_sD_b^{\beta}\left[R_{\mu}(t-s)^{\alpha-1}\right]\right]\mathrm{d}s\right\}\\&+\left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)(t-\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}\left(R_{\mu}(t-\tau)^{\alpha-1}\right)\right]\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right)=0,\quad(\sigma=1,2,\ldots,r),\end{aligned}\tag{37}$$

need to be satisfied.

*Definition 2.* If the Pfaff action (1) satisfies the equality

$$\Delta S_R=-\frac{1}{\Gamma(\alpha)}\int_a^b\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\mathrm{d}\tau,\tag{38}$$

where $\Delta G=\varepsilon_{\sigma}G^{\sigma}$ and $G^{\sigma}=G^{\sigma}(\tau,a^{\nu})$ is the gauge function, then transformation (6) is said to be Noether quasi-symmetric for system (5).

According to Definition 2, using formulas (10) and (13), we have the following.

*Criterion 3.* . If transformation (6) is Noether quasi-symmetric, then the equation,

$$\begin{aligned}&\left(\frac{\partial R_{\mu}}{\partial a^{\nu}}{}_aD_{\tau}^{\beta}a^{\mu}-\frac{\partial B}{\partial a^{\nu}}\right)\Delta a^{\nu}+\left(\frac{\partial R_{\mu}}{\partial\tau}{}_aD_{\tau}^{\beta}a^{\mu}-\frac{\partial B}{\partial\tau}\right)\Delta\tau\\&+\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau+R_{\mu}\left[{}_aD_{\tau}^{\beta}\Delta a^{\mu}-{}_aD_{\tau}^{\beta}\left(\dot{a}^{\mu}\Delta\tau\right)+{}_aD_{\tau}^{\beta}\dot{a}^{\mu}\Delta\tau\right]\\&-\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)\frac{\alpha-1}{t-\tau}\Delta\tau=-\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)(t-\tau)^{1-\alpha},\end{aligned}\tag{39}$$

needs to be satisfied.

Equation (39) can be written as $r$ equations:

$$\begin{aligned}&\left(\frac{\partial R_{\mu}}{\partial a^{\nu}}{}_aD_{\tau}^{\beta}a^{\mu}-\frac{\partial B}{\partial a^{\nu}}\right)\xi_{\nu}^{\sigma}+\left(\frac{\partial R_{\mu}}{\partial\tau}{}_aD_{\tau}^{\beta}a^{\mu}-\frac{\partial B}{\partial\tau}\right)\xi_0^{\sigma}\\&+\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)\dot{\xi}_0^{\sigma}+R_{\mu}\left({}_aD_{\tau}^{\beta}\xi_{\mu}^{\sigma}-{}_aD_{\tau}^{\beta}\left(\dot{a}^{\mu}\xi_0^{\sigma}\right)+{}_aD_{\tau}^{\beta}\dot{a}^{\mu}\xi_0^{\sigma}\right)\\&-\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)\frac{\alpha-1}{t-\tau}\xi_0^{\sigma}=-\dot{G}^{\sigma}(t-\tau)^{1-\alpha},\quad(\sigma=1,2,\ldots,r).\end{aligned}\tag{40}$$

If $r = 1$, equation (40) also gives the fractional Noether identity based on ERLFI.

*Criterion 4.* If transformation (7) is Noether quasi-symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)(t-\tau)^{\alpha-1}\xi_0^{\sigma}+\int_a^{\tau}\left[R_{\mu a}D_s^{\beta}\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right)(t-s)^{\alpha-1}-\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right){}_sD_b^{\beta}\left[R_{\mu}(t-s)^{\alpha-1}\right]\right]\mathrm{d}s\right\}\\&+\left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)(t-\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}\left(R_{\mu}(t-\tau)^{\alpha-1}\right)\right]\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_0^{\sigma}\right)=-\dot{G}^{\sigma},\quad(\sigma=1,2,\ldots,r),\end{aligned}\tag{41}$$

need to be satisfied.

*3.2. Fractional Noether Symmetries Based on EEFI*

*Definition 3.* If the Pfaff action (14) satisfies the equality

$$\Delta S_E = 0, \tag{42}$$

then transformation (6) is said to be Noether symmetric for system (18).

According to Definition 3, using formulas (21) and (23), we have

*Criterion 5.* If transformation (6) is Noether symmetric, then the equation,

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu + \left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau}\right)\Delta\tau + \left(R_{\mu a}D_\tau^\beta a^\mu - B\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau \\ &+ R_\mu\left({}_aD_\tau^\beta \Delta a^\mu - {}_aD_\tau^\beta\left(\dot{a}^\mu\Delta\tau\right) + {}_aD_\tau^\beta \dot{a}^\mu\Delta\tau\right) + \left(R_{\mu a}D_\tau^\beta a^\mu - B\right)\frac{(\alpha-1)\sinh\tau}{\cosh t - \cosh\tau}\Delta\tau = 0,\end{aligned} \tag{43}$$

needs to be satisfied.

Equation (43) can be written as $r$ equations:

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial a^\nu}\right)\xi_\nu^\sigma + \left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu - \frac{\partial B}{\partial \tau}\right)\xi_0^\sigma + \left(R_{\mu a}D_\tau^\beta a^\mu - B\right)\dot{\xi}_0^\sigma \\ &+ R_\mu\left({}_aD_\tau^\beta \xi_\mu^\sigma - {}_aD_\tau^\beta\left(\dot{a}^\mu\xi_0^\sigma\right) + {}_aD_\tau^\beta \dot{a}^\mu\xi_0^\sigma\right) + \left(R_{\mu a}D_\tau^\beta a^\mu - B\right)\frac{(\alpha-1)\sinh\tau}{\cosh t - \cosh\tau}\xi_0^\sigma = 0, \quad (\sigma = 1, 2, \ldots, r).\end{aligned} \tag{44}$$

If $r = 1$, equation (44) gives the fractional Noether identity based on EEFI.

*Criterion 6.* If transformation (7) is Noether symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)(\cosh t - \cosh\tau)^{\alpha-1}\xi_0^\sigma + \int_a^\tau\left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)(\cosh t - \cosh s)^{\alpha-1}\right.\right. \\ &\left.\left. - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right){}_sD_b^\beta\left[R_\mu(\cosh t - \cosh s)^{\alpha-1}\right]\right]\mathrm{d}s\right\} \\ &+ \left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu - \frac{\partial B}{\partial a^\mu}\right)(\cosh t - \cosh\tau)^{\alpha-1} + {}_\tau D_b^\beta\left(R_\mu(\cosh t - \cosh\tau)^{\alpha-1}\right)\right]\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right) = 0, \quad (\sigma = 1, 2, \ldots, r),\end{aligned} \tag{45}$$

need to be satisfied.

*Definition 4.* If the Pfaff action (14) satisfies the equality

$$\Delta S_E = -\frac{1}{\Gamma(\alpha)}\int_a^b \frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\mathrm{d}\tau, \tag{46}$$

where $\Delta G = \varepsilon_\sigma G^\sigma$ and $G^\sigma = G^\sigma(\tau, a^\nu)$ is the gauge function, then transformation (6) is said to be Noether quasi-symmetric for system (18).

According to Definition 4, using formulas (21) and (23), we have the following.

*Criterion 7.* If transformation (6) is Noether quasi-symmetric, then the equation,

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\Delta\tau\\&+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{d}{d\tau}\Delta\tau+R_\mu\left({}_aD_\tau^\beta\Delta a^\mu-{}_aD_\tau^\beta\left(\dot a^\mu\Delta\tau\right)+{}_aD_\tau^\beta\dot a^\mu\Delta\tau\right)\\&+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}\Delta\tau=-\frac{d}{d\tau}\left(\Delta G\right)\left(\cosh t-\cosh\tau\right)^{1-\alpha},\end{aligned}\tag{47}$$

needs to be satisfied.

Equation (47) can be written as $r$ equations:

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^\sigma+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\xi_0^\sigma\\&+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\dot\xi_0^\sigma+R_\mu\left({}_aD_\tau^\beta\xi_\mu^\sigma-{}_aD_\tau^\beta\left(\dot a^\mu\xi_0^\sigma\right)+{}_aD_\tau^\beta\dot a^\mu\xi_0^\sigma\right)\\&+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}\xi_0^\sigma=-\dot G^\sigma\left(\cosh t-\cosh\tau\right)^{1-\alpha},\quad(\sigma=1,2,\ldots,r).\end{aligned}\tag{48}$$

If $r=1$, equation (48) also gives the fractional Noether identity based on EEFI.

*Criterion 8.* If transformation (7) is Noether quasi-symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\left(\cosh t-\cosh\tau\right)^{\alpha-1}\xi_0^\sigma+\int_a^\tau\left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma-\dot a^\mu\xi_0^\sigma\right)\left(\cosh t-\cosh s\right)^{\alpha-1}\right.\right.\\&\left.\left.-\left(\xi_\mu^\sigma-\dot a^\mu\xi_0^\sigma\right){}_sD_b^\beta\left[R_\mu\left(\cosh t-\cosh s\right)^{\alpha-1}\right]\right]\mathrm{d}s\right\}\\&+\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\left(\cosh t-\cosh\tau\right)^{\alpha-1}+{}_\tau D_b^\beta\left(R_\mu\left(\cosh t-\cosh\tau\right)^{\alpha-1}\right)\right]\left(\xi_\mu^\sigma-\dot a^\mu\xi_0^\sigma\right)=-\dot G^\sigma,\quad(\sigma=1,2,\ldots,r),\end{aligned}\tag{49}$$

need to be satisfied.

### 3.3. Fractional Noether Symmetries Based on EPFI

*Definition 5.* If the Pfaff action 24 satisfies the equality

$$\Delta S_P=0,\tag{50}$$

then transformation (6) is said to be Noether symmetric for system (28).

According to Definition 5, using formulas (31) and (33), we have the following.

*Criterion 9.* If transformation (6) is Noether symmetric, then the equation,

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\Delta\tau+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\\&+R_\mu\left({}_aD_\tau^\beta\Delta a^\mu-{}_aD_\tau^\beta\left(\dot a^\mu\Delta\tau\right)+{}_aD_\tau^\beta\dot a^\mu\Delta\tau\right)-\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{\alpha-1}{\tan\left((\alpha-1)(t-\tau)+(\pi/2)\right)}\Delta\tau=0,\end{aligned}\tag{51}$$

needs to be satisfied.

Equation (51) can be written as $r$ equations:

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^\sigma+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\xi_0^\sigma+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\dot{\xi}_0^\sigma\\&+R_\mu\left({}_aD_\tau^\beta\xi_\mu^\sigma-{}_aD_\tau^\beta\left(\dot{a}^\mu\xi_0^\sigma\right)+{}_aD_\tau^\beta\dot{a}^\mu\xi_0^\sigma\right)-\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\\&\frac{\alpha-1}{\tan\left((\alpha-1)(t-\tau)+(\pi/2)\right)}\xi_0^\sigma=0,\quad(\sigma=1,2,\ldots,r).\end{aligned}\tag{52}$$

If $r=1$, equation (52) gives the fractional Noether identity based on EPFI.

*Criterion 10.* If transformation (7) is Noether symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\xi_0^\sigma+\int_a^\tau\left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right.\right.\\&\left.\left.-\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right){}_sD_b^\beta\left[R_\mu\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right]\right]\mathrm{d}s\right\}\\&+\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)+{}_\tau D_b^\beta\left(R_\mu\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right)\right]\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)=0,\quad(\sigma=1,2,\ldots,r),\end{aligned}\tag{53}$$

need to be satisfied.

*Definition 6.* If the Pfaff action (24) satisfies the equality

$$\Delta S_P=-\frac{1}{\Gamma(\alpha)}\int_a^b\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\mathrm{d}\tau,\tag{54}$$

where $\Delta G=\varepsilon_\sigma G^\sigma$ and $G^\sigma=G^\sigma(\tau,a^\nu)$ is the gauge function, then transformation (6) is said to be Noether quasi-symmetric for system (28).

According to Definition 6, using formulas (21) and (23), we have the following.

*Criterion 11.* If transformation (6) is Noether quasi-symmetric, then the equation,

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\Delta\tau+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\\&+R_\mu\left({}_aD_\tau^\beta\Delta a^\mu-{}_aD_\tau^\beta\left(\dot{a}^\mu\Delta\tau\right)+{}_aD_\tau^\beta\dot{a}^\mu\Delta\tau\right)-\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{\alpha-1}{\tan\left((\alpha-1)(t-\tau)+(\pi/2)\right)}\\&\Delta\tau=-\frac{(\mathrm{d}/\mathrm{d}\tau)(\Delta G)}{\sin\left((\alpha-1)(t-\tau)+(\pi/2)\right)},\end{aligned}\tag{55}$$

needs to be satisfied.

Equation (55) can be written as $r$ equations:

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial a^\nu}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^\sigma+\left(\frac{\partial R_\mu}{\partial \tau}{}_aD_\tau^\beta a^\mu-\frac{\partial B}{\partial \tau}\right)\xi_0^\sigma+\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\dot{\xi}_0^\sigma\\&+R_\mu\left({}_aD_\tau^\beta\xi_\mu^\sigma-{}_aD_\tau^\beta\left(\dot{a}^\mu\xi_0^\sigma\right)+{}_aD_\tau^\beta\dot{a}^\mu\xi_0^\sigma\right)-\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\frac{\alpha-1}{\tan\left((\alpha-1)(t-\tau)+(\pi/2)\right)}\\&\xi_0^\sigma=-\frac{\dot{G}^\sigma}{\sin\left((\alpha-1)(t-\tau)+(\pi/2)\right)}.\end{aligned}\tag{56}$$

If $r=1$, equation (56) gives the fractional Noether identity based on EPFI.

*Criterion 12.* If transformation (7) is Noether quasi-symmetric, then the following $r$ equations,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\xi_0^\sigma+\int_a^\tau\left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right.\right.\\&\left.\left.-\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right){}_sD_b^\beta\left[R_\mu\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right]\right]\mathrm{d}s\right\}\\&+\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)+{}_\tau D_b^\beta\left(R_\mu\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right)\right]\\&\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)=-\dot{G}^\sigma,\quad(\sigma=1,2,\ldots,r),\end{aligned}\tag{57}$$

need to be satisfied.

## 4. Fractional Noether's Theorems under Quasi-Fractional Dynamics Models

Now, we prove Noether's theorems for fractional Birkhoffian systems under three quasi-fractional dynamics models.

### 4.1. Fractional Noether's Theorems Based on ERLFI

**Theorem 1.** *If transformation (7) is Noether symmetric of system (5) based on ERLFI, then*

$$I^\sigma=\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)(t-\tau)^{\alpha-1}\xi_0^\sigma+\int_a^\tau\left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)(t-s)^{\alpha-1}-\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right){}_sD_b^\beta\left[R_\mu(t-s)^{\alpha-1}\right]\right]\mathrm{d}s=c^\sigma,\quad(\sigma=1,2,\ldots,r),\tag{58}$$

*are $r$ linearly independent conserved quantities.*

*Proof.* From Definition 1, we get $\Delta S_R=0$, namely,

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^b\varepsilon_\sigma\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(R_{\mu a}D_\tau^\beta a^\mu-B\right)(t-\tau)^{\alpha-1}\xi_0^\sigma+\int_a^\tau\left(R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)(t-s)^{\alpha-1}-\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right){}_sD_b^\beta\left(R_\mu(t-s)^{\alpha-1}\right)\right)\mathrm{d}s\right]\right.\\&\left.+\left[\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)(t-\tau)^{\alpha-1}+{}_\tau D_b^\beta\left(R_\mu(t-\tau)^{\alpha-1}\right)\right]\left(\xi_\mu^\sigma-\dot{a}^\mu\xi_0^\sigma\right)\mathrm{d}\tau\right\}=0.\end{aligned}\tag{59}$$

By substituting (5) into the above formula and considering the arbitrariness of the integral interval and the independence of $\varepsilon_\sigma$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)(t-\tau)^{\alpha-1}\xi_0^\sigma + \int_a^\tau \left(R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)(t-s)^{\alpha-1} - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)_s D_b^\beta\left(R_\mu(t-s)^{\alpha-1}\right)\right)\mathrm{d}s\right] = 0. \tag{60}$$

So, Theorem 1 is proved. □

**Theorem 2.** *If transformation (7) is Noether quasi-symmetric of system (5) based on ERLFI, then*

$$\begin{aligned} I^\sigma = &\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)(t-\tau)^{\alpha-1}\xi_0^\sigma + \int_a^\tau \left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)(t-s)^{\alpha-1} - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)_s D_b^\beta\left[R_\mu(t-s)^{\alpha-1}\right]\right]\mathrm{d}s \\ &+ G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r), \end{aligned} \tag{61}$$

*are r linearly independent conserved quantities.*

*Proof.* Combining Definition 2 and formula (13), using equation (5), and considering the arbitrariness of the integral interval and the independence of $\varepsilon_\sigma$, the conclusion is obtained. □

*4.2. Fractional Noether's Theorems Based on EEFI*

**Theorem 3.** *If transformation (7) is Noether symmetric of system (18) based on EEFI, then*

$$\begin{aligned} I^\sigma = &\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)(\cosh t - \cosh\tau)^{\alpha-1}\xi_0^\sigma + \int_a^\tau \left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)(\cosh t - \cosh s)^{\alpha-1}\right. \\ &\left. - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)_s D_b^\beta\left[R_\mu(\cosh t - \cos hs)^{\alpha-1}\right]\right]\mathrm{d}s = c^\sigma, \quad (\sigma = 1, 2, \ldots, r), \end{aligned} \tag{62}$$

*are r linearly independent conserved quantities.*

**Theorem 4.** *If transformation (7) is Noether quasi-symmetric of system (18) based on EEFI, then*

$$\begin{aligned} I^\sigma = &\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)(\cosh t - \cosh\tau)^{\alpha-1}\xi_0^\sigma + \int_a^\tau \left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)(\cosh t - \cosh s)^{\alpha-1}\right. \\ &\left. - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)_s D_b^\beta\left[R_\mu(\cosh t - \cosh s)^{\alpha-1}\right]\right]\mathrm{d}s + G^\sigma = c^\sigma, \quad (\sigma = 1, 2, \ldots, r), \end{aligned} \tag{63}$$

*are r linearly independent conserved quantities.*

*4.3. Fractional Noether's Theorems Based on EPFI*

**Theorem 5.** *If transformation (7) is Noether symmetric of system (28) based on EPFI, then*

$$\begin{aligned} I^\sigma = &\left(R_{\mu a}D_\tau^\beta a^\mu - B\right)\sin\left((\alpha-1)(t-\tau) + \frac{\pi}{2}\right)\xi_0^\sigma + \int_a^\tau \left[R_{\mu a}D_s^\beta\left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)\sin\left((\alpha-1)(t-s) + \frac{\pi}{2}\right)\right. \\ &\left. - \left(\xi_\mu^\sigma - \dot{a}^\mu\xi_0^\sigma\right)_s D_b^\beta\left[R_\mu\sin\left((\alpha-1)(t-s) + \frac{\pi}{2}\right)\right]\right]\mathrm{d}s = c^\sigma, \quad (\sigma = 1, 2, \ldots, r). \end{aligned} \tag{64}$$

*are r linearly independent conserved quantities.*

**Theorem 6.** *If transformation (7) is Noether quasi-symmetric of system (28) based on EPFI, then*

$$I^{\sigma}=\left(R_{\mu a}D_{\tau}^{\beta}a^{\mu}-B\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\xi_{0}^{\sigma}+\int_{a}^{\tau}\left[R_{\mu a}D_{\tau}^{\beta}\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_{0}^{\sigma}\right)\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right.$$
$$\left.-\left(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_{0}^{\sigma}\right){}_{s}D_{b}^{\beta}\left[R_{\mu}\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right]\right]\mathrm{d}s+G^{\sigma}=c^{\sigma},\quad(\sigma=1,2,\ldots,r),\tag{65}$$

*are r linearly independent conserved quantities.*

Obviously, if $\beta\longrightarrow 1$, then Theorems 1–6 give Noether's theorems for quasi-fractional Birkhoffian systems. If $\alpha\longrightarrow 1$ and $\beta\longrightarrow 1$, Theorems 1–6 give Noether's theorems for classical Birkhoffian systems [58].

## 5. Examples

*5.1. Example 1.* Consider a fractional Birkhoffian system based on ERLFI. The Pfaff action is

$$S_{R}=\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left\{a^{2}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{3}-a^{2}a^{3}\right\}(t-\tau)^{\alpha-1}d\tau,\tag{66}$$

where the Birkhoffian is $B=a^{2}a^{3}$, and Birkhoff's functions are $R_{1}=a^{2}, R_{2}=0, R_{3}=a^{4}$, and $R_{4}=0$.

From equation (5), Birkhoff's equations are

$$\begin{aligned}{}_{\tau}D_{b}^{\beta}\left[a^{2}(t-\tau)^{\alpha-1}\right]&=0,\\ {}_{a}D_{\tau}^{\beta}a^{1}-a^{3}&=0,\\ -a^{2}(t-\tau)^{\alpha-1}+{}_{\tau}D_{b}^{\beta}\left[a^{4}(t-\tau)^{\alpha-1}\right]&=0,\\ {}_{a}D_{\tau}^{\beta}a^{3}&=0.\end{aligned}\tag{67}$$

According to (40), the Noether identity gives

$$\begin{aligned}&\left({}_{a}D_{\tau}^{\beta}a^{1}-a^{3}\right)\xi_{2}^{\sigma}-a^{2}\xi_{3}^{\sigma}+{}_{a}D_{\tau}^{\beta}a^{3}\xi_{4}^{\sigma}+a^{2}\left[{}_{a}D_{\tau}^{\beta}\xi_{1}^{\sigma}-{}_{a}D_{\tau}^{\beta}\left(\dot{a}^{1}\xi_{0}^{\sigma}\right)+{}_{a}D_{\tau}^{\beta}\dot{a}^{1}\xi_{0}^{\sigma}\right]\\&+a^{4}\left[{}_{a}D_{\tau}^{\beta}\xi_{3}^{\sigma}-{}_{a}D_{\tau}^{\beta}\left(\dot{a}^{3}\xi_{0}^{\sigma}\right)+{}_{a}D_{\tau}^{\beta}\dot{a}^{3}\xi_{0}^{\sigma}\right]+\left[a^{2}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{3}-a^{2}a^{3}\right]\dot{\xi}_{0}^{\sigma}-\frac{\alpha-1}{t-\tau}\\&\cdot\left(a^{2}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{3}-a^{2}a^{3}\right)\xi_{0}^{\sigma}=-\dot{G}^{\sigma}(t-\tau)^{1-\alpha}.\end{aligned}\tag{68}$$

Let

$$\begin{aligned}\xi_{0}^{\sigma}&=1,\\ \xi_{1}^{\sigma}&=a^{1},\\ \xi_{2}^{\sigma}&=1,\\ \xi_{3}^{\sigma}&=a^{3},\\ \xi_{4}^{\sigma}&=1,\\ G^{\sigma}&=0.\end{aligned}\tag{69}$$

By Theorem 2, we obtain

$$I=\left(a^{2}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{3}-a^{3}a^{2}\right)(t-\tau)^{\alpha-1}=\text{const.}\tag{70}$$

The conserved quantity (70) corresponds the Noether symmetry (69).

When $\beta\longrightarrow 1$, formula (70) is reduced to

$$I=\left(a^{2}a^{1}+a^{4}a^{3}-a^{2}a^{3}\right)(t-\tau)^{\alpha-1}=\text{const.}\tag{71}$$

Formula (71) is the conserved quantity of Birkhoffian system based on ERLFI.

When $\beta\longrightarrow 1$ and $\alpha\longrightarrow 1$, formula (70) is reduced to

$$I=a^{2}a^{1}+a^{4}a^{3}-a^{2}a^{3}=\text{const.}\tag{72}$$

Formula (72) is the classical conserved quantity.

*5.2. Example 2.* Consider a fractional Birkhoffian system based on EEFI. The Pfaff action is

$$S_E = \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left(a^2 + a^3\right) {}_aD_\tau^\beta a^1 + a^4 {}_aD_\tau^\beta a^3 - \left(a^2 a^3 + \left(a^3\right)^2\right) \right\} \cdot (\cosh t - \cosh \tau)^{\alpha - 1} \mathrm{d}\tau, \tag{73}$$

where $B = a^2 a^3 + (a^3)^2$, $R_1 = a^2 + a^3, R_2 = 0, R_3 = a^4$, and $R_4 = 0$.

From equation (18), Birkhoff's equations are

$$\begin{aligned} {}_\tau D_b^\beta \left[\left(a^2 + a^3\right)(\cosh t - \cosh \tau)^{\alpha - 1}\right] &= 0, \\ {}_aD_\tau^\beta a^1 - a^3 &= 0, \\ \left({}_aD_\tau^\beta a^1 - a^2 - 2a^3\right)(\cosh t - \cosh \tau)^{\alpha - 1} + {}_\tau D_b^\beta \left[a^4 (\cosh t - \cosh \tau)^{\alpha - 1}\right] &= 0, \\ {}_aD_\tau^\beta a^3 &= 0. \end{aligned} \tag{74}$$

According to (48), the Noether identity is

$$\begin{aligned} &\left({}_aD_\tau^\beta a^1 - a^3\right)\xi_2^\sigma + \left({}_aD_\tau^\beta a^1 - a^2 - 2a^3\right)\xi_3^\sigma + {}_aD_\tau^\beta a^3 \xi_4^\sigma + \left(a^2 + a^3\right)\left[{}_aD_\tau^\beta \xi_1^\sigma - {}_aD_\tau^\beta\left(\dot{a}^1 \xi_0^\sigma\right) + {}_aD_\tau^\beta \dot{a}^1 \xi_0^\sigma\right] \\ &+ a^4\left[{}_aD_\tau^\beta \xi_3^\sigma - {}_aD_\tau^\beta\left(\dot{a}^3 \xi_0^\sigma\right) + {}_aD_\tau^\beta \dot{a}^3 \xi_0^\sigma\right] + \left[\left(a^2 + a^3\right){}_aD_\tau^\beta a^1 + a^4 {}_aD_\tau^\beta a^3 - a^2 a^3 - \left(a^3\right)^2\right]\dot{\xi}_0^\sigma \\ &+ \frac{(\alpha - 1)\sinh \tau}{\cosh t - \cosh \tau}\left(\left(a^2 + a^3\right){}_aD_\tau^\beta a^1 + a^4 {}_aD_\tau^\beta a^3 - a^2 a^3 - \left(a^3\right)^2\right)\xi_0^\sigma = -\dot{G}^\sigma (\cosh t - \cosh \tau)^{1 - \alpha}. \end{aligned} \tag{75}$$

Let

$$\begin{aligned} \xi_0 &= 1, \\ \xi_1 &= a^1, \\ \xi_2 &= 0, \\ \xi_3 &= a^3, \\ \xi_4 &= 0, \\ G^\sigma &= 0. \end{aligned} \tag{76}$$

By Theorem 3, we obtain

$$\begin{aligned} I = &\left[\left(a^2 + a^3\right){}_aD_\tau^\beta a^1 + a^4 {}_aD_\tau^\beta a^3 - a^2 a^3 - \left(a^3\right)^2\right] \\ &\cdot (\cosh t - \cosh \tau)^{\alpha - 1} = \text{const.} \end{aligned} \tag{77}$$

The conserved quantity (77) corresponds to the Noether symmetry (76).

If $\beta \longrightarrow 1$, then we obtain

$$\begin{aligned} I = &\left[\left(a^2 + a^3\right)a^1 + a^4 a^3 - a^2 a^3 - \left(a^3\right)^2\right] \\ &\cdot (\cosh t - \cosh \tau)^{\alpha - 1} = \text{const.} \end{aligned} \tag{78}$$

Formula (78) is the conserved quantity of Birkhoffian system based on EEFI.

If $\beta \longrightarrow 1$ and $\alpha \longrightarrow 1$, then we obtain

$$I = \left(a^2 + a^3\right)a^1 + a^4 a^3 - a^2 a^3 - \left(a^3\right)^2 = \text{const.} \tag{79}$$

Formula (79) is the classical conserved quantity.

*5.3. Example 3.* Consider a fractional Birkhoffian system based on EPFI. The Pfaff action is

$$S_P = \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \left[a^3 {}_aD_\tau^\beta a^1 + a^4 {}_aD_\tau^\beta a^2 - \frac{1}{2}\left(a^3\right)^2 - \frac{1}{2}\left(a^4\right)^2\right] \cdot \sin\left((\alpha - 1)(t - \tau) + \frac{\pi}{2}\right) \right\} \mathrm{d}\tau, \tag{80}$$

where $B = (1/2)(a^3)^2 + (1/2)(a^4)^2$, $R_1 = a^3, R_2 = a^4$, and $R_3 = R_4 = 0$.

From equation (28), Birkhoff's equations are

$$
\begin{aligned}
{}_{\tau}D_{b}^{\beta}\left[a^{3}\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right]&=0,\\
{}_{\tau}D_{b}^{\beta}\left[a^{4}\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right]&=0,\\
{}_{a}D_{\tau}^{\beta}a^{1}-a^{3}&=0,\\
{}_{a}D_{\tau}^{\beta}a^{2}-a^{4}&=0.
\end{aligned}
\tag{81}
$$

According to (56), the Noether identity is

$$
\begin{aligned}
&\left({}_{a}D_{\tau}^{\beta}a^{1}-a^{3}\right)\xi_{3}+\left({}_{a}D_{\tau}^{\beta}a^{2}-a^{4}\right)\xi_{4}+a^{3}\left({}_{a}D_{\tau}^{\beta}\xi_{1}^{\sigma}-{}_{a}D_{\tau}^{\beta}\left(\dot{a}^{1}\xi_{0}^{\sigma}\right)+{}_{a}D_{\tau}^{\beta}\dot{a}^{1}\xi_{0}^{\sigma}\right)\\
&\quad+a^{4}\left({}_{a}D_{\tau}^{\beta}\xi_{2}^{\sigma}-{}_{a}D_{\tau}^{\beta}\left(\dot{a}^{2}\xi_{0}^{\sigma}\right)+{}_{a}D_{\tau}^{\beta}\dot{a}^{2}\xi_{0}^{\sigma}\right)+\left(a^{3}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{2}-\left(\frac{1}{2}\left(a^{3}\right)^{2}+\frac{1}{2}\left(a^{4}\right)^{2}\right)\right)\dot{\xi}_{0}^{\sigma}\\
&\quad-\frac{\alpha-1}{\tan\left((\alpha-1)(t-\tau)+(\pi/2)\right)}\left(a^{3}{}_{a}D_{\tau}^{\beta}a^{1}+a^{4}{}_{a}D_{\tau}^{\beta}a^{2}-\left(\frac{1}{2}\left(a^{3}\right)^{2}+\frac{1}{2}\left(a^{4}\right)^{2}\right)\right)\xi_{0}^{\sigma}\\
&=-\dot{G}^{\sigma}\frac{1}{\sin\left((\alpha-1)(t-\tau)+(\pi/2)\right)}.
\end{aligned}
\tag{82}
$$

Let

$$
\begin{aligned}
\xi_{0}&=0,\\
\xi_{1}&=a^{2},\\
\xi_{2}&=-a^{1},\\
\xi_{3}&=\xi_{4}=0,\\
G^{\sigma}&=0.
\end{aligned}
\tag{83}
$$

By Theorem 5, we obtain

$$
\begin{aligned}
I=\int_{a}^{\tau}&\left\{a^{3}{}_{a}D_{\tau}^{\beta}a^{2}\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right\}\\
&-a^{4}{}_{a}D_{\tau}^{\beta}a^{1}\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\\
&-a^{2}{}_{s}D_{b}^{\beta}\left[a^{3}\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right]\\
&+a^{1}{}_{s}D_{b}^{\beta}\left[a^{4}\sin\left((\alpha-1)(t-s)+\frac{\pi}{2}\right)\right]\Big\}\mathrm{d}s\\
=&\ \text{const.}
\end{aligned}
\tag{84}
$$

The conserved quantity (84) corresponds to the Noether symmetry (83).

When $\beta\longrightarrow 1$, formula (84) becomes

$$
I=\left(a^{3}a^{2}-a^{4}a^{1}\right)\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)=\text{const.}
\tag{85}
$$

Formula (85) is the conserved quantity of Birkhoffian system based on EPFI.

When $\beta\longrightarrow 1$ and $\alpha\longrightarrow 1$, formula (84) becomes

$$
I=a^{3}a^{2}-a^{4}a^{1}=\text{const.}
\tag{86}
$$

Formula (86) is the classical conserved quantity.

## 6. Conclusions

By introducing fractional calculus into the dynamic modeling of nonconservative systems, the dynamic behavior and physical process of complex systems can be described more accurately, which provides the possibility for the quantization of nonconservative problems. Compared with fractional models, the quasi-fractional model greatly simplifies the calculation of complex fractional-order calculus, so it can be used to study complex nonconservative dynamic systems more conveniently. The dynamics of Birkhoffian system is an extension of Hamiltonian mechanics, and the fractional Birkhoffian system is an extension of integer Birkhoffian system. Therefore, fractional Birkhoffian dynamics is a research field worthy of further study and full of vitality.

The main contributions of this paper are as follows. Firstly, based on three quasi-fractional dynamics models, the fractional Pfaff–Birkhoff principles and fractional Birkhoff's equations are established, in which the Pfaff action contains fractional-order derivative terms. Secondly, the fractional Noether symmetry is explored, and its definitions and criteria are

established. Thirdly, Noether's theorems for fractional Birkhoffian systems under three quasi-fractional dynamics models are proved, and fractional conservation laws are obtained.

Obviously, the results of the following two systems are special cases of this paper: (1) the quasi-fractional Birkhoffian systems based on quasi-fractional dynamics models, in which the Pfaff action contains only integer-order derivative terms; (2) the classical Birkhoffian systems under integer-order models. Therefore, our study is of great significance.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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