

## Research Article

# Graphs Associated with the Ideals of a Numerical Semigroup Having Metric Dimension 2

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Let  $\Lambda$  be a numerical semigroup and  $I \subset \Lambda$  be an irreducible ideal of  $\Lambda$ . The graph  $G_I(\Lambda)$  assigned to an ideal  $I$  of  $\Lambda$  is a graph with elements of  $(\Lambda \setminus I)^*$  as vertices, and any two vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in I$ . In this work, we give a complete characterization (up to isomorphism) of the graph  $G_I(\Lambda)$  having metric dimension 2.

## 1. Introduction

In algebraic combinatorics, the study of graphs associated with algebraic objects is one of the most important and fascinating fields of research. During the last couple of decades, a lot of research is carried out in this field. There are many papers on assigning graphs to rings, groups, and semigroups [1–6]. Several authors [7–13] studied different properties of these graphs including diameter, girth, domination, metric dimension, central sets, and planarity.

We start by defining some basic concept related to graph theory. A graph  $G = (V(G), E(G))$  has a vertex set  $V(G)$  and the edge set  $E(G)$ . The cardinality of the vertex set and edge set is called the order and size of  $G$ , respectively. A path in  $G$  is a sequence of edges  $u_1u_2, u_2u_3, \dots, u_{k-1}u_k$ . A graph  $G$  is connected if every pair of vertices  $x, y \in V(G)$  is connected by a path. The distance between two vertices  $x, y \in V(G)$  is denoted by  $d(x, y)$  and is the length of the shortest path between them. The diameter of  $G$  is denoted by  $d(G)$  and is defined as the largest distance between the vertices of  $G$ . Let  $U = \{u_1, u_2, \dots, u_r\}$  be an ordered subset of  $V(G)$ . Then, the  $r$ -tuple  $(d(u, u_1), d(u, u_2), \dots, d(u, u_r))$  is the representation  $u$  with respect to  $U$ . The vertex  $u$  is said to be resolved by  $U$  if  $(d(u, u_1), d(u, u_2), \dots, d(u, u_r)) \neq (d(v, u_1), d(v, u_2), \dots,$

$d(v, u_r))$  for any vertex  $v \in V(G)$ . The set  $U$  is called resolving set of  $G$  if distinct vertices of  $G$  have distinct representations with respect to  $U$ , and it is called basis of  $G$  if it is a resolving set with minimal cardinality. The metric dimension of  $G$ , denoted by  $\mu(G)$ , is the cardinality of basis. The concept of metric dimension was introduced by Slater [14] and later studied by Harary and Melter [15]. It has many applications, for example, robot navigation [16], pharmaceutical chemistry [17, 18], sonar and coast guard long range navigation [14], and combinatorial optimization [19].

Let  $\mathbb{N}$  be set of nonnegative integers. A subset  $\Lambda \subset \mathbb{N}$  is said to be numerical semigroup if the following holds:

- (1)  $0 \in \Lambda$
- (2)  $x + y \in \Lambda$  for all  $x, y \in \Lambda$
- (3)  $\mathbb{N} \setminus \Lambda$  is finite

It is easy to observe that the numerical semigroup is a commutative monoid. Thus, the set of numerical semigroups classifies the set of all submonoids of  $(\mathbb{N}, +)$ . The elements of the set  $\mathbb{N} \setminus \Lambda$  are called gaps of  $\Lambda$ , and the largest element of this set is known as Frobenius number. Note that every numerical semigroup is finitely generated; that is, there exist a set  $A = \{a_1, a_2, \dots, a_t\}$  such that  $\Lambda = \langle A \rangle = \{n_1a_1 + \dots,$

$n_t a_t; n_1, \dots, n_t \in \mathbb{N}$ ). Moreover, every numerical semigroup has a unique minimal system of generators. The cardinality of the minimal system of generators is called embedding dimension of  $\Lambda$ . It is denoted by  $e_\Lambda$ . A subset  $I$  of numerical semigroup  $\Lambda$  is ideal (integral ideal) of  $\Lambda$  if for all  $x \in I$  and  $s \in \Lambda$  and the element  $x + s \in I$ . An ideal  $I$  is called irreducibly ideal if it cannot be written as intersections of two or more than two ideals which contained it properly. For more details on theory of numerical semigroup, the interested readers can refer to [20].

Recently, several authors studied the metric dimension of the graphs associated with the algebraic objects. Solymaniyarniab et al. [21] gave some metric dimension formula for annihilator graphs. Bailey et al. [22] studied the constructions of resolving sets of Kneseer and Johnson graphs and provided bounds on their metric dimension. Faisal et al. [23] studied the metric dimension of the commuting graph of a dihedral group. The metric dimension of a zero-divisor graph of a commutative ring was studied in [13], while the metric dimension of a total graph of a finite commutative ring was studied in [24]. For more results on the metric dimension, we refer the readers to [25–30].

## 2. Notation and Preliminaries

Let  $\Lambda = \langle \mathcal{A} \rangle$  be a numerical semigroup, where  $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$  is the minimal system of generators of  $\Lambda$ . Then, every  $x \in \Lambda$  has a representation of the form  $u_1 a_1 + u_2 a_2 + \dots + u_n a_n$ , where  $u_1, u_2, \dots, u_n$  are nonnegative integers. Let  $1 \leq p \leq n$  be a fixed integer. We say that an element  $x \in \Lambda$  has a  $p$ -representation if there exist  $a_{i_1}, a_{i_2}, \dots, a_{i_p} \in \mathcal{A}$  and  $u_{i_1}, u_{i_2}, \dots, u_{i_p}$  positive integers such that  $x = u_{i_1} a_{i_1} + u_{i_2} a_{i_2} + \dots + u_{i_p} a_{i_p}$ ; that is,  $x$  can be written as linear combination of exactly  $p$  generator of  $\Lambda$ . Let  $\Lambda_p$  denote the set containing all the elements  $x \in \Lambda$ , which have a  $p$  representation. It is easy to see that

$$\Lambda = \bigcup_{p=1}^n \Lambda_p. \quad (1)$$

Note that an element  $x \in \Lambda$  may have more than one  $p$  representations. For an element  $x \in \Lambda_p$ , we use the notation  $\Sigma_p$  if it has a unique  $p$  representation and  $\Sigma_{p,1}, \Sigma_{p,2}, \dots, \Sigma_{p,r}$  if it has  $r$  number of  $p$  representations. Let  $\Sigma_p \in \Lambda_p$ , then there exist two  $p$ -tuples, the coefficients  $p$  tuple  $(u_{i_1}, u_{i_2}, \dots, u_{i_p}) \in \mathbb{Z}_{>0}^p$ , and the generators  $p$ -tuple  $(a_{i_1}, a_{i_2}, \dots, a_{i_p}) \in \mathbb{Z}_{>0}^p$  such that  $\Sigma_p = u_{i_1} a_{i_1} + u_{i_2} a_{i_2} + \dots + u_{i_p} a_{i_p}$ . We denote the coefficient and generators  $p$  tuple of an element  $\Sigma_p$  by  $c(\Sigma_p)$  and  $g(\Sigma_p)$ , respectively. Also, the  $j$ -th component of  $c(\Sigma_p)$  and  $g(\Sigma_p)$  is denoted by  $c_j(\Sigma_p)$  and  $g_j(\Sigma_p)$ , respectively. By using the above notations, for any  $x \in \Lambda$ , we define

$$\begin{aligned} \Lambda_p(x) &= \{\Sigma_p: \Sigma_p = x\}, \\ \Lambda(x) &= \bigcup_{p=1}^n \Lambda_p(x). \end{aligned} \quad (2)$$

For a  $p$ -representation  $\Sigma_p = u_{i_1} a_{i_1} + u_{i_2} a_{i_2} + \dots + u_{i_p} a_{i_p}$ , we set

$$\mathcal{B}(\Sigma_p) = \left\{ v_{i_1} a_{i_1} + v_{i_2} a_{i_2} + \dots + v_{i_p} a_{i_p}; 0 \leq v_{i_j} \leq u_{i_j}, 1 \leq j \leq p \right\}. \quad (3)$$

**Lemma 1.** *With the notations defined above, we have*

$$\mathcal{B}(x) = \bigcup_{\Sigma_p \in \Lambda(x)} \mathcal{B}(\Sigma_p). \quad (4)$$

*Proof.* The proof of this lemma follows from the definition of  $\mathcal{B}(x)$ .

Let  $\Lambda$  be a numerical semigroup and  $I \subset \Lambda$  be irreducible ideal of  $\Lambda$ . Binyamin et al. [31] assigned a graph to numerical semigroup  $\Lambda$  and studied its properties. Peng Xu et al. [32] assign a graph  $G_I(\Lambda)$  to the ideal  $I$  of numerical semigroup  $\Lambda$  with vertex set  $V(G_I(\Lambda)) = (\Lambda \setminus I)^*$  and two vertices  $x, y$  are adjacent if and only if  $x + y \in I$ . Barucci [33] showed that every irreducible ideal  $I$  of numerical semigroup  $\Lambda$  can be expressed in the form  $\Lambda \setminus B(x)$ , where  $B(x) = \{y \in \Lambda: x - y \in \Lambda\}$ , for some  $x \in \Lambda$ . Hence, the vertex set of the graph  $G_I(\Lambda)$  is the set  $\{v_i: i \in B^*(x)\}$  for some  $x \in \Lambda$ . Peng Xu et al. [32] proved that the graph  $G_I(\Lambda)$  is always connected and diameter 2. The aim of this paper is to find all the graphs  $G_I(\Lambda)$  having metric dimension 2. The following result by Chartrand et al. [18] gives bound on the order of graph with given metric dimension  $k$  and diameter  $d$ .  $\square$

**Theorem 1.** *Let  $G$  be a graph with metric dimension  $k$  and  $|V(G)| = n$ . Let  $d$  be the diameter of  $G$ . Then,  $|V(G)| \leq d^k + k$ .*

Hence, to find graphs  $G_I(\Lambda)$  with metric dimension 2, it is enough to classify all graphs  $G_I(\Lambda)$  of order less than or equal to 6. In the next section, we give bounds for the graphs  $G_I(\Lambda)$  of orders 4 and 5.

### 2.1. Bounds for the Graphs $G_I(\Lambda)$ of Orders 4 and 5

**Lemma 2.** *Let  $\Lambda = \langle \mathcal{A} \rangle$  be a numerical semigroup of embedding dimension  $n \geq 2$ . Then,  $|G_I(\Lambda)| \neq 4$ , if one of the following holds:*

- (1)  $\Lambda_p(x) \neq \emptyset$  for some  $p \geq 3$ .
- (2)  $|\Lambda_1(x)| \geq 3$ .
- (3)  $|\Lambda_2(x)| \geq 2$ .
- (4)  $\Lambda_1(x) = \emptyset$  and  $|\Lambda_2(x)| = 1$ .
- (5)  $|\Lambda_1(x)| = 2$  and  $|\Lambda_2(x)| = 1$ .

*Proof*

- (1) If  $\Lambda_p(x) \neq \emptyset$  for some  $p \geq 3$ , then there is a  $p$ -representation  $\Sigma_p$  of  $x$  in  $\Lambda_p(x)$ . This gives  $g_1(\Sigma_p), g_2(\Sigma_p), g_3(\Sigma_p), g_1(\Sigma_p) + g_2(\Sigma_p), g_1(\Sigma_p) + g_3(\Sigma_p), g_2(\Sigma_p) + g_3(\Sigma_p), x \in \mathcal{B}^*(\Sigma_p) \subseteq \mathcal{B}^*(x)$ . This implies  $|G_I(\Lambda)| \neq 4$ .

- (2) If  $|\Lambda_1(x)| \geq 3$ , then there are  $\Sigma_{1,1}, \Sigma_{1,2}, \dots, \Sigma_{1,r} \in \Lambda_1(x)$  with  $r \geq 3$ . We assume that  $g(\Sigma_{1,1}) < g(\Sigma_{1,2}) < \dots < g(\Sigma_{1,r})$  and then  $c(\Sigma_{1,1}) \geq 5$ . This gives  $g(\Sigma_{1,1}), 2g(\Sigma_{1,1}), 3g(\Sigma_{1,1}), 4g(\Sigma_{1,1}), 5g(\Sigma_{1,1}) \in \mathcal{B}^*(\Sigma_{1,1})$ , and therefore,  $|G_I(\Lambda)| \neq 4$ .
- (3) If  $|\Lambda_2(x)| \geq 2$ , then we have  $\Sigma_{2,1}, \Sigma_{2,2}, \dots, \Sigma_{2,s} \in \Lambda_2(x)$  with  $s \geq 2$ . One can easily see that  $\mathcal{B}^*(\Sigma_{2,1}) \cup \mathcal{B}^*(\Sigma_{2,2}) \cup \dots \cup \mathcal{B}^*(\Sigma_{2,s})$  must contain  $g_1(\Sigma_{2,1}), g_2(\Sigma_{2,1}), g_1(\Sigma_{2,2}), g_2(\Sigma_{2,2}), g_1(\Sigma_{2,1}) + g_2(\Sigma_{2,1})$  and  $g_1(\Sigma_{2,2}) + g_2(\Sigma_{2,2})$ . Therefore,  $|G_I(\Lambda)| \neq 4$ .
- (4) Lemma 1: If  $\Lambda_1(x) = \emptyset$  and  $|\Lambda_2(x)| = 1$  then there is the unique 2-representation  $\Sigma_2$  of  $x$ . Now if  $\Lambda_p(x) \neq \emptyset$  for some  $p \geq 3$  then from (1), it follows that  $|G_I(\Lambda)| \neq 4$ , and if  $\Lambda_p(x) = \emptyset$  for all  $p \geq 3$ , then gives  $\mathcal{B}^*(x) = \mathcal{B}^*(\Sigma_2)$ . So if  $c(\Sigma_2) = (1, 1)$ , then  $|G_I(\Lambda)| = 3$ ; otherwise,  $|G_I(\Lambda)| > 4$ . Consequently,  $|G_I(\Lambda)| \neq 4$ .
- (5) If  $|\Lambda_1(x)| = 2$  and  $|\Lambda_2(x)| = 1$ , then we can assume  $\Sigma_{1,1}, \Sigma_{1,2} \in \Lambda_1(x)$  and  $\Sigma_2 \in \Lambda_2(x)$ . This gives  $g(\Sigma_{1,1}), g(\Sigma_{1,2}), g_1(\Sigma_2), g_2(\Sigma_2), g_1(\Sigma_2) + g_2(\Sigma_2)$  are in  $\mathcal{B}^*(x)$ , and therefore,  $|G_I(\Lambda)| \neq 4$ .  $\square$

**Lemma 3.** Let  $\Lambda = \langle \mathcal{A} \rangle$  be a numerical semigroup of embedding dimension  $n \geq 2$ . Then,  $|G_I(\Lambda)| \neq 5$ , if one of the following holds:

- (1)  $\Lambda_p(x) \neq \emptyset$  for some  $p \geq 3$ .
- (2)  $|\Lambda_1(x)| \geq 2$ .
- (3)  $|\Lambda_2(x)| \geq 3$ .
- (4)  $|\Lambda_1(x)| = 1$  and  $|\Lambda_2(x)| = 2$ .

$$\begin{aligned} \mathcal{B}^*(x) &= \mathcal{B}^*(\Sigma_{1,1}) \cup \mathcal{B}^*(\Sigma_{1,2}) = \{g(\Sigma_{1,1}), 2g(\Sigma_{1,1}), \dots, c(\Sigma_{1,1}) \cdot g(\Sigma_{1,1})\} \cup \{g(\Sigma_{1,2}), 2g(\Sigma_{1,2}), \dots, c(\Sigma_{1,2}) \cdot g(\Sigma_{1,2})\} \\ &= \{g(\Sigma_{1,1}), 2g(\Sigma_{1,1}), \dots, c(\Sigma_{1,1}) \cdot g(\Sigma_{1,1}), g(\Sigma_{1,2}), 2g(\Sigma_{1,2}), \dots, c(\Sigma_{1,2}) \cdot g(\Sigma_{1,2})\}. \end{aligned} \quad (5)$$

We show that  $\{\mathcal{B}^*(\Sigma_{1,1}) \cap \mathcal{B}^*(\Sigma_{1,2})\} \setminus \{c(\Sigma_{1,2}) \cdot g(\Sigma_{1,2})\} = \emptyset$ . Let  $p \cdot g(\Sigma_{1,1}) = q \cdot g(\Sigma_{1,2})$  for some  $q < p < c(\Sigma_{1,1})$  with  $q = 2, 3, \dots, c(\Sigma_{1,2}) - 1$ . Then,  $p \cdot g(\Sigma_{1,1}) + (c(\Sigma_{1,1}) - p) \cdot g(\Sigma_{1,1}) = c(\Sigma_{1,1}) \cdot g(\Sigma_{1,1})$ , and we get  $q \cdot g(\Sigma_{1,2}) + (c(\Sigma_{1,1}) - p) \cdot g(\Sigma_{1,1}) = x$ . This gives  $\Lambda_2(x) \neq \emptyset$ , a contradiction. Therefore, we have  $|\mathcal{B}^*(x)| = c(\Sigma_{1,1}) + c(\Sigma_{1,2}) - 1$ . As  $|G_I(\Lambda)| = 4$ ,  $c(\Sigma_{1,1}) = 3$  and  $g(\Sigma_{1,2}) = 2$  is the only possibility. This gives case (2).

Let  $|\Lambda_1(x)| = 1 = |\Lambda_2(x)|$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 3$ . Then, we can assume  $\Sigma_1 \in \Lambda_1(x)$  and  $\Sigma_2 \in \Lambda_2(x)$  are the only possible 1-representation and 2-representation of  $x$ , respectively. By (2) in Lemma 2, we have  $c(\Sigma_2) = (1, 1)$ . In this case, it is easy to see that  $\mathcal{B}^*(x) = \{g(\Sigma_1), 2g(\Sigma_1), \dots, c(\Sigma_1) \cdot g(\Sigma_1), g_1(\Sigma_2), g_2(\Sigma_2)\}$ . Then,  $|\mathcal{B}^*(x)| = 4$  gives  $c(\Sigma_1) = 2$  and we get case (3).  $\square$

*Proof.* This lemma can be proved in a similar way as we proved Lemma 2.  $\square$

## 2.2. Computation of Irreducible Ideals for the Graphs $G_I(\Lambda)$ of Orders 4 and 5

**Lemma 4.** Let  $\Lambda = \langle \mathcal{A} \rangle$  be a numerical semigroup of embedding dimension  $n \geq 2$ . If  $|G_I(\Lambda)| = 4$ , then  $x$  is one of the following:

- (1)  $x = 4g(\Sigma_1)$ .
- (2)  $x = 3g(\Sigma_{1,1})$  and  $x = 2g(\Sigma_{1,2})$ .
- (3)  $x = 2g(\Sigma_1)$  and  $x = g_1(\Sigma_2) + g_2(\Sigma_2)$ .

*Proof.* If  $|G_I(\Lambda)| = 4$ , then from Lemma 2, it follows that  $x \in \Lambda$  satisfies one of the following conditions:

- $|\Lambda_1(x)| \leq 2$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .
- $|\Lambda_1(x)| = 1, |\Lambda_2(x)| = 1$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 3$ .

If  $|\Lambda_1(x)| = 1$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ , then  $x$  has a unique 1-representation  $\Sigma_1$ . By Lemma 1, we get  $\mathcal{B}^*(x) = \mathcal{B}^*(\Sigma_1) = \{g(\Sigma_1), 2g(\Sigma_1), \dots, c(\Sigma_1) \cdot g(\Sigma_1)\}$ . As  $|G_I(\Lambda)| = 4$ , it follows that  $c(\Sigma_1) = 4$ . This gives case (1).

Now if  $|\Lambda_1(x)| = 2$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ , then there are exactly two 1-representations, say  $\Sigma_{1,1}$  and  $\Sigma_{1,2}$  of  $x$ . Assume that  $g(\Sigma_{1,1}) < g(\Sigma_{1,2})$ , then  $c(\Sigma_{1,2}) < c(\Sigma_{1,1})$  and  $c(\Sigma_{1,1})$  is not a multiple of  $c(\Sigma_{1,2})$ . Then, it follows from Lemma 1 that

**Lemma 5.** Let  $\Lambda = \langle \mathcal{A} \rangle$  be a numerical semigroup of embedding dimension  $n \geq 2$ . If  $|G_I(\Lambda)| = 5$ , then  $x$  is one of the following:

- (1)  $x = 5g(\Sigma_1)$ .
- (2)  $x = 2g_1(\Sigma_2) + g_2(\Sigma_2)$ .
- (3)  $x = g_1(\Sigma_{2,1}) + g_2(\Sigma_{2,1})$  and  $x = g_1(\Sigma_{2,2}) + g_2(\Sigma_{2,2})$ .
- (4)  $x = 3g(\Sigma_1)$  and  $x = g_1(\Sigma_2) + g_2(\Sigma_2)$ .

*Proof.* Given that  $|G_I(\Lambda)| = 5$ , then from Lemma 5, it follows that  $x \in \Lambda$  satisfies one of the following conditions:

- $|\Lambda_1(x)| = 1$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .
- $|\Lambda_2(x)| \leq 2$  and  $\Lambda_p(x) = \emptyset, \forall p \neq 2$ .
- $|\Lambda_1(x)| = 1, |\Lambda_2(x)| = 1$  and  $\Lambda_p(x) = \emptyset, \forall p \geq 3$ .

These possibilities can be checked in a similar way as we did in Lemma 4 to get the required result.  $\square$

### 3. Graphs $G_I(\Lambda)$ with Metric Dimension 2

**Theorem 2.** *There are exactly 5 nonisomorphic graphs  $G_I(\Lambda)$  with metric dimension 2.*

We prove Theorem 2 in a sequence of following lemmas.

**Lemma 6.** *There are exactly 2 nonisomorphic graphs  $G_I(\Lambda)$  with 4 or less vertices and metric dimension 2.*

*Proof.* It is trivial to note that no such graph exists for  $|G_I(\Lambda)| = 2, 3$ .

Now if  $|G_I(\Lambda)| = 4$ , then from Lemma 4, we have the following possibilities:

- (1)  $x = 4g(\Sigma_1)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .
- (2)  $x = 3g(\Sigma_{1,1}) = 2g(\Sigma_{1,2})$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .
- (3)  $x = 2g(\Sigma_1) = g_1(\Sigma_2) + g_2(\Sigma_2)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 3$ .

If (1) holds, then  $I = \Lambda \setminus \mathcal{B}^*(4g(\Sigma_1))$ , and therefore,  $G_I(\Lambda)$  is isomorphic to the graph given in Figure 1. So metric dimension of  $G_I(\Lambda)$  is 2.

Now if (2) or (3) holds, then either  $I = \Lambda \setminus \mathcal{B}^*(3g(\Sigma_{1,1}))$  or  $I = \Lambda \setminus \mathcal{B}^*(2g(\Sigma_1))$ . In both cases,  $G_I(\Lambda)$  is isomorphic to the graph given in Figure 2, and therefore, metric dimension of  $G_I(\Lambda)$  is 2.  $\square$

**Lemma 7.** *There are exactly 3 nonisomorphic graphs  $G_I(\Lambda)$  with 5 vertices and metric dimension 2.*

*Proof.* If  $|G_I(\Lambda)| = 5$ , then from Lemma 5, we have the following possibilities:

- (1)  $x = 5g(\Sigma_1)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .
- (2)  $x = 2g_1(\Sigma_2) + g_2(\Sigma_2)$  with  $\Lambda_p(x) = \emptyset, \forall p \neq 2$ .
- (3)  $x = g_1(\Sigma_{2,1}) + g_2(\Sigma_{2,1}) = g_1(\Sigma_{2,2}) + g_2(\Sigma_{2,2})$  with  $\Lambda_p(x) = \emptyset, \forall p \neq 2$ .
- (4)  $x = 3g(\Sigma_1)$  and  $x = g_1(\Sigma_2) + g_2(\Sigma_2)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 3$ .

If (1) holds, then  $I = \Lambda \setminus \mathcal{B}^*(5g(\Sigma_1))$ , and therefore,  $G_I(\Lambda)$  is isomorphic to the graph given in Figure 3.

Now, if (2) holds, then  $I = \Lambda \setminus \mathcal{B}^*(2g_1(\Sigma_2) + g_2(\Sigma_2))$ , and therefore,  $G_I(\Lambda)$  is isomorphic to the graph given in Figure 4.

If (3) or (4) holds, then  $I = \Lambda \setminus \mathcal{B}^*(g_1(\Sigma_{2,1}) + g_2(\Sigma_{2,1}))$  or  $I = \Lambda \setminus \mathcal{B}^*(3g(\Sigma_1))$ . In both cases,  $G_I(\Lambda)$  is isomorphic to the graph given in Figure 5.

For all these 3 cases, one can easily show that metric dimension of  $G_I(\Lambda)$  is 2.

Finally, it is required to check all the graphs  $G_I(\Lambda)$  of order six having metric dimension 2. Binyamin et al. [34] proved that if  $|G_I(\Lambda)| = 6$ , then  $G_I(\Lambda)$  is isomorphic to one of the graphs given in Table 1. Now, it is easy to see that all the graphs given in Table 1 have metric dimension 3.  $\square$

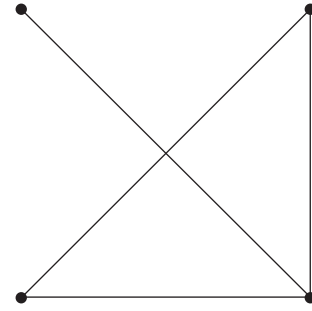


FIGURE 1: Graphs  $G_I(\Lambda)$  When  $x = 4g(\Sigma_1)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .

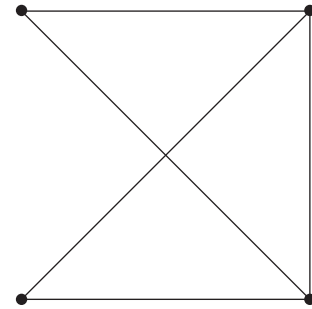


FIGURE 2: Graph  $G_I(\Lambda)$  for the remaining two cases.

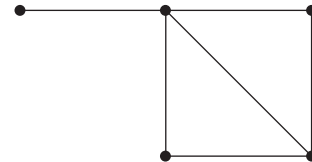


FIGURE 3: Graphs  $G_I(\Lambda)$  When  $x = 5g(\Sigma_1)$  with  $\Lambda_p(x) = \emptyset, \forall p \geq 2$ .

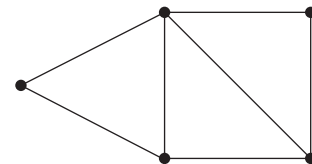


FIGURE 4: Graphs  $G_I(\Lambda)$  When  $x = 2g(\Sigma_2)$  with  $\Lambda_p(x) = \emptyset, \forall p \neq 2$ .

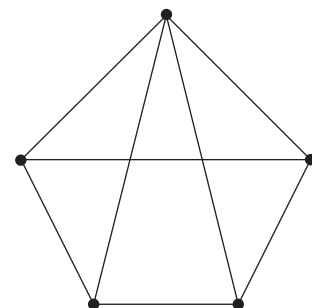


FIGURE 5: Graph  $G_I(\Lambda)$  for the remaining two cases.

TABLE 1: Graph  $G_I(\Lambda)$  of order 6 up to isomorphism.

Type	Degree sequence	Graph
1	(1, 2, 3, 3, 4, 5)	
2	(2, 3, 3, 4, 5, 5)	
3	(2, 3, 4, 4, 4, 5)	
4	(3, 3, 4, 4, 4, 5)	
5	(3, 4, 4, 4, 4, 5)	

TABLE 1: Continued.

Type	Degree sequence	Graph
6	(4, 4, 4, 4, 5, 5)	

**Data Availability**

No data are required for the study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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