Research Article

The Maximal ABC Index of the Corona of Two Graphs

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Received 11 November 2020; Revised 1 March 2021; Accepted 17 March 2021; Published 2 April 2021

Academic Editor: Bekir Sahin

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Let $G_1 \circ G_2$ be the corona of two graphs G_1 and G_2 which is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining the i th vertex of G_1 to every vertex in the i th copy of G_2 . The atom-bond connectivity index (ABC index) of a graph G is defined as $ABC(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u) + d_G(v) - 2)/d_G(u)d_G(v)}$, where $E(G)$ is the edge set of G and $d_G(u)$ and $d_G(v)$ are degrees of vertices u and v , respectively. For the ABC indices of $G_1 \circ G_2$ with G_1 and G_2 being connected graphs, we get the following results. (1) Let G_1 and G_2 be connected graphs. The ABC index of $G_1 \circ G_2$ attains the maximum value if and only if both G_1 and G_2 are complete graphs. If the ABC index of $G_1 \circ G_2$ attains the minimum value, then G_1 and G_2 must be trees. (2) Let T_1 and T_2 be trees. Then, the ABC index of $T_1 \circ T_2$ attains the maximum value if and only if T_1 is a path and T_2 is a star.

1. Introduction

Graph theory has been applied in many engineering fields such as mechanical design and manufacturing and chemical engineering. In the graph theory, the link in the mechanism can be regarded as the vertex, kinematic pair can be regarded as an edge, and then the topological configuration of the mechanism is abstracted as the graph. Therefore, the nature and characteristics of the mechanism can be analyzed by relevant graph theory. In chemical engineering, if a chemical molecule is regarded as a two-dimensional graph, the graph's vertices represent atoms, and edges represent chemical bonds, then the graph determines the topological properties of the given molecule.

Molecular descriptors play an important role in chemistry and pharmacology. Among these molecular descriptors, so-called topological indices play a significant role. Topological indices are the mathematical tools that correlate the chemical structure with various physical properties, chemical reactivity, or biological activity numerically. And the topological indices have been widely applied in the study of the stability of alkanes and the strain energy of cycloalkanes. In the field of pharmaceutical chemistry and

bioinformatics, topological index can be used to encode the chemical structure. This encode strategy provides the annotation, comparison, rapid collection, mining, and retrieval of chemical structures within large databases. Afterward, topological indices can be used to look for quantitative structure-activity relationships and quantitative structure-property relationships. In QSAR/QSPR studies, the biological activities of compounds can be predicted according to their topological indices, such as Zagreb, Randic, and the atom-bond connectivity indices.

The topological indices can be classified by the structural properties of graphs used for their calculation. The atom-bond connectivity index, which was proposed by Estrada et al. in 1998, is a vertex-degree-based graph topological index.

We consider finite undirected connected graphs without loops or multiple edges. Let G_1 and G_2 be two such graphs. The corona of G_1 and G_2 , denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining the i th vertex of G_1 to every vertex in the i th copy of G_2 (as shown in Figure 1). Corona graphs were introduced by Frucht and Harary in 1970 [1].

FIGURE 1: Corona $G_1 \circ G_2$ of G_1 and G_2 . (a) G_1 . (b) G_2 . (c) $G_1 \circ G_2$.

Let $G = (V, E)$ be a connected graph with vertex set $V = V(G)$ and arc set $E = E(G)$. The atom-bond connectivity index (ABC index) of G is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u)d_G(v)}}, \quad (1)$$

where $d_G(u)$ (or d_u) and $d_G(v)$ (or d_v) are degrees of vertices u and v , respectively.

Let G be a graph. For an edge $e = uv \in E(G)$ and an edge set $E_0 \subseteq E(G)$, denote

$$ABC_G(e) = \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}, \quad (2)$$

$$ABC_G(E_0) = \sum_{e \in E_0} ABC_G(e).$$

We say that $ABC_G(e)$ is the ABC index of e and $ABC_G(E_0)$ is the ABC index of E_0 .

Throughout the paper, we use K_n , S_n , and P_n to denote the complete graph, the star, and the path of order n , respectively.

The ABC index attracted a lot of attention in the last few years. Several properties of ABC index were established. In particular, if a new edge is inserted into G , then ABC index necessarily increases (see Lemma 1).

Lemma 1 (see [2]). *Let G be a simple graph with nonadjacent vertices i and j . Then,*

$$ABC(G + \{ij\}) > ABC(G). \quad (3)$$

It is evident that K_n has the maximal ABC index, whereas the connected graph with the minimal ABC index must be a tree (see [2, 3]). The smallest ABC index of trees with n pendant vertices was characterized in [4]. In contrast to the minimal case, the tree with the maximal ABC index was easily identified as the star (see [5]). In [6], the maximum and minimum ABC indices of all unicyclic graphs and unicyclic chemical graphs were obtained, and the corresponding extremal graphs were also characterized. In [7, 8], the maximum values of the ABC indices in the class of all n -vertex bicyclic and tricyclic graphs were presented, respectively.

Recently, some researchers have paid more attention to the ABC spectral radius which is associated with ABC energy.

Chen [9] characterized the graphs with extremal ABC spectral radius for a class of given graphs. Lin et al. [10] determined the trees with the third, fourth, and fifth largest ABC spectral radii.

For corona graph, Bian et al. [11] considered some Wiener-type indices of the corona graphs. Lu and Xue [12] studied the Kirchhoff index of two corona graphs. In [13], the lower and upper bounds for ABC indices of edge corona product of graphs were given. In [14], the extremal edge-version ABC index of some graph operations was given. For more information, see [15–19].

Motivated by this, it is interesting to determine the extremal graphs among the set $\{G_1 \circ G_2 | G_1 \text{ and } G_2 \text{ are connected graphs (trees)}\}$. From Lemma 1, the problem is simple if G_1 and G_2 are connected graphs, so we give some conclusions directly. We mainly consider the problem for the case that G_1 and G_2 are trees. The main results are as follows.

- (1) Let G_1 and G_2 be connected graphs. The ABC index of $G_1 \circ G_2$ attains the maximum value if and only if both G_1 and G_2 are complete graphs. If the ABC index of $G_1 \circ G_2$ attains the minimum value, then G_1 and G_2 must be trees.
- (2) Let T_1 and T_2 be trees. The ABC index of $T_1 \circ T_2$ attains the maximum value if and only if T_1 is a path and T_2 is a star.

2. Preliminaries

The following lemmas are known somehow (for example, see [7, 8]).

Lemma 2. *Let $f(x, y) = \sqrt{(x + y - 2/xy)}$ with $x, y > 1$. Then, $f(x, y)$ strictly decreases with x for fixed $y \geq 2$, and the function $\partial f / \partial x$ increases with x . Similarly, the function $f(x, y)$ strictly decreases with y for fixed $x \geq 2$, and the function $\partial f / \partial y$ increases with y .*

Lemma 3. *If G is a graph with $\delta(G) \geq 2$ ($\delta(G)$ is the minimum degree of G), then for any $xy \in E(G)$, $\sqrt{(d_x + d_y - 2/d_x d_y)} \leq (\sqrt{2}/2)$. Equality holds if and only if $d_x = 2$ or $d_y = 2$.*

Lemma 4. *Let $f(x, y) = \sqrt{(x + y - 2/xy)}$ with $x, y > 1$ and $g(x, y, z) = f(x, y) + f(z, y) - f(x + 1, y) - f(z - 1, y)$.*

If $x \geq z \geq 2$, then $g(x, y, z) < 0$.

3. The Bounds of ABC Indices of $G_1 \circ G_2$

Let G_1 and G_2 be two simple connected graphs of orders n_1 and n_2 , respectively, and $G = G_1 \circ G_2$. For a vertex $v \in V(G_1)$, we use G_2^v to denote the copy of G_2 in G which attaches to the vertex v of G_1 . Denote

$$\begin{aligned} E_1(G) &= E(G_1), \\ E_2(G) &= \cup_{v \in V(G_1)} \{vu | u \in V(G_2^v)\}, \\ E_3(G) &= \cup_{v \in V(G_1)} E(G_2^v). \end{aligned} \quad (5)$$

Then, the following four conditions hold:

- (C1) $V(G) = V(G_1) \cup (\cup_{v \in V(G_1)} V(G_2^v))$.
- (C2) $E(G) = E_1(G) \cup E_2(G) \cup E_3(G)$, and $E_k(G) \cap E_l(G) = \emptyset$ for $k, l = 1, 2, 3$ and $k \neq l$.
- (C3) For $v \in V(G_1)$ and $u \in V(G_2^v)$, $d_G(v) = d_{G_1}(v) + n_2$ and $d_G(u) = d_{G_2}(u) + 1$.
- (C4) $ABC(G) = ABC_G(E_1(G)) + ABC_G(E_2(G)) + ABC_G(E_3(G))$.

From (C4) and (C3), the following theorem is obtained immediately, and we omit the proof.

Theorem 1. Let G_1 and G_2 be two simple connected graphs of orders n_1 and n_2 , respectively, and $G = G_1 \circ G_2$. Then,

$$\begin{aligned} ABC(G) &= \sum_{uv \in E(G_1)} \sqrt{\frac{d_{G_1}(u) + d_{G_1}(v) + 2n_2 - 2}{(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}} \\ &+ \sum_{v \in V(G_1), u \in V(G_2^v)} \sqrt{\frac{d_{G_1}(v) + d_{G_2}(u) + n_2 - 1}{(d_{G_1}(v) + n_2)(d_{G_2}(u) + 1)}} \\ &+ n_1 \sum_{uv \in E(G_2)} \sqrt{\frac{d_{G_2}(u) + d_{G_2}(v)}{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}}. \end{aligned} \quad (6)$$

By Lemma 1 and Theorem 1, the following results are clear.

Lemma 5. Let G_1 and G_2 be two simple connected graphs of orders $n_1 \geq 3$ and $n_2 \geq 3$, respectively. Then, for any edge $x_1x_2 \in E(G_1)$,

$$ABC(G_1 \circ G_2) > ABC[(G_1 - x_1x_2) \circ G_2]. \quad (7)$$

Lemma 6. Let G_1 and G_2 be two simple connected graphs of orders $n_1 \geq 3$ and $n_2 \geq 3$, respectively. Then, for any edge $x_1x_2 \in E(G_2)$,

$$ABC(G_1 \circ G_2) > ABC[G_1 \circ (G_2 - x_1x_2)]. \quad (8)$$

Theorem 2. Let G_1 and G_2 be two simple connected graphs of orders $n_1 \geq 3$ and $n_2 \geq 3$, respectively. Then,

$$\begin{aligned} ABC(G_1 \circ G_2) &\leq ABC(K_{n_1} \circ K_{n_2}) \\ &= \frac{n_1(n_1 - 1)}{2} \frac{\sqrt{2(n_1 + n_2 - 1) - 2}}{n_1 + n_2 - 1} \\ &+ n_1 n_2 \sqrt{\frac{n_1 + 2n_2 - 3}{n_2(n_1 + n_2 - 1)}} \\ &+ \frac{n_1 n_2(n_2 - 1)}{2} \frac{\sqrt{2n_2 - 2}}{n_2}, \end{aligned} \quad (9)$$

and the equality holds if and only if $G_1 = K_{n_1}$ and $G_2 = K_{n_2}$.

Lemmas 5 and 6 show that deleting an edge in one of the graphs (G_1 and G_2) will decrease the ABC index of $G_1 \circ G_2$. Consequently, we get the necessary condition for the ABC indices of $G_1 \circ G_2$ to be the minimum.

Theorem 3. If the ABC index of $G_1 \circ G_2$ attains the minimum value, then G_1 and G_2 must be trees.

4. The Upper Bound of ABC Indices of $T_1 \circ T_2$

Since the minimal ABC trees are not unique (see [2, 5]), it seems to be difficult to characterize the corona of two trees with minimal ABC index. We leave the problem as a future task. In this section, we are going to give the upper bound of ABC indices for $T_1 \circ T_2$.

Lemma 7. Let T_1 be any tree of order $n_1 \geq 3$ and T_2 be a tree of order $n_2 \geq 4$ as depicted in Figure 2, where u_{1i} and u_{2j} are pendant vertices for $i = 1, \dots, s$ and $j = 1, \dots, t$ with $s \geq 1$, $t \geq 1$, $u_1 \neq u_2$, and $d_{T_2}(u_2) \geq d_{T_2}(u_1)$. Let $T_3 = T_2 - u_1u_{1s} + u_2u_{1s}$. Then,

$$ABC(E_2(T_1 \circ T_2)) < ABC(E_2(T_1 \circ T_3)). \quad (10)$$

Proof. Denote $T = T_1 \circ T_2$, and $T' = T_1 \circ T_3$. Let v be any vertex of T_1 . We can get the result if we prove that

$$\sum_{u \in V(T_2)} ABC_T(vu) < \sum_{u \in V(T_3)} ABC_{T'}(vu). \quad (11)$$

Note that both trees T_2 and T_3 have the same vertex set, and except u_1 and u_2 , the other vertices have the same edges in T_2 and T_3 . So,

FIGURE 2: Tress T_2 and T_3 . (a) Tree T_2 . (b) Tree T_3 .

$$\begin{aligned}
& \sum_{u \in V(T_2^v)} ABC_T(vu) - \sum_{u \in V(T_3^v)} ABC_{T'}(vu) \\
&= \sqrt{\frac{(d_{T_1}(v) + n_2) + (d_{T_2}(u_1) + 1) - 2}{(d_{T_1}(v) + n_2)(d_{T_2}(u_1) + 1)}} \\
&+ \sqrt{\frac{(d_{T_1}(v) + n_2) + (d_{T_2}(u_2) + 1) - 2}{(d_{T_1}(v) + n_2)(d_{T_2}(u_2) + 1)}} \\
&- \sqrt{\frac{(d_{T_1}(v) + n_2) + (d_{T_3}(u_1) + 1) - 2}{(d_{T_1}(v) + n_2)(d_{T_3}(u_1) + 1)}} \\
&- \sqrt{\frac{(d_{T_1}(v) + n_2) + (d_{T_3}(u_2) + 1) - 2}{(d_{T_1}(v) + n_2)(d_{T_3}(u_2) + 1)}}. \quad (12)
\end{aligned}$$

Note that $d_{T_3}(u_1) = d_{T_2}(u_1) - 1$, $d_{T_3}(u_2) = d_{T_2}(u_2) + 1$, and $d_{T_2}(u_2) \geq d_{T_2}(u_1) \geq 2$. Then, by Lemma 4 (taking $x = d_{T_2}(u_2) + 1$, $y = d_{T_1}(v) + n_2$, and $z = d_{T_2}(u_1) + 1$),

$$\sum_{u \in V(T_2^v)} ABC_T(vu) - \sum_{u \in V(T_3^v)} ABC_{T'}(vu) < 0. \quad (13)$$

The lemma holds.

Using Lemma 7 repeatedly, we can obtain the following result. $\square$

Corollary 1. Let T_1 and T_2 be two trees of orders $n_1 \geq 3$ and $n_2 \geq 3$, respectively. Then,

$$ABC_T(E_2(T_1 \circ T_2)) \leq ABC_{T'}(E_2(T_1 \circ S_{n_2})), \quad (14)$$

and the equality holds if and only if $T_2 = S_{n_2}$.

Theorem 4. Let T_1 and T_2 be two trees of orders $n_1 \geq 3$ and $n_2 \geq 3$, respectively. Then,

$$ABC(T_1 \circ T_2) \leq ABC(T_1 \circ S_{n_2}), \quad (15)$$

and the equality holds if and only if $T_2 = S_{n_2}$.

Proof. Denote $T = T_1 \circ T_2$ and $T' = T_1 \circ S_{n_2}$. Note $ABC_T(E_1(T)) = ABC_{T'}(E_1(T'))$. By Corollary 1, we can get the result if we show $ABC_T(E_3(T)) \leq ABC_{T'}(E_3(T'))$.

Note that

$$\begin{aligned}
ABC_T(E_3(T)) &= \sum_{uv \in E(T_2)} \sqrt{\frac{(d_{T_2}(u) + 1) + (d_{T_2}(v) + 1) - 2}{(d_{T_2}(u) + 1)(d_{T_2}(v) + 1)}}, \\
ABC_{T'}(E_3(T')) &= \sum_{uv \in E(S_{n_2})} \sqrt{\frac{(d_{S_{n_2}}(u) + 1) + (d_{S_{n_2}}(v) + 1) - 2}{(d_{S_{n_2}}(u) + 1)(d_{S_{n_2}}(v) + 1)}} \\
&= (n_2 - 1) \sqrt{\frac{1}{2}}. \quad (16)
\end{aligned}$$

By Lemma 3,

$$ABC_T(E_3(T)) \leq \sum_{uv \in E(T_2)} \sqrt{\frac{1}{2}} = (n_2 - 1) \sqrt{\frac{1}{2}} = ABC_{T'}(E_3(T')), \quad (17)$$

and equality holds if and only if $d_{T_2}(u) + 1 = 2$ or $d_{T_2}(v) + 1 = 2$ for each edge $uv \in E(T_2)$, that is, $T_2 = S_{n_2}$. The result follows now. $\square$

Theorem 5. Let T_1 and T_1' be two trees as depicted in Figure 3, which contain T_0 as a subtree, where $d_v = d_{T_1}(v) \geq 3$, and $l \geq k \geq 1$. Let $n_2 \geq 4$. Then,

FIGURE 3: Trees T_1 and T'_1 . (a) Tree T_1 . (b) Tree T'_1 .

$$ABC(T_1 \circ S_{n_2}) < ABC(T'_1 \circ S_{n_2}). \quad (18)$$

Proof. Denote $T = T_1 \circ S_{n_2}$ and $T' = T'_1 \circ S_{n_2}$. Note that $ABC_T(E_3(T)) = ABC_{T'}(E_3(T'))$. We can get the result if we show

$$ABC_T(E_1(T)) + ABC_T(E_2(T)) < ABC_{T'}(E_1(T')) + ABC_{T'}(E_2(T')). \quad (19)$$

Firstly, we consider $ABC_T(E_2(T)) - ABC_{T'}(E_2(T'))$. Let u be any vertex of T_1 (T'_1). If $w \in V(S_{n_2})$ is a pendant vertex of S_{n_2} , then the ABC index of uw is $\sqrt{1/2}$. If u is a pendant vertex of T_1 and T'_1 and $w \in V(S_{n_2})$ is the central vertex of S_{n_2} , then the ABC index of uw has no change. So, we only need to calculate the ABC index of uw , where u is not a pendant vertex of T'_1 and w is the central vertex of S_{n_2} .

Denote $a = n_2 + 2$, $b = d_v + n_2$, $A = N_{T_0}(v)$. Then, $b > a \geq 6$ and

$$\begin{aligned} & ABC_T(E_2(T)) - ABC_{T'}(E_2(T')) \\ &= \sqrt{\frac{(d_v + n_2) + n_2 - 2}{(d_v + n_2)n_2}} + \sqrt{\frac{(1 + n_2) + n_2 - 2}{(1 + n_2)n_2}} \\ &\quad - \sqrt{\frac{(d_v + n_2 - 1) + n_2 - 2}{(d_v + n_2 - 1)n_2}} - \sqrt{\frac{(2 + n_2) + n_2 - 2}{(2 + n_2)n_2}} \\ &= \sqrt{\frac{a + b - 4}{b(a - 2)}} + \sqrt{\frac{2a - 5}{(a - 1)(a - 2)}} - \sqrt{\frac{a + b - 5}{(b - 1)(a - 2)}} - \sqrt{\frac{2}{a}}. \end{aligned} \quad (20)$$

Combining the situation of $ABC_T(E_1(T)) - ABC_{T'}(E_1(T'))$, we consider the following three cases. $\square$

Case 1. $k = l = 1$.

By Lemma 2,

$$\begin{aligned} & ABC_T(E_1(T)) - ABC_{T'}(E_1(T')) \\ &= 2\sqrt{\frac{(1 + n_2) + (d_v + n_2) - 2}{(1 + n_2)(d_v + n_2)}} \\ &\quad + \sum_{x \in A} \sqrt{\frac{(d_x + n_2) + (d_v + n_2) - 2}{(d_x + n_2)(d_v + n_2)}} \\ &\quad - \sqrt{\frac{(2 + n_2) + (d_v + n_2 - 1) - 2}{(2 + n_2)(d_v + n_2 - 1)}} \\ &\quad - \sqrt{\frac{(1 + n_2) + (2 + n_2) - 2}{(1 + n_2)(2 + n_2)}} \\ &\quad - \sum_{x \in A} \sqrt{\frac{(d_x + n_2) + (d_v + n_2 - 1) - 2}{(d_x + n_2)(d_v + n_2 - 1)}} \\ &< 2\sqrt{\frac{(1 + n_2) + (d_v + n_2) - 2}{(1 + n_2)(d_v + n_2)}} \\ &\quad - \sqrt{\frac{(2 + n_2) + (d_v + n_2 - 1) - 2}{(2 + n_2)(d_v + n_2 - 1)}} \\ &\quad - \sqrt{\frac{(1 + n_2) + (2 + n_2) - 2}{(1 + n_2)(2 + n_2)}} \\ &= 2\sqrt{\frac{a + b - 3}{b(a - 1)}} - \sqrt{\frac{a + b - 3}{a(b - 1)}} - \sqrt{\frac{2a - 3}{a(a - 1)}}, \\ & ABC_T(E_1(T)) + ABC_T(E_2(T)) - ABC_{T'}(E_1(T')) \\ &\quad - ABC_{T'}(E_2(T')) \\ &< 2\sqrt{\frac{a + b - 3}{b(a - 1)}} - \sqrt{\frac{a + b - 3}{a(b - 1)}} - \sqrt{\frac{2a - 3}{a(a - 1)}} \\ &\quad + \sqrt{\frac{a + b - 4}{b(a - 2)}} + \sqrt{\frac{2a - 5}{(a - 1)(a - 2)}} \\ &\quad - \sqrt{\frac{a + b - 5}{(b - 1)(a - 2)}} - \sqrt{\frac{2}{a}} \\ &\triangleq f_1(a, b). \end{aligned} \quad (21)$$

The function $f_1(a, b)$ has the following properties :

- (1) The function $f_1(a, b)$ is a strictly decreasing function on b for fixed $a \geq 6$.

We can see the result from the graph of $f_1(a, b)$. For example, the graphs of $f_1(6, b)$ and $f_1(10, b)$ are as follows (as depicted in Figures 4 and 5).

So,

$$\begin{aligned} f_1(a, b) &\leq f_1(a, a+1) \\ &= 2\sqrt{\frac{2}{a+1}} - \sqrt{\frac{2a-2}{a^2}} - \sqrt{\frac{2a-3}{a(a-1)}} \\ &\quad + \sqrt{\frac{2a-3}{(a+1)(a-2)}} + \sqrt{\frac{2a-5}{(a-1)(a-2)}} - 2\sqrt{\frac{2}{a}}. \end{aligned} \quad (22)$$

- (2) The maximum value of the function $f_1(a, a+1)$ is attained when a is sufficiently large.

We can get this result from the graph (as depicted in Figure 6) of $f_1(a, a+1)$.

Then,

$$f_1(a, a+1) < \lim_{a \rightarrow +\infty} f_1(a, a+1) = 0. \quad (23)$$

So, for $b > a \geq 6$, $f_1(a, b) < 0$ and the theorem follows for Case 1.

Case 2. $k = 1$ and $l \geq 2$.

By Lemma 2,

$$\begin{aligned} &ABC_T(E_1(T)) - ABC_{T'}(E_1(T')) \\ &= \sqrt{\frac{(d_v + n_2) + (1 + n_2) - 2}{(d_v + n_2)(1 + n_2)}} + \sqrt{\frac{(d_v + n_2) + (2 + n_2) - 2}{(d_v + n_2)(2 + n_2)}} \\ &\quad + \sum_{x \in A} \sqrt{\frac{(d_v + n_2) + (d_x + n_2) - 2}{(d_v + n_2)(d_x + n_2)}} \\ &\quad - \sqrt{\frac{(d_v + n_2 - 1) + (2 + n_2) - 2}{(d_v + n_2 - 1)(2 + n_2)}} - \sqrt{\frac{(2 + n_2) + (2 + n_2) - 2}{(2 + n_2)(2 + n_2)}} \\ &\quad - \sum_{x \in A} \sqrt{\frac{(d_v + n_2 - 1) + (d_x + n_2) - 2}{(d_v + n_2 - 1)(d_x + n_2)}} \\ &< \sqrt{\frac{(d_v + n_2) + (1 + n_2) - 2}{(d_v + n_2)(1 + n_2)}} + \sqrt{\frac{(d_v + n_2) + (2 + n_2) - 2}{(d_v + n_2)(2 + n_2)}} \\ &\quad - \sqrt{\frac{(d_v + n_2 - 1) + (2 + n_2) - 2}{(d_v + n_2 - 1)(2 + n_2)}} - \sqrt{\frac{(2 + n_2) + (2 + n_2) - 2}{(2 + n_2)(2 + n_2)}} \\ &= \sqrt{\frac{a+b-3}{b(a-1)}} + \sqrt{\frac{a+b-2}{ab}} - \sqrt{\frac{a+b-3}{a(b-1)}} - \sqrt{\frac{2a-2}{a^2}}, \\ &ABC_T(E_1(T)) + ABC_T(E_2(T)) - ABC_{T'}(E_1(T')) \\ &\quad - ABC_{T'}(E_2(T')) \\ &< \sqrt{\frac{a+b-3}{b(a-1)}} + \sqrt{\frac{a+b-2}{ab}} - \sqrt{\frac{a+b-3}{a(b-1)}} - \sqrt{\frac{2a-2}{a^2}} \\ &\quad + \sqrt{\frac{a+b-4}{b(a-2)}} + \sqrt{\frac{2a-5}{(a-1)(a-2)}} - \sqrt{\frac{a+b-5}{(b-1)(a-2)}} - \sqrt{\frac{2}{a}} \\ &\triangleq f_2(a, b). \end{aligned} \quad (24)$$

FIGURE 4: $f_1(6, b)$ with $b > 6$. FIGURE 5: $f_1(10, b)$ with $b > 10$. FIGURE 6: $f_1(a, a+1)$.

Similar to the discussions in Case 1, we have that the function $f_2(a, b)$ is a strictly decreasing function on b for fixed $a \geq 6$, and the maximum value of the function $f_2(a, a+1)$ is attained when a is sufficiently large. So,

$$f_2(a, b) \leq f_2(a, a+1) < \lim_{a \rightarrow +\infty} f_2(a, a+1) = 0. \quad (25)$$

The theorem holds for Case 2.

Case 3. $l \geq k \geq 2$.

By Lemma 2,

$$\begin{aligned} & ABC_T(E_1(T)) - ABC_{T'}(E_1(T')) \\ &= 2\sqrt{\frac{(d_v + n_2) + (2 + n_2) - 2}{(d_v + n_2)(2 + n_2)}} + \sqrt{\frac{(2 + n_2) + (1 + n_2) - 2}{(2 + n_2)(1 + n_2)}} \\ &+ \sum_{x \in A} \sqrt{\frac{(d_v + n_2) + (d_x + n_2) - 2}{(d_v + n_2)(d_x + n_2)}} \\ &- \sqrt{\frac{(d_v + n_2 - 1) + (2 + n_2) - 2}{(d_v + n_2 - 1)(2 + n_2)}} - 2\sqrt{\frac{(2 + n_2) + (2 + n_2) - 2}{(2 + n_2)(2 + n_2)}} \\ &- \sum_{x \in A} \sqrt{\frac{(d_v + n_2 - 1) + (d_x + n_2) - 2}{(d_v + n_2 - 1)(d_x + n_2)}} \\ &< 2\sqrt{\frac{(d_v + n_2) + (2 + n_2) - 2}{(d_v + n_2)(2 + n_2)}} + \sqrt{\frac{(2 + n_2) + (1 + n_2) - 2}{(2 + n_2)(1 + n_2)}} \\ &- \sqrt{\frac{(d_v + n_2 - 1) + (2 + n_2) - 2}{(d_v + n_2 - 1)(2 + n_2)}} - 2\sqrt{\frac{(2 + n_2) + (2 + n_2) - 2}{(2 + n_2)(2 + n_2)}} \\ &= 2\sqrt{\frac{a+b-2}{ab}} + \sqrt{\frac{2a-3}{a(a-1)}} - \sqrt{\frac{a+b-3}{a(b-1)}} - 2\sqrt{\frac{2a-2}{a^2}}, \\ &ABC_T(E_1(T)) + ABC_T(E_2(T)) - ABC_{T'}(E_1(T')) \\ &- ABC_{T'}(E_2(T')) \\ &< 2\sqrt{\frac{a+b-2}{ab}} + \sqrt{\frac{2a-3}{a(a-1)}} - \sqrt{\frac{a+b-3}{a(b-1)}} - 2\sqrt{\frac{2a-2}{a^2}} \\ &+ \sqrt{\frac{a+b-4}{b(a-2)}} + \sqrt{\frac{2a-5}{(a-1)(a-2)}} - \sqrt{\frac{a+b-5}{(b-1)(a-2)}} - \sqrt{\frac{2}{a}} \\ &\triangleq f_3(a, b). \end{aligned} \quad (26)$$

Similar to the discussions in Case 1, we have that the function $f_3(a, b)$ is a strictly decreasing function on b for fixed $a \geq 6$, and the maximum value of the function $f_3(a, a+1)$ is attained when a is sufficiently large. So,

$$f_3(a, b) \leq f_3(a, a+1) < \lim_{a \rightarrow +\infty} f_3(a, a+1) = 0. \quad (27)$$

Then, the theorem holds for Case 3.

By Theorems 4, 5, and 1, we get the main result of this section.

Theorem 6. Let T_1 and T_2 be trees of orders $n_1 \geq 3$ and $n_2 \geq 4$, respectively. Then,

$$\begin{aligned} ABC(T_1 \circ T_2) \leq & ABC(P_{n_1} \circ S_{n_2}) = 2\sqrt{\frac{2n_2 + 1}{(n_2 + 1)(n_2 + 2)}} \\ & + (n_1 - 3)\sqrt{\frac{2n_2 + 2}{(n_2 + 2)(n_2 + 2)}} \\ & + 2\sqrt{\frac{2n_2 - 1}{n_2(n_2 + 1)}} \\ & + (n_1 - 2)\sqrt{\frac{2}{n_2 + 2}} + 2n_1(n_2 - 1)\sqrt{\frac{1}{2}}, \end{aligned} \quad (28)$$

and the equality holds if and only if $T_1 = P_{n_1}$ and $T_2 = S_{n_2}$.

The values of ABC indices of $T_1 \circ T_2$ for $n_1 < 3, n_2 < 4$ are as follows.

Theorem 7. Let T_1 and T_2 be trees of orders $n_1 < 3$ and $n_2 < 4$, respectively. Then,

- (1) $n_1 = 1, n_2 = 1, ABC(T_1 \circ T_2) = 0$.
- (2) $n_1 = 1, n_2 = 2, ABC(T_1 \circ T_2) = (3\sqrt{2}/2)$.
- (3) $n_1 = 1, n_2 = 3,$
 $ABC(T_1 \circ T_2) = ABC(T_1 \circ P_3) = 2\sqrt{2} + (2/3)$.
- (4) $n_1 = 2, n_2 = 1, ABC(T_1 \circ T_2) = (3\sqrt{2}/2)$.
- (5) $n_1 = 2, n_2 = 2, ABC(T_1 \circ T_2) = 3\sqrt{2} + (2/3)$.
- (6) $n_1 = 2, n_2 = 3,$
 $ABC(T_1 \circ T_2) = ABC(P_2 \circ P_3) = 4\sqrt{2} + 2\sqrt{(5/12)} + (\sqrt{6}/4)$.

5. Conclusions

In this paper, we mainly determined the upper bounds of the ABC indices of $T_1 \circ T_2$. It is evident that the ABC index of $G_1 \circ G_2$ attains the maximum value if and only if both G_1 and G_2 are complete graphs. If the ABC index of $G_1 \circ G_2$ attains the minimum value, then G_1 and G_2 must be trees. We deduced that the ABC index of $T_1 \circ T_2$ attains the maximum value if and only if T_1 is a path and T_2 is a star. We will discuss the minimum value of the ABC indices of the corona of two trees in near future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This study was supported by the Shanxi Province Science Foundation for Youths (no. 201901D211227).

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