Research Article

Estimations of Upper Bounds for n -th Order Differentiable Functions Involving χ -Riemann–Liouville Integrals via γ -Preinvex Functions

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A new fractional integral identity is obtained involving n -th order differentiable functions and χ -Riemann–Liouville fractional integrals. Then, some associated estimates of upper bounds involving γ -preinvex functions are obtained. In order to relate some unrelated results, several special cases are discussed.

1. Introduction and Preliminaries

A set $\mathcal{C} \subset \mathbb{R}$ is said to be convex, if

$$(1 - \nu)b_1 + \nu b_2 \in \mathcal{C}, \quad \forall b_1, b_2 \in \mathcal{C}, \nu \in [0, 1]. \quad (1)$$

A function $f: \mathcal{C} \rightarrow \mathbb{R}$ is said to be convex, if

$$\begin{aligned} f((1 - \nu)b_1 + \nu b_2) \\ \leq (1 - \nu)f(b_1) + \nu f(b_2), \quad \forall b_1, b_2 \in \mathcal{C}, \nu \in [0, 1]. \end{aligned} \quad (2)$$

In recent years, several new generalizations of the classical concepts of convexity have been proposed in the literature. For example, Hanson [1] introduced the notion of differentiable invex functions, without calling them as invex, in connection with their special global optimum behaviour. It was Craven [2] who introduced the term invex for calling this class of functions, due to their property described as invariance by convexity. The concept of invex sets is defined as follows.

A set $\mathcal{C}_\mu \subset \mathbb{R}$ is said to be invex with respect to bi-function $\mu(\cdot, \cdot): \mathcal{C}_\mu \times \mathcal{C}_\mu \rightarrow \mathbb{R}$, if

$$b_1 + \nu\mu(b_2, b_1) \in \mathcal{C}_\mu, \quad \forall b_1, b_2 \in \mathcal{C}_\mu, \nu \in [0, 1]. \quad (3)$$

Remark 1. If $\mu(b_2, b_1) = b_2 - b_1$, then the concept of invex sets reduces to classical convexity. Thus, it is true that every convex set is also an invex set with respect to $\mu(b_2, b_1) = b_2 - b_1$, but the converse is not necessarily true. For more details, see [3].

Weir and Mond [4] introduced the concept of preinvex functions as follows.

A function $f: \mathcal{C}_\mu \rightarrow \mathbb{R}$ is said to be preinvex with respect to bi-function $\mu(\cdot, \cdot): \mathcal{C}_\mu \times \mathcal{C}_\mu \rightarrow \mathbb{R}$, if

$$\begin{aligned} f(b_1 + \nu\mu(b_2, b_1)) \\ \leq (1 - \nu)f(b_1) + \nu f(b_2), \quad \forall b_1, b_2 \in \mathcal{C}_\mu, \nu \in [0, 1]. \end{aligned} \quad (4)$$

Note that if we take $\mu(b_2, b_1) = b_2 - b_1$, then from the class of preinvex functions, we recapture the class of classical convex functions.

Having inspiration from the research work of [5–7], Awan et al. [8] introduced the notion of γ -preinvex functions.

Let $\gamma: (0, 1) \rightarrow (0, \infty)$ be a real function. A function $f: \mathcal{C}_\mu \rightarrow \mathbb{R}$ is said to be γ -preinvex, if

$$\begin{aligned}
& F(b_1 + \nu\mu(b_2, b_1)) \\
& \leq (1 - \nu)\gamma(1 - \nu)F(b_1) \\
& \quad + \nu\gamma(\nu)F(b_2), \quad \forall b_1, b_2 \in \mathcal{C}_\mu, \nu \in [0, 1].
\end{aligned} \tag{5}$$

It has been observed that the class of γ -preinvex functions generalizes several other classes of preinvexity and convexity. For example:

- (1) If we take $\gamma(\nu) = 1$, then we have the class of classical preinvex function.
- (2) If we take $\gamma(\nu) = \nu^{-1}$, then we have the definition of P -preinvex function (see [6]).
- (3) If we take $\gamma(\nu) = \nu^{s-1}$ where $s \in (0, 1)$, then we have the class of s -preinvex functions of Breckner type (see [6]).
- (4) If we take $\gamma(\nu) = \nu^{-s-1}$, then we have the class of s -Godunova–Levin–Dragomir type of preinvex functions (see [9]).
- (5) If we take $\gamma(\nu) = 1 - \nu$, then we have the definition of tgs -preinvex functions (see [8]).

It is obvious that if we take $\mu(b_2, b_1) = b_2 - b_1$ in the above discussed special cases, then we can recapture the classes of classical convexity.

Theory of convexity has played significant role in the development of theory of inequalities. Many famously known results in theory of inequalities can easily be obtained using the convexity property of the functions. The Hermite–Hadamard inequality which provides us a necessary and sufficient condition for a function to be convex is one of the most studied results pertaining to convexity. It reads as follows.

Let $F: \mathcal{C} = [b_1, b_2] \rightarrow \mathbb{R}$ be a convex function; then,

$$F\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} F(x) dx \leq \frac{F(b_1) + F(b_2)}{2}. \tag{6}$$

In recent years, several new extensions and generalizations for the Hermite–Hadamard inequality have been obtained in the literature. For example, Noor [10] obtained a new refinement of the Hermite–Hadamard inequality using the class of preinvex functions. Awan et al. [8] obtained its new version by utilizing the class of γ -preinvex functions. The authors have also discussed various applications for some special means. Sarikaya et al. [11] introduced a new dimension by introducing the fractional analogue of the Hermite–Hadamard inequality. The idea of Sarikaya and his co-authors has attracted many inequalities experts and consequently a variety of new fractional analogues of classical inequalities have been obtained in the literature using different variants of classical concepts of fractional calculus and also by different generalizations of classical convexity. For example, Hwang et al. [12] obtained different refinements and extensions of the Hermite–Hadamard inequality via fractional integrals. Turhan et al. [13] obtained Hermite–Hadamard type of inequalities

via n -times differentiable convex functions involving Riemann–Liouville fractional integrals. Wu et al. [14] obtained fractional analogues of k -th order differentiable functions involving Riemann–Liouville integrals via higher order strongly h -preinvex functions. Zhang et al. [15] obtained new k -fractional integral inequalities containing multiple parameters via generalized (s, m) -preinvexity. The aim of this paper is to derive a new integral identity involving n -times differentiable functions and χ -Riemann–Liouville fractional integrals. Some associated estimates of upper bounds involving γ -preinvex functions are also obtained. In order to relate some unrelated results, several special cases are discussed. This shows that our results are more generalized and quite unifying. In order to show the significance of our obtained results, we also present applications to special means of real numbers. We hope that the ideas of this paper will inspire interested readers working in this field.

Before we proceed further, let us recall some basic preliminaries from fractional calculus. These preliminaries will be helpful during the study of this paper.

Definition 1. (see [16]). Let $F \in L_1[b_1, b_2]$. The Riemann–Liouville fractional integrals $J_{b_1^+}^\alpha F$ and $J_{b_2^-}^\alpha F$ of order $\alpha > 0$ with $b_1 \geq 0$ are defined by

$$\begin{aligned}
J_{b_1^+}^\alpha F(x) &= \frac{1}{\Gamma(\alpha)} \int_{b_1}^x (x - \nu)^{\alpha-1} F(\nu) d\nu, \quad x > b_1, \\
J_{b_2^-}^\alpha F(x) &= \frac{1}{\Gamma(\alpha)} \int_x^{b_2} (\nu - x)^{\alpha-1} F(\nu) d\nu, \quad x < b_2,
\end{aligned} \tag{7}$$

respectively, and $\Gamma(\alpha)$ is the gamma function. Also, we define $J_{b_1^+}^0 F(x) = J_{b_2^-}^0 F(x) = F(x)$.

Mobeen and Habibullah extended the notion of Riemann–Liouville fractional integrals and introduced the concept of χ -Riemann–Liouville fractional integrals.

Definition 2 (see [17]). Let $F \in L_1[b_1, b_2]$. The χ -Riemann–Liouville fractional integrals ${}_\chi J_{b_1^+}^\alpha F$ and ${}_\chi J_{b_2^-}^\alpha F$ of order $\alpha, \chi > 0$ with $b_1 \geq 0$ are given as follows:

$$\begin{aligned}
{}_\chi J_{b_1^+}^\alpha F(x) &= \frac{1}{\chi \Gamma_\chi(\alpha)} \int_{b_1}^x (x - \nu)^{(\alpha/\chi)-1} F(\nu) d\nu, \quad x > b_1, \\
{}_\chi J_{b_2^-}^\alpha F(x) &= \frac{1}{\chi \Gamma_\chi(\alpha)} \int_x^{b_2} (\nu - x)^{\alpha/\chi-1} F(\nu) d\nu, \quad x < b_2,
\end{aligned} \tag{8}$$

respectively, where

$$\Gamma_\chi(x) = \int_0^\infty \nu^{x-1} e^{-(\nu^\chi/\chi)} d\nu, \tag{9}$$

is the χ -gamma function (see also [18]).

Note that when $\chi \rightarrow 1$, χ -Riemann–Liouville fractional integral reduces to classical Riemann–Liouville fractional integral [16].

χ -beta function is defined as

$$B_\chi(x, y) = \frac{1}{\chi} \int_0^1 v^{(x/\chi)-1} (1-v)^{(y/\chi)-1} dv = \frac{\Gamma_\chi(x) \Gamma_\chi(y)}{\Gamma_\chi(x+y)}, \quad x > 0, y > 0. \quad (10)$$

2. Auxiliary Result

We now derive a key lemma which will be helpful in obtaining the coming results of the paper.

Lemma 1. Let $n \geq 1$ and $F: [b_1, b_1 + \mu(b_2, b_1)] \rightarrow \mathbb{R}$ be n -times differentiable function. If $\mu(b_2, b_1) > 0$ and $F^{(n)} \in L[b_1, b_1 + \mu(b_2, b_1)]$, then

$$\begin{aligned} & \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\ & + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \\ & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \int_0^1 [(-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) dv, \end{aligned} \quad (11)$$

where

$$\mathcal{E}(\alpha, \chi, n) = \frac{\mu^n(b_2, b_1) \chi^n}{(\alpha + \chi)(\alpha + 2\chi), \dots, (\alpha + n\chi)}. \quad (12)$$

Proof. We prove this result by using the mathematical induction principle. The case for $n = 1$ is obvious. Suppose Lemma 1 holds for $n - 1$, that is,

$$\begin{aligned} & \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\ & + \sum_{j=1}^{n-2} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \\ & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-2) \int_0^1 [(-1)^{n-2} (1-v)^{(\alpha/\chi)+n-2} - v^{(\alpha/\chi)+n-2}] F^{(n-1)}(b_1 + (1-v)\mu(b_2, b_1)) dv. \end{aligned} \quad (13)$$

Now, we prove (11) for n . Integrating by parts, we have

$$\begin{aligned} & \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \int_0^1 [(-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) dv \\ & = \mathcal{E}(\alpha, \chi, n-1) \frac{F^{(n-1)}(b_1) + (-1)^{n-1} F^{(n-1)}(b_1 + \mu(b_2, b_1))}{2} \\ & + \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-2) \int_0^1 [(-1)^{n-2} (1-v)^{(\alpha/\chi)+n-2} - v^{(\alpha/\chi)+n-2}] F^{(n-1)}(b_1 + (1-v)\mu(b_2, b_1)) dv \\ & = \mathcal{E}(\alpha, \chi, n-1) \frac{F^{(n-1)}(b_1) + (-1)^{n-1} F^{(n-1)}(b_1 + \mu(b_2, b_1))}{2} \\ & + \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{n-2} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \\
& = \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\
& + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right].
\end{aligned} \tag{14}$$

This completes the proof. $\square$

(i) If we take $n = 1$ in (11), then we have the following identity:

Remark 2.

$$\begin{aligned}
& \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\
& = \frac{\mu(b_2, b_1)}{2} \int_0^1 \left[(1-v)^{(\alpha/\chi)} - v^{(\alpha/\chi)} \right] F'(b_1 + (1-v)\mu(b_2, b_1)) dv.
\end{aligned} \tag{15}$$

(ii) If we take $n = 2$ to (11), then

$$\begin{aligned}
& \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\
& + \frac{\mu(b_2, b_1)\chi}{2(\alpha + \chi)} \left[\frac{F'(b_1) - F'(b_1 + \mu(b_2, b_1))}{2} \right] \\
& = \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 \left[-(1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1} \right] F''(b_1 + (1-v)\mu(b_2, b_1)) dv \\
& = \frac{\mu(b_2, b_1)\chi}{2(\alpha + \chi)} \left[\frac{F'(b_1 + \mu(b_2, b_1)) - F'(b_1)}{2} \right] + \frac{\mu^2(b_2, b_1)}{2(\alpha + \chi)} \int_0^1 \left[-(1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1} \right] F''(b_1 + (1-v)\mu(b_2, b_1)) dv.
\end{aligned} \tag{16}$$

On the other hand, one can easily see that

$$\begin{aligned}
& \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \\
& = \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 \left[1 - (1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1} \right] F''(b_1 + (1-v)\mu(b_2, b_1)) dv \\
& = \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 F''(b_1 + (1-v)\mu(b_2, b_1)) dv + \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 \left[-(1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1} \right] F''(b_1 + (1-v)\mu(b_2, b_1)) dv
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mu(b_2, b_1)\chi}{2(\alpha + \chi)} \int_{b_1}^{b_1 + \mu(b_2, b_1)} F''(s) ds + \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 [-(1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1}] F''(b_1 + (1-v)\mu(b_2, b_1)) dv \\
&= \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} [F'(b_1 + \mu(b_2, b_1)) - F'(b_1)] + \frac{\mu^2(b_2, b_1)\chi}{2(\alpha + \chi)} \int_0^1 [-(1-v)^{(\alpha/\chi)+1} - v^{(\alpha/\chi)+1}] F''(b_1 + (1-v)\mu(b_2, b_1)) dv.
\end{aligned} \tag{17}$$

One can easily see that (16) and (17) are identical.

(iii) If we take $\chi = 1$ to (11), then we have

$$\begin{aligned}
&\left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma(\alpha + 1)}{2\mu^\alpha(b_2, b_1)} \left[I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
&= \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, n-1) \int_0^1 [(-1)^{n-1} (1-v)^{\alpha+n-1} - v^{\alpha+n-1}] F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) dv,
\end{aligned} \tag{18}$$

where

$$\mathcal{C}(\alpha, n) = \frac{\mu^n(b_2, b_1)}{(\alpha + 1)(\alpha + 2), \dots, (\alpha + n)}. \tag{19}$$

Theorem 1. Let $n \geq 1$ and $F: [b_1, b_1 + \mu(b_2, b_1)] \longrightarrow \mathbb{R}$ be n -times differentiable function with $\mu(b_2, b_1) > 0$. If $F^{(n)} \in L[b_1, b_1 + \mu(b_2, b_1)]$ and $|F^{(n)}|$ is α -preinvex function, then

3. Main Results

We now derive our main results of the paper.

$$\begin{aligned}
&\left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
&\leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} &\left[\begin{aligned} &|F^{(n)}(b_1)| \int_0^1 v \gamma(v) [(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1}] dv \\ &+ |F^{(n)}(b_2)| \int_0^1 (1-v) \gamma(1-v) \left[(1-v)^{\frac{\alpha}{\chi}+n-1} + v^{\frac{\alpha}{\chi}+n-1} \right] dv \end{aligned} \right], & n \text{ is even,} \\ &\left[\begin{aligned} &|F^{(n)}(b_1)| \int_0^{(1/2)} v \gamma(v) [(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] dv \\ &+ |F^{(n)}(b_2)| \int_0^{(1/2)} (1-v) \gamma(1-v) [(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] dv \\ &+ |F^{(n)}(b_1)| \int_{(1/2)}^1 v \gamma(v) [v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1}] dv \\ &+ |F^{(n)}(b_2)| \int_{(1/2)}^1 (1-v) \gamma(1-v) [v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1}] dv \end{aligned} \right], & n \text{ is odd.} \end{aligned} \right\}
\end{aligned} \tag{20}$$

Proof. Using Lemma 1 and the γ -preinvexity of $|F^{(n)}|$, we have

$$\begin{aligned}
 & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{b_1 + \mu(b_2, b_1)}^\alpha F(b_1) \right] \right. \\
 & \quad \left. + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv, & n \text{ is even,} \\ \int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv & n \text{ is odd,} \end{cases} \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is even,} \\ \int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is odd,} \end{cases} \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right| \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is even,} \\ \int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is odd,} \end{cases} \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left[(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right] \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is even,} \\ \int_0^{1/2} \left[(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right] \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv \\ \quad + \int_{1/2}^1 \left[v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1} \right] \left[v\gamma(v) \left| F^{(n)}(b_1) \right| + (1-v)\gamma(1-v) \left| F^{(n)}(b_2) \right| \right] dv, & n \text{ is odd,} \end{cases} \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \left[\begin{aligned} & \left| F^{(n)}(b_1) \right| \int_0^1 v\gamma(v) \left[(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right] dv \\ & + \left| F^{(n)}(b_2) \right| \int_0^1 (1-v)\gamma(1-v) \left[(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right] dv \end{aligned} \right], & n \text{ is even,} \\ \left[\begin{aligned} & \left| F^{(n)}(b_1) \right| \int_0^{1/2} v\gamma(v) \left[(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right] dv \\ & + \left| F^{(n)}(b_2) \right| \int_0^{1/2} (1-v)\gamma(1-v) \left[(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right] dv \\ & + \left| F^{(n)}(b_1) \right| \int_{1/2}^1 v\gamma(v) \left[v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1} \right] dv \\ & + \left| F^{(n)}(b_2) \right| \int_{1/2}^1 (1-v)\gamma(1-v) \left[v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1} \right] dv \end{aligned} \right], & n \text{ is odd.} \end{cases}
 \end{aligned}$$

(21)

This completes the proof. $\square$

We now discuss some special cases of Theorem 1.

- (i) If $\gamma(v) = 1$, then Theorem 1 reduces to the following result for the class of preinvex function.

Corollary 1. Under the assumptions of Theorem 1, if $|F^{(n)}|$ is classical preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mathcal{C}(\alpha, \chi, n)}{2} \begin{cases} \left[|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right], & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi) + n - 1}} \right) \left[|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right], & n \text{ is odd.} \end{cases} \end{aligned} \quad (22)$$

- (ii) If $\gamma(v) = v^{-1}$, then Theorem 1 reduces to the following result for the class of P -preinvex function.

Corollary 2. Under the assumptions of Theorem 1, if $|F^{(n)}|$ is P -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \mathcal{C}(\alpha, \chi, n) \begin{cases} \left[|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right], & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi) + n - 1}} \right) \left[|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right], & n \text{ is odd.} \end{cases} \end{aligned} \quad (23)$$

- (iii) If $\gamma(v) = v^{s-1}$, then Theorem 1 reduces to the following result for the class of s -preinvex functions of Breckner type.

Corollary 3. Under the assumptions of Theorem 1, if $|F^{(n)}|$ is s -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right|$$

$$\begin{aligned}
& + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \Bigg| \\
& \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \begin{cases} \left[L(\alpha, n, \chi) |F^{(n)}(b_1)| + M(\alpha, n, \chi) |F^{(n)}(b_2)| \right], & n \text{ is even,} \\ N(\alpha, n, \chi) \left[N(\alpha, n, \chi) |F^{(n)}(b_1)| + R(\alpha, n, \chi) |F^{(n)}(b_2)| \right], & n \text{ is odd,} \end{cases}
\end{aligned} \tag{24}$$

where

$$L(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi + s\chi} + \chi \mathcal{B}_\chi(sk + \chi, \alpha + nk), \tag{25}$$

$$M(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi + s\chi} + \chi \mathcal{B}_\chi(\alpha + n\chi, s\chi + \chi), \tag{26}$$

$$N(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi + s\chi} - \frac{\chi}{2^{(\alpha/\chi)+n+s-1}(\alpha + n\chi + s\chi)} + 2\chi_{1/2} \mathcal{B}_\chi(s\chi + \chi, \alpha + n\chi) - \chi \mathcal{B}_\chi(s\chi + \chi, \alpha + n\chi), \tag{27}$$

$$R(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi + s\chi} - \frac{\chi}{2^{(\alpha/\chi)+n+s-1}(\alpha + n\chi + s\chi)} - 2\chi_{1/2} \mathcal{B}_\chi(s\chi + \chi, \alpha + n\chi) + \chi \mathcal{B}_\chi(s\chi + \chi, \alpha + n\chi). \tag{28}$$

(iv) If $\gamma(v) = v^{-s-1}$, then Theorem 1 reduces to the following result in the class of s -Godunova–Levin preinvex function.

Corollary 4. Under the assumptions of Theorem 1, if $|F^{(n)}|$ is s -Godunova–Levin preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned}
& \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\
& \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
& \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \begin{cases} \left[L^*(\alpha, n, \chi) |F^{(n)}(b_1)| + M^*(\alpha, n, \chi) |F^{(n)}(b_2)| \right], & n \text{ is even,} \\ \left[N^*(\alpha, n, \chi) |F^{(n)}(b_1)| + R^*(\alpha, n, \chi) |F^{(n)}(b_2)| \right], & n \text{ is odd,} \end{cases}
\end{aligned} \tag{29}$$

where

$$L^*(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi - s\chi} + \chi \mathcal{B}_\chi(\alpha + n\chi, \chi - s\chi), \tag{30}$$

$$M^*(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi - s\chi} + \chi \mathcal{B}_\chi(\chi - s\chi, \alpha + n\chi), \tag{31}$$

$$N^*(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi - s\chi} - \frac{\chi}{2^{(\alpha/\chi)+n-s-1}(\alpha + n\chi - s\chi)} + 2\chi_{1/2} \mathcal{B}_\chi(\chi - s\chi, \alpha + n\chi) - \chi \mathcal{B}_\chi(\chi - s\chi, \alpha + n\chi), \tag{32}$$

$$R^*(\alpha, n, \chi) = \frac{\chi}{\alpha + n\chi - s\chi} - \frac{\chi}{2^{(\alpha/\chi)+n-s-1}(\alpha + n\chi - s\chi)} - 2\chi_{1/2}\mathcal{B}_\chi(\chi - s\chi, \alpha + n\chi) + \chi\mathcal{B}_\chi(\chi - s\chi, \alpha + n\chi). \quad (33)$$

(v) If $\gamma(v) = 1 - v$, then Theorem 1 reduces to the following result for the class of tgs-preinvex function.

Corollary 5. Under the assumptions of Theorem 1, if $|F^{(n)}|$ is tgs-preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)\chi^2 \mathcal{E}(\alpha, \chi, n-1)}{2(\chi + n\chi + 1)(\chi + n\chi + 2)} \begin{cases} 2 \left(|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right), & n \text{ is even,} \\ \left(1 - \frac{\alpha + n\chi + 3\chi}{2^{(\alpha/\chi)+n+1}\chi} \right) \left(|F^{(n)}(b_1)| + |F^{(n)}(b_2)| \right), & n \text{ is odd.} \end{cases} \end{aligned} \quad (34)$$

Theorem 2. Let $n \geq 1$ and $F: [b_1, b_1 + \mu(b_2, b_1)] \longrightarrow \mathbb{R}$ be n -times differentiable function with $\mu(b_2, b_1) > 0$. If $F^{(n)} \in L[b_1, b_1 + \mu(b_2, b_1)]$ and $|F^{(n)}|^q$ is $\alpha\gamma$ -preinvex function, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \left(|F^{(n)}(b_1)|^q \int_0^1 v\gamma(v)dv + |F^{(n)}(b_2)|^q \int_0^1 (1-v)\gamma(1-v)dv \right)^{1/q}, & n \text{ is even,} \\ \left[\left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi)+n-1)}} \right) \right)^{1/p} \right. \\ \quad \times \left[\left(|F^{(n)}(b_1)|^q \int_0^{1/2} v\gamma(v)dv + |F^{(n)}(b_2)|^q \int_0^{1/2} (1-v)\gamma(1-v)dv \right)^{1/q} \right. \\ \quad \left. \left. + \left(|F^{(n)}(b_1)|^q \int_{1/2}^1 v\gamma(v)dv + |F^{(n)}(b_2)|^q \int_{1/2}^1 (1-v)\gamma(1-v)dv \right)^{1/q} \right] \right], & n \text{ is odd.} \end{cases} \end{aligned} \quad (35)$$

Proof. Using Lemma 1, Hölder's integral inequality, and γ -preinvexity of $|F^{(n)}|^q$, we have

$$\begin{aligned}
 & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{b_1 + \mu(b_2, b_1)}^\alpha F(b_1) \right] \right. \\
 & \quad \left. + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \int_0^1 \left| (-1)^{n-1} (1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| -(1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu, & n \text{ is even,} \\ \int_0^1 \left| (1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu, & n \text{ is odd,} \end{cases} \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| (1-\nu)^{(\alpha/\chi)+n-1} + \nu^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu, & n \text{ is even,} \\ \int_0^1 \left| (1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu, & n \text{ is odd,} \end{cases} \\
 & = \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left[(1-\nu)^{(\alpha/\chi)+n-1} + \nu^{(\alpha/\chi)+n-1} \right] \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu, & n \text{ is even,} \\ \left[\int_0^{1/2} \left[(1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right] \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu \right. \\ \quad \left. + \int_{1/2}^1 \left[\nu^{(\alpha/\chi)+n-1} - (1-\nu)^{(\alpha/\chi)+n-1} \right] \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right| d\nu \right], & n \text{ is odd,} \end{cases} \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \left(\int_0^1 \left[(1-\nu)^{(\alpha/\chi)+n-1} + \nu^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \left(\int_0^1 \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right|^q d\nu \right)^{1/q}, & n \text{ is even,} \\ \left[\left(\int_0^{1/2} \left[(1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \left(\int_0^{1/2} \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right|^q d\nu \right)^{1/q} \right. \\ \quad \left. + \left(\int_{1/2}^1 \left[\nu^{(\alpha/\chi)+n-1} - (1-\nu)^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \left(\int_{1/2}^1 \left| F^{(n)}(b_1 + (1-\nu)\mu(b_2, b_1)) \right|^q d\nu \right)^{1/q} \right], & n \text{ is odd,} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
& \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} & \left[\begin{aligned} & \left(\int_0^1 \left[(1-\nu)^{(\alpha/\chi)+n-1} + \nu^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \\ & \times \left(\int_0^1 \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \end{aligned} \right], & n \text{ is even,} \\ & \left[\begin{aligned} & \left(\int_0^{1/2} \left[(1-\nu)^{(\alpha/\chi)+n-1} - \nu^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \\ & \times \left(\int_0^{1/2} \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \\ & + \left(\int_{1/2}^1 \left[\nu^{(\alpha/\chi)+n-1} - (1-\nu)^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \\ & \times \left(\int_{1/2}^1 \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \end{aligned} \right], & n \text{ is odd,} \end{aligned} \right\} \\
& \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} & \left[\begin{aligned} & \left(\int_0^1 \left[(1-\nu)^{(\alpha/\chi)+n-1} + \nu^{(\alpha/\chi)+n-1} \right]^p d\nu \right)^{1/p} \\ & \times \left(\int_0^1 \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \end{aligned} \right], & n \text{ is even,} \\ & \left[\begin{aligned} & \left(\int_0^{1/2} \left[(1-\nu)^{p((\alpha/\chi)+n-1)} - \nu^{p((\alpha/\chi)+n-1)} \right] d\nu \right)^{1/p} \\ & \times \left(\int_0^{1/2} \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \\ & + \left(\int_{1/2}^1 \left[\nu^{p((\alpha/\chi)+n-1)} - (1-\nu)^{p((\alpha/\chi)+n-1)} \right] d\nu \right)^{1/p} \\ & \times \left(\int_{1/2}^1 \left[\nu \gamma(\nu) |F^{(n)}(b_1)|^q + (1-\nu) \gamma(1-\nu) |F^{(n)}(b_2)|^q \right] d\nu \right)^{1/q} \end{aligned} \right], & n \text{ is odd,} \end{aligned} \right\} \\
& = \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} & \left(|F^{(n)}(b_1)|^q \int_0^1 \nu \gamma(\nu) d\nu + |F^{(n)}(b_2)|^q \int_0^1 (1-\nu) \gamma(1-\nu) d\nu \right)^{1/q}, & n \text{ is even,} \\ & \left[\begin{aligned} & \left(\frac{1}{p((\alpha/\chi)+n-1)+1} \left(1 - \frac{1}{2^{p((\alpha/\chi)+n-1)}} \right) \right)^{1/p} \\ & \times \left[\left(|F^{(n)}(b_1)|^q \int_0^{1/2} \nu \gamma(\nu) d\nu + |F^{(n)}(b_2)|^q \int_0^{1/2} (1-\nu) \gamma(1-\nu) d\nu \right)^{1/q} \right. \\ & \left. + \left(|F^{(n)}(b_1)|^q \int_{1/2}^1 \nu \gamma(\nu) d\nu + |F^{(n)}(b_2)|^q \int_{1/2}^1 (1-\nu) \gamma(1-\nu) d\nu \right)^{1/q} \right] \end{aligned} \right], & n \text{ is odd.} \end{aligned} \right\}
\end{aligned}$$

This completes the proof. $\square$

We now discuss some special cases of Theorem 2.

- (i) If $\gamma(v) = 1$, then Theorem 2 reduces to the following result for the class of classical preinvex function.

Corollary 6. Under the assumptions of Theorem 2, if $|F^{(n)}|^q$ is classical preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{b_1 + \mu(b_2, b_1)}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} & \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{2} \right)^{1/q}, & n \text{ is even,} \\ & \left[\left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi) + n - 1)}} \right) \right)^{1/p} \right. \\ & \quad \times \left[\left(\frac{|F^{(n)}(b_1)|^q + 3|F^{(n)}(b_2)|^q}{8} \right)^{1/q} \right. \\ & \quad \left. \left. + \left(\frac{3|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{8} \right)^{1/q} \right] \right], & n \text{ is odd.} \end{aligned} \right. \end{aligned} \quad (37)$$

- (ii) If $\gamma(v) = v^{-1}$, then Theorem 2 reduces to the following result for the class of P -preinvex function.

Corollary 7. Under the assumptions of Theorem 2, if $|F^{(n)}|^q$ is P -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{b_1 + \mu(b_2, b_1)}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{aligned} & \left(|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is even,} \\ & \left[2 \left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi) + n - 1)}} \right) \right)^{1/p} \right. \\ & \quad \times \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{2} \right)^{1/q} \right], & n \text{ is odd.} \end{aligned} \right. \end{aligned} \quad (38)$$

(iii) If $\gamma(v) = v^{s-1}$, then Theorem 2 reduces to the following result for the class of s -preinvex function.

Corollary 8. Under the assumptions of Theorem 2, if $|F^{(n)}|^q$ is s -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{array}{ll} \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{s+1} \right)^{1/q}, & n \text{ is even,} \\ \left[\begin{array}{l} \left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi) + n - 1)}} \right) \right)^{1/p} \\ \times \frac{1}{(s+1)^{1/q}} \left[\left(\frac{1}{2^{s+1}} |F^{(n)}(b_1)|^q + \left(1 - \frac{1}{2^{s+1}} \right) |F^{(n)}(b_2)|^q \right)^{1/q} \right. \\ \left. + \left(\left(1 - \frac{1}{2^{s+1}} \right) |F^{(n)}(b_1)|^q + \frac{1}{2^{s+1}} |F^{(n)}(b_2)|^q \right)^{1/q} \right] \end{array} \right], & n \text{ is odd.} \end{array} \right. \end{aligned} \quad (39)$$

(iv) If $\gamma(v) = v^{-s-1}$, then Theorem 2 reduces to the following result for the class of s -Godunova–Levin preinvex function.

Corollary 9. Under the assumptions of Theorem 2, if $|F^{(n)}|^q$ is s -Godunova–Levin preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \begin{array}{ll} \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{1-s} \right)^{1/q}, & n \text{ is even,} \\ \left[\begin{array}{l} \left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi) + n - 1)}} \right) \right)^{1/p} \\ \times \frac{1}{(1-s)^{1/q}} \left[\left(\frac{1}{2^{1-s}} |F^{(n)}(b_1)|^q + \left(1 - \frac{1}{2^{1-s}} \right) |F^{(n)}(b_2)|^q \right)^{1/q} \right. \\ \left. + \left(\left(1 - \frac{1}{2^{1-s}} \right) |F^{(n)}(b_1)|^q + \frac{1}{2^{1-s}} |F^{(n)}(b_2)|^q \right)^{1/q} \right] \end{array} \right], & n \text{ is odd.} \end{array} \right. \end{aligned} \quad (40)$$

(v) If $\gamma(v) = 1 - v$, then Theorem 2 reduces to the following result for the class of tgs -preinvex function.

Corollary 10. Under the assumptions of Theorem 2, if $|F^{(n)}|^q$ is tgs -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \begin{cases} \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{6} \right)^{1/q}, & n \text{ is even,} \\ \left[2 \left(\frac{1}{p((\alpha/\chi) + n - 1) + 1} \left(1 - \frac{1}{2^{p((\alpha/\chi) + n - 1)}} \right) \right)^{1/p} \right. \\ \quad \left. \times \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{12} \right)^{1/q} \right], & n \text{ is odd.} \end{cases} \end{aligned} \quad (41)$$

Theorem 3. Let $n \geq 1$ and $F: [b_1, b_1 + \mu(b_2, b_1)] \rightarrow \mathbb{R}$ be n -times differentiable function with $\mu(b_2, b_1) > 0$. If $F^{(n)} \in L[b_1, b_1 + \mu(b_2, b_1)]$ and $|F^{(n)}|^q$ is $\alpha\gamma$ -preinvex function, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \begin{cases} \left[\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \right. \\ \quad \left. \times \left(\frac{|F^{(n)}(b_1)|^q \int_0^1 v\gamma(v) [(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1}] dv}{+ |F^{(n)}(b_2)|^q \int_0^1 (1-v)\gamma(1-v) [(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1}] dv} \right)^{1/q} \right], & n \text{ is even,} \\ \left[\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \right. \\ \quad |F^{(n)}(b_1)|^q \int_0^{1/2} v\gamma(v) [(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] dv \\ \quad \times \left(\frac{|F^{(n)}(b_2)|^q \int_0^{1/2} (1-v)\gamma(1-v) [(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1}] dv}{+ |F^{(n)}(b_1)|^q \int_{1/2}^1 v\gamma(v) [v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1}] dv} \right)^{1/q} \\ \quad \left. + |F^{(n)}(b_2)|^q \int_{1/2}^1 (1-v)\gamma(1-v) [v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1}] dv \right], & n \text{ is odd.} \end{cases} \end{aligned} \quad (42)$$

Proof. Using Lemma 1, power mean integral inequality, and γ -preinvexity of $|F^{(n)}|^q$, we have

$$\begin{aligned}
 & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\
 & \quad \left. + \sum_{j=1}^{n-1} \mathcal{E}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv, & n \text{ is even,} \\ \int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right| dv, & n \text{ is odd,} \end{cases} \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \left[\left(\int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| dv \right)^{1-(1/q)} \right. \\ \quad \left. \times \left(\int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right|^q dv \right)^{1/q} \right], & n \text{ is even,} \\ \left[\left(\int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| dv \right)^{1-(1/q)} \right. \\ \quad \left. \times \left(\int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \left| F^{(n)}(b_1 + (1-v)\mu(b_2, b_1)) \right|^q dv \right)^{1/q} \right], & n \text{ is odd,} \end{cases} \\
 & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{E}(\alpha, \chi, n-1) \begin{cases} \left[\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \right. \\ \quad \times \left(\int_0^1 \left| (-1)^{n-1} (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \right. \\ \quad \left. \times [v\gamma(v)|F^{(n)}(b_1)|^q + (1-v)\gamma(1-v)|F^{(n)}(b_2)|^q] dv \right)^{1/q} \Big], & n \text{ is even,} \\ \left[\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \right. \\ \quad \times \left(\int_0^1 \left| (1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right| \right. \\ \quad \left. \times [v\gamma(v)|F^{(n)}(b_1)|^q + (1-v)\gamma(1-v)|F^{(n)}(b_2)|^q] dv \right)^{1/q} \Big], & n \text{ is odd,} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left\{ \left[\begin{aligned} &\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \\ &\times \left(\begin{aligned} &|F^{(n)}(b_1)|^q \int_0^1 v\gamma(v) \left[(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right] dv \\ &+ |F^{(n)}(b_2)|^q \int_0^1 (1-v)\gamma(1-v) \left[(1-v)^{(\alpha/\chi)+n-1} + v^{(\alpha/\chi)+n-1} \right] dv \end{aligned} \right)^{1/q} \end{aligned} \right], \quad n \text{ is even,} \\
&\left[\begin{aligned} &\left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \\ &\times \left(\begin{aligned} &|F^{(n)}(b_1)|^q \int_0^{1/2} v\gamma(v) \left[(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right] dv \\ &+ |F^{(n)}(b_2)|^q \int_0^{1/2} (1-v)\gamma(1-v) \left[(1-v)^{(\alpha/\chi)+n-1} - v^{(\alpha/\chi)+n-1} \right] dv \\ &+ |F^{(n)}(b_1)|^q \int_{1/2}^1 v\gamma(v) \left[v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1} \right] dv \\ &+ |F^{(n)}(b_2)|^q \int_{1/2}^1 (1-v)\gamma(1-v) \left[v^{(\alpha/\chi)+n-1} - (1-v)^{(\alpha/\chi)+n-1} \right] dv \end{aligned} \right)^{1/q} \end{aligned} \right], \quad n \text{ is odd.} \end{aligned} \right\} \quad (43)
\end{aligned}$$

This completes the proof. $\square$

Now we will discuss some special cases of Theorem 3.

- (i) If $\gamma(v) = 1$, then Theorem 3 reduces to the following result for the class of classical preinvex function.

Corollary 11. Under the assumptions of Theorem 3, if $|F^{(n)}|^q$ is classical preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned}
&\left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[{}_\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + {}_\chi I_{b_1 + \mu(b_2, b_1)}^\alpha F(b_1) \right] \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\
&\leq \mathcal{C}(\alpha, \chi, n) \left\{ \begin{aligned} &\left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{2} \right)^{1/q}, \quad n \text{ is even,} \\ &\left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right) \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{2} \right)^{1/q}, \quad n \text{ is odd.} \end{aligned} \right. \quad (44)
\end{aligned}$$

(ii) If $\gamma(v) = v^{-1}$, then Theorem 3 reduces to the following result for the class of P -preinvex function.

Corollary 12. Under the assumptions of Theorem 3, if $|F^{(n)}|^q$ is P -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \mathcal{C}(\alpha, \chi, n) \begin{cases} \left(|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right) \left(|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is odd.} \end{cases} \end{aligned} \quad (45)$$

(iii) If $\gamma(v) = v^{s-1}$, then Theorem 3 reduces to the following result for the class of s -preinvex function.

Corollary 13. Under the assumptions of Theorem 3, if $|F^{(n)}|^q$ is s -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \\ & \quad \cdot \begin{cases} \left(L(\alpha, n, \chi) |F^{(n)}(b_1)|^q + M(\alpha, n, \chi) |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \left(N(\alpha, n, \chi) |F^{(n)}(b_1)|^q + R(\alpha, n, \chi) |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is odd,} \end{cases} \end{aligned} \quad (46)$$

where $L(\alpha, n, \chi)$, $M(\alpha, n, \chi)$, $N(\alpha, n, \chi)$, and $R(\alpha, n, \chi)$ are given by (25), (26), (27), and (28), respectively.

(iv) If $\gamma(v) = v^{-s-1}$, then Theorem 3 reduces to the following result for the class of s -Godunova-Levin preinvex function.

Corollary 14. Under the assumptions of Theorem 3, if $|F^{(n)}|^q$ is s -Godunova–Levin preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1)}{2} \mathcal{C}(\alpha, \chi, n-1) \left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \\ & \cdot \begin{cases} \left(L^*(\alpha, n, \chi) |F^{(n)}(b_1)|^q + M^*(\alpha, n, \chi) |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \left(N^*(\alpha, n, \chi) |F^{(n)}(b_1)|^q + R^*(\alpha, n, \chi) |F^{(n)}(b_2)|^q \right)^{1/q}, & n \text{ is odd,} \end{cases} \end{aligned} \quad (47)$$

where $L^*(\alpha, n, \chi)$, $M^*(\alpha, n, \chi)$, $N^*(\alpha, n, \chi)$, and $R^*(\alpha, n, \chi)$ are given by (30), (31), (32), and (33), respectively.

(v) If $\gamma(v) = 1 - v$, then Theorem 3 reduces to the following result for the class of tgs -preinvex function.

Corollary 15. Under the assumptions of Theorem 3, if $|F^{(n)}|^q$ is tgs -preinvex function on $[b_1, b_1 + \mu(b_2, b_1)]$ with respect to $\mu(\cdot, \cdot)$, then

$$\begin{aligned} & \left| \frac{F(b_1) + F(b_1 + \mu(b_2, b_1))}{2} - \frac{\Gamma_\chi(\alpha + \chi)}{2\mu^{(\alpha/\chi)}(b_2, b_1)} \left[\chi I_{b_1^+}^\alpha F(b_1 + \mu(b_2, b_1)) + \chi I_{(b_1 + \mu(b_2, b_1))^-}^\alpha F(b_1) \right] \right. \\ & \quad \left. + \sum_{j=1}^{n-1} \mathcal{C}(\alpha, \chi, j) \left[\frac{F^{(j)}(b_1) + (-1)^j F^{(j)}(b_1 + \mu(b_2, b_1))}{2} \right] \right| \\ & \leq \frac{\mu(b_2, b_1) \chi^2 \mathcal{C}(\alpha, \chi, n-1)}{2} \left(\frac{2\chi}{\alpha + n\chi} \right)^{1-(1/q)} \\ & \cdot \begin{cases} 2 \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{(\chi + n\chi + 1)(\chi + n\chi + 2)} \right)^{1/q}, & n \text{ is even,} \\ \left(1 - \frac{1}{2^{(\alpha/\chi)+n-1}} \right)^{1-(1/q)} \left(1 - \frac{\alpha + n\chi + 3\chi}{2^{(\alpha/\chi)+n+2}\chi} \right)^{1/q} \left(\frac{|F^{(n)}(b_1)|^q + |F^{(n)}(b_2)|^q}{(\chi + n\chi + 1)(\chi + n\chi + 2)} \right)^{1/q}, & n \text{ is odd.} \end{cases} \end{aligned} \quad (48)$$

4. Applications

In this section, we present some applications to special means. We now recall some special means for two different positive real numbers $b_1 < b_2$:

- (i) The arithmetic mean: $\mathcal{A}(b_1, b_2) := (b_1 + b_2)/2$.
- (ii) The logarithmic mean: $\mathcal{L}(b_1, b_2) := (b_2 - b_1) / \ln b_2 - \ln b_1$.
- (iii) The p -logarithmic mean: $\mathcal{L}_p(b_1, b_2) := ((b_2^{p+1} - b_1^{p+1}) / (p+1)(b_2 - b_1))^{1/p}$, $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 1. For $b_1, b_2 \in \mathbb{R}_+, 0 < b_1 < b_2$, and $s \in (0, 1)$, the following inequality holds:

$$\left| \mathcal{A}(b_1^s, b_1^s) - \mathcal{L}_p^p(b_1, b_2) + \frac{s(b_2 - b_1)}{2} \mathcal{A}(b_1^{s-1}, -b_2^{s-1}) \right| \leq \frac{(b_2 - b_1)^2 s(1-s)(s^2 + 3s + 4)}{4(s+1)(s+2)(s+3)} \left[|b_1^{s-1}| + |b_1^{s-2}| \right]. \quad (49)$$

Proof. Taking $n = 2, \chi = \alpha = 1$, and $\mu(b_2, b_1) = b_2 - b_1$ in inequality (24), we have

$$\left| \frac{F(b_1) + F(b_1)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} F(u) du + \frac{b_2 - b_1}{2} \left[\frac{F'(b_1) - F'(b_2)}{2} \right] \right| \leq \frac{(b_2 - b_1)^2 (s^2 + 3s + 4)}{4(s+1)(s+2)(s+3)} \left[|F''(b_1)| + |F''(b_2)| \right]. \quad (50)$$

Now for the function $F: [0, 1] \rightarrow [0, 1], F(x) = x^s$, $F''(x) = s(s-1)x^{s-2}$ is a s -convex function. This completes the proof. $\square$

where $(1/p) + (1/q) = 1$.

Proposition 2. For $b_1, b_2 \in \mathbb{R}_+, 0 < b_1 < b_2$, and $s \in (0, 1)$, the following inequality holds:

$$\left| \mathcal{A}(b_1^s, b_1^s) - \mathcal{L}_p^p(b_1, b_2) + \frac{s(b_2 - b_1)}{2} \mathcal{A}(b_1^{s-1}, -b_2^{s-1}) \right| \leq \frac{(b_2 - b_1)^2 s^q (1-s)^q}{4} \left(\frac{|b_1^{s-2}|^q + |b_2^{s-2}|^q}{s+1} \right)^{1/q}, \quad (51)$$

Proof. Taking $n = 2, \chi = \alpha = 1$, and $\mu(b_2, b_1) = b_2 - b_1$ in inequality (39), we have

$$\left| \frac{F(b_1) + F(b_1)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} F(u) du + \frac{b_2 - b_1}{2} \left[\frac{F'(b_1) - F'(b_2)}{2} \right] \right| \leq \frac{(b_2 - b_1)^2}{4} \left(\frac{|F''(b_1)|^q + |F''(b_2)|^q}{s+1} \right)^{1/q}. \quad (52)$$

Now for the function $F: [0, 1] \rightarrow [0, 1], F(x) = x^s$, $|F''(x)|^q = s^q(1-s)^q|x^{s-2}|^q$ is a s -convex function. This completes the proof. $\square$

Proposition 3. For $b_1, b_2 \in \mathbb{R}_+, 0 < b_1 < b_2$, and $s \in (0, 1)$, the following inequality holds:

$$\left| \mathcal{A}(b_1^s, b_1^s) - \mathcal{L}_p^p(b_1, b_2) + \frac{s(b_2 - b_1)}{2} \mathcal{A}(b_1^{s-1}, -b_2^{s-1}) \right| \leq \frac{(b_2 - b_1)^2 (s - s^2)}{4} (s^2 + 3s + 4)^{1/q} \left(\frac{2}{3} \right)^{1-(1/q)} \left(\frac{|b_1^{s-2}|^q + |b_2^{s-2}|^q}{(s+1)(s+2)(s+3)} \right)^{1/q}, \quad (53)$$

where $q \geq 1$.

Proof. Taking $n = 2$, $\chi = \alpha = 1$, and $\mu(b_2, b_1) = b_2 - b_1$ in inequality (46), we have

$$\left| \frac{F(b_1) + F(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} F(u) du + \frac{b_2 - b_1}{2} \left[\frac{F'(b_1) - F'(b_2)}{2} \right] \right| \leq \frac{(b_2 - b_1)^2}{4} (s^2 + 3s + 4)^{1/q} \left(\frac{2}{3} \right)^{1-(1/q)} \left(\frac{|F''(b_1)|^q + |F''(b_2)|^q}{(s+1)(s+2)(s+3)} \right)^{1/q}. \quad (54)$$

Now for the function $F: [0, 1] \rightarrow [0, 1]$, $F(x) = x^s$, $|F''(x)|^q = s^q(1-s)^q|x^{s-2}|^q$ is a s -convex function. This completes the proof. $\square$

5. Conclusion

We have derived a new fractional integral identity by using the χ -fractional integral and n -times differentiable functions. Using this identity as an auxiliary result, we have obtained some new associated upper bounds involving γ -preinvex functions. Several new special cases are also discussed in detail which show that our results represent significant generalizations and are quite unifying. In order to show the significance of our theocratical results, we also presented applications to special means of our obtained results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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