Research Article

On Fault-Tolerant Partition Dimension of Homogeneous Caterpillar Graphs

Kamran Azhar, Sohail Zafar, and Agha Kashif

University of Management and Technology (UMT), Lahore, Pakistan

Correspondence should be addressed to Agha Kashif; kashif.khan@umt.edu.pk

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Metric-related parameters in graph theory have several applications in robotics, navigation, and chemical strata. An important such parameter is the partition dimension of graphs that plays an important role in engineering, computer science, and chemistry. In the context of chemical and pharmaceutical engineering, these parameters are used for unique representation of chemical compounds and their structural analysis. The structure of benzenoid hydrocarbon molecules is represented in the form of caterpillar trees and studied for various attributes including UV absorption spectrum, molecular susceptibility, anisotropy, and heat of atomization. Several classes of trees have been studied for partition dimension; however, in this regard, the advanced variant, the fault-tolerant partition dimension, remains to be explored. In this paper, we computed fault-tolerant partition dimension for homogeneous caterpillars \( C(p; 1) \), \( C(p; 2) \), and \( C(p; 3) \) for \( p \geq 5 \), \( p \geq 3 \), and \( p \geq 4 \), respectively, and it is found to be constant. Further numerical examples and an application are furnished to elaborate the accuracy and significance of the work.

1. Introduction and Basic Terminologies

Graph theory is a widely excelling branch of mathematics that is used to model and simplify the solution of daily-life problems. Richly engaged area of research now-a-days is the application of mathematics in chemistry. Graph theory provides simple rules to obtain many qualitative predictions about the structure and reactivity of various compounds. In the chemical graph, vertices of the graph correspond to the atoms of the molecule and edges between the vertices correspond to the chemical bonds. The caterpillar graph plays an important role in chemical graph theory for studying the combinatorial and physical properties of benzenoid hydrocarbons. They have promising uses in data reduction, modeling of interactions, computational chemistry, and ordering of graphs [1]. The partition dimension for various classes of trees such as stars, caterpillars, and homogeneous firecrackers have been computed; however, the values of partition dimensions for most kind of trees are still to be solved completely [2, 3]. Among different parameters of graph theory, partition dimension of the graph is a unique and important parameter and has applications in network discovery and verification [4], mastermind games [5], and image processing [6].

In 2000, the concept of partition dimension of the graph was initiated by Chartrand et al. [7] as another variant of the metric dimension of graphs. The metric dimension of graph was first presented by Slater [8] and later by Harary et al. [9]. Consider \( \Psi \) to be a connected graph of order \( n \), where \( V(\Psi) \) and \( E(\Psi) \) are the set of vertices and edges, respectively. If two vertices \( w, z \in V(\Psi) \), then the length of shortest path between \( w \) and \( z \) in \( \Psi \) is the distance between these vertices and is denoted by \( d(w, z) \). The distance between a vertex \( z \) and \( J \subseteq V(\Psi) \) is defined as \( \min\{d(z, y) | y \in J \} \) and is denoted by \( d(z, J) \). For a vertex \( z \in V(\Psi) \), \( N(z) \) will denote the open neighbourhood of \( z \) in \( \Psi \), i.e., \( N(z) = \{q \in V(\Psi) : q \text{ is adjacent to } z \} \) and a closed neighbourhood of \( z \) will be denoted by \( N[z] = N(z) \cup \{z\} \) [10]. Consider \( \mu = \{z_1, z_2, \ldots, z_t\} \subseteq V(\Psi) \) to be an ordered subset of \( V(\Psi) \). The representation of \( z \) with respect to \( \mu \) is \( t \)-tuple \( (d(z, z_1), d(z, z_2), \ldots, d(z, z_t)) \), denoted by \( r(z|\mu) \). The subset \( \mu \) is called a resolving set of \( \Psi \), if representation of \( z \) with respect to \( \mu \) is distinct for all \( z \in V(\Psi) \). The metric dimension of \( \Psi \) is defined as \( \min\{|\mu|: \mu \text{ is resolving set of } \Psi\} \).
and is denoted by $\beta(\Psi)$. In 2000, [11] observed the application of $\beta(\Psi)$ in pharmaceutical chemistry. Zehui et al. computed the metric dimension of the families of generalized Petersen graphs in [12]. Hussain et al. studied the metric dimension of 1-pentagonal carbon nanocone networks in [13].

In 2008, Hernando et al. [14] initiated the concept of fault-tolerant metric dimension of graphs. The resolving set $\mu$ of $V(\Psi)$ is called fault-tolerant if $\mu, \{a\}$ is also a resolving set for each $a \in \mu$. The fault-tolerant metric dimension of $\Psi$ is the minimum cardinality of fault-tolerant resolving set $\mu$ and is denoted by $\beta'(\Psi)$. Raza et al. presented bounds on the fault-tolerant metric dimension of three infinite families of regular graphs [15] and also computed fault-tolerant metric dimension of convex polytopes [16]. Seyedi et al. studied the metric dimension of 1-pentagonal carbon chemistry. Zehui et al. computed the metric dimension of $C_{p;1}$ and $C_{p;2}$, and $C_{p;3}$ and show that they have constant fault-tolerant partition dimension. Some basic results on $\mathcal{F}(\Psi)$ are stated as follows.

Salman et al. revealed the following basic results on $\mathcal{F}(\Omega)$.

Proposition 1 (see [32]). For $n \geq 2$,

(a) $pd(\Omega) \leq \mathcal{F}(\Omega)$

(b) $\mathcal{F}(\Omega) = n \iff \Omega = K_n$ or $\Omega = K_n - e$

Proposition 2 (see [28]). (a) For $n \geq 2$, $\mathcal{F}(\Omega) \leq \beta'(\Omega) + 1$

(b) For $n \geq 3$, $3 \leq \mathcal{F}(\Omega) \leq n$

The remaining part of the article is structured in the following manner: Section 2 is devoted for the computation of $\mathcal{F}(C(p;\delta))$, where $C(p;\delta)$ is a homogeneous caterpillar. In Section 3, we have concluded the paper by giving future research direction and application showing significance of the current work.

2. Fault-Tolerant Partition Dimension of the Homogeneous Caterpillar Graph

A caterpillar graph is a tree having a central path with $p$ vertices $\{c_1, c_2, \ldots, c_p\}$. Leaves $\delta$ are pendant vertices those are attached to every vertex of the central path. If an equal number of leaf vertices are attached to each $c_i$ where $1 \leq i \leq p$, then caterpillar is called the homogeneous caterpillar and is denoted by $C(p;\delta)$. The set $V(C(p;\delta)) = C \cup A_{ij}$, where $C = \{c_i : 1 \leq i \leq p\}$ and $A_{ij} = \{a_{ij} : 1 \leq i \leq p, 1 \leq j \leq \delta\}$, and $E(C(p;\delta)) = \{c_i a_{ij} : 1 \leq i \leq p, 1 \leq j \leq \delta\}$ are the vertex set and edge set of homogeneous caterpillar $C(p;\delta)$, respectively. The graph of homogeneous caterpillar $C(5;3)$ is shown in Figure 1.

Lemma 1 (see [18]). Let $C(p;\delta)$ be a homogeneous caterpillar with $p, \delta \geq 1$. Then, $pd(C(p;\delta)) = 3$ if and only if $(\delta = 1$ and $p \geq 3)$ or $(\delta = 2$ and $p \geq 2)$ or $(\delta = 3$ and $p \leq 3)$.

Lemma 2 (see [3]). Let $C(p;\delta)$ be a homogeneous caterpillar with any integers $p, \delta \geq 1$. Then, $pd(C(p;\delta)) = 4$ if and only if $(\delta = 3$ and $p \geq 4)$ or $(\delta = 4$ and $p \leq 4)$.

The following theorems will allow us to compute $\mathcal{F}(C(p;\delta))$.

Theorem 1. Let $C(p;\delta)$ be a homogeneous caterpillar. If $\delta = 1$ and $2 \leq p \leq 4$, then $\mathcal{F}(C(p;1)) = 3$.

Proof. Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ be a partition with 3 partition classes of the vertices of $C(p;1)$. We have the following:

Case (i): for $p = 2$

The $r(\nu(\Omega))$ of $C(2;1)$ with respect to $\Omega_1 = \{c_1, c_2\}$, $\Omega_2 = \{a_{11}\}$, and $\Omega_3 = \{a_{21}\}$ is as follows:
r(c_\sigma|\Omega) = \begin{cases} 
(0, 1, 2), & \text{for } \sigma = 1, \\
(0, 2, 1), & \text{for } \sigma = 2, \\
(1, 0, 3), & \text{for } \sigma = 1, \\
(1, 3, 0), & \text{for } \sigma = 2.
\end{cases} 
(1)

r(a_{i1}|\Omega) = \begin{cases} 
(0, 2, 2), & \text{for } \sigma = 1, \\
(0, 1, 1), & \text{for } \sigma = 2, \\
(1, 0, 2), & \text{for } \sigma = 3, \\
(0, 3, 3), & \text{for } \sigma = 1, \\
(1, 2, 0), & \text{for } \sigma = 2, \\
(2, 0, 3), & \text{for } \sigma = 3.
\end{cases} 
(2)

Case (ii): for \( p = 3 \).
Consider \( \Omega_1 = \{c_1, c_2, c_{a1}\} \), \( \Omega_2 = \{c_3, c_{a3}\} \) and \( \Omega_3 = \{a_{21}\} \). The \( r(\nu|\Omega) \) of \( C(3; 1) \) is as follows:

r(c_\sigma|\Omega) = \begin{cases} 
(0, 2, 2), & \text{for } \sigma = 1, \\
(0, 1, 1), & \text{for } \sigma = 2, \\
(1, 0, 2), & \text{for } \sigma = 3, \\
(0, 3, 3), & \text{for } \sigma = 1, \\
(1, 2, 0), & \text{for } \sigma = 2, \\
(2, 0, 3), & \text{for } \sigma = 3.
\end{cases} 
(3)

Case (iii): for \( p = 4 \).
Consider \( \Omega_1 = \{c_1, c_2, c_{a1}\} \), \( \Omega_2 = \{c_3, c_{a3}, a_2\} \), and \( \Omega_3 = \{a_{21}, a_{31}\} \). The \( r(\nu|\Omega) \) of \( C(4; 1) \) is as follows:

As all the vertices have distinct representations, \( \Omega \) is fault-tolerant resolving partition of \( C(p; 1) \); therefore, \( \mathcal{F}(C(p; 1)) \geq 3 \). It follows from Proposition 1 (a) and Lemma 1 that \( \mathcal{F}(C(p; 1)) = 3 \), which completes the proof.

Theorem 2. Let \( C(p; \delta) \) be a homogeneous caterpillar; if \( \delta \geq 1 \) and \( p \geq 5 \), or \( \delta = 2 \) and \( p \geq 3 \), then \( \mathcal{F}(C(p; \delta)) \geq 4 \).

Proof. In order to prove that \( \mathcal{F}(C(p; \delta)) \geq 4 \), we show that \( \mathcal{F}(C(p; \delta)) \neq 3 \). Suppose on the contrary that \( \Omega = \{\Omega_1, \Omega_2, \Omega_3\} \) is a fault-tolerant partition basis of \( C(p; \delta) \). One of the partition sets \( \Omega_1, \Omega_2, \) or \( \Omega_3 \) contains at least one vertex of degree 3. Without loss of generality, we assume that \( \nu \) is a vertex of degree 3 that belongs to \( \Omega_1 \), and \( N(\nu) = \{z_1, z_2, z_3\} \). Suppose \( |\Omega_1| = 1 \) and \( N(\nu) \subseteq \Omega_2 \cup \Omega_3 \), \( |N(\nu) \cap \Omega_2| \geq 2 \), or \( |N(\nu) \cap \Omega_3| \geq 2 \). Without loss of generality, we assume that at least two vertices \( g, h \in N(\nu) \cap \Omega_2 \). As \( r(g|\Omega) = (1, 0, c_1) \) and \( r(h|\Omega) = (1, 0, c_2) \) have two identical coordinates, hence a contradiction. Now, we suppose that \( |\Omega_1| \geq 2 \). We discuss the following cases:

Case 1: if \( N(\nu) \cap \Omega_1 = \{z_1, z_2, z_3\} \), then \( r(\nu\Omega) = (0, b_1, c_0) \), \( r(z_1|\Omega) = (0, b_1, c_1) \), \( r(z_2|\Omega) = (0, b_2, c_2) \) and \( r(z_3|\Omega) = (0, b_3, c_3) \). As \( b_0 - 1 \leq b_1, b_2, b_3 \leq b_0 + 1 \), by Pigeonhole principle, it is observed that two vertices have two identical coordinates in their representation, which leads to a contradiction.

Case 2: if \( N(\nu) \cap \Omega_1 = \{z_1, z_2\} \) and one vertex \( z_3 \in \Omega_2 \), then \( r(\nu|\Omega) = (0, 1, c_0) \), \( r(z_1|\Omega) = (0, b_1, c_1) \), \( r(z_2|\Omega) = (0, b_2, c_2) \) and \( r(z_3|\Omega) = (1, 0, c_3) \). Since \( 1 \leq b_1, b_2 \leq 2 \), the representation of two vertices will again be identical at two places, which is a contradiction.

Case 3: if \( N(\nu) \cap \Omega_1 = \{z_1\} \) and two vertices \( z_2, z_3 \in \Omega_2 \), then \( r(\nu|\Omega) = (0, 1, c_0) \), \( r(z_2|\Omega) = (0, b_1, c_1) \), \( r(z_3|\Omega) = (1, 0, c_3) \) and \( r(z_3|\Omega) = (1, 0, c_3) \). As \( r(z_2|\Omega) \) and \( r(z_3|\Omega) \) have two identical coordinates, hence a contradiction.

Case 4: if \( N(\nu) \cap \Omega_1 = \{z_1\} \), \( z_2 \in \Omega_2 \) and \( z_3 \in \Omega_3 \). We have \( r(\nu|\Omega) = (0, 1, 1) \).

Case 4(a): for \( \delta = 1 \) and \( p \geq 5 \).
Consider \( N(z_1) = \{v, s_1\} \), \( N(z_2) = \{v\} \) and \( N(z_3) = \{v, s_2, s_3\} \). Let \( s_1 \in \Omega_1 \) and \( s_2, s_3 \in \Omega_3 \); then, \( r(s_1|\Omega) = (2, q_1, 0) \) and \( r(s_2|\Omega) = (2, q_2, 0) \), which is a contradiction. Now, let \( s_2, s_3 \in \Omega_2 \), then \( r(s_2|\Omega) = (2, 0, 1) \) and \( r(s_2|\Omega) = (2, 0, 1) \), a contradiction.

Case 4(b): for \( \delta = 2 \) and \( p \geq 3 \).
Consider \( N(z_1) = \{v\} \), \( N(z_2) = \{v\} \) and \( N(z_3) = \{v, s_2, s_3\} \). Let \( s_1, s_2, s_3 \in \Omega_1 \); then, \( r(s_1|\Omega) = (2, q_1, 0) \), \( r(s_2|\Omega) = (2, q_2, 0) \) and \( r(s_3|\Omega) = (2, q_3, 0) \), which leads to a contradiction. Now, let \( s_1, s_2 \in \Omega_2 \) and \( s_3 \in \Omega_3 \); then \( r(s_1|\Omega) = (2, 0, 1) \) and \( r(s_2|\Omega) = (2, 0, 1) \), which leads to a contradiction.

Case 5: Let \( N(\nu) \cap \Omega = \emptyset \) and at least two vertices from \( N(\nu) \) belong to \( \Omega_2 \). Without loss of generality, we suppose that \( z_1, z_2 \in \Omega_2 \); then \( r(\nu|\Omega) = (0, 1, c_0) \), \( r(z_1|\Omega) = (1, 0, c_1) \), \( r(z_2|\Omega) = (1, 0, c_2) \). Again, \( r(z_1|\Omega) \) and \( r(z_2|\Omega) \) have two identical coordinates, which leads to a contradiction.

It is obvious from this discussion that \( \mathcal{F}(C(p; \delta)) \geq 4 \), which completes the proof.
Theorem 3. Let $C(p; \delta)$ be a homogeneous caterpillar; if $\delta = 1$ and $p \geq 3$, then $\mathcal{F}(C(p; 1)) = 4$.

Proof. Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ be a partition set of $V(C(p; 1))$. The $r(v|\Omega)$ of $(C(p; 1))$, taking $\Omega_1 = \{c_i: 1 \leq i \leq p\} \cup \{a_{i1}\}$, $\Omega_2 = \{a_{i1}: 2 \leq j \leq p - 2\}$, $\Omega_3 = \{a_{p-1}\}$, and $\Omega_4 = \{a_{p}\}$, is as follows:

$$r(c_i|\Omega) = \begin{cases} (0, 0, 1, 1), & \text{for } \sigma = 1, \\ (0, 0, 2, 2), & \text{for } \sigma = 2, \\ (0, 1, 1, 1), & \text{for } \sigma = 3, \\ (0, 1, 0, 2), & \text{for } \sigma = 4. \end{cases}$$

$$r(a_{i1}|\Omega) = \begin{cases} (0, 2, 2, 0), & \text{for } \sigma = 1, \\ (0, 0, 2, 2), & \text{for } \sigma = 2, \\ (1, 1, 0, 2), & \text{for } \sigma = 3, \\ (2, 1, 0, 2), & \text{for } \sigma = 4. \end{cases}$$

$$r(a_{i2}|\Omega) = \begin{cases} (0, 0, 2, 2), & \text{for } \sigma = 1, \\ (0, 2, 0, 2), & \text{for } \sigma = 2, \\ (1, 2, 1, 0), & \text{for } \sigma = 3, \\ (2, 2, 1, 0), & \text{for } \sigma = 4. \end{cases}$$

As all the representations are distinct, $\Omega$ is a fault-tolerant resolving partition of $C(p; 1)$; therefore, $\mathcal{F}(C(p; 2)) \leq 4$. Also, from Theorem 2, $\mathcal{F}(C(p; 1)) \geq 4$. This completes the proof.

Theorem 4. Let $C(p; \delta)$ be a homogeneous caterpillar; if $\delta = 2$ and $p = 2$, then $\mathcal{F}(C(2; 2)) = 3$.

Proof. Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ be a partition set of $V(C(2; 2))$. The $r(v|\Omega)$ of $C(2; 2)$, taking $\Omega_1 = \{c_1, a_{11}\}$, $\Omega_2 = \{c_2, a_{21}\}$, and $\Omega_3 = \{a_{12}, a_{22}\}$, is as follows:

$$r(c_i|\Omega) = \begin{cases} (0, 0, 1, 1), & \text{for } \sigma = 1, \\ (1, 0, 1, 0), & \text{for } \sigma = 2, \end{cases}$$

$$r(a_{i1}|\Omega) = \begin{cases} (0, 2, 2, 0), & \text{for } \sigma = 1, \\ (2, 0, 2, 2), & \text{for } \sigma = 2, \end{cases}$$

$$r(a_{i2}|\Omega) = \begin{cases} (1, 2, 1, 0), & \text{for } \sigma = 1, \\ (2, 1, 0, 2), & \text{for } \sigma = 2. \end{cases}$$

It is obvious from the representations in (5) that $\Omega$ is a fault-tolerant resolving partition of $C(2; 2)$; therefore, $\mathcal{F}(C(2; 2)) \leq 3$. It follows from Proposition 1 (a) and Lemma 1 that $\mathcal{F}(C(2; 2)) = 3$, which completes the proof.

Theorem 5. Let $C(p; \delta)$ be a homogeneous caterpillar; if $\delta = 2$ and $p \geq 3$, then $\mathcal{F}(C(p; 2)) = 4$.

Proof. Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ be a partition with 4 partition classes of the vertices of $C(p; 2)$. The $r(v|\Omega)$ of $C(p; 2)$ with respect to $\Omega_1 = \{c_i: 1 \leq i \leq p\} \cup \{a_{i1}, a_{i2}: 3 \leq i \leq p - 1\}$, $\Omega_2 = \{a_{12}, a_{21}\}$, $\Omega_3 = \{a_{p-1}\}$, and $\Omega_4 = \{a_{p}\}$ is as follows:

$$r(c_i|\Omega) = \begin{cases} (0, 0, 1, 2, p), & \text{for } \sigma = 1, \\ (0, 0, 2, 1, p - 1), & \text{for } \sigma = 2 p \leq 2. \end{cases}$$

$$r(a_{i1}|\Omega) = \begin{cases} (0, 2, 3, p + 1), & \text{for } \sigma = 1, \\ (1, 0, 2, p), & \text{for } \sigma = 2, \end{cases}$$

$$r(a_{i2}|\Omega) = \begin{cases} (0, 0, 2, 1, p - 1), & \text{for } \sigma = 1, \\ (1, 0, 3, p + 1), & \text{for } \sigma = 2. \end{cases}$$

$$r(a_{p}|\Omega) = \begin{cases} (1, 2, 0, p), & \text{for } \sigma = 1, \\ (1, 0, 2, p), & \text{for } \sigma = 2. \end{cases}$$

The representations in (7) shows that $\Omega$ is a fault-tolerant resolving partition of $C(5; 2)$. The following lemma will be used in computing $\mathcal{F}(C(p; 3))$.

Lemma 3. If $C(p; 3)$ is a homogeneous caterpillar, then we develop the relations of distances of $V(C(p; 3))$.

Proof. The relations of distances of the vertices $C = \{c_1, c_2, \ldots, c_p\}$ and
Lemma 3 (c), representation of two vertices will have three identical coordinates, which is a contradiction.}

Case 2: we discuss cases when two partitioning sets are a subset of union of 2 sets of vertices:

Case 2(a): when two partitioning sets of \( \Omega \) are subsets of \( A_{13} \cup A_{32} \) and at least two vertices from \( a_{11}, a_{12}, \) and \( a_{13} \) belong to the remaining two partition sets, then by Lemma 3 (b), representation of two vertices will have three identical coordinates, which is a contradiction.

\[
\mathcal{F}(C(p; 3)) = \begin{cases} 
4, & \text{if } 2 \leq p \leq 3, \\
5, & \text{if } p \geq 4.
\end{cases}
\] (8)

\begin{align*}
r(c_1, \Omega) &= \begin{cases} 
(0, 1, 1, p - \sigma + 1, p - \sigma + 1), & \text{for } 1 \leq \sigma \leq p - 1, \\
(0, 2, 2, 1, 1), & \text{for } \sigma = p,
\end{cases} \\
r(a_{21}, \Omega) &= \begin{cases} 
(0, 2, 2, p - \sigma + 2, p - \sigma + 2), & \text{for } 1 \leq \sigma \leq p - 1, \\
(0, 3, 3, 2, 2), & \text{for } \sigma = p.
\end{cases} \\
r(a_{22}, \Omega) &= \begin{cases} 
(1, 0, 2, p - \sigma + 2, p - \sigma + 2), & \text{for } 1 \leq \sigma \leq p - 1, \\
(1, 3, 3, 0, 2), & \text{for } \sigma = p.
\end{cases} \\
r(a_{23}, \Omega) &= \begin{cases} 
(1, 2, 0, p - \sigma + 2, p - \sigma + 2), & \text{for } 1 \leq \sigma \leq p - 1, \\
(1, 3, 3, 2, 0), & \text{for } \sigma = p.
\end{cases}
\end{align*}

(9)
Example 2. Consider the homogeneous caterpillar $C(4; 3)$, shown in Figure 3.

Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\}$ be a partition with 5 partition classes of the vertices of $C(4; 3)$. The representations of vertices of $C(4; 3)$, considering $\Omega_1 = \{c_i: 1 \leq i \leq 4\} \cup \{a_{11}, a_{21}, a_{31}, a_{41}\}$, $\Omega_2 = \{a_{12}, a_{22}, a_{32}\}$, $\Omega_3 = \{a_{13}, a_{23}, a_{33}\}$, $\Omega_4 = \{a_{42}\}$, and $\Omega_5 = \{a_{43}\}$, are as follows:

\[
\begin{align*}
    r(c_1|\Omega) &= (0, 1, 1, 4, 4), \\
    r(c_2|\Omega) &= (0, 1, 1, 3, 3), \\
    r(c_3|\Omega) &= (0, 1, 1, 2, 2), \\
    r(c_4|\Omega) &= (0, 2, 2, 1, 1), \\
    r(a_{11}|\Omega) &= (0, 2, 2, 5, 5), \\
    r(a_{12}|\Omega) &= (1, 0, 2, 5, 5), \\
    r(a_{13}|\Omega) &= (1, 2, 0, 5, 5), \\
    r(a_{21}|\Omega) &= (0, 2, 2, 4, 4), \\
    r(a_{22}|\Omega) &= (1, 0, 2, 4, 4), \\
    r(a_{23}|\Omega) &= (1, 2, 0, 4, 4), \\
    r(a_{31}|\Omega) &= (0, 2, 2, 3, 3), \\
    r(a_{32}|\Omega) &= (1, 0, 2, 3, 3), \\
    r(a_{33}|\Omega) &= (1, 2, 0, 3, 3), \\
    r(a_{41}|\Omega) &= (0, 3, 3, 2, 2), \\
    r(a_{42}|\Omega) &= (1, 3, 3, 0, 2), \\
    r(a_{43}|\Omega) &= (1, 3, 3, 2, 0).
\end{align*}
\]

The representations in (10) shows that $\Omega$ is a fault-tolerant resolving partition of $C(4; 3)$.

3. Conclusion

In this paper, we have computed that $F(C(p; \delta))$ for $\delta = 1, 2$, and 3 is between 3 and 5. The obtained results led us to the conclusion that the structures of homogeneous caterpillars $C(p; 1)$, $C(p; 2)$, and $C(p; 3)$ have constant fault-tolerant partition dimension for $p \geq 5$, $p > 3$, and $p > 4$, respectively. Future research can focus on computing the fault-tolerant partition dimension for the classes of homogeneous caterpillar $C(p; \delta)$, when $\delta = 4, 5$, and 6.

Here, we include an application of routing optimization problem that shows the significance of the current work. Consider a company that wants to pick passengers from different locations in a certain area using minimum resources and avoiding repeated visits. If locations are considered as nodes and roads connecting them as edges of a graph, then locations can be grouped together that require a single vehicle to pick the passenger. The minimum number of grouping required to represent each location uniquely can be realised as the partition dimension problem of the graph. Also, the minimum number of grouping required to uniquely represent each location even if one of the groups is inaccessible relates to fault-tolerant partition dimension of the graph.

Data Availability

The data used to support the findings of this study are included within the article.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


