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Research Article

Fuzzy Observer-Based H_2/H_∞ Output-Feedback Control for Stochastic Nonlinear Systems with Multiplicative Noise

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A design method is established for the mixed H_2/H_∞ output-feedback control of stochastic nonlinear systems with multiplicative noise. Firstly, using T-S fuzzy rules, we obtain a fuzzy model to approximate the original nonlinear system. Then, by Schur's complement, the suboptimal H_2/H_∞ output-feedback control design is transformed into a two-step convex optimization problem. A numerical example is given to show the effectiveness of the proposed method.

1. Introduction

One of the objectives of system control is to design a controller for the object model so that the closed-loop system achieves good performance while ensuring internal stability [1–4]. H_2 control and H_∞ control have been attractive subjects since they are of great practical significance in the field of engineering [5–9]. H_2 control has a high request of model accuracy and generally does not consider the influence of model error. However, in practical control systems, the system could not exclude the implication of uncertain factors. Being put forward by Zames in 1981, nowadays the H_∞ design idea has grown into an important robust control theory to eliminate external interference [10]. Accordingly, the combination of H_2 and H_∞ control design methods will ensure the robustness and optimality of the controlled system at the same time (we refer the readers to [11–14]).

It is noticeable that randomness is ubiquitous in the real world [15]. For example, there is a great deal of randomness in financial risk managements. Correspondingly, stochastic H_2/H_∞ control has been an attractive subject in recent decades. Chen et al. [16] implemented a detailed study on stochastic H_2/H_∞ control problems for linear systems with state-dependent noise. Subsequently, [17–20] reported research progress on H_2/H_∞ control of Markov jump systems.

On the other hand, nonlinearity is a universal phenomenon existing in engineering systems [21, 22]. Giving an example, a buck-boost circuit is rich in nonlinear dynamics. Generally speaking, control problems of nonlinear systems are more complicated than those of linear systems [23–25]. Linearizing the nonlinear system has become mature technology to treat the nonlinear problems. [26] introduced a suitable linear model gained by the T-S fuzzy rule to approximate a nonlinear system. [27] designed a mixed H_2/H_∞ controller for nonlinear systems based on fuzzy observer.

According to all above, robust control for stochastic nonlinear systems is definitely worthy both from the theoretical and practical application views. Compared with [27], in which the considered system model does not contain multiplicative noise, it is clear that H_2/H_∞ control for nonlinear systems with multiplicative noise has broader application prospects. The other contribution of this paper is that the suboptimal H_2/H_∞ output-feedback control design is transformed into a two-step convex optimization problem, which is convenient for solving by MATLAB efficiently.

This article is organized as follows. Section 2 builds up an approximate model of the original nonlinear system by the T-S fuzzy rule. Section 3 designs a fuzzy observer-based H_2/H_∞ output-feedback controller by solving a two-step convex optimization problem. A numerical example is given to illustrate the efficiency of the proposed design method in Section 4. Section 5 gives the summary of this paper.

For convenience, we adopt the following notations:

$\mathrm{tr}(A)$: the trace of matrix A

A^T: the transpose of matrix A

$A \geq 0\,(A > 0)$: a positive semidefinite (positive definite) matrix A

I: the identity matrix

$\|x\|$: the Euclidean 2-norm of the n-dimensional real vector x

2. Problem Description

Consider the following nonlinear random perturbation system:

$$\begin{cases} \mathrm{d}x(t) = [f(x(t)) + g(x(t))u(t)]\mathrm{d}t + C(t)x(t)\mathrm{d}w_1, \\ y(t) = h(x(t)) + v(t), \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^{n\times 1}$ is the state vector, $u(t)$ is the input of the system, and $y(t)$ is the measured output. We assume that w_1 is a one-dimensional standard Wiener process. $f(x(t))$, $g(x(t))$, and $h(x(t))$ are supposed to be smooth functions. System (1) is influenced by $v(t)$ which is a bounded measurement noise, that is, $E\int_0^T vv'\mathrm{d}t = R_0 \geq 0$.

Using T-S fuzzy rule, we establish a linear fuzzy model for the stochastic model (1). Specifically, by the fuzzy rule, R_i: if $m_1(t)$ is $F_{i1}, \ldots, m_g(t)$ is F_{ig}, $i = 1, 2, \ldots, L$, then we have

$$\begin{cases} \mathrm{d}x(t) = [A_i(t)x(t) + B_i(t)u(t)]\mathrm{d}t + C(t)x(t)\mathrm{d}w_1, \\ y(t) = D_i(t)x(t) + v(t), \end{cases} \tag{2}$$

where R_i represents the i th rule, L demotes the rule number, $m(t) = [m_1(t), \ldots, m_g(t)]^{\mathrm{T}}$ are the measurable prerequisite variables, F_{ij} is the fuzzy set, and A_i, B_i, $D_i\,(i = 1, 2, \ldots, L)$ are matrices with right dimensions.

By using the single point blur method, the product of reasoning, and the average weighted fuzzification, the following form of fuzzy model is obtained:

$$\begin{cases} \mathrm{d}x(t) = \dfrac{\sum_{i=1}^{L}\mu_i(m(t))[A_i(t)x(t) + B_i(t)u(t)]\mathrm{d}t}{\sum_{i=1}^{L}\mu_i(m(t))} + C(t)x(t)\mathrm{d}w_1 = \displaystyle\sum_{i=1}^{L} h_i(m(t))[A_i(t)x(t) + B_i(t)u(t)]\mathrm{d}t + C(t)x(t)\mathrm{d}w_1, \\ y(t) = \dfrac{\sum_{i=1}^{L}\mu_i(m(t))D_i(t)x(t)}{\sum_{i=1}^{L}\mu_i(m(t))} + v(t) = \displaystyle\sum_{i=1}^{L} h_i(m(t))D_i(t)x(t) + v(t), \end{cases} \tag{3}$$

where $\mu_i(m(t)) = \prod_{j=1}^{g} F_{ij}(m_j(t))$, $h_i(m(t)) = \mu_i(m(t)) / \sum_{i=1}^{L}\mu_i(m(t))$, $m(t) = [m_1(t), \ldots, m_g(t)]^T$, and $F_{ij}(m_j(t))$ is the grade of membership of $m_j(t)$ in F_{ij}. We suppose that $\mu_i(m(t)) \geq 0$, $i = 1, 2, \ldots, L$. It is easy to see

$$\begin{aligned} &h_i(m(t)) \geq 0, \\ &\sum_{i=1}^{L} h_i(m(t)) = 1. \end{aligned} \tag{4}$$

Thus, system (1) is equivalent to the following system:

$$\begin{cases} \mathrm{d}x(t) = \displaystyle\sum_{i=1}^{L} h_i(m(t))[A_i(t)x(t) + B_i(t)u(t)]\mathrm{d}t + C(t)x(t)\mathrm{d}w_1 + (\Delta f + \Delta g)\mathrm{d}t, \\ y(t) = \displaystyle\sum_{i=1}^{L} h_i(m(t))D_i(t)x(t) + v(t) + \Delta h, \end{cases} \tag{5}$$

where

$$\Delta f = \left[f(x(t)) - \sum_{i=1}^{L} h_i(m(t)) A_i x(t) \right],$$
$$\Delta g = \left[g(x(t)) - \sum_{i=1}^{L} h_i(m(t)) B_i u(t) \right], \tag{6}$$
$$\Delta h = \left[h(x(t)) - \sum_{i=1}^{L} h_i(m(t)) D_i x(t) \right]$$

represent the approximate error between system (3) and nonlinear model (1).

Select the finite-dimension compensator shown below:

$$\begin{cases} \mathrm{d}\hat{x}(t) = \sum_{i=1}^{L} h_i(m(t)) \left[A_i(t)\hat{x}(t) + B_i(t)u(t) + L_i \left(y - \sum_{i=1}^{L} h_i(m(t)) D_i(t)\hat{x}(t) \right) \right] \mathrm{d}t, \\ u(t) = \sum_{j=1}^{L} h_j(m(t)) k_j \hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \end{cases} \tag{7}$$

where $u(t) = \sum_{j=1}^{L} h_j(m(t)) k_j \hat{x}(t)$ is the fuzzy controller and $k_j\,(j = 1, 2, \ldots, L)$ is the control parameter.

Setting $\tilde{x} = x - \hat{x}$ and $\overline{x} = [\hat{x}, \tilde{x}]^T$, we get the following closed-loop system:

$$\mathrm{d}\overline{x} = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(m(t)) h_j(m(t)) \left(\overline{A}_{ij}\overline{x} + \overline{B}_i v \right) \mathrm{d}t + \overline{C}\overline{x}\mathrm{d}w_1 + (\overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h})\mathrm{d}t, \tag{8}$$

where

$$\overline{A}_{ij} = \begin{bmatrix} A_i + B_i k_j & L_i D_i \\ 0 & A_i - L_i D_i \end{bmatrix},$$
$$\overline{B}_i = \begin{bmatrix} L_i \\ -L_i \end{bmatrix}, \overline{C} = \begin{bmatrix} 0 & 0 \\ C & C \end{bmatrix}, \tag{9}$$
$$\overline{\Delta f} = \begin{bmatrix} 0 \\ \Delta f \end{bmatrix}, \overline{\Delta g} = \begin{bmatrix} 0 \\ \Delta g \end{bmatrix}, \overline{\Delta h} = \left[\sum_{i=1}^{L} h_i(m(t)) L_i \Delta h - \sum_{i=1}^{L} h_i(m(t)) L_i \Delta h \right].$$

Next we consider H_∞ control performance. Given $\gamma > 0$ and weighting matrix $Q_1 > 0$, if $(\overline{x}(0) = 0)$,

$$E \int_0^T \overline{x}^T Q_1 \overline{x} \mathrm{d}t < \gamma^2 E \int_0^T v^T v \mathrm{d}t \triangleq \gamma^2 tr(R_0), \tag{10}$$

then we call the H_∞ performance is satisfied.

H_∞ control aims to eliminate the influence of external interference, but the performance of the closed-loop system may not be ideal. Therefore, a mixed H_2/H_∞ control design based on fuzzy observer will be implemented. H_2 performance is defined as follows:

$$J(\overline{x}, u) = E \int_0^T \left(\overline{x}^T Q_2 \overline{x} + u^T R_1 u \right) \mathrm{d}t, \tag{11}$$

where $Q_2 > 0$ and $R_1 > 0$.

3. Output-Feedback Control Design Based on Fuzzy Observer

In the previous work, using the T-S fuzzy rule, we got a fuzzy model (3) and the approximate error between nonlinear system (1) and the fuzzy model. This section attempts to design an output-feedback control satisfying H_∞ performance and H_2 performance for fuzzy model (3).

Let the following inequalities be true:

$$\begin{aligned} &\|\Delta f\| \leq \|\Delta A x(t)\|, \\ &\|\Delta g\| \leq \left\|\sum_{i=1}^{L} h_i(m(t))\Delta B k_j \hat{x}(t)\right\|, \\ &\left\|\sum_{i=1}^{L} h_i(m(t)) L_i \Delta h\right\| \leq \left\|\sum_{i=1}^{L} h_i(m(t)) L_i \Delta D x(t)\right\|. \end{aligned} \tag{12}$$

By computation and the above inequalities, we have

$$\begin{aligned}
(\overline{\Delta f})^T(\overline{\Delta f}) &= (\Delta f)^T(\Delta f) \\
&= \left(f(x(t)) - \sum_{i=1}^{L} h_i(m(t))A_i x(t)\right)^T \times \left((f x(t)) - \sum_{i=1}^{L} h_i(m(t))A_i x(t)\right) \\
&\leq (\Delta A x(t))^T(\Delta A x(t)) \\
&= (\Delta A\hat{x}(t) + \Delta A\tilde{x}(t))^T \times (\Delta A\hat{x}(t) + \Delta A\tilde{x}(t)) \\
&= ([\Delta A, \Delta A]\overline{x}(t))^T \times ([\Delta A, \Delta A]\overline{x}(t)) \\
&= (\Phi\overline{x}(t))^T(\Phi\overline{x}(t)), \\
(\overline{\Delta g})^T(\overline{\Delta g}) &= (\Delta g)^T(\Delta g) \\
&\leq \left(\sum_{j=1}^{L} h_j(m(t))\Delta B k_j \hat{x}(t)\right)^T \times \left(\sum_{j=1}^{L} h_j(m(t))\Delta B k_j \hat{x}(t)\right) \\
&= \left(\sum_{j=1}^{L} h_j(m(t))\left[\Delta B k_j, 0\right]\overline{x}(t)\right)^T \times \left(\sum_{j=1}^{L} h_j(m(t))\left[\Delta B k_j, 0\right]\overline{x}(t)\right) \\
&= \left(\sum_{j=1}^{L} h_j(m(t))\Omega_j\overline{x}(t)\right)^T \left(\sum_{j=1}^{L} h_j(m(t))\Omega_j\overline{x}(t)\right) \\
&\leq \sum_{j=1}^{L} h_j(m(t))\overline{x}^T(t)\Omega_j^T\Omega_j\overline{x}(t), \\
(\overline{\Delta h})^T(\overline{\Delta h}) &= 2\left(\sum_{i=1}^{L} h_i(m(t))L_i\Delta h\right)^T \times \left(\sum_{i=1}^{L} h_i(m(t))L_i\Delta h\right) \\
&\leq 2\left(\sum_{i=1}^{L} h_i(m(t))L_i\Delta D x(t)\right)^T \times \left(\sum_{i=1}^{L} h_i(m(t))L_i\Delta D x(t)\right) \\
&= 2\left(\sum_{i=1}^{L} h_i(m(t))[L_i\Delta D, L_i\Delta D]\overline{x}(t)\right)^T \times \left(\sum_{i=1}^{L} h_i(m(t))[L_i\Delta D, L_i\Delta D]\overline{x}(t)\right) \\
&= 2\left(\sum_{i=1}^{L} h_i(m(t))\Xi_i\overline{x}(t)\right)^T \left(\sum_{i=1}^{L} h_i(m(t))\Xi_i\overline{x}(t)\right) \\
&\leq 2\sum_{i=1}^{L} h_i(m(t))\overline{x}^T(t)\Xi_i^T\Xi_i\overline{x}(t),
\end{aligned} \tag{13}$$

where $\Phi = [\Delta A, \Delta A]$, $\Omega_j = [\Delta B k_j, 0]$, $\Xi_i = [L_i\Delta D, L_i\Delta D]$, $j = 1, 2, \ldots, L, i = 1, 2, \ldots, L$.

For the smooth progress of subsequent work, let us choose a Lyapunov function for system (8):

$$V(\overline{x}(t)) = \overline{x}^T(t)P\overline{x}(t), \tag{14}$$

where P is a weighted matrix with appropriate dimensions.

By integrating (14), we have

$$\begin{aligned}
E\int_0^T d(V(\overline{x}(t))) &= E\int_0^T d\left(\overline{x}^T P\overline{x}\right) \\
&= \sum_{i=1}^{L}\sum_{j=1}^{L} h(m(t))h_j(m(t))E_i\int_0^T\Big[\left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right)^T P\overline{x} + \overline{x}^T P\left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right) \\
&\quad + \overline{x}^T\overline{C}^T P\overline{C}\overline{x} + v^T\overline{B}_i^T P\overline{x} + \overline{x}^T P\overline{B}_i v\Big]\mathrm{d}t \\
&\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \overline{x}^T\left(3P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i\right)\overline{x}\mathrm{d}t \\
&\quad + \sum_{i=1}^{L} h_i(m(t))E\int_0^T\left(v^T\overline{B}_i^T P\overline{x} + \overline{x}^T P\overline{B}_i v\right)\mathrm{d}t \\
&= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \overline{x}^T 3P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i\overline{x}\mathrm{d}t \\
&\quad - E\int_0^T\left(\gamma^{-1}\overline{x}^T P\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right) - \gamma v^T\right)\cdot\left(\gamma^{-1}\overline{x}^T P\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right) - \gamma v^T\right)^T\mathrm{d}t \\
&\quad + E\int_0^T\left[\gamma^{-2}\overline{x}^T P\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)\cdot\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)^T P\overline{x} + \gamma^2 v^T v\right]\mathrm{d}t \\
&\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \overline{x}^T 3P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i \\
&\quad + \gamma^{-2}P\overline{B}_i\overline{B}_i^T\overline{x}\mathrm{d}t + E\int_0^T \gamma^2 v^T v\mathrm{d}t.
\end{aligned} \tag{15}$$

Based on (15), we can derive the following theorem.

Theorem 1. *If there exists a $P > 0$ satisfying the following inequalities:*

$$3P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + \gamma^{-2}P\overline{B}_i\overline{B}_i^T P + Q_1 + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i < 0, \tag{16}$$

$$4P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + \widetilde{k}_j^T R_1\widetilde{k}_j + Q_2 + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i < 0, \tag{17}$$

where $\widetilde{k}_j = \left[k_j, 0\right]$, then

(a) H_∞ control performance (10) is fulfilled.

(b) H_2 performance (11) has a upper bound, that is,

$$\begin{aligned}
J\left(\overline{x}, u = k_j\hat{x}\right) &\leq \|\overline{x}(0)\|^2\mathrm{tr}(P) \\
&\quad + \mathrm{tr}\left[\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)R_0\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)^T\right].
\end{aligned} \tag{18}$$

Proof. For given $Q_1 > 0$, by Schur's complement and (15), one can see that

$$\begin{aligned}
&E\int_0^T \overline{x}^T Q_1 \overline{x}\mathrm{d}t - \gamma^2 E\int_0^T v^T v\mathrm{d}t \\
&\leq E\int_0^T \left(\overline{x}^T Q_1 \overline{x} - \gamma^2 v^T v\right)\mathrm{d}t + \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \Big[\overline{x}^T P\left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right) \\
&\quad + \left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right)^T P\overline{x} + \overline{x}^T P\overline{B}_i v + v^T \overline{B}_i^T P\overline{x} + \overline{x}^T \overline{C}^T P\overline{C}\overline{x}\Big]\mathrm{d}t \\
&\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \Big[\overline{x}^T\left(3P^2 + \overline{A}_{ij}^T P + P\overline{A}_{ij} + \overline{C}^T P\overline{C} + Q_1 + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i\right)\overline{x} \\
&\quad + \overline{x}^T P\overline{B}_i v + v^T\overline{B}_i^T P\overline{x} - \gamma^2 v^T v\Big]\mathrm{d}t \\
&= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \begin{bmatrix}\overline{x}\\ v\end{bmatrix}^T \begin{bmatrix} X & P\overline{B}_i \\ \overline{B}_i^T P & -\gamma^2 I\end{bmatrix}\begin{bmatrix}\overline{x}\\ v\end{bmatrix}\mathrm{d}t < 0,
\end{aligned} \tag{19}$$

where $X = 3P^2 + P\overline{A}_{ij} + \overline{A}_{ij}^T P + \overline{C}^T P\overline{C} + Q_1 + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i$. Therefore, $E\int_0^T \overline{x}^T Q_1\overline{x}\mathrm{d}t < \gamma^2 E\int_0^T v^T v\mathrm{d}t$ is directly derived, i.e., conclusion (a) is valid.

Now let us prove (b). Under the constraint of (8), with the help of the method of completing square, we assert that

$$\begin{aligned}
J\left(\overline{x}, u = k_j\widehat{x}\right) &= E\int_0^T \left(\overline{x}^T Q_2\overline{x} + u^T R_1 u\right)\mathrm{d}t \\
&\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \Big[\overline{x}^T Q_2\overline{x} + \left(k_j\widehat{x}\right)^T\widetilde{R}_2\left(k_j\widehat{x}\right) + \overline{x}^T P\left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right) \\
&\quad + \left(\overline{A}_{ij}\overline{x} + \overline{\Delta f} + \overline{\Delta g} + \overline{\Delta h}\right)^T P\overline{x} + \overline{x}^T P\overline{B}_i v + v^T\overline{B}_i^T P\overline{x} + \overline{x}^T\overline{C}^T P\overline{C}\overline{x}\Big]\mathrm{d}t + \overline{x}^T(0)P\overline{x}(0) \\
&\leq \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \overline{x}^T\Big[\Big(3P^2 + \overline{A}_{ij}^T P + P\overline{A}_{ij} + \overline{C}^T P\overline{C} + Q_2 + \widetilde{k}_j^T R_1\widetilde{k}_j + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i\Big)\overline{x}dt \\
&\quad + E\int_0^T v^T\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)^T\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)v\mathrm{d}t + \overline{x}^T(0)P\overline{x}(0) \\
&\leq \overline{x}^T(0)P\overline{x}(0) + E\int_0^T v^T\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)^T\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)v\mathrm{d}t \\
&\quad + \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(m(t))h_j(m(t))E\int_0^T \overline{x}^T\left(4P^2 + \overline{A}_{ij}^T P + P\overline{A}_{ij} + \overline{C}^T P\overline{C} + Q_2 + \widetilde{k}_j^T R_2\widetilde{k}_j + \Phi^T\Phi + \Omega_j^T\Omega_j + 2\Xi_i^T\Xi_i\right)\overline{x}\mathrm{d}t \\
&\leq \|\overline{x}(0)\|^2\mathrm{tr}(P) + \mathrm{tr}\left[\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)R_0\left(\sum_{i=1}^{L} h_i(m(t))\overline{B}_i\right)^T\right],
\end{aligned} \tag{20}$$

which shows that (b) holds. The proof of this theorem is concluded.

According to Theorem 1, suboptimal H_2/H_∞ control design has been transformed into solving the optimization problem $\min_{P>0}\mathrm{tr}(P)$ under the constraint of (16) and (17). However, because P_{22}, L_i, and W_{11} are coupled in some components, the optimization problem is not convex. So, we need to convert it into convex optimization problems.

Express P, Q_1, and Q_2 as follows:

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}, Q_1 = \begin{bmatrix} Q_{11}^{(1)} & 0 \\ 0 & Q_{22}^{(1)} \end{bmatrix}, Q_2 = \begin{bmatrix} Q_{11}^{(2)} & 0 \\ 0 & Q_{22}^{(2)} \end{bmatrix}. \tag{21}$$

Plugging these representations into (16) and (17), one gets

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned}
A_{11} &= 3P_{11}^2 + P_{11}\left(A_i + B_i k_j\right) + \left(A_i + B_i k_j\right)^T P_{11} + C^T P_{22} C + \gamma^{-2} P_{11} L_i L_i^T P_{11} + Q_{11}^{(1)} + \Delta A^T \Delta A + k_j^T \Delta B^T \Delta B k_j + 2\Delta D^T L_i^T L_i \Delta D, \\
A_{12} &= P_{11} L_i D_i + C^T P_{22} C - \gamma^{-2} P_{11} L_i L_i^T P_{22} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
A_{21} &= D_i^T L_i^T P_{11} + C^T P_{22} C - \gamma^{-2} P_{22} L_i L_i^T P_{11} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
A_{22} &= 3P_{22}^2 + P_{22}\left(A_i - L_i D_i\right) + \left(A_i - L_i D_i\right)^T P_{22} + C^T P_{22} C + \gamma^{-2} P_{22} L_i L_i^T P_{22} + Q_{22}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
B_{11} &= 4P_{11}^2 + P_{11}\left(A_i + B_i k_j\right) + \left(A_i + B_i k_j\right)^T P_{11} + C^T P_{22} C + Q_{11}^{(2)} + k_j^T \widetilde{R}_2 k_j + \Delta A^T \Delta A + k_j^T \Delta B^T \Delta B k_j + 2\Delta D^T L_i^T L_i \Delta D, \\
B_{12} &= P_{11} L_i D_i + C^T P_{22} C + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
B_{21} &= D_i^T L_i^T P_{11} + C^T P_{22} C + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
B_{22} &= 4P_{22}^2 + P_{22}\left(A_i - L_i D_i\right) + \left(A_i - L_i D_i\right)^T P_{22} + C^T P_{22} C + Q_{22}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D.
\end{aligned} \tag{24}$$

Next, let $W = \operatorname{diag}(W_{11}, I)$ with $W_{11} = P_{11}^{-1}$. Multiplying both sides of (22) and (23) by W and setting $Z_i = P_{22} L_i$, $Y_j = k_j W_{11}$, we have

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned}
C_{11} &= 3I + A_i W_{11} + W_{11} A_i^T + B_i Y_j + Y_j^T B_i^T + W_{11} C^T P_{22} C W_{11} + \gamma^{-2} L_i L_i^T + W_{11} Q_{11}^{(1)} W_{11} + W_{11} \Delta A^T \Delta A W_{11} + Y_j^T \Delta B^T \Delta B Y_j \\
&\quad + 2W_{11} \Delta D^T L_i^T L_i \Delta D W_{11}, \\
C_{12} &= L_i D_i + W_{11} C^T P_{22} C - \gamma^{-2} L_i Z_i^T + W_{11} \Delta A^T \Delta A + 2W_{11} \Delta D^T L_i^T L_i \Delta D, \\
C_{21} &= D_i^T L_i^T + C^T P_{22} C W_{11} - \gamma^{-2} Z_i L_i^T + \Delta A^T \Delta A W_{11} + 2\Delta D^T L_i^T L_i \Delta D W_{11}, \\
C_{22} &= 3I + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + \gamma^{-2} Z_i Z_i^T + Q_{22}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
D_{11} &= 4I + A_i W_{11} + W_{11} A_i^T + B_i Y_j + Y_j^T B_i^T + W_{11} C^T P_{22} C W_{11} + W_{11} Q_{11}^{(2)} W_{11} + Y_j^T \widetilde{R}_2 Y_j + Y_j^T \Delta B^T \Delta B Y_j + W_{11} \Delta A^T \Delta A W_{11} \\
&\quad + 2W_{11} \Delta D^T L_i^T L_i \Delta D W_{11}, \\
D_{12} &= L_i D_i + W_{11} C^T P_{22} C + W_{11} \Delta A^T \Delta A + 2W_{11} \Delta D^T L_i^T L_i \Delta D, \\
D_{21} &= D_i^T L_i^T + C^T P_{22} C W_{11} + \Delta A^T \Delta A W_{11} + 2\Delta D^T L_i^T L_i \Delta D W_{11}, \\
D_{22} &= 4P_{22}^2 + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + Q_{22}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D.
\end{aligned} \tag{27}$$

By Schur's complement, (25) and (26) can be rewritten as

$$\begin{bmatrix} M_{11} & W_{11} & Y_j^T & M_{14} & 0 \\ W_{11} & M_{22} & 0 & 0 & 0 \\ Y_j^T & 0 & M_{33} & 0 & 0 \\ M_{14}^T & 0 & 0 & M_{44} & L_i \\ 0 & 0 & 0 & L_i^T & -\gamma^2 I \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} N_{11} & W_{11} & Y_j^T & N_{14} \\ W_{11} & N_{22} & 0 & 0 \\ Y_j & 0 & N_{33} & 0 \\ N_{14}^T & 0 & 0 & N_{44} \end{bmatrix} < 0,$$

where

$$\begin{aligned}
M_{11} &= 3I + A_i W_{11} + W_{11} A_i^T + B_i Y_j + Y_j^T B_i^T + Y_j^T \Delta B^T \Delta B Y_j, \\
M_{14} &= L_i D_i + W_{11}^T C^T P_{22} C - \gamma^{-2} L_i Z_i^T + W_{11} \Delta A^T \Delta A + 2 W_{11} \Delta D^T L_i^T L_i \Delta D, \\
M_{22} &= -\left(Q_{11}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D\right)^{-1}, \\
M_{33} &= -\left(\Delta B^T \Delta B\right)^{-1}, \\
M_{44} &= 3P_{22}^2 + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + \gamma^{-2} Z_i Z_i^T + Q_{22}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D, \\
N_{11} &= 4I + A_i W_{11} + W_{11} A_i^T + B_i Y_j + Y_j^T B_i^T, \\
N_{14} &= L_i D_i + W_{11} C^T P_{22} C + W_{11} \Delta A^T \Delta A + 2 W_{11} \Delta D^T L_i^T L_i \Delta D, \\
N_{22} &= -\left(Q_{11}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D\right)^{-1}, \\
N_{33} &= -\left(R_1 + \Delta B^T \Delta B\right)^{-1}, \\
N_{44} &= 4P_{22}^2 + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + Q_{22}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D.
\end{aligned} \tag{29}$$

Noticing that if (16) and (17) are true, then $M_{44} < 0$ and $N_{44} < 0$, the following two inequalities hold:

$$\begin{aligned}
3P_{22}^2 + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + \gamma^{-2} Z_i Z_i^T + Q_{22}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D < 0, \\
4P_{22}^2 + P_{22} A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + Q_{22}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D < 0,
\end{aligned} \tag{30}$$

which can be written as the following LMIs:

$$\begin{bmatrix} G_1 & P_{22} & Z_i \\ P_{22} & \dfrac{-1}{3I} & 0 \\ Z_i^T & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} G_2 & P_{22} \\ P_{22} & \dfrac{-1}{4I} \end{bmatrix} < 0, \tag{32}$$

where

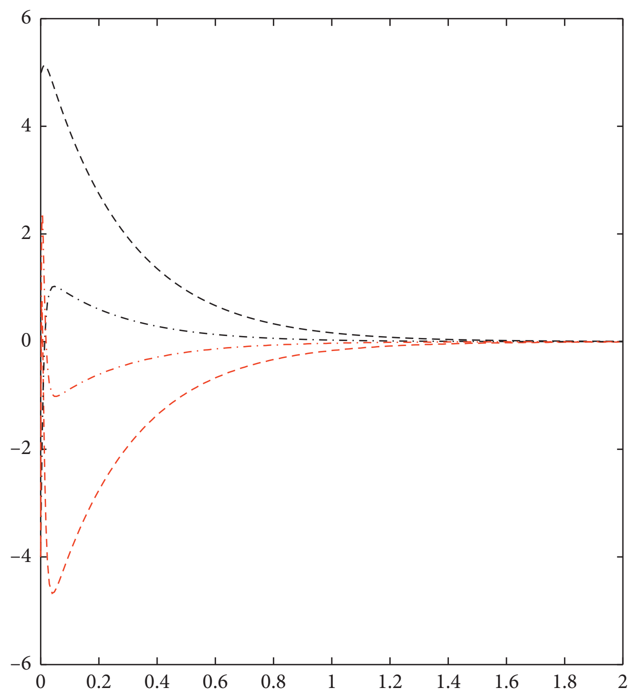

FIGURE 1: The trajectories of states x_1 and x_2.

$$G_1 = P_{22}A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + Q_{22}^{(1)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D,$$
$$G_2 = P_{22}A_i + A_i^T P_{22} - Z_i D_i - D_i^T Z_i^T + C^T P_{22} C + Q_{22}^{(2)} + \Delta A^T \Delta A + 2\Delta D^T L_i^T L_i \Delta D. \quad (33)$$

Therefore, the observer-based suboptimal stochastic H_2/H_∞ control design can be transformed into solving a two-step convex optimization problem.

The first step: under the constraint of (31) and (32), solve the convex optimization problem:

$$\min_{P_{22}>0} \operatorname{tr}(P_{22}). \quad (34)$$

It can be obtained that P_{22}, Z_i and $L_i = P_{22}^{-1} Z_i$.

The second step: under the constraint of (25) and (26), solve the following convex optimization problem:

$$\min_{W_{11}>0} \operatorname{tr}(W_{11}). \quad (35)$$

We can get $P_{11} = W_{11}^{-1}$ and the feedback gain k_j. A suboptimal solution $P = \operatorname{diag}(P_{11}, P_{22})$ and compensator (7) are achieved.

To sum up, we state the following main result. □

Theorem 2. *If the above convex optimization problems (34) and (35) have solutions, then $L_i = P_{22}^{-1} Z_i$ and $k_j = Y_j W_{11}^{-1}$. Moreover, we have $u^*(t) = Y_j W_{11}^{-1} \hat{x}(t)$, and $J^*(\overline{x}, u^*) = tr(W_{11}^{-1} + P_{22})\|\overline{x}(0)\|^2 + tr[(\sum_{i=1}^{h} h_i\ (m(t))\overline{B}_i) R_0 (\sum_{i=1}^{L}\ h_i(m(t))\overline{B}_i)^T]$.*

4. A Numerical Example

For system (5), we define the fuzzy number as "big and small" and assume its coefficient matrices are

$$A_1 = \begin{bmatrix} 0.2 & 3.2 \\ 2.8 & 0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.6 & 2.6 \\ 3.4 & 0.3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ -0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \quad (36)$$
$$C = \begin{bmatrix} 2.5 & 3 \\ 2.8 & 2 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Give

$$\Delta A = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.2 \end{bmatrix}, \Delta B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \quad (37)$$

We have $\Delta l = 0$, $\Delta h = 0$ and $\Delta C = 0$, $\Delta D = 0$.

Choose $\gamma^2 = 0.9$. Using LMI toolbox in MATLAB, we get

$$P_{22} = \begin{bmatrix} 0.3140 & -0.3253 \\ -0.3253 & 0.3761 \end{bmatrix}, W_{11} = \begin{bmatrix} 4.9206 & -4.9454 \\ -4.9454 & 4.9958 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} 1.0971 \\ 1.1117 \end{bmatrix}, Z_2 = \begin{bmatrix} 1.0752 \\ 1.1196 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 0.9422 & 34.3978 \end{bmatrix}, Y_2 = \begin{bmatrix} -1.1508 & 29.9927 \end{bmatrix}. \tag{38}$$

According to $L_i = P_{22}^{-1} Z_i$, $k_j = Y_j W_{11}^{-1}$, the parameters of observer and controller are

$$L_1 = \begin{bmatrix} 63.0966 \\ 57.5249 \end{bmatrix}, L_2 = \begin{bmatrix} 62.6370 \\ 57.1486 \end{bmatrix},$$
$$k_1 = \begin{bmatrix} 1.4025 & 1.3952 \end{bmatrix} \times 10^3, k_2 = \begin{bmatrix} 1.1374 & 1.1319 \end{bmatrix} \times 10^3. \tag{39}$$

Taking the controller $u^*(t)$ into account, the simulation results are shown in Figure 1. It is shown that the system can achieve the desired control effects under the fuzzy controller.

5. Conclusions

In this paper, the mixed H_2/H_∞ output-feedback control problem for stochastic nonlinear systems in a finite horizon has been studied. Firstly, the nonlinear system is transformed into a linear fuzzy model by T-S rules, and the error between the original system and the fuzzy one has been considered. A fuzzy observer-based two-step convex optimization method has been proposed to treat the suboptimal H_2/H_∞ problem. The method is simple and effective. The closed-loop system can guarantee the robustness and minimize the energy output. Since time delays exist widely in practical systems, how to generalize the obtained H_2/H_∞ output-feedback controller design method to stochastic nonlinear systems with delays is one of the directions of future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] Z. Ji, H. Lin, S. Cao, Q. Qi, and H. Ma, "The complexity in complete graphic characterizations of multiagent controllability," *IEEE Transactions on Cybernetics*, vol. 51, no. 1, pp. 64–76, 2021.

[2] J. Qu, Z. Ji, and Y. Shi, "The graphical conditions for controllability of multiagent systems under equitable partition," *IEEE Transactions on Cybernetics*, vol. 99, pp. 1–12, 2020.

[3] R. Peng, C. Jiang, and R. Guo, "Stabilization of a class of fractional order systems with both uncertainty and disturbance," *IEEE Access*, vol. 9, pp. 42697–42706, 2021.

[4] L. Liu, B. Li, and R. Guo, "Consensus control for networked manipulators with switched parameters and topologies," *IEEE Access*, vol. 9, pp. 9209–9217, 2021.

[5] H. Hua, Y. Qin, J. Geng, C. Hao, and J. Cao, "Robust mixed H_2/H_∞ controller design for energy routers in energy internet," *Energies*, vol. 12, no. 3, p. 340, 2019.

[6] Q. Zhu and Y. Ma, "H_2/H_∞antivertical controller based on particle swarm optimization (PSO) using active T-foils and trim tabs for a fast catamaran," *Mathematical Problems in Engineering*, vol. 2019, Article ID 4868473, 17 pages, 2019.

[7] W. Zhang, L. Ma, and T. Zhang, "Discrete-time mean-field stochastic H_2/H_∞ control," *Journal of Systems Science and Complexity*, vol. 30, no. 4, pp. 765–781, 2017.

[8] B. Bor-Sen Chen, C. Chung-Shi Tseng, and H. Huey-Jian Uang, "Mixed H_2/H_∞ fuzzy output-feedback control design for nonlinear dynamic systems: an LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 3, pp. 249–265, 2000.

[9] T. Hou, W. Zhang, and H. Ma, "Finite horizon H_2/H_∞ control for discrete-time stochastic systems with Markovian jumps and multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1185–1191, 2000.

[10] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and p H_∞ roblems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.

[11] D. J. N. Limebeer, B. D. O. Anderson, and B. Hendel, "A nash game approach to mixed H_2/H_∞ control," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 69–82, 1994.

[12] B.-S. Chen, H.-C. Lee, and C.-F. Wu, "Pareto optimal filter design for nonlinear stochastic fuzzy systems via multiobjective H_2/H_∞ optimization," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 2, pp. 387–399, 2015.

[13] C.-S. Wu, B.-S. Chen, and Y.-W. Jan, "Unified design for H_2, H_∞ and mixed control of spacecraft," *Journal of Guidance, Control, and Dynamics*, vol. 22, no. 6, pp. 884–896, 1999.

[14] C.-S. Wu and B.-S. Chen, "Adaptive attitude control of spacecraft: mixed approach," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 4, pp. 755–766, 2001.

[15] P. Shi, X. Su, and F. Li, "Dissipativity-based filtering for fuzzy switched systems with sochastic perturbation," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1694–1699, 2016.

[16] B.-S. Chen and W. Zhang, "Stochatic H_2/H_∞ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[17] M. Wang, Q. Meng, and Y. Shen, "H_2/H_∞control for stochastic jump-diffusion systems with Markovian switching," *Journal of Systems Science and Complexity*, vol. 34, no. 3, pp. 924–954, 2021.

[18] Z. Yan, Y. Song, and J. H. Park, "Finite-time H_2/H_∞ control for linear it $\hat{o}$ stochastic Markovian jump systems: mode-dependent approach," *IET Control Theory & Applications*, vol. 14, no. 20, pp. 3557–3567, 2021.

[19] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa, "Output feedback control of Markov jump

linear systems in continuous-time," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 944–949, 2000.
[20] T. Hou, Y. Liu, and F. Deng, "Finite horizon H_2/H_∞ control for SDEs with infinite Markovian jumps," *Nonlinear Analysis: Hybrid Systems*, vol. 34, pp. 108–120, 2019.
[21] C.-F. Wu, B.-S. Chen, and W. Zhang, "Multiobjective investment policy for a nonlinear stochastic financial system: a fuzzy approach," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 2, pp. 460–474, 2017.
[22] X. Zhao, Y. Yin, L. Zhang, and H. Yang, "Control of switched nonlinear systems via T-S fuzzy modeling," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 1, pp. 235–241, 2016.
[23] W. Li, Z. Xie, J. Zhao, P. K. Wong, H. Wang, and X. Wang, "Static-output-feedback based robust fuzzy wheelbase preview control for uncertain active suspensions with time delay and finite frequency constraint," *IEEE/CAA Journal of Automatica Sinica*, vol. 8, no. 3, pp. 664–678, 2021.
[24] X. Cai, S. Zhong, J. Wang, and K. Shi, "Robust H_∞ control for uncertain delayed T-S fuzzy systems with stochastic packet dropouts," *Applied Mathematics and Computation*, vol. 385, 2020.
[25] B.-S. Chen and S.-J. Ho, "Multiobjective tracking control design of T-S fuzzy systems: fuzzy pareto optimal approach," *Fuzzy Sets and Systems*, vol. 290, no. 1, pp. 39–55, 2016.
[26] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. SMC-15, no. 1, pp. 116–132, 1985.
[27] C.-F. Wu, B.-S. Chen, and W. Zhang, "Multiobjective H_2/H_∞ control design of the nonlinear mean-field stochastic jump-diffusion systems via fuzzy approach," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 4, pp. 686–700, 2019.