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Research Article

Replacement Analysis of Mutually Exclusive Projects of Unequal Lives Using Kelly Specific Real Option Criterion

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This paper presents a model to address the uncertainty inherent in replacement problems, whereby a firm must select between mutually exclusive projects of unequal lifespans by applying the Kelly criterion (which is not well known to the engineering economics community) within a binomial lattice option-pricing environment. Assuming that only the interest rate, among many factors, is uncertain, Brown and Davis performed an economic analysis of this problem by employing a real option-pricing method and argued that their model yields results opposite to those yielded by the traditional approach. However, the results yielded by the model proposed herein are consistent with those by the traditional approach, unlike Brown and Davis's model. The conclusion is that since the investment time horizon is infinite, a firm rationale pertaining to the selection of the best method for the investment problem of such types does not exist.

1. Introduction

The Kelly criterion, proposed by John Kelly [1] in 1956, is a formula pertaining to money management system that allows investors to maximize the expected value of a logarithmic return of their wealth, provided that they invest the optimal fraction of their investment capital over a long duration. Since its introduction, the Kelly criterion has been primarily investigated in financial investment and gambling fields [2]. A comprehensive analysis into the real-world success afforded by the Kelly criterion is provided in Poundstone's well-known book [3], "Fortune's Formula." Poundstone describes Thorp earning a significant amount of money by applying the Kelly criterion to a stock market. Furthermore, MacLean, Thorp, and Ziemba [2] analyzed the extraordinary achievement of the legendary investor W. Buffett and claims that Buffett invests like a fully Kelly bettor. From a theoretical perspective, Brieman [4] developed fundamental mathematical properties and the expected log criterion rigorously. Thorp [5] proved that the Kelly criterion generates a significantly better return on investment than any other investment strategy and investigated the mathematical characteristics of the criterion. McEnally [6] presented a few properties pertaining to the geometric mean strategy and used simple numerical examples to explain them; the examples were based on Latane's independent study, which is equivalent to the Kelly criterion.

From the perspective of portfolio theory, Thorp [5, 7] applied the Kelly criterion to portfolio selection and contrasted it with the Markowitz mean-variance efficiency. He argued that Kelly weightings do not necessarily imply mean-variance efficiency. Markwitiz [8] argues that a log-optimal portfolio is a limiting mean-variance portfolio. Based on Latane's study instead of the Kelly criterion, Roll [9] investigated the relationship among the Kelly capital growth model, mean-variance, and capital asset pricing analysis (CAPM). He compared the Kelly criterion model to a CAPM and reported a similar correspondence between the two models. For example, if the covariance between an asset's expected return and the average return on all assets in a portfolio is zero, then both models provide an expected return equivalent to the risk-free interest rate. Otherwise, they provide an expected return that exceeds the risk-free interest rate.

Publications relevant to this study are scarce. Zhang [10] discussed the relationship between the optimal geometric mean return of stocks and their option value based on the Kelly criterion, in which it was assumed that a definite betting strategy that guarantees winning does not exist. Furthermore, Zhang proved that the optimal geometric return, along with the optimal fraction required for investment, depends on the objective probabilities. However, the ratio of the optimal fraction required for investment does not rely on probabilities in a situation in which no arbitrage opportunities exist and, furthermore, is a single-step binomial lattice option-pricing (BLOP) model that can appropriately represent the investment. Zhu [11] discussed the Kelly criterion within the option-pricing framework. He demonstrated that a pricing interval for options with a specified strike price exists such that having a nonnegative expected return in a single-period economy, as a replacement for the original investment, enables the investment performance to be assessed easily based on the Kelly criterion. He further argued that his assertion is valid only when the option price is outside the interval.

DeFigueiredo and Barr [12] discussed an online reputation system that provides insurance against trade fraud by leveraging existing relationships between players. In the discussion, the Kelly criterion was used to determine the minimum value for the ratio of the option value, along with its strike price, within the specified BLOP environment. Sæen [13] investigated the effect of estimation errors in the Kelly criterion on a portfolio's performance. He adopted a multistep BLOP model and demonstrated its relationship with the Kelly criterion. However, based on literature review, studies pertaining to the application of the Kelly criterion to the economic analysis of real investment projects (including replacement problems) do not exist. Wu and Chung [14] proposed a method for option trading to identify a profitable option portfolio by bidding the optimal fraction of the Kelly criterion. Compared with prior studies, they argued that their model is a novel approach for option trading which included the money management of position sizing. Based on the following phrase from Levitt's paper, "there are many parallels between trading in financial markets and sports betting. In both settings, investors with heterogeneous beliefs and information seek to profit through trading, as uncertainty is resolved over time," Johnstone [15] clarified and extended the formal connections between investment and betting within conventional binomial asset pricing models. He argued that any investment position in a binary asset can be replicated by a conventional bet made at odds within the market price of the asset. According to him, the relevant betting odds are based on the risk-neutral probability rather than actual probabilities. This current study is motivated by Johnstone's ideas.

Using the Kelly criterion and BLOP model, mutually exclusive projects of unequal lives with an implicit assumption of continued project replication over a long (infinite) duration are considered. In this study, a single and stationary replacement problem is focused: no technological change occurs, and the operating cost is constant over an analysis time horizon. However, for real option methods and the method proposed herein, it is assumed that the interest rate is relaxed to be uncertain.

The BLOP model yields results similar to those by the Black-Scholes model [16], in which assumptions (such as a geometric Brownian motion and the occurrence of infinitesimal trading) similar to those of the Kelly criterion are applied [3, 10, 11]. Therefore, it is both conceivable and meaningful to discuss the application of the Kelly criterion to the economic analysis of replacement problems within the BLOP environment. Brown and Davis [17] argued that conventional techniques can result in errors in a stochastic investment environment; hence, it is decided to use the real option method, which provides results opposite to those of standard techniques. Because it is assumed that the investment time horizon is infinite, no firm rationale exists for selecting the best method to solve the replacement problem. Therefore, the authors herein present and compare the results obtained from three different approaches: (i) conventional analysis techniques, (ii) Brown and Davis's real options method, and (iii) the method proposed herein. However, Eschenbach et al. [18] reported that if technological change is slow and/or the uncertainty in forecasts is significant, then the conventional analysis technique should be employed. If their suggestion is valid, then Brown and Davis's real option method should be reconsidered to solve the abovementioned problems.

The remainder of this paper is organized as follows: Section 2 presents the Kelly criterion within the context of the applicable BLOP environment. Section 3 presents a brief numerical example to demonstrate the validity of the economic analysis of replacement problems. Finally, Section 4 provides the concluding remarks.

2. Development of the Kelly Criterion within BLOP Environment Context

As it is believed that, among the many option-pricing models, the characteristics of the BLOP model match well with those of the Kelly criterion, this section presents a discussion of the economic analysis of replacement problems from the viewpoint of the Kelly criterion within the BLOP environment context.

2.1. Relationship between Kelly Criterion and BLOP Model. As reported by Zhang [10], the ratio of the optimal fractions of stocks (and their options), when assumed to generate a binary outcome, is independent of the objective probability distribution of the return. This statement implies that a few options can be used to replace the underlying asset without changing the optimal geometric mean return, as well as to determine the objective probability distribution of the return thereof. This argument is mathematically expressed as follows:

$$\frac{f_{opt}^{s}}{f_{opt}^{o}} = \frac{e_1^o}{e_1^s} = \frac{e_2^o}{e_2^s}, \tag{1}$$

where f_{opt}^{s} is optimal faction of capital on hand to invest in a stock; f_{opt}^{o} is optimal fraction of capital on hand to invest in an option; e_1^s is excess return from a winning game in a stock investment; e_1^o is excess return from a winning game in an option investment; e_2^s is excess return from a losing game in a stock investment; and e_2^o is excess return from a losing game in an option investment.

Equation (1) shows that the Kelly criterion can be used to replicate any investment position in the BLOP environment [19]. When replicating an investment position, the betting odds are based on subjective probabilities equivalent to risk-neutral probabilities in option-pricing theories. The risk-neutral probabilities are both artificial and subjective and cannot be observed in the real world. These probabilities are used to derive betting odds and find arbitrage-free asset prices without referring to objective probabilities. This enables one to manage the options on Kelly bets, which can be valued using the BLOP model. In other words, investing in an option is equivalent to investing in its underlying asset (based on the Kelly criterion).

Before discussing the relationship between the Kelly criterion and the BLOP model, several important mathematical formulas used to evaluate an option value using the model are presented at first, along with the basic concept of the Kelly criterion. Equation (2) shows the well-known option-pricing formula with a risk-neutral probability.

$$C = \frac{p' C_u + (1 - p')C_d}{1 + r_f}, \tag{2}$$

where C is the call option value; p' is the risk-neutral probability; C_u is the call option value when stock price increases; C_d is the call option value when stock price decreases, and r_f is the risk-free interest rate.

The risk-neutral probability in Equation (2), denoted by p, is generally expressed as follows:

$$p' = \frac{S\left(1 + r_f\right) - S_d}{S_u - S_d} = \frac{\left(1 + r_f\right) - d}{u - d}, \tag{3}$$

where S_u is the stock price when a favorable outcome is yielded, S_d is the stock price when an unfavorable outcome is yielded, u is the upward movement rate of stock price, and d is the downward movement rate of stock price

Using Equation (3), a risk-free rate of return can be obtained; however, the upward and downward movement rates must be calculated. Equation (4) is used to calculate the upward movement rate. The downward movement rate is the reciprocal of the upward movement rate, expressed as follows:

$$u = e^{\sigma \sqrt{\delta_t}}, \tag{4}$$

where σ is the volatility of stock return and δ_t is the time duration of a step measured in years

In the following discussion, the basic concept of the Kelly criterion [1, 2, 12] is introduced. Assuming a game/investment with an initial capital amount of X_0 that is played over a long duration with a fraction of capital on hand, denoted by f, where $0 \le f \le 1$, the accumulated capital after N bets is expressed as

$$X_N = X_0 (1 + f)^S (1 - f)^L, \tag{5}$$

where S is the number of wins and L is the number of losses [20]. In this context, gamblers/investors never enter a complete ruin situation, which is analogous to $P_r (X_N = 0) = 0$. The primary objective of the Kelly criterion is to identify the optimal fraction (bet size) of f^* by dividing both sides of Equation (5) by X_0 and taking its logarithm as follows:

$$G(f) = \ln\left[\frac{X_N}{X_o}\right]^{(1/N)} = \frac{S}{N}\ln(1 + f) + \frac{L}{N}\ln(1 - f), \tag{6}$$

where $N = S + L$. Here, $G(f)$ yields the exponential rate of increase per play. It is noteworthy that

$$e^{N \ln\left[\frac{X_N}{X_0}\right]^{(1/N)}} = \frac{X_N}{X_0}. \tag{7}$$

The expression of $G(f)$ is the basic expression for the Kelly criterion. In fact, Kelly maximized the expected value of the growth rate coefficient, $g(f)$, that is,

$$\begin{aligned} g(f) = E\left[\frac{1}{N}\ln\left(\frac{X_N}{X_0}\right)\right] &= \lim_{N \longrightarrow \infty}\left\{\frac{S}{N}\ln(1 + f)\right. \\ &\left.+ \frac{L}{N}\ln(1 - f)\right\} = p\ln(1 + f) + q\ln(1 - f). \end{aligned} \tag{8}$$

Variable p is an objective winning probability that can be observed in a marketplace, and $q = 1 - p$.

Taking the derivative of Equation (8) with respect to f, the optimal fraction (bet size) of f that maximizes $g(f)$ can be obtained as follows:

$$g'(f) = \frac{p}{1 + f} - \frac{q}{1 - f} = \frac{p - q - f(p + q)}{(1 + f)(1 - f)} = 0. \tag{9}$$

Setting Equation (9) equal to zero and rearranging it in terms of f yield the optimal fraction of the current capital that should be bet on each round of play to maximize the growth rate of Equation (8). Therefore,

$$f^* = p - q. \tag{10}$$

Furthermore,

$$g'(f) = \frac{-p}{(1 + f)^2} - \frac{q}{(1 - f)^2}. \tag{11}$$

As such, p and $q \ge 0$ and $g''(f) < 0$, such that $g(f)$ is a strictly concave function for $f \in [0, 1)$. Because $g(f)$ is concave over [0, 1) and f^* satisfies $g''(f) = 0$, $g(f)$ has a subsequent global maximum at f^*, and

$$\begin{aligned} g_{\max} &= p\ln(1 + p - q) + q\ln(1 - p + q) \\ &= \ln 2 + p\ln p + q\ln q. \end{aligned} \tag{12}$$

Thus far, the Kelly criterion in terms of even payoff events has been discussed. However, this criterion can be extended to uneven payoff events. Suppose that investors win b units for every unit wager. Furthermore, suppose that, on each bet, $bp - q > 0$, which implies that the events are favorable for investors. Therefore, the expected value of the growth rate coefficient of $g(f)$ for uneven payoff events can be expressed as

$$g(f) = E\ln\left[\frac{X_N}{X_0}\right] = p\ln(1+bf) + q\ln(1-f). \tag{13}$$

By adopting a procedure similar to that described above, the optimal fraction of the current capital to invest can be obtained as follows:

$$f^* = p - \frac{q}{b}. \tag{14}$$

More information regarding the Kelly criterion is available in the publication by Maclean [2].

2.2. Odds System for Replicating Investment with Bets. To derive a gross payoff α equivalent to the amount of capital to bet, which is associated with risk-neutral probabilities and odds, as defined with respect to the number of wins (successes) and losses (failures), the following is considered. Because $N = S + L$, a favorable odd O_f, an unfavorable odd O_a, and a winning probability p' are defined as follows:

$$\begin{aligned} O_f &= \frac{S}{L}, \\ O_a &= \frac{L}{S}, \\ p' &= \frac{S}{(S+L)}. \end{aligned} \tag{15}$$

Since the odds are described based on risk-neutral probabilities, the winning probability in Equation (16) is regarded as a risk-neutral probability in the BLOP model. Because the odds are specified as the number of O_f in Equation (15), the probability can be represented as a ratio of O_f as follows:

$$p' = \frac{O_f}{(O_f+1)} = \frac{1}{(O_a+1)}. \tag{16}$$

The reciprocal of Equation (16) represents the extent to which a gross payoff α is applicable to the winning game, and it is expressed as

$$\alpha = \frac{1}{p'} = \frac{(O_f+1)}{O_f} = 1 + \frac{1}{O_f} = 1 + O_a. \tag{17}$$

Subsequently, by referring to Figure 1, it is considered that an investment project whose current value is S dollars is replicated with bets that constitute a portfolio. As shown in Figure 1, investors can simultaneously invest in risky assets (e.g., a stock) and risk-free assets (e.g., a government bond). If they fail to invest in risky assets, then they will be left with the amount required to invest in the risk-free asset. This event occurs when the price of risky assets decreases. Otherwise, they will be left with an amount equal to the amount required to invest in the risky asset. In this regard, the portfolio of interest involves allocating $f_u^* = S - (S_d/1 + r_f)$ in the risky asset and $f_d^* = (S_d/1 + r_f)$ in the risk-free asset, where S_d represents the price of the risky asset when an unfavorable outcome is yielded [21]. The value of f^* is expressed as an absolute monetary value instead of a ratio.

For the investment, the total amount of payoffs is determined by the product of the optimal fraction of investment capital and α earned in one dollar. Because α is the present value at $t = 0$, this value should be compounded by $(1 + r_f)$ to qualify as α for one unit of time elapsed. Therefore, the total amount of payoffs corresponding to the occurrence of the favorable outcome Tp_f is expressed as follows [7]:

$$\begin{aligned} TP_f = \alpha_u \cdot f_u^* &= \left\{\frac{S_u - S_d}{(1+r_f)S - S_d}(1+r_f)\right\} \\ &\cdot \left\{\frac{(1+r_f)S - S_d}{(1+r_f)}\right\} = S_u - S_d. \end{aligned} \tag{18}$$

However, when the unfavorable outcome TP_u occurs, it is expressed as follows:

$$\begin{aligned} TP_u = \alpha_d \cdot f_d^* &= \left\{\frac{(1+r_f)}{1-p'}\right\} \cdot \left\{\frac{S_d}{(1+r_f)}\right\} \\ &= \left\{\frac{S_u - S_d}{S_u - S(1+r_f)}(1+r_f)\right\} \cdot \left\{\frac{S_d}{(1+r_f)}\right\} \\ &= \frac{(S_u - S_d)S_d}{S_u - S(1+r_f)}. \end{aligned} \tag{19}$$

The calculated present value of the average of the two values yielded by Equations (18) and (19) yields the value of the original investment at $t = 0$, as shown below

$$\frac{\{p'(S_u - S_d) + (1-p')((S_u - S_d)S_d/S_u - S(1+r_f))\}}{1+r_f} = S. \tag{20}$$

The result of Equation (20) indicates that the portfolio with bets completely replicates the original investment whose value at $t = 0$ is S. To validate this result, a brief numerical example is provided, in which $S = 50$, $S_u = 70$, and $r_f = 10\%$. To calculate the total amount of payoffs, the risk-neutral probability in the example using Equation (3) should be determined firstly as follows:

$$p' = \frac{50(1+0.1) - 30}{70 - 30} = \frac{(1+0.1) - 0.6}{1.4 - 0.6} = 0.625. \tag{21}$$

In this example, the odds for and against are expressed, respectively, as follows:

$$\alpha_u = \frac{(1+0.1)}{0.625} = 1.76,$$
$$\alpha_d = \frac{(1+0.1)}{(1-0.625)} = 2.933. \tag{22}$$

Therefore, the optimal amounts of capital to invest in risky and risk-free assets are expressed, respectively as follows:

$$f_u^* = 50 - \frac{30}{(1+0.1)} = 22.7272,$$
$$f_d^* = \frac{30}{(1+0.1)} = 27.2727. \tag{23}$$

The present value of the total amount of payoffs earned from the two events is calculated as follows:

$$\frac{(0.625)(1.76)(22.7272) + (1-0.625)(2.933)(27.2727)}{(1+0.1)} = 50. \tag{24}$$

In fact, the same result can be obtained directly using (20) as follows:

$$\frac{\{(0.625)(70-30) + (1-0.625)(((70-30)30/70 - 50(1+0.1)))\}}{(1+0.1)} = 50. \tag{25}$$

This numerical example demonstrates that the investment can be replicated with the bets proportioned via the Kelly criterion in the BLOP model.

2.3. Optimal Fraction of Kelly Criterion in the BLOP Model. Next, an investment setting is considered, where the initial investment amount is X_0 and a fixed fraction f of capital is invested in the upstate event, whereas the remaining fraction is invested in the downstate event. Therefore, the expected accumulated capital after N investment is expressed as

$$X_N = X_0 (f\alpha_u)^S \{(1-f)\alpha_d\}^L. \tag{26}$$

Dividing both sides of Equation (5) by X_0 and applying the logarithm yield the expected growth rate of the investment capital as follows [15]:

$$\frac{X_N}{X_0} = (f\alpha_u)^S \{(1-f)\alpha_d\}^L,$$
$$g = \ln\left(\frac{X_N}{X_0}\right) = p\ln(f\alpha_u) + (1-p)\ln\{(1-f)\alpha_d\} \tag{27}$$
$$= p\ln\left(f\frac{1+r_f}{p'}\right) + (1-p)\ln\left\{(1-f)\frac{1+r_f}{1-p'}\right\}.$$

The probability of p in Equation (27) is the objective probability that each event can have. To determine the optimal fraction f^* of capital invested in the Kelly criterion, the derivative of the last expression in the equation with respect to f should be taken. Subsequently, the appropriate value can be taken by solving $(\partial g/\partial f) = (p/f) - (1-p/1-f) = 0$.

$$f^* = p. \tag{28}$$

Replacing p with f in the last expression of equation (27) yields the expected maximum exponential growth rate g_{max} of the investment capital as follows:

$$g_{\max} = p\ln\left(p\frac{1+r_f}{p'}\right) + (1-p)\ln\left\{(1-p)\frac{1+r_f}{1-p'}\right\}. \tag{29}$$

The final value v_f and the average rate of return $\overline{r}$ on investment are expressed as follows, respectively:

$$v_f = X_0 e^{g_{\max}}, \tag{30}$$

and

$$\overline{r} = e^{g_{\max}} - 1. \tag{31}$$

Equation (28) implies that investors should invest $f^* = p$ and $1 - f^* = 1 - p$ of their investment capital in the up and downstate events, respectively, disregarding the odds for each item. As shown by the equation, the value of f^* is restricted by an objective probability of p, which is between zero and one. The values of r_f and S are simply observed in the market, and the variance of stock prices can be estimated easily because a significant amount of stock prices can be obtained in the market. Hence, the risk-neutral probability of p' in Equation (29) is determined without difficulty. However, the investor must postulate two quantities at $t = 0$: (1) the objective probability of p that an upstate event can occur and (2) the initial investment capital.

The example described in the previous section is considered to validate the procedure discussed in this section. Suppose that the objective probability of the upstate event occurring is 0.7 and the initial investment capital is equal to the current stock price of $S = 50$. Therefore, the expected maximum exponential growth rate, final value, and average rate of return on investment are expressed as

$$g_{\max} = (0.7)\ln(0.7)\frac{1+0.1}{0.625} + (1-0.7)\ln \cdot \left\{(1-0.7)\frac{1+0.1}{1-0.625}\right\} = 0.1077, \tag{32}$$

and

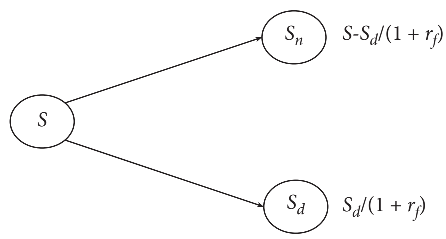

Figure 1: Portfolio of investment with bets.

$$v_f = 50e^{0.1077},$$
$$\overline{r} = e^{0.10.77} - 1 = 0.11371. \tag{33}$$

3. Numerical Examples

Replacement problems with infinite planning horizons are typically encountered in companies. In this section, the approach proposed herein is applied to Brown and Davis's example [17] and the results obtained are compared with those of Brown and Davis's. In this example, Brown and Davis consider two projects that have different lifespans and interest rates, where these variables serve as sources of uncertainty which affect the problem. Next, the manner by which Brown and Davis solved this problem is discussed. Projects A and B were defined with estimated cash flows, as shown in Table 1, with different timelines, that is, three and five years, respectively.

3.1. Traditional Approach. An initial interest rate of 7% was used for the traditional economic analysis. First, the net present values (NPV) of the two projects based on a single life cycle should be calculated as follows: $NPV_A^1(7\%) = 11.08$ and $NPV_B^1(7\%) = 17.40$. Next, assuming that the project will be repeatedly replaced with an identical one over the infinite planning horizon, their NPV_s over the planning horizon, denoted by $NPV_{(A/B)}^{\infty}$, must be determined. These values are determined by their single life cycle NPV_s (obtained above) as follows: $NPV_A^{\infty} = (7\%) = 60.30$ and $NPV_B^{\infty} = (7\%) = 60.63$. Because $NPV_A^{\infty} < NPV_B^{\infty}$, it is economically desirable for a firm to undertake project B rather than project A.

3.2. Brown and Davis's Real Option Approach. Next, the uncertainty over the interest rate is considered and its effect on the final decision based on the real option-pricing concept is investigated. In the example, it is assumed that the initial value of 7% will be maintained over the first three years and then change to either 6% or 8% with an equal objective probability. An objective probability is defined as the probability that the interest rate decreases to 6%, resulting in an increase in the project value. Once it changes to either level, it will remain therein indefinitely. Dixit and Pindyck [16] extensively discussed the problems and assumptions similar to those in this example and its derivatives, including those of Brown and Davis [17], which have been created and applied in a number of investment decision problems. Most of these problems were solved by applying real option valuation techniques [12] directly or by considering the optionality embedded in cash flows. The latter case employs only a real options concept to solve this problem. In this case, an example from the perspective of the BLOP model will be discussed.

Table 1: Estimated cash flows for projects A and B.

Year	Project A	Project B
0	−100	−99
1	5	11
2	5	11
3	125	11
4		11
5		111

Table 2 shows NPV_A^1, NPV_B^1, NPV_A^{∞}, and NPV_B^{∞} with the interest rate varying from 1% to 10%. The table reveals the dominating project over a certain range of interest rates. As shown in the table, $NPV_A^{\infty}(6\%) = 87.29$, $NPV_B^{\infty}(6\%) = 87.29$, $NPV_A^{\infty}(8\%) = 39.51$, and $NPV_B^{\infty}(8\%) = 40.63$. If the interest rate changes to 6%, then the next optimal alternative is to select project A. However, if the interest rate changes to 8%, then project B becomes the optimal alternative.

According to Brown and Davis's real option-pricing approach, if project A is selected at first, then NPV from this decision is the sum of NPV_s from project A over the next three years and the optimal choice in the three years, which is expressed by the first one of the two values below. However, if project B is selected first, then its NPV is expressed as the second option as follows:

$$\begin{aligned}
NPV_A^{\max}(\text{uncertainty}) &= NPV_A^1(7\%) + (0.5)\frac{\{NPV_A^{\infty}(6\%) + NPV_B^{\infty}(8\%)\}}{(1+0.07)^3},\\
&= 11.08 + (0.5)\frac{\{88.04 + 40.63\}}{(1+0.07)^3} = 63.60,\\
NPV_B^{\max}(\text{uncertainty}) &= NPV_B^1(7\%) + (0.5)\frac{\{NPV_B^{\infty}(6\%) + NPV_B^{\infty}(8\%)\}}{(1+0.07)^5},\\
&= 17.40 + (0.5)\frac{\{88.04 + 40.63\}}{(1+0.07)^5} = 63.27.
\end{aligned} \tag{34}$$

Table 2: NPV_s for single life cycle and infinite planning horizon for A and B.

MARR (%)	NPV (A, 3)	NPV (B, 5)	NPV (A, ∞)	NPV (B, ∞)
1	31.18	49.53	1060.04	1020.60
2	27.50	43.42	476.75	460.61
3	23.96	37.64	282.35	273.95
4	20.56	32.16	185.17	180.62
5	17.28	26.98	126.88	124.62
6	14.12	22.06	88.04	87.29
7	11.08	17.40	60.30	60.63
8	8.15	12.98	39.51	40.63
9	5.32	8.78	23.35	25.08
10	2.59	4.79	10.42	12.64

Since NPV_A^{max} (uncertainty) > NPV_B^{max} (uncertainty), it is recommended that the firm undertakes project A instead of project B. Comparing this with the previous decision, it is recognized that the decision is changed from project B to project A, causing an increase in the firm's NPV from 60.63 to 63.60.

3.3. Kelly Criterion Approach within the BLOP Environment. In this section, the same problem from the perspective of Kelly within the context of the applicable BLOP environment is discussed. For the discussion, a binomial lattice model for projects A and B is developed at first, as shown in Figure 2, using the information provided in Table 2. To complete the figure, it may be considered such that NPV_s provided in the table with an interest rate of 7% be the current project value and those with interest rates of 6% and 8% be the up and down states, respectively.

Similar to Brown and Davis's approach, the two cases in the example will be discussed subsequently. The first case in which an identical project will be repeated over an infinite planning horizon in the context of the applicable BLOP environment via the Kelly criterion is considered. Using Equations (29) and (30), the final values of the two projects can be determined. However, before resolving the equation, the risk-neutral probability, denoted by p', for each project using Equations (3) should be obtained. Subsequently, the two risk-neutral probabilities are determined to be $p_A' = 0.4290$ and $p_B' = 0.4286$ when the objective probability is 50%. As such, $g_{\max}^{A}$ and g_{max}^{B} for the two projects are 0.2131 and 0.3486, respectively. Substituting these values into Equation (30) yields the final values of the two projects: $V_A = 74.6218$ (illustrated in the following) and $V_B = 85.9177$.

Based on the final values, it can be inferred that project B is superior to project A; hence, the firm should undertake project B instead of project A. This is congruent with the decision in the conventional approach but in contrast to the decision based on Brown and Davis's real option approach. However, it is noteworthy that this phenomenon cannot be generalized to all problems. Table 3 shows the final values of the two projects with different objective probabilities, assuming that the risk-neutral probability is maintained over the entire range of objective probabilities. This assumption is reasonably valid because a 1% difference adjacent to a base interest rate, such as 7% in the example, does not

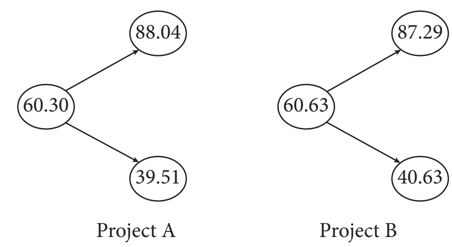

Figure 2: Binomial lattice model with identical projects.

significantly affect the risk-neutral probability. As shown in the table, project B dominates project A over the entire range of objective probabilities. When the objective probability is 50%, the procedure to calculate the final values of projects A and B is as follows:

(1) The risk-neutral probability for project A is expressed as $p_A^{'} = (60.30 - 39.51/88.04 - 39.51) = 0.4290$ and $p_B = (60.30 - 40.63/87.29 - 40.63) = 0.4286$.

(2) The expected maximum exponential growth rate of project A is expressed as $g_{max}^{A} = (0.5)\ln(0.5(1.07^3/0.4290)) + (0.5)\ln(0.5(1.07^3/1 - 0.4290)) = 0.2131$ and $g_{max}^{B} = (0.5)\ln(0.5(1.07^5/0.4286)) + (0.5)\ln(0.5(1.07^5/1 - 0.4286)) = 0.3486$.

(3) The final value of project A is expressed as $V_A = e^{0.2131}(60.30) = 74.6218$ and $V_B = e^{0.3486}(60.63) = 85.9177$.

Next, the second case is considered where either project A or project B is selected first. As illustrated in Brown and Davis's real option approach, the decision concerning the next optimal project (after the first cycle of planning horizons for the first selected project has been determined) is made based on information regarding the movement of the interest rate. Recapitulating their illustration here is convenient and useful for understanding the following discussion. Because $NPV_A^{\infty}(6\%) = 88.04 > NPV_B^{\infty}(6\%) = 87.29$ and $NPV_A^{\infty}(8\%) = 39.51 > NPV_B^{\infty}(8\%) = 40.63$, project A with $NPV_A^{\infty}(6\%) = 88.04$ and project B with $NPV_B^{\infty}(8\%) = 40.63$ must be selected to calculate the final project value, each with an objective probability of 50% at interest rates of 6% and 8%, respectively. These selections with NPV_s of the two projects allow us to construct another binomial lattice model for each option, as shown in Figure 3. Table 4 shows the final values of the two projects in a case in which project A or project B is selected at the outset of repeatable replacements. When the objective probability is 50%, the procedure to calculate the final values of projects A and B is as follows:

(1) The risk-neutral probability for project A is expressed as $p_A' = (60.30 - 40.63/88.04 - 40.63) = 0.4148$ and $p_B = (60.63 - 40.63/88.04 - 40.63) = 0.4219$.

(2) The expected maximum exponential growth rates of projects A and B are expressed as $g_{max}^{A} = (0.5)\ln(0.5(1.07^3/0.4149)) + (0.5)\ln(0.5(1.07^3/1- 0.4149)) = 0.2176$ and $g_{max}^{B} =$

Table 3: NPV_s for single life cycle and infinite planning horizon for A and B.

p (%)	g_{max}^{A}	g_{max}^{B}	V_A	V_B
5	0.5791	0.7139	107.6416	123.7789
10	0.4669	0.6017	96.2100	110.6422
15	0.3835	0.5184	88.5176	101.8037
20	0.3201	0.4551	83.0798	95.5572
25	0.2725	0.4075	79.2148	91.1188
30	0.2383	0.3734	76.5487	88.0589
35	0.2160	0.3512	74.8615	86.1247
40	0.2047	0.3400	74.0225	85.1660
45	0.2039	0.3392	73.9608	85.1017
50	0.2132	0.3486	74.6508	85.9022
55	0.2325	0.3680	76.1057	87.5833
60	0.2619	0.3975	78.3781	90.2053
65	0.3017	0.4374	81.5653	93.8807
70	0.3526	0.4884	85.8224	98.7883
75	0.4154	0.5513	91.3870	105.2018
80	0.4917	0.6276	98.6255	113.5433
85	0.5836	0.7196	108.1281	124.4930
90	0.6956	0.8316	120.9330	139.2467
95	0.8364	0.9726	139.2260	160.3223
99	0.9904	1.1266	162.3976	187.0166

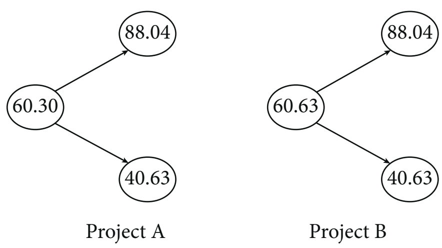

Figure 3: Binomial lattice model with the first selected project.

Table 4: Final values of A and B with different probabilities when either one is first initiated.

p (%)	g_{max}^{A}	g_{max}^{B}	V_A	V_B
5	0.5576	0.7035	97.0470	104.7531
10	0.4482	0.5927	88.1400	95.5906
15	0.3678	0.5108	82.1843	89.4444
20	0.3073	0.4489	78.0100	85.1177
25	0.2625	0.4027	75.0811	82.0620
30	0.2312	0.3700	73.1064	79.9774
35	0.2118	0.3491	71.9151	78.6878
40	0.2034	0.3393	71.4077	78.0897
45	0.2055	0.3400	71.5322	78.1280
50	0.2177	0.3507	72.2730	78.7844
55	0.2399	0.3714	73.6465	80.0725
60	0.2722	0.4023	75.7015	82.0384
65	0.3149	0.4436	78.5242	84.7654
70	0.3687	0.4960	82.2500	88.3854
75	0.4344	0.5603	87.0843	93.0988
80	0.5136	0.6380	93.3422	99.2144
85	0.6084	0.7314	101.5298	107.2288
90	0.7233	0.8448	112.5344	118.0139
95	0.8670	0.9871	128.2196	133.4030
99	1.0232	1.1422	148.0321	152.8683

$(0.5)\ln(0.5(1.07^5/0.4219)) + (0.5)\ln(0.5(1.07^5/1-0.4219)) = 0.3507$.

(3) The final values of projects A and B are expressed as $V_A = e^{0.2176}(60.30) = 74.9583$ and $V_B = e^{0.3507}(60.63) = 86.0983$.

Because $V_A = 74.9583 < V_B = 86.0983$, it is recommended that the firm undertakes project B. In this case, project B dominates project A over the entire range of the objective probabilities. This is in contrast to the decision made using Brown and Davis's real option approach but is congruent with the more conventional approach. As expected, the final values in the second case were greater, even when the improved amount was not much greater than that of the first case.

4. Conclusions

Herein, a replacement problem with an infinite planning horizon based on three different analysis techniques has been discussed: (1) a conventional analysis technique, (2) a real option-pricing technique, and (3) the Kelly criterion technique in the context of an applicable BLOP environment. This study applied each of these techniques to a specific example obtained from Brown and Davis's study. It is discovered that the third technique yielded the same result as the first but yielded a different result when compared with that of the second technique.

The finding above may apply to the specific example. However, because $NPV_A^{\infty} = 60.30 < NPV_A^{max}(uncertainty) = 60.63 < V_B = 86.0983$, these values were prioritized in the three techniques. It is believed that the third technique dominated over the other two primarily because of the objective winning probability p. As shown in equation (28), the objective winning probability equals the optimal betting ratio f^*. Therefore, the approach to adequately determine the objective winning probability in advance must be determined. This probability may be obtained based on well-documented data, management's subjective judgment, and so forth. Once the objective winning probability is determined, the risk-neutral probability is vital to the method proposed herein.

The major disadvantage of applying the method proposed herein is that it hinders the understanding of the concepts of real option-pricing theories and the Kelly criterion. Since the beginning of 2000s, the real option theory has been extensively investigated in both theoretical and practical aspects. However, it has not been applied widely as an investment project analysis in firms worldwide. For example, 4.6% of Korean firms, 0.5% of Japanese firms, and 26.6% of Fortune 500 and FEI companies use real option-pricing theories [22]. As described earlier, the Kelly criterion has been investigated and applied primarily in gambling games and finance, whereas almost no research regarding the Kelly criterion has been conducted from a real investment perspective. Therefore, it is believed that the proposed model in this study will only be adopted after a significant amount of time. In addition, depending on the variety of conditions and assumptions, many different replacement problems may occur. The problem addressed in this study is extremely simple and hence easy to solve. In fact, for the proposed model to be practically useful, further investigations involving more sophisticated and complex replacement problems are to be conducted. As such, they will be performed in future studies.

Data Availability

Data are available upon request to the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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