Research Article

Some New Codes on the k -Fibonacci Sequence

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In this paper, we define and study the k -Fibonacci sequence matrix for $k \geq 3$. By these obtained results, we introduce some new codes on k -Fibonacci sequence.

1. Introduction

The Fibonacci sequence F_n^2 is defined by the recurrence relation $F_0^2 = 0$, $F_1^2 = 1$, and $F_n^2 = F_{n-1}^2 + F_{n-2}^2$, $n \geq 2$. The Fibonacci sequence and its generalization sequence (k -Fibonacci sequence number and k -Pell sequence) are famous sequences in mathematics. For information about these sequences, see [1–4].

Definition 1. For $k \geq 3$, the k -Fibonacci sequence, denoted by $\{F_n^k\}_{-\infty}^{\infty}$, is defined by

$$\begin{cases} F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k, & n \geq 0, \\ F_{n-1}^k = F_{n+k-1}^k - (F_{n+k-2}^k + \cdots + F_n^k), & n < 0, \end{cases} \quad (1)$$

with initial conditions $F_i^k = 0$, $0 \leq i \leq k-2$, and $F_{k-1}^k = 1$ (see [5]).

For example, let $k = 3$, we have

$$\begin{cases} F_n^3 = F_{n-1}^3 + F_{n-2}^3 + F_{n-3}^3, & n \geq 0, \\ F_{n-1}^3 = F_{n+2}^3 - (F_{n+1}^3 + F_n^3), & n < 0. \end{cases} \quad (2)$$

So, $\{F_n^3\}_{-\infty}^{\infty} = \{\cdots, -1, 1, 0, 0, 1, 1, 2, 4, 7, \cdots\}$.

By the definition of the k -Fibonacci sequence, the proof of the following lemma is trivial.

Lemma 1. For $k \geq 3$, the following relations are satisfied about the k -Fibonacci sequence:

- (i) $F_i^k = 0$, $i \in \mathbb{N}$, $-k \leq i \leq -2$
- (ii) $F_{-1}^k = 1$
- (iii) $F_{-k-1}^k = -1$

Apostolic and Fraenkle introduced the Fibonacci code which is used in source coding as well as in cryptography (see [6]). For information on the Fibonacci code, see in [7, 8]. The Fibonacci Q_p -matrices were introduced (see [9]), and in [10], for $p = 1$, a coding theory is obtained on the Fibonacci Q_p -matrices. Also, a Fibonacci coding method is introduced by using Fibonacci polynomials and Fibonacci sequence (see [8, 11–14]). Here, we introduce some coding methods on k -Fibonacci sequence.

In Section 2, we define the k -Fibonacci sequence matrix and calculate the determinant of it. Section 3 and Section 4 are devoted to obtain some codes on k -Fibonacci sequence.

2. k -Fibonacci Sequence Matrix

The 2-Fibonacci sequence matrix (the Fibonacci Q_1 -matrices) is defined in [15]. The aim of this section is to define the k -Fibonacci sequence matrix and get to the determinant of this matrix. First, we define the k -Fibonacci sequence matrix for $k = 3$ and obtain the determinant of it.

Definition 2. The 3-Fibonacci sequence matrix, denoted by $Q_{(n,3)}$, is defined:

$$Q_{(n,3)} = \begin{bmatrix} F_{n+2}^3 & F_{n+1}^3 & F_n^3 \\ F_{n+1}^3 & F_n^3 & F_{n-1}^3 \\ F_n^3 & F_{n-1}^3 & F_{n-2}^3 \end{bmatrix}, \quad (3)$$

where F_n^3 is the element of the 3-Fibonacci sequence.

For example, $n = 10$, and we have

$$Q_{(10,3)} = \begin{bmatrix} F_{12}^3 & F_{11}^3 & F_{10}^3 \\ F_{11}^3 & F_{10}^3 & F_9^3 \\ F_{10}^3 & F_9^3 & F_8^3 \end{bmatrix} = \begin{bmatrix} 274 & 149 & 81 \\ 149 & 81 & 44 \\ 81 & 44 & 24 \end{bmatrix}. \quad (4)$$

Now, we get the determinant of the 3-Fibonacci sequence matrix. By the definition of F_n^3 , we have

$$F_{n-3}^3 = F_n^3 - (F_{n-1}^3 + F_{n-2}^3). \quad (5)$$

Theorem 1. *The determinant of $Q_{(n,3)}$ is equal to -1 .*

Proof. For $n = 3m + i$ where $0 \leq i \leq 2$, we have

$$\begin{bmatrix} F_{3(m-1)+i+2}^3 & F_{3m+i+1}^3 & F_{3m+i}^3 \\ F_{3(m-1)+i+1}^3 & F_{3m+i}^3 & F_{3m+i-1}^3 \\ F_{3(m-1)+i}^3 & F_{3m+i-1}^3 & F_{3m+i-2}^3 \end{bmatrix} \longrightarrow \begin{bmatrix} F_{3(m-1)+i+2}^3 & F_{3(m-1)+i+1}^3 & F_{3m+i}^3 \\ F_{3(m-1)+i+1}^3 & F_{3(m-1)+i}^3 & F_{3m+i-1}^3 \\ F_{3(m-1)+i}^3 & F_{3(m-1)+i-1}^3 & F_{3m+i-2}^3 \end{bmatrix}. \quad (8)$$

(iii) Add column 1 and column 2, then subtract this summation from column 3, and replace to it. Thus, by relation (5), we have

$$\begin{bmatrix} F_{3(m-1)+i+2}^3 & F_{3(m-1)+i+1}^3 & F_{3m+i}^3 \\ F_{3(m-1)+i+1}^3 & F_{3(m-1)+i}^3 & F_{3m+i-1}^3 \\ F_{3(m-1)+i}^3 & F_{3(m-1)+i-1}^3 & F_{3m+i-2}^3 \end{bmatrix} \longrightarrow \begin{bmatrix} F_{3(m-1)+i+2}^3 & F_{3(m-1)+i+1}^3 & F_{3(m-1)+i}^3 \\ F_{3(m-1)+i+1}^3 & F_{3(m-1)+i}^3 & F_{3(m-1)+i-1}^3 \\ F_{3(m-1)+i}^3 & F_{3(m-1)+i-1}^3 & F_{3(m-1)+i-2}^3 \end{bmatrix}. \quad (9)$$

Then, we have

$$Q_{(3m+i,3)} \longrightarrow Q_{(3(m-1)+i,3)} \quad (10)$$

Continuing this process $m - 1$ steps on $Q_{(3(m-1)+i,3)}$, we obtain the following:

$$Q_{(3(m-1)+i,3)} \longrightarrow Q_{(i,3)} = \begin{bmatrix} F_{i+2}^3 & F_{i+1}^3 & F_i^3 \\ F_{i+1}^3 & F_i^3 & F_{i-1}^3 \\ F_i^3 & F_{i-1}^3 & F_{i-2}^3 \end{bmatrix}. \quad (11)$$

Therefore, $\det Q_{(n,3)} = \det Q_{(i,3)}$. It is sufficient that we get the determinant of $Q_{(i,3)}$ where $0 \leq i \leq 2$. If $i = 0$, then by Lemma 1, we obtain

$$Q_{(3m+i,3)} = \begin{bmatrix} F_{3m+i+2}^3 & F_{3m+i+1}^3 & F_{3m+i}^3 \\ F_{3m+i+1}^3 & F_{3m+i}^3 & F_{3m+i-1}^3 \\ F_{3m+i}^3 & F_{3m+i-1}^3 & F_{3m+i-2}^3 \end{bmatrix}. \quad (6)$$

We can perform specific column operations to reduce $Q_{(3m+i,3)}$ to $Q_{(i,3)}$, where $0 \leq i \leq 2$. The operations that we use in this process are as follows:

(i) Add column 2 and column 3, then subtract this summation from column 1, and replace it to the result column 1 ($c_1 - (c_2 + c_3) \rightarrow c_1$). So, by relation (5), we have

$$Q_{(3m+i,3)} \longrightarrow \begin{bmatrix} F_{3(m-1)+i+2}^3 & F_{3m+i+1}^3 & F_{3m+i}^3 \\ F_{3(m-1)+i+1}^3 & F_{3m+i}^3 & F_{3m+i-1}^3 \\ F_{3(m-1)+i}^3 & F_{3m+i-1}^3 & F_{3m+i-2}^3 \end{bmatrix}. \quad (7)$$

(ii) Add column 1 and column 3, then subtract it from column 2, and replace to the result column 2. Then, by relation (5), we get

$$Q_{(0,3)} = \begin{bmatrix} F_2^3 & F_1^3 & F_0^3 \\ F_1^3 & F_0^3 & F_{-1}^3 \\ F_0^3 & F_{-1}^3 & F_{-2}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (12)$$

So, $\det Q_{(0,3)} = -1$. If $i = 1$, then by Lemma 1, we have

$$Q_{(1,3)} = \begin{bmatrix} F_3^3 & F_2^3 & F_1^3 \\ F_2^3 & F_1^3 & F_0^3 \\ F_1^3 & F_0^3 & F_{-1}^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (13)$$

Therefore, $\det Q_{(1,3)} = -1$.

Similarly, for $i = 2$, we get $\det Q_{(2,3)} = -1$.

Consequently, the theorem is proven. $\square$

Example 1. $\det Q_{(10,3)} = -1$. Since $n = 3 \times 3 + 1$, we have

$$Q_{(10,3)} = \begin{bmatrix} F_{12}^3 & F_{11}^3 & F_{10}^3 \\ F_{11}^3 & F_{10}^3 & F_9^3 \\ F_{10}^3 & F_9^3 & F_8^3 \end{bmatrix} = \begin{bmatrix} 274 & 149 & 81 \\ 149 & 81 & 44 \\ 81 & 44 & 24 \end{bmatrix}. \quad (14)$$

We get the following:

Step 1:

$$Q_{(10,3)} \longrightarrow Q_{(7,3)} = \begin{bmatrix} F_9^3 & F_8^3 & F_7^3 \\ F_8^3 & F_7^3 & F_6^3 \\ F_7^3 & F_6^3 & F_5^3 \end{bmatrix} = \begin{bmatrix} 44 & 24 & 13 \\ 24 & 13 & 7 \\ 13 & 7 & 4 \end{bmatrix}. \quad (15)$$

Step 2:

$$Q_{(7,3)} \longrightarrow Q_{(4,3)} = \begin{bmatrix} F_6^3 & F_5^3 & F_4^3 \\ F_5^3 & F_4^3 & F_3^3 \\ F_4^3 & F_3^3 & F_2^3 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \quad (16)$$

Step 3:

$$Q_{(4,3)} \longrightarrow Q_{(1,3)} = \begin{bmatrix} F_3^3 & F_2^3 & F_1^3 \\ F_2^3 & F_1^3 & F_0^3 \\ F_1^3 & F_0^3 & F_{-1}^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (17)$$

Therefore, $\det Q_{(10,3)} = \det Q_{(1,3)} = -1$.

Now, we are ready to generalize the idea of the 3-Fibonacci sequence matrix to the k -Fibonacci sequence matrices ($k > 3$).

Definition 3. The k -Fibonacci sequence matrices of size $k \times k$ ($k \geq 4$), denoted by $Q_{(n,k)}$, are defined as follows:

$$Q_{(n,k)} = \begin{bmatrix} F_{n+(k-1)}^k & F_{n+(k-2)}^k & \cdots & F_n^k \\ F_{n+(k-2)}^k & F_{n+(k-3)}^k & \cdots & F_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_n^k & F_{n-1}^k & \cdots & F_{n-(k-1)}^k \end{bmatrix}, \quad (18)$$

where F_i^k is the element of the k -Fibonacci sequence.

Example 3. By the definition of $Q_{(n,k)}$, we have

$$Q_{(8,4)} = \begin{bmatrix} F_{11}^4 & F_{10}^4 & F_9^4 & F_8^4 \\ F_{10}^4 & F_9^4 & F_8^4 & F_7^4 \\ F_9^4 & F_8^4 & F_7^4 & F_6^4 \\ F_8^4 & F_7^4 & F_6^4 & F_5^4 \end{bmatrix} = \begin{bmatrix} 108 & 56 & 29 & 15 \\ 56 & 29 & 15 & 8 \\ 29 & 15 & 8 & 4 \\ 15 & 8 & 4 & 2 \end{bmatrix}. \quad (19)$$

In Theorem 1, we get the determinant of the 3-Fibonacci sequence matrix. Now, we obtain the determinant of the k -Fibonacci sequence matrices for $k \geq 4$. By the definition of F_n^k , we have

$$F_{n-k}^k = F_n^k - (F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k-1}^k). \quad (20)$$

Theorem 2. $\det Q_{(n,k)}$ is equal to -1 .

Proof. For $n = mk + i$, $0 \leq i \leq k-1$, we have

$$Q_{(mk+i,k)} = \begin{bmatrix} F_{mk+i+(k-1)}^k & F_{mk+i+(k-2)}^k & \cdots & F_{mk+i}^k \\ F_{mk+i+(k-2)}^k & F_{mk+i+(k-3)}^k & \cdots & F_{mk+i-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_{mk+i}^k & F_{mk+i-1}^k & \cdots & F_{mk+i-(k-1)}^k \end{bmatrix}. \quad (21)$$

We can perform specific column operations to reduce $Q_{(mk+i,k)}$ to $Q_{(i,k)}$, where $0 \leq i \leq k-1$. The operations that we use in this process are as follows.

Add column 2 until column k , then subtract this summation from column 1, and replace it to the result column 1 ($c_1 - (c_2 + c_3 + \cdots + c_k) \rightarrow c_1$). Then, add column 1 until column k except column 2, then subtract it from column 2, and replace it to ($c_2 - (c_1 + c_3 + \cdots + c_k) \rightarrow c_2$). Continuing the process until column k and by relation (20), we have

$$Q_{(mk+i,k)} \longrightarrow Q_{((m-1)k+i,k)} = \begin{bmatrix} F_{(m-1)k+i+(k-1)}^k & F_{(m-1)k+i+(k-2)}^k & \cdots & F_{(m-1)k+i}^k \\ F_{(m-1)k+i+(k-2)}^k & F_{(m-1)k+i+(k-3)}^k & \cdots & F_{(m-1)k+i-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_{(m-1)k+i}^k & F_{(m-1)k+i-1}^k & \cdots & F_{(m-1)k+i-(k-1)}^k \end{bmatrix}. \quad (22)$$

Thus, performing above the operations until $m-1$ steps, we get

$$Q_{((m-1)k+i,k)} \longrightarrow Q_{(i,k)} = \begin{bmatrix} F_{i+(k-1)}^k & F_{i+(k-2)}^k & \cdots & F_i^k \\ F_{i+(k-2)}^k & F_{i+(k-3)}^k & \cdots & F_{i-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_i^k & F_{i-1}^k & \cdots & F_{i-(k-1)}^k \end{bmatrix}. \quad (23)$$

Therefore, $\det Q_{(n,k)} = \det Q_{(i,k)}$, $0 \leq i \leq k-1$.

Let $i = 0$, by Lemma 1, we have $F_{-1}^k = F_{k-1}^k = -1$ and $F_i^k = 0$, $-k+1 \leq i \leq k-2$. So, we get

$$Q_{(0,k)} = \begin{bmatrix} F_{k-1}^k & F_{k-2}^k & \cdots & F_0^k \\ F_{k-2}^k & F_{k-3}^k & \cdots & F_{-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_0^k & F_{-1}^k & \cdots & F_{-k+1}^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}. \quad (24)$$

By elementary matrix operations, we get $\det Q_{(0,k)} = -1$. The other cases are proven similarly. $\square$

3. A Code on the k -Fibonacci Sequence

In this section, we introduce a coding method by using the k -Fibonacci sequence and obtain error detection and correction of it. First, we need the following definition.

Definition 4. For $k = 2$, $n \geq 4$ or $k \geq 3$, $n \geq k+1$, let F_n^k and F_{n+1}^k be two consecutive elements of the k -Fibonacci sequence. The sequence $\{T_s\}_{s=1}^{F_{n+2}^k}$ is defined as follows:

$$T_s \equiv s(F_{n+1}^k + 1) \pmod{F_{n+2}^k} \quad (25)$$

For $u = F_{n+2}^k$ and $1 \leq i, j \leq u$, by using definition T_s , we define the matrix $T^u = (t_{ij})_{u \times u}$ as follows:

$$t_{ij} = \begin{cases} 1, & \text{if } i = T_j, \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

Example 3. Suppose $k = 3$ and $n = 4$. Since, $F_5^3 = 4$ and $u = F_6^3 = 7$, we have

$$\begin{aligned} T_1 &\equiv 1(F_5^3 + 1) = 1 \times 5 \equiv 5 \pmod{7}, \\ T_2 &\equiv 2(F_5^3 + 1) = 2 \times 5 \equiv 3 \pmod{7}, \\ T_3 &\equiv 3(F_5^3 + 1) = 3 \times 5 \equiv 1 \pmod{7}, \\ T_4 &\equiv 4(F_5^3 + 1) = 4 \times 5 \equiv 6 \pmod{7}, \\ T_5 &\equiv 5(F_5^3 + 1) = 5 \times 5 \equiv 4 \pmod{7}, \\ T_6 &\equiv 6(F_5^3 + 1) = 6 \times 5 \equiv 2 \pmod{7}, \\ T_7 &\equiv 7(F_5^3 + 1) = 7 \times 5 \equiv 7 \pmod{7}. \end{aligned} \quad (27)$$

Then,

$$t_{ij} = \begin{cases} 1, & \text{if } (i, j) = (5, 1), (3, 2), (1, 3), (6, 4), (4, 5), (2, 6), (7, 7), \\ 0, & \text{otherwise.} \end{cases} \quad (28)$$

That is,

$$T^7 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

Now, by using above notations, we can explain a method coding, which is named T^u -code.

For $u = F_{n+2}^k$, $1 \leq i \leq u^2$ and a message matrix M of size $u \times u$,

$$M = \begin{bmatrix} m_1 & m_2 & \cdots & m_u \\ \vdots & \vdots & \ddots & \vdots \\ m_{u^2-u+1} & m_{u^2-u+2} & \cdots & m_{u^2} \end{bmatrix}, \quad (30)$$

where $m_i \geq 0$. We consider a transformation $E = M \times T^u$ as the T^u -coding method. Then, $M = E \times (T^u)^{-1}$ is the T^u -decoding method.

Example 4. Suppose $k = 2$ and $n = 3$, we have $F_4^2 = 3$, $F_5^2 = 5$ and $T_1 \equiv 4 \pmod{5}$, $T_2 \equiv 3 \pmod{5}$, $T_3 \equiv 2 \pmod{5}$, $T_4 \equiv 1 \pmod{5}$, and $T_5 \equiv 5 \pmod{5}$. Then,

$$T^5 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (31)$$

Let "mathematics is beautiful" be a message. Then, by Table 1 and $0 :=$ blank space, we have

$$\begin{aligned} M &= \begin{bmatrix} m & a & t & h & e \\ m & a & t & i & c \\ s & o & i & s & o \\ b & e & a & u & t \\ i & f & u & l & o \end{bmatrix} = \begin{bmatrix} F_{17}^2 & F_5^2 & F_{24}^2 & F_{12}^2 & F_9^2 \\ F_{17}^2 & F_5^2 & F_{24}^2 & F_{13}^2 & F_7^2 \\ F_{23}^2 & F_{31}^2 & F_{13}^2 & F_{23}^2 & F_{31}^2 \\ F_6^2 & F_9^2 & F_5^2 & F_{25}^2 & F_{24}^2 \\ F_{13}^2 & F_{24}^2 & F_{25}^2 & F_{16}^2 & F_{31}^2 \end{bmatrix} \\ &= \begin{bmatrix} 1597 & 5 & 46368 & 144 & 34 \\ 1597 & 5 & 46368 & 233 & 13 \\ 28657 & 1346269 & 233 & 28657 & 1346269 \\ 8 & 34 & 5 & 75025 & 46368 \\ 233 & 46368 & 75025 & 987 & 1346269 \end{bmatrix}. \end{aligned} \quad (32)$$

Hence, we get

$$\begin{aligned}
 E = M \times T^5 &= \begin{bmatrix} 1597 & 5 & 46368 & 144 & 34 \\ 1597 & 5 & 46368 & 233 & 13 \\ 28657 & 1346269 & 233 & 28657 & 1346269 \\ 8 & 34 & 5 & 75025 & 46368 \\ 233 & 46368 & 75025 & 987 & 1346269 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 144 & 46368 & 5 & 1597 & 34 \\ 233 & 46368 & 5 & 1597 & 13 \\ 28657 & 233 & 1346269 & 28657 & 1346269 \\ 75025 & 5 & 34 & 8 & 46368 \\ 987 & 75025 & 46368 & 233 & 1346269 \end{bmatrix}.
 \end{aligned} \tag{33}$$

So, “htameiamcsi s uaebluti” is the code matrix E .

Note that $M = E \times (T^5)^{-1}$ is the T^5 -decoding method.

By using the elementary operations on matrix T^u , one can prove the following Lemma.

Lemma 2. For $u \geq 4$, $\det T^u = \pm 1$. In particular, $\det T^5 = 1$ and $\det T^4 = \det T^7 = -1$.

Note 1. This lemma follows that for $E = M \times T^u$, we have

$$\det E = \det(M \times T^u) = \det M \times \det T^u = \pm \det M. \tag{34}$$

In particular, when $u = 4, 7$, we get $\det E = -\det M$, and for $u = 5$, we have $\det E = \det M$.

By using the above facts, we find the error detection and correction for T^u -code where $u = 4, 5, 7$.

The code matrix E may contain “single,” “double,” ..., “ u^2 -fold” errors. Thus, we get

$$1C_{u^2} + 2C_{u^2} + \dots + u^2 C_{u^2} = 2^{u^2} - 1. \tag{35}$$

where $rC_s = s!/r! \times (s-r)!$.

It is clear that the “single” errors in the code matrix E are as follows:

$$(1) \begin{bmatrix} x_1 & e_{12} & \dots & e_{1u} \\ e_{21} & e_{22} & \dots & e_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ e_{u1} & e_{u2} & \dots & e_{uu} \end{bmatrix}, (2) \begin{bmatrix} e_{11} & x_2 & \dots & e_{1u} \\ e_{21} & e_{22} & \dots & e_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ e_{u1} & e_{u2} & \dots & e_{uu} \end{bmatrix}, \dots, (u^2) \begin{bmatrix} e_{11} & e_{12} & \dots & e_{1u} \\ e_{21} & e_{22} & \dots & e_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ e_{u1} & e_{u2} & \dots & x_{u^2} \end{bmatrix}, \tag{36}$$

where x_i , $1 \leq i \leq u^2$ are possible “destroyed” elements. In this case, there are u^2 errors in the matrix code E .

Suppose $u = 4$ and 7 . For correction, by using the algebraic equations and the relation $\det E = -\det M$, we have

TABLE 1: English alphabet.

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	0
F_{n+1}^k	F_{n+2}^k	F_{n+3}^k	F_{n+4}^k	F_{n+5}^k	F_{n+6}^k	F_{n+7}^k	F_{n+8}^k	F_{n+9}^k	F_{n+10}^k	F_{n+11}^k	F_{n+12}^k	F_{n+13}^k	F_{n+14}^k	F_{n+15}^k	F_{n+16}^k	F_{n+17}^k	F_{n+18}^k	F_{n+19}^k	F_{n+20}^k	F_{n+21}^k	F_{n+22}^k	F_{n+23}^k	F_{n+24}^k	F_{n+25}^k	F_{n+26}^k	F_{n+27}^k

$$\begin{aligned}
& x_1 \det \begin{bmatrix} e_{22} & \cdots & e_{2u} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu} \end{bmatrix} - e_{12} \det \begin{bmatrix} e_{21} & \cdots & e_{2u} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu} \end{bmatrix} + \cdots + (-1)^u e_{1u} \det \begin{bmatrix} e_{21} & \cdots & e_{2u-1} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu-1} \end{bmatrix} = -\det M, \\
& e_{11} \det \begin{bmatrix} e_{22} & \cdots & e_{2u} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu} \end{bmatrix} - x_2 \det \begin{bmatrix} e_{21} & \cdots & e_{2u} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu} \end{bmatrix} + \cdots + (-1)^u e_{1u} \det \begin{bmatrix} e_{21} & \cdots & e_{2u-1} \\ \vdots & & \vdots \\ e_{u2} & \cdots & e_{uu-1} \end{bmatrix} = -\det M, \\
& \vdots \\
& e_{u1} \det \begin{bmatrix} e_{12} & \cdots & e_{1u} \\ \vdots & & \vdots \\ e_{u-12} & \cdots & e_{u-1u-1} \end{bmatrix} + \cdots + (-1)^{u^2-1} e_{u^2-1} \det \begin{bmatrix} e_{11} & \cdots & e_{1u} \\ \vdots & & \vdots \\ e_{u-11} & \cdots & e_{u-1u} \end{bmatrix} \\
& + (-1)^{u^2} x_{u^2} \det \begin{bmatrix} e_{11} & \cdots & e_{1u-1} \\ \vdots & & \vdots \\ e_{u-11} & \cdots & e_{u-1u} \end{bmatrix} = -\det M.
\end{aligned} \tag{37}$$

By above relations, we can obtain “single” correction. Using an analogous argument, we can find the error correction for T^5 -code.

4. Coding and Decoding on the k -Fibonacci Sequence Matrix

Here, we find coding and decoding on the k -Fibonacci sequence matrix $Q_{(n,k)}$ and get its error detection and correction. For $k \geq 3$ and an initial message $M_{k \times k}$, we will name a transformation $E = M \times Q_{(n,k)}$ as the k -Fibonacci coding and a transformation $M = E \times Q_{(n,k)}^{-1}$ as the k -Fibonacci decoding. Also, the matrix E is as a code matrix. Now, we explain the above method by an example.

Example 4. Suppose $k = 4$, $n = 5$, and

$$M = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 4 & 5 & 7 \end{bmatrix}. \tag{38}$$

We have

$$Q_{(5,4)} = \begin{bmatrix} F_8^4 & F_7^4 & F_6^4 & F_5^4 \\ F_7^4 & F_6^4 & F_5^4 & F_4^4 \\ F_6^4 & F_5^4 & F_4^4 & F_3^4 \\ F_5^4 & F_4^4 & F_3^4 & F_2^4 \end{bmatrix} = \begin{bmatrix} 15 & 8 & 4 & 2 \\ 8 & 4 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}. \tag{39}$$

By the above notations, we have

$$E = M \times Q_{(5,4)} = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 4 & 5 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 4 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 57 & 29 & 17 & 8 \\ 76 & 38 & 21 & 9 \\ 25 & 13 & 7 & 4 \\ 111 & 57 & 32 & 15 \end{bmatrix}. \tag{40}$$

Also, we obtain

$$M = E \times Q_{(5,4)}^{-1} = \begin{bmatrix} 57 & 29 & 17 & 8 \\ 76 & 38 & 21 & 9 \\ 25 & 13 & 7 & 4 \\ 111 & 57 & 32 & 15 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 4 & 5 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 3 & 2 \\ 1 & 0 & 2 & 1 \\ 3 & 4 & 5 & 7 \end{bmatrix}. \tag{41}$$

The error-correction algorithm for the Fibonacci Q_2 -matrices is described in [16]. Now, we explain error detection and correction for the 3-Fibonacci coding.

$$E = M \times Q_{(n,3)} = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \begin{bmatrix} F_{n+2}^3 & F_{n+1}^3 & F_n^3 \\ F_{n+1}^3 & F_n^3 & F_{n-1}^3 \\ F_n^3 & F_{n-1}^3 & F_{n-2}^3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \quad (42)$$

$$M = E \times Q_{(n,3)}^{-1} = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix} \begin{bmatrix} F_{n+2}^3 & F_{n+1}^3 & F_n^3 \\ F_{n+1}^3 & F_n^3 & F_{n-1}^3 \\ F_n^3 & F_{n-1}^3 & F_{n-2}^3 \end{bmatrix}^{-1} = \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix}.$$

Also,

$$\det Q_{(n,3)} = F_{n+2}^3 \left[F_n^3 F_{n-2}^3 - (F_{n-1}^3)^2 \right] - F_{n+1}^3 \left[F_{n+1}^3 F_{n-2}^3 - F_{n-1}^3 F_n^3 \right] + F_n^3 \left[F_{n+1}^3 F_{n-1}^3 - (F_n^3)^2 \right] = -1,$$

$$M = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix} \begin{bmatrix} (F_{n-1}^3)^2 - F_n^3 F_{n-2}^3 & F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 & (F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \\ F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 & (F_n^3)^2 - F_{n+2}^3 F_{n-2}^3 & F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3 \\ (F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 & F_{n+2}^3 F_{n-1}^3 - F_{n+1}^3 F_n^3 & (F_{n+1}^3)^2 - F_{n+2}^3 F_n^3 \end{bmatrix}. \quad (43)$$

Since $m_i \geq 0$, $1 \leq i \leq 9$, we have

$$m_1 = e_1 \left[(F_{n-1}^3)^2 - F_n^3 F_{n-2}^3 \right] + e_2 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_3 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] > 0, \quad (44)$$

$$m_2 = e_1 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_2 \left[(F_n^3)^2 - F_{n+2}^3 F_{n-2}^3 \right] + e_3 \left[F_{n+2}^3 F_{n-1}^3 - F_{n+1}^3 F_n^3 \right] > 0, \quad (45)$$

$$m_3 = e_1 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] + e_2 \left[F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3 \right] + e_3 \left[(F_{n+1}^3)^2 - F_{n+2}^3 F_n^3 \right] > 0, \quad (46)$$

$$m_4 = e_4 \left[(F_{n-1}^3)^2 - F_n^3 F_{n-2}^3 \right] + e_5 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_6 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] > 0,$$

$$m_5 = e_4 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_5 \left[(F_n^3)^2 - F_{n+2}^3 F_{n-2}^3 \right] + e_6 \left[F_{n+2}^3 F_{n-1}^3 - F_{n+1}^3 F_n^3 \right] > 0,$$

$$m_6 = e_4 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] + e_5 \left[F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3 \right] + e_6 \left[(F_{n+1}^3)^2 - F_{n+2}^3 F_n^3 \right] > 0, \quad (47)$$

$$m_7 = e_7 \left[(F_{n-1}^3)^2 - F_n^3 F_{n-2}^3 \right] + e_8 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_9 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] > 0,$$

$$m_8 = e_7 \left[F_{n+1}^3 F_{n-2}^3 - F_n^3 F_{n-1}^3 \right] + e_8 \left[(F_n^3)^2 - F_{n+2}^3 F_{n-2}^3 \right] + e_9 \left[F_{n+2}^3 F_{n-1}^3 - F_{n+1}^3 F_n^3 \right] > 0,$$

$$m_9 = e_7 \left[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right] + e_8 \left[F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3 \right] + e_9 \left[(F_{n+1}^3)^2 - F_{n+2}^3 F_n^3 \right] > 0.$$

From (44), we get

$$e_1 (F_{n-1}^3)^2 + e_2 F_{n+1}^3 F_{n-2}^3 + e_3 (F_n^3)^2 > e_1 F_n^3 F_{n-2}^3 + e_2 F_n^3 F_{n-1}^3 + e_3 F_{n+1}^3 F_{n-1}^3. \quad (48)$$

By using (45), we have

$$e_1 F_{n+1}^3 F_{n-2}^3 + e_2 (F_n^3)^2 + e_3 F_{n+2}^3 F_{n-1}^3 > e_1 F_n^3 F_{n-1}^3 + e_2 F_{n+2}^3 F_{n-2}^3 + e_3 F_{n+1}^3 F_n^3. \quad (49)$$

From (46), we get

$$e_1(F_n^3)^2 + e_2F_{n+2}^3F_{n-1}^3 + e_3(F_{n+1}^3)^2 > e_1F_{n+1}^3F_{n-1}^3 - e_2F_n^3F_{n+1}^3 + e_3F_{n+2}^3F_n^3. \quad (50)$$

By relations (48)–(50), we have

$$e_2[F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3] + [F_n^3F_{n-2}^3] - (F_{n-1}^3)^2 < e_3[(F_n^3)^2 - F_{n+1}^3F_{n-1}^3], \quad (51)$$

$$e_2[(F_{n+2}^3F_{n-2}^3 - F_n^3)^2] + [F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3] < e_3[F_{n+2}^3F_{n-1}^3 - F_{n+1}^3F_n^3], \quad (52)$$

$$e_2[F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3] + [(F_n^3)^2 - F_{n+1}^3F_{n-1}^3] > e_3[F_{n+2}^3F_n^3 - (F_{n+1}^3)^2]. \quad (53)$$

Let $A_1 = (F_n^3)^2 - F_{n+1}^3F_{n-1}^3$, $A_2 = F_{n+2}^3F_{n-1}^3 - F_{n+1}^3F_n^3$, and $A_3 = F_{n+2}^3F_n^3 - (F_{n+1}^3)^2$. Now, there exist $3^3 = 27$ cases for $A_1 > 0$, $A_2 > 0$, $A_3 > 0$. We calculate two cases and the rest of the cases get in similar ways.

Case 1. Let $A_1 > 0$, $A_2 > 0$, and $A_3 > 0$. By using (51) and (53), we have

$$\begin{aligned} \frac{e_2[F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3] + [F_n^3F_{n-2}^3] - (F_{n-1}^3)^2}{e_1A_1} &< \frac{e_3}{e_1}, \\ \frac{e_2[F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3] + [(F_n^3)^2 - F_{n+1}^3F_{n-1}^3]}{e_1A_3} &> \frac{e_3}{e_1}. \end{aligned} \quad (54)$$

Then, we get

$$\frac{e_2[F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3] + [F_n^3F_{n-2}^3] - (F_{n-1}^3)^2}{e_1A_1} < \frac{e_2[F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3] + [(F_n^3)^2 - F_{n+1}^3F_{n-1}^3]}{e_1A_3}. \quad (55)$$

So,

$$\frac{A_3(F_n^3F_{n-2}^3 - (F_{n-1}^3)^2) - A_1((F_n^3)^2 - F_{n+1}^3F_{n-1}^3)}{A_1A_3} < \frac{e_2(A_1([F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3]) - A_3(F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3))}{e_1A_1A_3}. \quad (56)$$

Since $A_1 = (F_n^3)^2 - F_{n+1}^3F_{n-1}^3$, $A_3 = F_{n+2}^3F_n^3 - (F_{n+1}^3)^2$ and by using relation (43), we have

$$\begin{aligned} A_3(F_n^3F_{n-2}^3 - (F_{n-1}^3)^2) - A_1((F_n^3)^2 - F_{n+1}^3F_{n-1}^3) &= F_n^3 \det Q_{(n,3)}, \\ A_1([F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3]) - A_3(F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3) &= F_{n+1}^3 \det Q_{(n,3)}. \end{aligned} \quad (57)$$

Therefore,

$$\frac{e_1}{e_2} < \frac{F_{n+1}^3}{F_n^3}. \quad (58)$$

Now, we consider (52) and (53). Then,

$$\begin{aligned} \frac{e_2[(F_{n+2}^3F_{n-2}^3 - F_n^3)^2] + [F_n^3F_{n-1}^3 - F_{n+1}^3F_{n-2}^3]}{e_1A_2} &< \frac{e_3}{e_1}, \\ \frac{e_2[F_{n+2}^3F_{n-1}^3 - F_n^3F_{n+1}^3] + [(F_n^3)^2 - F_{n+1}^3F_{n-1}^3]}{e_1A_3} &> \frac{e_3}{e_1}. \end{aligned} \quad (59)$$

Hence, we have

$$\begin{aligned} & \frac{e_2 \left[(F_{n+2}^3 F_{n-2}^3 - F_n^3)^2 \right]}{e_1 A_2} + \frac{[F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]}{A_2} > \frac{e_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]}{e_1 A_3} + \frac{[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3]}{A_3}, \\ & \frac{A_3 (F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3) - A_2 \left((F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right)}{A_2 A_3} < \frac{e_2 (A_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3] - A_3 (F_{n+2}^3 F_{n-2}^3 - (F_n^3)^2))}{e_1 A_2 A_3}. \end{aligned} \quad (60)$$

We get,

$$\begin{aligned} & A_3 (F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3) - A_2 \left((F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right) = -F_{n+1}^3 \det Q_{(n,3)}, \\ & A_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3] - A_3 (F_{n+2}^3 F_{n-2}^3 - (F_n^3)^2) = F_{n+2}^3 \det Q_{(n,3)}. \end{aligned} \quad (61)$$

So,

$$\frac{e_1}{e_2} > \frac{F_{n+2}^3}{F_{n+1}^3}. \quad (62)$$

From (58) and (62), we have

$$\frac{F_{n+2}^3}{F_{n+1}^3} < \frac{e_1}{e_2} < \frac{F_{n+1}^3}{F_n^3}. \quad (63)$$

Similarly, one can prove that

$$\begin{aligned} & \frac{F_{n+2}^3}{F_{n+1}^3} < \frac{e_2}{e_3} < \frac{F_{n+1}^3}{F_n^3}, \\ & \frac{F_{n+2}^3}{F_n^3} < \frac{e_1}{e_3} < \frac{F_{n+1}^3}{F_{n-1}^3}. \end{aligned} \quad (64)$$

Case 2. Let $A_1 < 0$, $A_2 < 0$, and $A_3 < 0$. By using (51) and (53), we have

$$\begin{aligned} & \frac{e_2 [F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]}{e_1 A_1} + \frac{[F_n^3 F_{n-2}^3] - (F_{n-1}^3)^2}{A_1} > \frac{e_3}{e_1}, \\ & \frac{e_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]}{e_1 A_3} + \frac{[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3]}{A_3} < \frac{e_3}{e_1}. \end{aligned} \quad (65)$$

Then, we get

$$\frac{e_2 [F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]}{e_1 A_1} + \frac{[F_n^3 F_{n-2}^3] - (F_{n-1}^3)^2}{A_1} > \frac{e_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]}{e_1 A_3} + \frac{[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3]}{A_3}. \quad (66)$$

So,

$$\frac{A_3 (F_n^3 F_{n-2}^3 - (F_{n-1}^3)^2) - A_1 \left((F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right)}{A_1 A_3} > \frac{e_2 (A_1 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3] - A_3 (F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3))}{e_1 A_1 A_3}. \quad (67)$$

By using (43), we have

$$\begin{aligned} & A_3 (F_n^3 F_{n-2}^3 - (F_{n-1}^3)^2) - A_1 \left((F_n^3)^2 - F_{n+1}^3 F_{n-1}^3 \right) = F_n^3 \det Q_{(n,3)}, \\ & A_1 ([F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3] - A_3 (F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3)) = F_{n+1}^3 \det Q_{(n,3)}. \end{aligned} \quad (68)$$

Therefore, we have

$$\frac{e_1}{e_2} > \frac{F_{n+1}^3}{F_n^3}. \quad (69)$$

In the following, by relations (52) and (53), we get

$$\begin{aligned} \frac{e_2 \left[(F_{n+2}^3 F_{n-2}^3 - F_n^3)^2 \right]}{e_1 A_2} + \frac{[F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]}{A_2} &> \frac{e_3}{e_1}, \\ \frac{e_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]}{e_1 A_3} + \frac{[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3]}{A_3} &< \frac{e_3}{e_1}. \end{aligned} \quad (70)$$

$$\begin{aligned} \frac{e_2 \left[(F_{n+2}^3 F_{n-2}^3 - F_n^3)^2 \right]}{e_1 A_2} + \frac{[F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]}{A_2} &> \frac{e_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]}{e_1 A_3} + \frac{[(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3]}{A_3}, \\ \frac{A_3 ([F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]) - A_2 \left([(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3] \right)}{A_2 A_3} &> \frac{e_2 (A_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3]) - A_3 (F_{n+2}^3 F_{n-2}^3 - (F_n^3)^2)}{e_1 A_2 A_3}. \end{aligned} \quad (71)$$

We get

$$\begin{aligned} A_3 ([F_n^3 F_{n-1}^3 - F_{n+1}^3 F_{n-2}^3]) - A_2 \left([(F_n^3)^2 - F_{n+1}^3 F_{n-1}^3] \right) &= -F_{n+1}^3 \det Q_{(n,3)}, \\ A_2 [F_{n+2}^3 F_{n-1}^3 - F_n^3 F_{n+1}^3] - A_3 (F_{n+2}^3 F_{n-2}^3 - (F_n^3)^2) &= F_{n+2}^3 \det Q_{(n,3)}. \end{aligned} \quad (72)$$

So,

$$\frac{e_1}{e_2} < \frac{F_{n+2}^3}{F_{n+1}^3}. \quad (73)$$

By using relations (69) and (73), we have

$$\frac{F_{n+1}^3}{F_n^3} < \frac{e_1}{e_2} < \frac{F_{n+2}^3}{F_{n+1}^3}. \quad (74)$$

Similarly, we get

$$\begin{aligned} \frac{F_{n+1}^3}{F_n^3} &< \frac{e_2}{e_3} < \frac{F_{n+2}^3}{F_{n+1}^3}, \\ \frac{F_{n+1}^3}{F_{n-1}^3} &< \frac{e_1}{e_3} < \frac{F_{n+2}^3}{F_n^3}. \end{aligned} \quad (75)$$

Similarly, by the above argument, we get

Then, we have

$$\frac{F_{n+1}^3}{F_n^3} \leq \frac{e_1}{e_2} \leq \frac{F_{n+2}^3}{F_{n+1}^3}, \quad (76)$$

$$\frac{F_{n+1}^3}{F_n^3} \leq \frac{e_2}{e_3} \leq \frac{F_{n+2}^3}{F_{n+1}^3}, \quad (77)$$

$$\frac{F_{n+1}^3}{F_{n-1}^3} \leq \frac{e_1}{e_3} \leq \frac{F_{n+2}^3}{F_n^3}. \quad (78)$$

So, by above facts, we have

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}^3}{F_n^3} = \lim_{n \rightarrow \infty} \frac{F_{n+2}^3}{F_{n+1}^3} = \lambda,$$

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}^3}{F_{n-1}^3} = \lim_{n \rightarrow \infty} \frac{F_{n+2}^3}{F_n^3} = \lambda^2,$$

$$\frac{e_1}{e_2} \approx \lambda, \quad (79)$$

$$\frac{e_2}{e_3} \approx \lambda,$$

$$\frac{e_1}{e_3} \approx \lambda^2,$$

where $\lambda = 1.839$ is the golden ratio of the 3-Fibonacci sequence.

Hence, we have

$$\begin{aligned}
 \frac{e_4}{e_5} &\approx \lambda, \\
 \frac{e_5}{e_6} &\approx \lambda, \\
 \frac{e_4}{e_6} &\approx \lambda^2, \\
 \frac{e_7}{e_8} &\approx \lambda, \\
 \frac{e_8}{e_9} &\approx \lambda, \\
 \frac{e_7}{e_9} &\approx \lambda^2.
 \end{aligned} \tag{80}$$

Now, we are in position that the above results are generalized to the k -Fibonacci coding.

Here, for $k > 3$, we get relations among entries the code matrix E . Similar to $k = 3$, we can obtain the following relations among the first-row entries code matrix E .

$$\begin{aligned}
 \frac{e_1}{e_2} &\approx \mu_k, \quad \frac{e_2}{e_3} \approx \mu_k, \quad \dots, \quad \frac{e_{k-1}}{e_k} \approx \mu_k, \\
 \frac{e_1}{e_3} &\approx \mu_k^2, \quad \frac{e_2}{e_4} \approx \mu_k^2, \quad \dots, \quad \frac{e_{k-2}}{e_k} \approx \mu_k^2, \\
 &\vdots \\
 \frac{e_1}{e_k} &\approx \mu_k^{k-1},
 \end{aligned} \tag{81}$$

where μ_k is the golden ratio of the k -Fibonacci sequence.

In general, the following relations among the entries of each row of the code matrix $E = (e_{ij})_{k \times k}$ are obtained by

$$\frac{F_{n+k-2}^k}{F_{n+k-s-2}^k} \leq \frac{e_{ij}}{e_{ij+s}} \leq \frac{F_{n+k-1}^k}{F_{n+k-s-1}^k}, \tag{82}$$

where $i, j = 1, 2, \dots, k, s = 1, 2, \dots, k-1$ and $2 \leq j+s \leq k, n > k$.

Now, we calculate the determinant of the code matrix E .

For the coding matrix $Q_{(n,k)}$, $E = M \times Q_{(n,k)}$ and $\det Q_{(n,k)} = -1$, and we have

$$\det E = -\det M. \tag{83}$$

Here, we calculate the error detection and correction for the k -Fibonacci coding.

Let $k = 3$. According to the matrix E of the order 3×3 , we have “single,” “double,” ..., “nine-fold” errors. The first assumption is that there exists only one error in the matrix E received from the communication channel. It is clear that there are nine different cases for it, as follows:

$$\begin{aligned}
 (1) & \begin{bmatrix} a & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (2) & \begin{bmatrix} e_1 & b & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (3) & \begin{bmatrix} e_1 & e_2 & c \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (4) & \begin{bmatrix} e_1 & e_2 & e_3 \\ d & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (5) & \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (6) & \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & f \\ e_7 & e_8 & e_9 \end{bmatrix}, \\
 (7) & \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ g & e_8 & e_9 \end{bmatrix}, \\
 (8) & \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & h & e_9 \end{bmatrix}, \\
 (9) & \begin{bmatrix} e_1 & e_2 & e_1 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & i \end{bmatrix},
 \end{aligned} \tag{84}$$

where $a, b, \dots, i$ are possible “destroyed” entries.

From $\det(E) = -\det(M)$, we have

$$\begin{aligned}
 (1) & a(e_5e_9 - e_6e_8) - e_2(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = -\det M, \\
 (2) & e_1(e_5e_9 - e_6e_8) - b(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = -\det M, \\
 & \vdots \\
 (9) & e_1(e_5i - e_6e_8) - e_2(e_4i - e_6e_7) + e_3(e_4e_8 - e_5e_7) = -\det M.
 \end{aligned} \tag{85}$$

In a similar way, we will obtain a double error for the matrix E . For example, we consider a bivariate case for matrix E as follows:

$$\begin{bmatrix} a & b & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \tag{86}$$

in which possible cases are $2C_9 = 36$. Similarly, we obtain “triple,” “four-fold,” ..., “nine-fold” errors, which the total number of cases is $1C_9 + 2C_9 + \dots + 9C_9 = 2^9 - 1 = 511$

errors. By using $\det E = -\det M$ and the relations (76)–(78), we can correct up to “single,” “double,” . . . , “eight” errors except “nine” errors. Therefore, we get that the correctable possibility of the method is equal to $510/511 = 0.9980 = \%99.80$.

In general, for $k > 3$, by the above method, the code matrix E may contain “single,” “double,” . . . , “ k^2 – fold” errors. Thus,

$$1C_{k^2} + 2C_{k^2} + \dots + k^2 C_{k^2} = 2^{k^2} - 1. \quad (87)$$

Therefore, there are $2^{k^2} - 1$ errors. By $\det E = -\det M$ and relation (82), we can give that the correct ability of the k –Fibonacci sequence matrix coding is equal to

$$\frac{2^{(k)^2} - 2}{2^{(k)^2} - 1}. \quad (88)$$

Then, for large value of k , the correct possibility of this method is

$$\frac{2^{(k)^2} - 2}{2^{(k)^2} - 1} \simeq 1 = \%100. \quad (89)$$

5. Conclusion

In this paper, we give some codes on the k –Fibonacci sequence. These coding methods are the applications of this sequence. Also, we obtain the following results:

- (1) For $u = 4, 5$ and 7 , we get error detection and correction.
- (2) For $k > 3$, the correct ability of the k –Fibonacci sequence matrix coding is $2^{(k)^2} - 2/2^{(k)^2} - 1 \simeq \%100$. In particular, for $k = 3$, the correct ability of this method is equal to $510/511 = \%99.80$.

Data Availability

There are no applications, analysis, or generation during the study. The results are related to the Ph. D. thesis.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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