Research Article

Three Topological Indices of Two New Variants of Graph Products

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Graph operations play an important role to constructing complex network structures from simple graphs, and these complex networks play vital roles in different fields such as computer science, chemistry, and social sciences. Computation of topological indices of these complex network structures via graph operation is an important task. In this study, we defined two new variants of graph products, namely, corona join and subdivision vertex join products and investigated exact expressions of the first and second Zagreb indices and first reformulated Zagreb index for these new products.

1. Introduction

In mathematics, the graph theory is the study of graphs which are mathematical structures used to model pairwise connection between objects. The graph theory is applied in the various fields such as computer science, biology, chemistry, social sciences, and operation research [1, 2].

Let $G = (V(G), E(G))$ be a simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and number of edges are called the order n and size m , respectively, of the graph G . A graph of order n and size m will be denoted by $G(n, m)$. For any vertex $v \in V(G)$, the degree of vertex v is the number of edges incident on the vertex v , and it is written as $d_G(v)$ or simply $d(v)$. For a simple graph G , the subdivision of the graph G is denoted by $S(G)$ and obtained by inserting a new vertex on every edge of G .

Topological index is a numeric value which is associated with a chemical structure of a certain chemical compound. This numeric value can help to predict certain properties of that chemical compound. Hundreds of topological indices have been introduced, but few of them gain attention of the

scientific community. Zagreb indices are among the oldest and useful topological indices. For a graph G , the first and second Zagreb indices are defined as

$$\begin{aligned} M_1(G) &= \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)], \\ M_2(G) &= \sum_{uv \in E(G)} d_G(u)d_G(v). \end{aligned} \quad (1)$$

In 1972, these topological indices were applied for the first time to find the total electron π -energy of molecular graphs [3]. Later, the Zagreb indices developed important applications in QSPR/QSAR studies, and a lot of research studies have been published on these [4–10].

Milićević et al. in 2004 reformulated the Zagreb indices in terms of edge degree which is defined as

$$\begin{aligned} EM_1(G) &= \sum_{e \in E(G)} d(e)^2, \\ EM_2(G) &= \sum_{e \sim f} d(e)d(f), \end{aligned} \quad (2)$$

where $d(e)$ shows the degree of the edge e in G , which is defined as $d(e) = d(u) + d(v) - 2$ with edge $e = uv$ and $e \sim f$ shows that the edge e and f are adjacent [11].

Graph operations, especially graph products, play a significant role not only in pure and applied mathematics but also in computer science, chemistry, electrical engineering, and pharmaceuticals. For instance, the Cartesian product provides a significant model for connecting computers [12].

Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two connected simple graphs. Corona product of graphs G_1 and G_2 , denoted by $G_1 \odot G_2$, is obtained by taking one copy of G_1 and n_1 copies of G_2 , and joining each vertex of i^{th} copy of G_2 to i^{th} vertex of G_1 [8]. In [13], authors introduced two variants of the corona product and discussed their spectral properties. The subdivision vertex variant of corona of G_1 and G_2 is attained from $S(G_1)$ and n_1 copies of G_2 by joining the i^{th} vertex of $V(G_1)$ to every vertex in the i^{th} copy of G_2 . Similarly, the subdivision-edge neighborhood corona is obtained by attaching the neighbors of the i^{th} vertex of $V(G_1)$ to every vertex in the i^{th} copy of G_2 .

The join graph of G_1 and G_2 is obtained by joining each vertex of G_1 to each vertex of G_2 , and it is denoted by $G_1 + G_2$ [14].

Khalifeh et al. [15] computed the first and second Zagreb indices of Cartesian product, composition, join, disjunction, and symmetric difference of graphs and applied the results on C_4 tube, torus, and multiwalled polyhex nanotorus. Authors in [16] investigated the upper bounds on the multiplicative Zagreb indices of some product of graphs. Azari and Iranmanesh [17] discussed the rooted product of graphs and found the exact expression of first and second Zagreb indices for this product. Jamil and Tomescu [18] found the exact formulas of the first reformulated Zagreb index for Cartesian product, composition, join, corona product, splice, link, and chain of graphs. Some graph operations and their topological indices are presented in [13, 15–28].

Now, we define variants of these graphs' product.

Definition 1. Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be simple connected graphs, and the corona join product of G_1 and G_2 is

obtained by taking one copy of G_1 , n_1 copies of G_2 , and joining each vertex of the i^{th} copy of G_2 with all vertices of G_1 . The corona join product of G_1 and G_2 is denoted by $G_1 \oplus G_2$ and shown in Figure 1.

Definition 2. For $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$, the subdivision vertex join is denoted by $G_1 \dot{+} G_2$ and obtained by joining the each new vertex of $S(G_1)$ to all vertices of G_2 . Figure 2 shows the illustration of subdivision vertex join for P_3 and P_2 .

2. Main Results

In this section, we present the main results. The following lemmas are useful to obtain the exact expressions of topological indices of new variants of graph products. The proofs of the following two lemmas are directly from the definitions of corona join product $G_1 \oplus G_2$ and subdivision vertex join $G_1 \dot{+} G_2$.

Lemma 1. Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two graphs; then, the degree behavior of vertices in the graph $G_1 \oplus G_2$ is

$$d_{G_1 \oplus G_2}(v) = \begin{cases} d_{G_1}(v) + n_1 n_2, & \text{if } v \in V(G_1), \\ d_{G_2}(v) + n_1, & \text{if } v \in V(G_2). \end{cases} \quad (3)$$

Lemma 2. Let we have three simple connected graphs $G_1 = (n_1, m_1)$ and $S(G_1) = (n'_1, m'_1)$; then, the degree behavior of vertices in the graph $G_1 \dot{+} G_2$ is

$$d_{G_1 \dot{+} G_2}(v) = \begin{cases} d_{G_1}(v), & \text{if } v \in V(G_1), \\ 2 + n_2, & \text{if } v \in V_s(G_1), \\ d_{G_2}(v) + m_1, & \text{if } v \in V(G_2). \end{cases} \quad (4)$$

The following result gives the formula of the first Zagreb index for $G_1 \oplus G_2$.

Theorem 1. Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two simple graphs; then, the first Zagreb index of corona join product $G_1 \oplus G_2$ is given as

$$M_1(G_1 \oplus G_2) = M_1(G_1) + n_1 M_1(G_2) + n_1^3 n_2 (n_2 + 1) + 4n_1 (m_1 n_2 + m_2 n_1). \quad (5)$$

Proof. From the definition of the first Zagreb index, we have

$$M_1(G_1 \oplus G_2) = \sum_{v \in V(G_1 \oplus G_2)} d_{G_1 \oplus G_2}(v)^2. \quad (6)$$

FIGURE 1: Corona join product $G_1 \oplus G_2$. FIGURE 2: Subdivision vertex join graph $G_1 + G_2$.

Now, we apply Lemma 1:

$$\begin{aligned}
 &= \sum_{v \in V(G_1)} (d_{G_1}(v) + n_1 n_2)^2 + \sum_{v \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_2}(v) + n_1)^2 \\
 &= \sum_{v \in V(G_1)} (d_{G_1}(v)^2 + (n_1 n_2)^2 + 2d_{G_1}(v)(n_1 n_2)) + \\
 &\quad \sum_{v \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_2}(v)^2 + n_1^2 + 2d_{G_2}(v)(n_1)) \\
 &= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + (n_1 n_2)^2 \sum_{v \in V(G_1)} 1 + 2(n_1 n_2) \sum_{v \in V(G_1)} d_{G_1}(v) + \sum_{v \in V(G_1)} 1 \\
 &\quad \sum_{v \in V(G_2)} d_{G_2}(v)^2 + n_1^2 \sum_{v \in V(G_1)} 1 \sum_{v \in V(G_2)} 1 + 2n_1 \sum_{v \in V(G_1)} 1 \sum_{v \in V(G_2)} d_{G_2}(v) \\
 &= M_1(G_1) + n_1^3 n_2^2 + 4m_1 n_1 n_2 + n_1 M_1(G_2) + n_1^3 n_2 + 4m_2 n_1^2 \\
 &= M_1(G_1) + n_1 M_1(G_2) + n_1^3 n_2 (n_2 + 1) + 4n_1 (m_1 n_2 + m_2 n_1),
 \end{aligned} \tag{7}$$

which is our required result.

The next theorem is about the exact expression of the second Zagreb index for $G_1 \oplus G_2$. $\square$

Theorem 2. For simple graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$, the second Zagreb index of corona join product $G_1 \oplus G_2$ is given as

$$\begin{aligned}
 M_2(G_1 \oplus G_2) &= n_1 [n_2 M_1(G_1) + n_1 M_1(G_2)] \\
 &\quad + M_2(G_1) + n_1 M_2(G_2) + m_2 n_1 (n_1^2 + 4m_1) \\
 &\quad + n_1^2 n_2 [m_1 n_2 + 2m_1 + 2m_2 n_1 + n_1^2 n_2].
 \end{aligned} \tag{8}$$

Proof. From the definition of the second Zagreb index, we have

$$M_2(G_1 \oplus G_2) = \sum_{uv \in E(G_1 \oplus G_2)} d_{G_1 \oplus G_2}(u) \cdot d_{G_1 \oplus G_2}(v). \quad (9)$$

Now, we apply the Lemma 1:

$$\begin{aligned}
&= \sum_{uv \in E(G_1)} (d_{G_1}(u) + n_1 n_2) (d_{G_1}(v) + n_1 n_2) \\
&\quad + \sum_{v \in V(G_1)} \sum_{uv \in E(G_2)} (d_{G_2}(u) + n_1) (d_{G_2}(v) + n_1) \\
&+ \sum_{v \in V(G_1)} \sum_{uv \in E(G_1 \oplus G_2)} \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_1}(u) + n_1 n_2) \\
&\quad (d_{G_2}(v) + n_1) \\
&= \sum_{uv \in E(G_1)} d_{G_1}(u) d_{G_1}(v) + n_1 n_2 \sum_{uv \in E(G_1)} [d_{G_1}(u) + d_{G_1}(v)] + (n_1 n_2)^2 \\
&\quad \sum_{uv \in E(G_1)} 1 + n_1 \sum_{uv \in E(G_2)} d_{G_2}(u) d_{G_2}(v) + n_1^2 \sum_{uv \in E(G_2)} [d_{G_2}(u) + d_{G_2}(v)] + n_1^3 \\
&\quad \sum_{uv \in E(G_2)} 1 + n_1 \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} d_{G_2}(v) + n_1^2 \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} 1 + n_1^2 n_2 \sum_{v \in V(G_2)} d_{G_2}(v) \\
&\quad \sum_{u \in V(G_1)} 1 + n_1^3 n_2 \sum_{u \in V(G_1)} 1 \sum_{v \in V(G_2)} 1 \\
&= M_2(G_1) + n_1 n_2 M_1(G_1) + m_1 n_1^2 n_2^2 + n_1 M_2(G_2) + n_1^2 M_1(G_2) + m_2 n_1^3 \\
&\quad + 4m_1 m_2 n_1 + 2n_1^2 n_2 m_1 + 2n_1^3 n_2 m_2 + n_1^4 n_2^2 \\
&= n_1 [n_2 M_1(G_1) + n_1 M_1(G_2)] + M_2(G_1) + n_1 M_2(G_2) + m_2 n_1 (n_1^2 + 4m_1) + \\
&\quad n_1^2 n_2 [m_1 n_2 + 2m_1 + 2m_2 n_1 + n_1^2 n_2],
\end{aligned} \quad (10)$$

which is our required result.

□ **Theorem 3.** Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two simple graphs; then, the first reformulated Zagreb index of corona join product $G_1 \oplus G_2$ is

$$\begin{aligned}
&EM_1(G_1 \oplus G_2) \\
&= 5n_1 [n_2 M_1(G_1) + n_1 M_1(G_2)] + EM_1(G_1) + n_1 [EM_1(G_2) + 8m_1 m_2] \\
&\quad + 4m_1 n_1 n_2 (n_1 n_2 - 2) + 4m_2 n_1^2 (n_1 - 2) + (n_1 + n_1 n_2 - 2) \\
&\quad [4m_1 n_1 n_2 + 4m_2 n_1^2 + n_1^2 n_2 (n_1 + n_1 n_2 - 2)].
\end{aligned} \quad (11)$$

Proof. From the definition of the first reformulated Zagreb index and by Lemma 1, we have

$$\begin{aligned}
EM_1(G_1 \oplus G_2) &= \sum_{uv \in E(G_1 \oplus G_2)} [d_{G_1 \oplus G_2}(u) + d_{G_1 \oplus G_2}(v) - 2]^2 \\
&= \sum_{uv \in E(G_1)} [d_{G_1}(u) + n_1 n_2 + d_{G_1}(v) + n_1 n_2 - 2]^2 + n_1 \sum_{uv \in E(G_2)} [d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1 - 2]^2 \\
&\quad + n_1 \sum_{\substack{uv \in E(G_1 \oplus G_2) \\ u \in V(G_1) \\ v \in V(G_2)}} [d_{G_1}(u) + n_1 n_2 + d_{G_2}(v) + n_1 - 2]^2 \\
&= \sum_{uv \in E(G_1)} [(d_{G_1}(u) + d_{G_1}(v) - 2) + 2n_1 n_2]^2 + n_1 \sum_{uv \in E(G_2)} [(d_{G_2}(u) + d_{G_2}(v) - 2) + 2n_1]^2 \\
&\quad + n_1 \sum_{\substack{uv \in E(G_1 \oplus G_2) \\ u \in V(G_1) \\ v \in V(G_2)}} [(d_{G_1}(u) + d_{G_2}(v)) + (n_1 + n_1 n_2 - 2)]^2 \\
&= \sum_{uv \in E(G_1)} [(d_{G_1}(u) + d_{G_1}(v) - 2)^2 + 4n_1^2 n_2^2 + 4n_1 n_2 (d_{G_1}(u) + d_{G_1}(v) - 2)] \\
&\quad + n_1 \sum_{uv \in E(G_2)} [(d_{G_2}(u) + d_{G_2}(v) - 2)^2 + 4n_1^2 + 4n_1 (d_{G_2}(u) + d_{G_2}(v) - 2)] \\
&\quad + n_1 \sum_{uv \in E(G_1 \oplus G_2)} \sum_{\substack{u \in V(G_1) \\ v \in V(G_2)}} [(d_{G_1}(u) + d_{G_2}(v))^2 + (n_1 + n_1 n_2 - 2)^2 + 2(n_1 + n_1 n_2 - 2) \\
&\quad (d_{G_1}(u) + d_{G_2}(v))] \\
&= \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 + 4n_1^2 n_2^2 \sum_{uv \in E(G_1)} 1 + 4n_1 n_2 \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) \\
&\quad - 8n_1 n_2 \sum_{uv \in E(G_1)} 1 + n_1 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 + 4n_1^3 \sum_{uv \in E(G_2)} 1 \\
&\quad + 4n_1^2 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) - 8n_1^2 \sum_{uv \in E(G_2)} 1 + n_1 \sum_{\substack{uv \in E(G_1 \oplus G_2) \\ u \in V(G_1) \\ v \in V(G_2)}} (d_{G_1}(u) + d_{G_2}(v))^2 \\
&\quad + n_1 (n_1 + n_1 n_2 - 2)^2 \sum_{\substack{uv \in E(G_1 \oplus G_2) \\ u \in V(G_1) \\ v \in V(G_2)}} 1 + 2n_1 (n_1 + n_1 n_2 - 2) \sum_{\substack{uv \in E(G_1 \oplus G_2) \\ u \in V(G_1) \\ v \in V(G_2)}} (d_{G_1}(u) + d_{G_2}(v)) \\
&= \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^2 + 4n_1^2 n_2^2 \sum_{uv \in E(G_1)} 1 + 4n_1 n_2 \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) \\
&\quad - 8n_1 n_2 \sum_{uv \in E(G_1)} 1 + n_1 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v) - 2)^2 + 4n_1^3 \sum_{uv \in E(G_2)} 1 \\
&\quad + 4n_1^2 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) - 8n_1^2 \sum_{uv \in E(G_2)} 1 + n_1 \sum_{u \in V(G_1)} d_{G_1}(u)^2 \sum_{v \in V(G_2)} 1 + \\
&\quad n_1 \sum_{v \in V(G_2)} d_{G_2}(v)^2 \sum_{u \in V(G_1)} 1 + 2n_1 \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} d_{G_2}(v) + n_1 (n_1 + n_1 n_2 - 2)^2 \\
&\quad \sum_{u \in V(G_1)} 1 \sum_{v \in V(G_2)} 1 + 2n_1 (n_1 + n_1 n_2 - 2) \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} 1 + 2n_1 (n_1 + n_1 n_2 - 2) \\
&\quad \sum_{v \in V(G_2)} d_{G_2}(v) \sum_{u \in V(G_1)} 1
\end{aligned}$$

$$\begin{aligned}
&= EM_1(G_1) + 4n_1^2 n_2^2 m_1 + 4n_1 n_2 M_1(G_1) - 8n_1 n_2 m_1 + n_1 EM_1(G_1) + 4n_1^3 m_2 \\
&\quad + 4n_1^2 M_1(G_2) - 8n_1^2 m_2 + n_1 n_2 M_1(G_1) + n_1^2 M_1(G_2) + 8m_1 m_2 n_1 + n_1^2 n_2 (n_1 + n_1 n_2 - 2)^2 \\
&\quad + 4m_1 n_1 n_2 (n_1 + n_1 n_2 - 2) + 4m_2 n_1^2 (n_1 + n_1 n_2 - 2) \\
&= 5n_1 [n_2 M_1(G_1) + n_1 M_1(G_2)] + EM_1(G_1) + n_1 [EM_1(G_2) + 8m_1 m_2] \\
&\quad + 4m_1 n_1 n_2 (n_1 n_2 - 2) + 4m_2 n_1^2 (n_1 - 2) + (n_1 + n_1 n_2 - 2) \\
&\quad [4m_1 n_1 n_2 + 4m_2 n_1^2 + n_1^2 n_2 (n_1 + n_1 n_2 - 2)],
\end{aligned} \tag{12}$$

hence the required result.

In the following results, we discussed the Zagreb indices of the subdivision vertex join graph. $\square$

Theorem 4. For simple graphs $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$, the first Zagreb index of subdivision vertex join $G_1 \dot{+} G_2$ is given as

$$M_1(G_1 \dot{+} G_2) = M_1(G_1) + M_1(G_2) + n_2(n_2 + 2)^2 + 4m_1 m_2 + n_2 m_1^2. \tag{13}$$

Proof. Applying Lemma 2 in the definition of first Zagreb index, we have the following:

$$\begin{aligned}
M_1(G_1 \dot{+} G_2) &= \sum_{v \in V(G_1 \dot{+} G_2)} d_{G_1 \dot{+} G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V_s(G_1)} (2 + n_2)^2 + \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V_s(G_1)} (2 + n_2)^2 + \sum_{v \in V(G_2)} (d_{G_2}(v)^2 + m_1^2 + 2m_1 d_{G_2}(v)) \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + (2 + n_2)^2 \sum_{v \in V_s(G_1)} 1 + \sum_{v \in V(G_2)} d_{G_2}(v)^2 + m_1^2 \sum_{v \in V(G_2)} 1 \\
&\quad + 2m_1 \sum_{v \in V(G_2)} d_{G_2}(v) \\
&= M_1(G_1) + m_1(n_2 + 2)^2 + M_1(G_2) + n_2 m_1^2 + 4m_1 m_2 \\
&= M_1(G_1) + M_1(G_2) + m_1(n_2 + 2)^2 + 4m_1 m_2 + n_2 m_1^2,
\end{aligned} \tag{14}$$

which is our required result. $\square$

Theorem 5. Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ are two simple graphs and $G_1 \dot{+} G_2$ be their subdivision vertex join graph; then, the second Zagreb index of $G_1 \dot{+} G_2$ is given as

$$\begin{aligned}
M_2(G_1 \dot{+} G_2) &= \\
&= m_1 M_1(G_2) + M_2(G_2) + m_1^2 m_2 + m_1(2 + n_2)(2m_2 + m_1 n_2) \\
&\quad + (2 + n_2) \sum_{uv \in E(S(G_1))} d_{G_1}(u).
\end{aligned} \tag{15}$$

Proof. From the definition of the second Zagreb index, we have

$$M_2(G_1 \dot{+} G_2) = \sum_{uv \in E(G_1 \dot{+} G_2)} d_{G_1 \dot{+} G_2}(u) \cdot d_{G_1 \dot{+} G_2}(v). \quad (16)$$

Now, we apply Lemma 2:

$$\begin{aligned} &= \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} [d_{G_1}(u)(2+n_2)] + \sum_{uv \in E(G_2)} [(d_{G_2}(u) + m_1)(d_{G_2}(v) + m_1)] \\ &\quad + \sum_{\substack{uv \in E(G_1 \dot{+} G_2) \\ u \in V_s(G_1) \\ v \in V(G_2)}} [(2+n_2)(d_{G_2}(v) + m_1)] \\ &= (2+n_2) \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} d_{G_1}(u) + \sum_{uv \in E(G_2)} [d_{G_2}(u)d_{G_2}(v) + m_1(d_{G_2}(u) + d_{G_2}(v)) + m_1^2] \\ &\quad + (2+n_2) \sum_{u \in V_s(G_1)} 1 \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1) \\ &= (2+n_2) \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} d_{G_1}(u) + \sum_{uv \in E(G_2)} d_{G_2}(u)d_{G_2}(v) + m_1 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) \\ &\quad + m_1^2 \sum_{uv \in E(G_2)} 1 + (2+n_2) \sum_{u \in V_s(G_1)} 1 \left(\sum_{v \in V(G_2)} d_{G_2}(v) + m_1 \sum_{v \in V(G_2)} 1 \right) \\ &= (2+n_2) \sum_{uv \in E(S(G_1))} d_{G_1}(u) + M_2(G_2) + m_1 M_1(G_2) + m_1^2 m_2 \\ &\quad + m_1(2+n_2)(2m_2 + m_1 n_2) \\ &= m_1 M_1(G_2) + M_2(G_2) + m_1^2 m_2 + m_1(2+n_2)(2m_2 + m_1 n_2) + (2+n_2) \sum_{uv \in E(S(G_1))} d_{G_1}(u). \end{aligned} \quad (17)$$

Theorem 6. Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ are two graphs; then, the first reformulated Zagreb index of subdivision vertex join $G_1 + G_2$ is given as

$$\begin{aligned} EM_1(G_1 \dot{+} G_2) &= \\ &= 5m_1 M_1(G_2) + EM_1(G_2) + m_1 n_2 (n_2^2 + m_1^2) + n_2^2 (2m_1^2 + m_1') \\ &\quad + 4m_1 m_2 (2m_1 + n_2 - 2) + \sum_{uv \in E(S(G_1))} [d_{G_1}(u)^2 + 2n_2 d_{G_1}(u)]. \end{aligned} \quad (18)$$

Proof. From the definition of the first reformulated Zagreb index, we have

Now, we apply Lemma 2:

$$EM_1(G_1+G_2) = \sum_{uv \in E(G_1+G_2)} [d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2]^2. \quad (19)$$

$$\begin{aligned}
&= \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} [d_{G_1}(u) + 2 + n_2 - 2]^2 + \sum_{uv \in E(G_2)} [d_{G_2}(u) + m_1 + d_{G_2}(v) + m_1 - 2]^2 \\
&\quad + \sum_{\substack{uv \in E(G_1+G_2) \\ u \in V_s(G_1) \\ v \in V(G_2)}} [2 + n_2 + d_{G_2}(v) + m_1 - 2]^2 \\
&= \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} [d_{G_1}(u) + n_2]^2 + \sum_{uv \in E(G_2)} [(d_{G_2}(u) + d_{G_2}(v) - 2) + 2m_1]^2 \\
&\quad + \sum_{\substack{uv \in E(G_1+G_2) \\ u \in V_s(G_1) \\ v \in V(G_2)}} [n_2 + (d_{G_2}(v) + m_1)]^2 \\
&= \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} [d_{G_1}(u)^2 + n_2^2 + 2n_2d_{G_1}(u)] + \sum_{uv \in E(G_2)} [(d_{G_2}(u) + d_{G_2}(v) - 2)^2 \\
&\quad + (2m_1)^2 + 2(2m_1)(d_{G_2}(u) + d_{G_2}(v) - 2) + \sum_{\substack{uv \in E(G_1+G_2) \\ u \in V_s(G_1) \\ v \in V(G_2)}} [n_2^2 + (d_{G_2}(v) + m_1)^2 + 2n_2(d_{G_2}(v) + m_1)] \quad (20) \\
&= \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} d_{G_1}(u)^2 + n_2^2 \sum_{uv \in E(S(G_1))} 1 + 2n_2 \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} d_{G_1}(u) + \sum_{uv \in E(G_2)} d(e)^2 + \\
&\quad 4m_1^2 \sum_{uv \in E(G_2)} 1 + 4m_1 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) - 8m_1 \sum_{uv \in E(G_2)} 1 + n_2^2 \sum_{u \in V_s(G_1)} 1 \\
&\quad \sum_{v \in V(G_2)} 1 + \sum_{u \in V_s(G_1)} 1 \sum_{v \in V(G_2)} d_{G_2}(v)^2 + m_1^2 \sum_{u \in V_s(G_1)} 1 \sum_{v \in V(G_2)} 1 + 2m_1 \sum_{u \in V_s(G_1)} 1 \\
&\quad \sum_{v \in V(G_2)} d_{G_2}(v) + 2n_2 \sum_{u \in V_s(G_1)} 1 \sum_{v \in V(G_2)} d_{G_2}(v) + 2m_1n_2 \sum_{u \in V_s(G_1)} 1 \sum_{v \in V(G_2)} 1 = \\
&\quad \sum_{\substack{uv \in E(S(G_1)) \\ u \in V(G_1) \\ v \in V_s(G_1)}} [d_{G_1}(u)^2 + 2n_2d_{G_1}(u)] + m_1'n_2^2 + EM_1(G_2) + 4m_1^2m_2 + 4m_1M_1(G_2) \\
&\quad - 8m_1m_2 + m_1n_2^3 + m_1M_1(G_2) + m_1^3n_2 + 4m_1^2m_2 + 4m_1m_2n_2 + 2m_1^2n_2^2 \\
&= 5m_1M_1(G_2) + EM_1(G_2) + m_1n_2(n_2^2 + m_1^2) + n_2^2(2m_1^2 + m_1') \\
&\quad + 4m_1m_2(2m_1 + n_2 - 2) + \sum_{uv \in E(S(G_1))} [d_{G_1}(u)^2 + 2n_2d_{G_1}(u)],
\end{aligned}$$

which is our required result. $\square$

3. Conclusion

In this study, we proposed two new variants of special graph products and found their exact expressions for the first Zagreb index, second Zagreb index, and first reformulated Zagreb index. These new graph invariants can be used to construct cellular networks, for example, the factor cellular network or some hybrid cellular networks. In future, other degree and distance-based topological indices of these graph operations can be found. The obtained results may help to construct and investigate the topological indices of complex networks structures.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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