Research Article

Reliability Sensitivity Analysis Method for Mechanical Components

Yan-Fang Zhang ¹ and Yan-Lin Zhang ²

¹Department of Basic Course, Institute of Disaster Prevention, Sanhe 065201, China

²North China Institute of Science & Technology, Sanhe 065201, China

Correspondence should be addressed to Yan-Fang Zhang; zyf@cidp.edu.cn

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Based on the univariate dimension-reduction method (UDRM), Edgeworth series, and sensitivity analysis, a new method for reliability sensitivity analysis of mechanical components is proposed. The univariate dimension-reduction method is applied to calculate the response origin moments and their sensitivity with respect to distribution parameters (e.g., mean and standard deviation) of fundamental input random variables. Edgeworth series is used to estimate failure probability of mechanical components by using first few response central moments. The analytic formula of reliability sensitivity can be derived by calculating partial derivative of the failure probability P_f with respect to distribution parameters of basic random variables. The nonnormal random parameters need not to be transformed into equivalent normal ones. Three numerical examples are employed to illustrate the accuracy and efficiency of the proposed method by comparing the failure probability and reliability sensitivity results obtained by the proposed method with those obtained by Monte Carlo simulation (MCS).

1. Introduction

The responses of mechanical components or engineering structures are often random due to random inputs of them including loads, material properties, and geometry. Many reliability analysis methods have been developed to calculate the failure probabilities of these random structures [1–4]. In reliability analysis, reliability sensitivity is defined as the partial derivative of the failure probability with respect to the distribution parameters (e.g., mean and standard deviation) of fundamental random input variables. Reliability sensitivity provides information about the importance of each input random variable to a structure's failure probability. Reliability sensitivity analysis methods can be divided into the numerical simulation method and approximate analytic calculation method.

The numerical simulation method can be divided into many different methods due to different sampling methods including importance sampling, direction sampling, line sampling, subset simulation, and low-discrepancy sampling

[5–10]. These different simulation methods are all based on Monte Carlo simulation but have different sampling methods. Although the program of these methods is not complicated, lots of samples are needed to get precise results because the structural failure probability is often very small and the computational efficiency is therefore reduced.

Analytic methods for reliability sensitivity analysis are always based on analytic reliability methods. These reliability analysis methods are the mean-value first-order reliability method (MVFORM)/mean-value second-order reliability method (MVSORM) [11, 12], JC method, mean-value first-order saddlepoint approximation (MVFOSA) [13], and moment method. FORM/SORM needs to expand the performance function into first-order or second-order Taylor series at the most likely failure point. The expansion is widely used because of its relatively moderate precision and efficiency. The corresponding reliability sensitivity calculation methods have been derived from these reliability methods [14–23]. Analytic methods for reliability sensitivity analysis always have higher calculation efficiency than MCS.

In this study, a feasible method to compute the sensitivity of failure probability with respect to distribution parameters of basic random input variables is proposed. The method can be applied to estimate failure probability and reliability sensitivity based on Edgeworth series. The proposed reliability sensitivity analysis method includes the following steps. (1) Choose random input variables and determine their distributions and distribution parameters. (2) Calculate the first four response origin moments and their sensitivity with respect to distribution parameters of random input variables by UDRM. (3) Calculate failure probability by the reliability method based on Edgeworth series. (4) Calculate partial derivatives of failure probability with respect to response central moments. (5) Calculate partial derivatives of response central moments with respect to response origin moments. (6) Calculate reliability sensitivity defined by partial derivatives of failure probability with respect to distribution parameters of input random variables.

2. Response Origin Moments and Response Sensitivity

2.1. Response Origin Moments by UDRM. Structures subject to random input vector $\mathbf{X} = [X_1, X_2, \Lambda, X_N]^T \in R^N$, which characterizes uncertainty in loads, material properties, and geometry. Let $Y(\mathbf{X})$ represent a response of interest that depends on independent random variables $\mathbf{X} = [X_1, X_2, \Lambda, X_N]^T$; then, the l^{th} order origin moment of $Y(\mathbf{X})$ can be written as follows:

$$m_l = E[Y^l(\mathbf{X})] = \int_{R^N} Y^l(\mathbf{X}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad (1)$$

where m_l is the l^{th} order origin moment of $Y(\mathbf{X})$, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of $\mathbf{X}$, and $E(\cdot)$ is the expectation operator. According to the univariate dimension reduction method proposed by Rahman and Xu [24–26], $Y(\mathbf{X})$ can be approximately written as follows:

$$Y(\mathbf{X}) \approx \hat{Y}(\mathbf{X}) = \hat{Y}(X_1, X_2, \Lambda, X_N) = \sum_{j=1}^N Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) - (N-1)Y(\mu_1, \Lambda, \mu_N), \quad (2)$$

where $Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N)$ is the response which depends on the j^{th} random input X_j , $Y(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N)$ is the response where $\mathbf{X} = [X_1, X_2, \Lambda, X_N]^T$, and $\mu_j = E(X_j)$ is the mean of X_j .

Then, the l^{th} order origin moment m_l of $Y(\mathbf{X})$ can be written as follows:

$$m_l \cong E[\hat{Y}^l(\mathbf{X})] = E\left\{\left[\sum_{j=1}^N Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N) - (N-1)Y(\mu_1, \Lambda, \mu_N)\right]^l\right\}. \quad (3)$$

Applying the binomial formula on the right-hand side of equation (3), the l^{th} response origin moment m_l can be written as follows:

$$m_l \cong \sum_{i=0}^l C_l^i E\left[\sum_{j=1}^N Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)\right]^i [- (N-1)Y(\mu_1, \Lambda, \mu_N)]^{l-i}, \quad (4)$$

where $C(\cdot)$ is the combination operator.

Define

$$S_j^i = E\left\{\left[\sum_{i=1}^j Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)\right]^i\right\}, \quad j = 1, \dots, N, i = 1, \dots, l. \quad (5)$$

Equation (5) can be expressed using the recursive formula as follows [23]:

$$\begin{aligned}
S_1^i &= E[Y^i(X_1, \mu_2, \Lambda, \mu_N)], \quad (i = 1, \dots, N), \\
S_2^i &= \sum_{k=0}^i C_i^k S_1^i E[Y^{i-k}(\mu_1, X_2, \Lambda, \mu_N)], \quad (i = 1, \dots, N), \\
S_j^i &= \sum_{k=0}^i C_i^k S_{j-1}^i E[Y^{i-k}(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)], \quad (i = 1, \dots, N), \\
S_N^i &= \sum_{k=0}^i C_i^k S_{N-1}^i E[Y^{i-k}(\mu_1, \Lambda, \mu_{N-1}, X_N)], \quad (i = 1, \dots, N).
\end{aligned} \tag{6}$$

Then, the l^{th} origin moment of $Y(\mathbf{X})$ can be written as follows:

$$m_l \equiv E[\hat{Y}^l(\mathbf{X})] = \sum_{i=0}^l C_l^i S_N^i [-(N-1)Y(\mu_1, \Lambda, \mu_{N-1}, \mu_N)]^{l-i}. \tag{7}$$

From equations (6) and (7), the l^{th} response origin moment m_l can be calculated out by using the origin moments of univariate functions $Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)$ ($j = 1, 2, \Lambda, N$). The q^{th} origin moment of univariate function $Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)$ ($j = 1, 2, \Lambda, N$) can be written as follows:

$$m_{Y_j}^q = E[Y^q(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)] = \int Y^q(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) f_{X_j}(x_j) dx_j, \tag{8}$$

where $f_{X_j}(x_j)$ is the probability density function of random variable X_j .

The q^{th} origin moment $m_{Y_j}^q$ in equation (8) can be calculated out by the numerical integration method and can be written as follows:

$$m_{Y_j}^q = \int Y^q(\mu_1, \Lambda, \mu_{j-1}, x_j, \mu_{j+1}, \Lambda, \mu_N) f(x_j) dx_j = \sum_{k=0}^n W_j^{(k)} Y^q(\mu_1, \Lambda, \mu_{j-1}, x_j^{(k)}, \mu_{j+1}, \Lambda, \mu_N), \tag{9}$$

where $W_j^{(k)}$ is the k^{th} integral weight, $x_j^{(k)}$ is the k^{th} integration point, and n is the number of integration points.

If x_j has different probability density (e.g., normal, lognormal, and Weibull), the corresponding integral weights and integration points in equation (9) are different. The numerical integration points and the corresponding orthogonal polynomial for different probability densities are listed in Table 1 [27].

2.2. Sensitivity of Response Moment with Respect to Distribution Parameters of Inputs. Calculate the partial derivative on both side of equation (1) with respect to distribution parameters T_j^i of random input variables $\mathbf{X} = [X_1, X_2, \Lambda, X_N]^T$. The partial derivative can be written as follows:

$$\begin{aligned}
\frac{\partial m_l}{\partial T_j^i} &= \frac{\partial E(Y^l(\mathbf{X}))}{\partial T_j^i} = \frac{\partial (\int_{R^N} Y^l(\mathbf{X}) f(\mathbf{X}) d\mathbf{x})}{\partial T_j^i} = \int_{R^N} Y^l(\mathbf{X}) \frac{1}{f_j(x)} \cdot \frac{\partial f_j(x)}{\partial T_j^i} f(\mathbf{X}) d\mathbf{x} \\
&= E \left(Y^l(\mathbf{X}) \frac{1}{f_j(x)} \cdot \frac{\partial f_j(x)}{\partial T_j^i} \right) \\
&\approx E \left\{ \left[\sum_{j=1}^N \hat{Y}_j(\mathbf{X}_j) - (N-1)Y(\mu_1, \Lambda, \mu_N) \right]^l \frac{1}{f_j(x)} \cdot \frac{\partial f_j(x)}{\partial T_j^i} \right\}, \quad j = 1, 2, \dots, N, i = 1, 2,
\end{aligned} \tag{10}$$

TABLE 1: Orthogonal polynomial and Gauss integration.

Distribution	Orthogonal polynomial	Integral formula
Normal	Hermite	$(1/\sqrt{\pi}) \sum_{k=0}^n W_j^{(k)} Y^q (\sqrt{2} \sigma t_j^{(k)} + \mu)$
Lognormal	Hermite	$(1/\sqrt{\pi}) \sum_{k=0}^n W_j^{(k)} Y^q (e^{\sqrt{2} \sigma_1 t_j^{(k)} + \mu_1})$
Uniform	Legendre	$(1/2) \sum_{k=0}^n W_l^{(k)} Y^q ((b-a/2)t_j^{(k)} + (a+b/2))$
Exponential	Laguerre	$(1/\lambda) \sum_{k=0}^n W_j^{(k)} Y^q (t_j^{(k)}/\lambda)$
Weibull	Laguerre	$\sum_{k=0}^n W_j^{(k)} Y^q (a t_j^{1/\beta})$

where T_j^i ($j = 1, 2, \dots, N$, $i = 1, 2$), is the i^{th} distribution parameter of the j^{th} random input variable ($i = 1$, represents the mean; $i = 2$ represents standard deviation), $f_j(x)$ is the

probability density function of the j^{th} random variable X_j , and $\hat{Y}_j(\mathbf{X}_j) = Y(\mu_1, \Lambda, \mu_{j-1}, X_j, \mu_{j+1}, \Lambda, \mu_N)$.

Applying the binomial formula on the right-hand side of equation (10), $\partial m_l / \partial T_j^i$ can be written as follows:

$$\frac{\partial m_l}{\partial T_j^i} \approx \sum_{z=0}^l C_l^z E \left\{ \left[\sum_{j=1}^N \hat{Y}_j(\mathbf{X}_j) \right]^z \frac{1}{f_j(x)} \cdot \frac{\partial f_j(x)}{\partial T_j^i} \right\} [-(N-1)Y(\mu_1, \Lambda, \mu_N)]^{l-z}, \quad (11)$$

where $\partial m_l / \partial T_j^i$ can be calculated out by equation (11) and recursive formula similar to equations (5) and (6).

Suppose

$$k_j^i = \frac{1}{f_j(x)} \cdot \frac{\partial f_j(x)}{\partial T_j^i}, \quad j = 1, 2, \dots, N, i = 1, 2, \quad (12)$$

where k_j^i is the kernel function of the i^{th} distribution parameter T_j^i of j^{th} the random input variable.

The kernel function in equation (12) of X_j with different probability density functions can be obtained as follows: here, x_j is denoted by x .

If X is a normal distributed variable, its probability density function can be written as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-((x-\mu)^2/2\sigma^2)}, \quad (13)$$

where μ is the mean of X , and σ is the standard deviation of X .

The kernel function for normal variable X with respect to its distribution parameters (mean and standard deviation) can be derived from equation (12) directly and can be written as follows:

$$\begin{aligned} k_\mu &= \frac{x-\mu}{\sigma^2}, \\ k_\sigma &= \frac{1}{\sigma} \left(\left(\frac{x-\mu}{\sigma} \right)^2 - 1 \right), \end{aligned} \quad (14)$$

where k_μ and k_σ are the kernel functions with respect to mean and standard deviation, respectively. μ and σ are the mean and standard deviation of normal variable X , respectively.

If X is a lognormal distributed variable, its probability density function can be written as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_1 x} e^{-((\ln x - \mu_1)^2/2\sigma_1^2)}, \quad (15)$$

where μ_1 is the mean of $\ln X$, and σ_1 is the standard deviation of $\ln X$.

The kernel function of lognormal variable X with respect to μ_1 and σ_1 can be derived from equation (12) directly and can be written as follows:

$$\begin{aligned} k_{\mu_1} &= \frac{\ln(x) - \mu_1}{\sigma_1^2}, \\ k_{\sigma_1} &= \frac{[\ln(x) - \mu_1]^2 - \sigma_1^2}{\sigma_1^3}, \end{aligned} \quad (16)$$

where k_{μ_1} and k_{σ_1} are the kernel functions with respect to μ_1 and σ_1 , respectively.

Distribution parameters μ_1 and σ_1 of the lognormal distributed variable x can be written as follows:

$$\begin{aligned} \mu_1 &= \ln \left(\frac{\mu^2}{\sqrt{\sigma^2 + \mu^2}} \right), \\ \sigma_1 &= \sqrt{\ln \left(\frac{\sigma^2}{\mu^2 + 1} \right)}, \end{aligned} \quad (17)$$

where μ and σ are the mean and standard deviation of the lognormal distributed variable x , respectively.

Calculate partial derivative of μ_1 and σ_1 with respect to μ and σ , respectively, according to equation (16); then,

$$\begin{aligned}
\frac{\partial \mu_1}{\partial \mu} &= \frac{2\sigma^2 + \mu^2}{\mu(\sigma^2 + \mu^2)}, \\
\frac{\partial \mu_1}{\partial \sigma} &= -\frac{2\sigma}{\sigma^2 + \mu^2}, \\
\frac{\partial \sigma_1}{\partial \mu} &= \frac{\mu}{\sqrt{\ln(\sigma^2/\mu^2 + 1)} \cdot (\mu^2 + 1)}, \\
\frac{\partial \sigma_1}{\partial \sigma} &= \frac{1}{\sigma \sqrt{\ln(\sigma^2/\mu^2 + 1)}}.
\end{aligned} \tag{18}$$

Jacobi matrix $\mathbf{J}$ can be written as follows:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mu_1}{\partial \mu} & \frac{\partial \sigma_1}{\partial \mu} \\ \frac{\partial \mu_1}{\partial \sigma} & \frac{\partial \sigma_1}{\partial \sigma} \end{bmatrix}. \tag{19}$$

The kernel function of the lognormal distributed variable X with respect to its mean and standard deviation can be written as follows:

$$\begin{bmatrix} k_\mu \\ k_\sigma \end{bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial \mu} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \mu} k_{\sigma_1} \\ \frac{\partial \mu_1}{\partial \sigma} k_{\mu_1} + \frac{\partial \sigma_1}{\partial \sigma} k_{\sigma_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial \mu} & \frac{\partial \sigma_1}{\partial \mu} \\ \frac{\partial \mu_1}{\partial \sigma} & \frac{\partial \sigma_1}{\partial \sigma} \end{bmatrix} \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix} = \mathbf{J} \cdot \begin{bmatrix} k_{\mu_1} \\ k_{\sigma_1} \end{bmatrix}, \tag{20}$$

where k_μ and k_σ are the kernel functions with respect to mean and standard deviation of the lognormal distributed variable x , respectively.

If x is two parameter Weibull distributed variable, its probability density function can be written as follows:

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha} \right)^{\beta-1} e^{-(x/\alpha)^\beta}, \tag{21}$$

where β is the shape parameter, and α is the scale parameter.

The kernel function of the Weibull distributed variable X with respect to scale parameter α and shape parameter β can be derived from equation (12) directly, which can be written as follows:

$$\begin{aligned}
k_\alpha &= \frac{\beta}{\alpha} \left[\left(\frac{x}{\alpha} \right)^\beta - 1 \right], \\
k_\beta &= \frac{1}{\beta} + \left[1 - \left(\frac{x}{\beta} \right)^\beta \right] \ln \left(\frac{x}{\alpha} \right).
\end{aligned} \tag{22}$$

The mean and standard deviation μ and σ of X can be written as follows:

$$\begin{aligned}
\mu &= \alpha \cdot \Gamma \left(\frac{1}{\beta} + 1 \right), \\
\sigma &= \alpha \cdot \sqrt{\Gamma \left(\frac{2}{\beta} + 1 \right) - \left(\Gamma \left(\frac{1}{\beta} + 1 \right) \right)^2},
\end{aligned} \tag{23}$$

where μ and σ are the mean and standard deviation of X , respectively, and $\Gamma(\cdot)$ is the gamma function.

Calculate partial derivative of μ and σ with respect to α and β , respectively, from equation (21); then,

$$\begin{aligned}
\frac{\partial \mu}{\partial \alpha} &= \Gamma \left(\frac{1}{\beta} + 1 \right), \\
\frac{\partial \sigma}{\partial \alpha} &= \frac{\sigma}{\alpha}, \\
\frac{\partial \mu}{\partial \beta} &= -\frac{\alpha \Gamma((1/\beta) + 1) \Psi((1/\beta) + 1)}{\beta^2},
\end{aligned} \tag{24}$$

$$\frac{\partial \sigma}{\partial \beta} = \frac{\alpha^2 \left[\Gamma^2((1/\beta) + 1) \Psi((1/\beta) + 1) - \Gamma((2/\beta) + 1) \Psi((2/\beta) + 1) \right]}{\beta^2 \sigma}, \tag{25}$$

where $\Psi(\cdot)$ is the psi function.

Jacobi matrix $\mathbf{J}$ can be written as follows: $\mathbf{J} =$

$$\begin{bmatrix} \frac{\partial \mu}{\partial \alpha} & \frac{\partial \sigma}{\partial \alpha} \\ \frac{\partial \mu}{\partial \beta} & \frac{\partial \sigma}{\partial \beta} \end{bmatrix}; \text{ then, the inverse matrix of } \mathbf{J} \text{ can be written}$$

$$\text{as } \mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \alpha}{\partial \mu} & \frac{\partial \beta}{\partial \mu} \\ \frac{\partial \alpha}{\partial \sigma} & \frac{\partial \beta}{\partial \sigma} \end{bmatrix}.$$

The kernel function of the Weibull distributed variable X with respect to its mean and standard deviation can be written as follows:

$$\begin{bmatrix} k_\mu \\ k_\sigma \end{bmatrix} = \begin{bmatrix} \frac{\partial \alpha}{\partial \mu} k_\alpha + \frac{\partial \beta}{\partial \mu} k_\beta \\ \frac{\partial \alpha}{\partial \sigma} k_\alpha + \frac{\partial \beta}{\partial \sigma} k_\beta \end{bmatrix} = \begin{bmatrix} \frac{\partial \alpha}{\partial \mu} & \frac{\partial \beta}{\partial \mu} \\ \frac{\partial \alpha}{\partial \sigma} & \frac{\partial \beta}{\partial \sigma} \end{bmatrix} \cdot \begin{bmatrix} k_\alpha \\ k_\beta \end{bmatrix} = \mathbf{J}^{-1} \cdot \begin{bmatrix} k_\alpha \\ k_\beta \end{bmatrix}, \tag{26}$$

where k_α and k_β are the kernel functions with respect to scale parameter and shape parameter, respectively. α and β are the scale parameter and shape parameter of the two parameter Weibull distributed X , respectively.

3. Reliability and Reliability Sensitivity Based on Edgeworth Series

3.1. Reliability Based on Edgeworth Series. The failure probability of a mechanical component or a random structure can be calculated out by the following multidimensional integral:

$$P_f = P[g(\mathbf{X}) < 0] = \int_{g(\mathbf{X}) < 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \tag{27}$$

where P_f is the failure probability of a random structure, $\mathbf{X} = [X_1, X_2, \dots, X_N]^T \in R^N$ represents the N -dimensional random input variables of the random structure, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of random input variable $\mathbf{X}$, $g(\mathbf{X})$ is the performance function, $g(\mathbf{X}) < 0$ represents the failure domain, $g(\mathbf{X}) > 0$ represents the safety domain, and $g(\mathbf{X}) = 0$ represents the limit state.

In practice, it is difficult to obtain the analytical solution of the multidimensional integral in equation (25) because of the nonlinear integration boundary $g(\mathbf{X}) = 0$ and high dimension of $\mathbf{X}$. There are two classes of approaches available for estimating the failure probability P_f in equation (25), the analytical method and numerical simulation method. The numerical simulation method is the Monte Carlo simulation as known. Analytical methods (e.g., first-order reliability method (FORM)/second-order reliability method (SORM)),

higher-order moment method, and saddlepoint approximation) can always be applied to calculate failure probability P_f because of high computational efficiency and accuracy of them.

Suppose the first four central moments of performance function $g(\mathbf{X})$ are μ_g, ν_g, θ_g , and η_g . The failure probability P_f in equation (27) can be approximately estimated by Edgeworth series which can be written as follows:

$$P_f(\zeta) = \Phi(\zeta) - \phi(\zeta) \left[\frac{1}{6} \frac{\theta_g}{\sigma_g^3} H_2(\zeta) + \frac{1}{24} \left(\frac{\eta_g}{\sigma_g^4} - 3 \right) H_3(\zeta) + \frac{1}{72} \left(\frac{\theta_g}{\sigma_g^3} \right)^2 H_5(\zeta) \right], \quad (28)$$

where $\Phi(\cdot)$ is the standard normal distribution function, $\phi(\cdot)$ is the standard normal probability density, μ_g, ν_g, θ_g , and η_g are the mean, deviation, the third central moment, and the fourth central moment of performance function $g(\mathbf{X})$, σ_g is the standard deviation of $g(\mathbf{X})$, $\zeta = -\mu_g/\sigma_g$ is the negative reliability index, and $H_j(\cdot)$ is the j^{th} Hermite polynomial and can be written as follows:

$$\begin{aligned} H_{j+1}(\zeta) &= 2\zeta H_j(\zeta) - j H_{j-1}(\zeta), \\ H_0(\zeta) &= 1, \\ H_1(\zeta) &= 2\zeta. \end{aligned} \quad (29)$$

3.2. Reliability Sensitivity Based on Edgeworth Series. Partial derivative of failure probability P_f in equation (28) with respect to the first four central moments μ_g, ν_g, θ_g , and η_g of performance function $g(\mathbf{X})$ can be written as follows:

$$\frac{\partial P_f}{\partial \mu_g} = \frac{\partial P_f}{\partial \zeta} \frac{\partial \zeta}{\partial \mu_g}, \quad (30)$$

$$\frac{\partial P_f}{\partial \nu_g} = \left(\frac{\partial P_f}{\partial \zeta} \frac{\partial \zeta}{\partial \sigma_g} + \frac{\partial P_f}{\partial \sigma_g} \right) \frac{\partial \sigma_g}{\partial \nu_g}, \quad (31)$$

$$\frac{\partial P_f}{\partial \theta_g} = -\phi(\zeta) \left[\frac{1}{6 \cdot \sigma_g^3} H_2(\zeta) + \frac{1}{36} \frac{\theta_g}{\sigma_g^6} H_5(\zeta) \right], \quad (32)$$

$$\frac{\partial P_f}{\partial \eta_g} = -\phi(\zeta) \left[\frac{1}{24} \frac{\eta_g}{\sigma_g^4} H_3(\zeta) \right],$$

where

$$\begin{aligned} \frac{\partial P_f}{\partial \zeta} &= \phi(\zeta) + \left\{ \begin{aligned} &\zeta \phi(\zeta) \left[\frac{1}{6} \frac{\theta_g}{\sigma_g^3} H_2(\zeta) + \frac{1}{24} \left(\frac{\eta_g}{\sigma_g^4} - 3 \right) H_3(\zeta) + \frac{1}{72} \left(\frac{\theta_g}{\sigma_g^3} \right)^2 H_5(\zeta) \right] - \\ &\phi(\zeta) \left[\frac{1}{3} \frac{\theta_g}{\sigma_g^3} H_1(\zeta) + \frac{1}{8} \left(\frac{\eta_g}{\sigma_g^4} - 3 \right) H_2(\zeta) + \frac{5}{72} \left(\frac{\theta_g}{\sigma_g^3} \right)^2 H_4(\zeta) \right] \end{aligned} \right\}, \\ \frac{\partial \zeta}{\partial \mu_g} &= -\frac{1}{\sigma_g}, \end{aligned} \quad (33)$$

$$\frac{\partial \zeta}{\partial \sigma_g} = \frac{\mu_g}{\sigma_g^2},$$

$$\frac{\partial \sigma_g}{\partial \nu_g} = \frac{1}{2\sigma_g},$$

$$\frac{\partial P_f}{\partial \sigma_g} = \phi(\zeta) \left[\frac{1}{2} \frac{\theta_g}{\sigma_g^4} H_2(\zeta) + \frac{1}{6} \frac{\eta_g}{\sigma_g^5} H_3(\zeta) + \frac{1}{12} \left(\frac{\theta_g}{\sigma_g^{7/2}} \right)^2 H_5(\zeta) \right].$$

The first four central moments μ_g, ν_g, θ_g , and η_g of performance function $g(\mathbf{X})$ can be written as follows:

$$\begin{cases} \mu_g = m_1, \\ \nu_g = m_2 - m_1^2, \\ \theta_g = m_3 - 3m_1m_2 + 2m_1^3, \\ \eta_g = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4, \end{cases} \quad (34)$$

where m_l ($l = 1, 2, 3, 4$) are the first four origin moments of $g(\mathbf{X}) = 0$ which can be estimated by equation (7).

From equation (34), the partial derivative of central moments μ_g, ν_g, θ_g , and η_g of performance function $g(\mathbf{X})$ with respect to the distribution parameters T_j^i ($j = 1, 2, \dots, N, i = 1, 2$) ($i = 1$ represents mean; $i = 2$ represents standard deviation) of random inputs $\mathbf{X} = [X_1, X_2, \dots, X_N]^T$ can be written as follows:

$$\begin{cases} \frac{\partial \mu_g}{\partial T_j^i} = \frac{\partial m_1}{\partial T_j^i}, \\ \frac{\partial \nu_g}{\partial T_j^i} = \frac{\partial m_2}{\partial T_j^i} - 2m_1 \frac{\partial m_1}{\partial T_j^i}, \\ \frac{\partial \theta_g}{\partial T_j^i} = \frac{\partial m_3}{\partial T_j^i} - 3 \left(\frac{\partial m_1}{\partial T_j^i} m_2 + \frac{\partial m_2}{\partial T_j^i} m_1 \right) + 6m_1^2 \frac{\partial m_2}{\partial T_j^i}, \\ \frac{\partial \eta_g}{\partial T_j^i} = \frac{\partial m_4}{\partial T_j^i} - 4 \left(\frac{\partial m_1}{\partial T_j^i} m_3 + \frac{\partial m_3}{\partial T_j^i} m_1 \right) + 6 \left(2m_1 m_2 \frac{\partial m_1}{\partial T_j^i} + m_1^2 \frac{\partial m_2}{\partial T_j^i} \right) - 12m_1^3 \frac{\partial m_1}{\partial T_j^i}, \end{cases} \quad (i = 1, 2, j = 1, 2, \dots, N), \quad (35)$$

where $(\partial m_l / \partial T_j^i)$ ($l = 1, 2, 3, 4$) can be calculated out by equation (11).

The failure probability sensitivity defined as partial derivative of failure probability P_f with respect to distribution

parameters T_j^i of random inputs $\mathbf{X} = [X_1, X_2, \dots, X_N]^T$ can be written as follows:

$$\frac{\partial P_f}{\partial T_j^i} = \frac{\partial P_f}{\partial \mu_g} \frac{\partial \mu_g}{\partial T_j^i} + \frac{\partial P_f}{\partial \nu_g} \frac{\partial \nu_g}{\partial T_j^i} + \frac{\partial P_f}{\partial \theta_g} \frac{\partial \theta_g}{\partial T_j^i} + \frac{\partial P_f}{\partial \eta_g} \frac{\partial \eta_g}{\partial T_j^i}, \quad (j = 1, 2, \dots, N, i = 1, 2,). \quad (36)$$

From equation (36), $\partial P_f / \partial T_j^i$ can be calculated out according to equations (28)–(31) and equation (35). The flow process diagram of the computational procedure is shown in Figure 1.

4. Numerical Examples

4.1. Example 1: Nonlinear Performance Function with Normal Variables. This example considers a nonlinear performance function which is the stress limit state function of a multileaf spring written as follows:

$$g(X) = r - \frac{3Pl}{7bh^2}, \quad (37)$$

where basic random variables $\mathbf{X} = [r, p, l, b, h]$ are the independent and identically distributed Gaussian random variables, r is the material strength of the leaf spring, p is the load, and b, h , and l are the geometric dimension width,

thickness, and span of the leaf spring, respectively. The mean and standard deviation of random variables $\mathbf{X} = [r, p, l, b, h]$ are listed in Table 2 [22].

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 3 for comparison. The MCS results can be considered as accurate results and N is the number of samples.

In Table 3, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 1 indicates that the proposed method is suitable for problems with nonlinear performance functions.

4.2. Example 2: Linear Performance Function with Six Log-normal Variables. This example considers a linear performance function, a plastic collapse mechanism of a one-bay frame which has been used as example 5 by Der et al. [28] and Zhao and Ono [29], written as follows:

FIGURE 1: Flow process diagram for computations.

TABLE 2: Statistic moments and distributions of input variables of Example 1.

Distribution	Normal	Normal	Normal	Normal	Normal
Variables	r (MPa)	P (N)	l (mm)	b (mm)	h (mm)
Mean	614	5537	1100	70	6.5
Standard deviation	45.8	276.85	5.5	0.35	0.0325

TABLE 3: Failure probability and sensitivity results of Example 1.

	Proposed method	MCS ($N = 5 \times 106$)
P_f	3.6904×10^{-4}	3.7501×10^{-4}
$\partial P_f / \partial \mu_r$	-2.6160×10^{-5}	-2.6826×10^{-5}
$\partial P_f / \partial \sigma_r$	7.9061×10^{-5}	7.8365×10^{-5}
$\partial P_f / \partial \mu_h$	-3.5000×10^{-3}	-3.9000×10^{-3}
$\partial P_f / \partial \sigma_h$	1.1000×10^{-3}	9.4906×10^{-4}
$\partial P_f / \partial \mu_b$	-1.6736×10^{-4}	-1.5067×10^{-4}
$\partial P_f / \partial \sigma_b$	2.5960×10^{-5}	2.7601×10^{-5}
$\partial P_f / \partial \mu_l$	1.0535×10^{-5}	1.1517×10^{-5}
$\partial P_f / \partial \sigma_l$	1.5250×10^{-6}	1.8140×10^{-6}
$\partial P_f / \partial \mu_P$	2.0798×10^{-6}	2.1164×10^{-6}
$\partial P_f / \partial \sigma_P$	3.0267×10^{-6}	3.0777×10^{-6}

$$g(X) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6, \quad (38)$$

where basic random variables $\mathbf{X} = [x_1, x_2, x_3, x_4, x_5, x_6]$ are the independent and identically distributed lognormal random variables. The mean and standard deviation of random variables $\mathbf{X} = [x_1, x_2, x_3, x_4, x_5, x_6]$ are listed in Table 4.

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 5 for comparison.

In Table 5, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 2 indicates that the proposed method is suitable for problems with nonnormal random input variables.

4.3. Example 3: Nonlinear Performance Function with Non-normal Variables. This example considers a nonlinear performance function which describes the displacement response of a I section cantilever beam shown in Figure 2 written as follows:

$$g(X) = d - \frac{4Fl^3}{E[bh^3 - b(h-2t)^3 + a(h-2t)^3]}, \quad (39)$$

where basic random variables $\mathbf{X} = [a, t, b, h, l, F, d, E]$ are the independent random variables. Here, a is the thickness of web, t is the thickness of flange, b is the width of I -beam section, h is the height of I -beam section, l is the length of cantilever beam, f is the load force, d is the allowable maximum deformation, and E is the elastic modulus. The

TABLE 4: Statistic moments and distributions of input variables of Example 2.

Distribution	Lognormal	Lognormal	Lognormal	Lognormal	Lognormal	Lognormal
Variables	x_1	x_2	x_3	x_4	x_5	x_6
Mean	120	120	120	120	50	40
Standard deviation	12	12	12	12	15	12

TABLE 5: Failure probability and sensitivity results of Example 2.

	Proposed method	MCS ($N=5 \times 10^6$)
P_f	1.3600×10^{-2}	1.204×10^{-2}
$\partial P_f / \partial \mu_1$	-2.2290×10^{-4}	-1.942×10^{-4}
$\partial P_f / \partial \sigma_1$	3.4596×10^{-5}	4.104×10^{-5}
$\partial P_f / \partial \mu_2$	-4.4554×10^{-4}	-4.068×10^{-4}
$\partial P_f / \partial \sigma_2$	1.3341×10^{-4}	1.610×10^{-4}
$\partial P_f / \partial \mu_3$	-4.4554×10^{-4}	-4.072×10^{-4}
$\partial P_f / \partial \sigma_3$	1.3341×10^{-4}	1.429×10^{-4}
$\partial P_f / \partial \mu_4$	-2.2290×10^{-4}	-2.342×10^{-4}
$\partial P_f / \partial \sigma_4$	3.4596×10^{-5}	7.114×10^{-5}
$\partial P_f / \partial \mu_5$	9.5913×10^{-4}	9.157×10^{-4}
$\partial P_f / \partial \sigma_5$	2.9000×10^{-3}	2.482×10^{-3}
$\partial P_f / \partial \mu_6$	1.0000×10^{-3}	9.500×10^{-4}
$\partial P_f / \partial \sigma_6$	1.9000×10^{-3}	1.732×10^{-3}

FIGURE 2: Section cantilever beam.

TABLE 6: Statistic moments and distributions of input variables of Example 2.

Distribution	Normal	Normal	Normal	Normal	Normal	Weibull	Lognormal	Lognormal
Variables	a (mm)	t (mm)	b (mm)	h (mm)	l (mm)	F (N)	d (mm)	E (MPa)
Mean	10	14	88	205	2000	5.187×10^4	27	2.03×10^5
Standard deviation	0.05	0.07	0.44	1.025	10	213.865	0.135	1.015×10^3

distributions and distribution parameters (mean and standard deviation) of these input random variables are listed in Table 6.

The scale parameter and shape parameter of random variable F are $\alpha = 5.197 \times 10^4$ and $\beta = 310.36$, respectively.

The proposed method and MCS are applied to analyze the failure probability and reliability sensitivity, respectively, and their results are listed in Table 7 for comparison.

In Table 7, the results obtained by the proposed method have small errors compared with those obtained by MCS. Example 3 indicates that the proposed method is suitable for problems with nonlinear performance functions with non-normal random input variables. The nonnormal variables need not to be transformed into equivalent normal ones. But the coefficient of variation of the input random variables should be small (≤ 0.1 for Weibull distribution variables).

TABLE 7: Failure probability and sensitivity results of Example 3.

	Proposed method	MCS ($N=5 \times 10^6$)
P_f	9.7363×10^{-4}	2.8000×10^{-4}
$\partial P_f / \partial \mu_a$	-8.2470×10^{-4}	-5.0299×10^{-4}
$\partial P_f / \partial \sigma_a$	1.0579×10^{-4}	1.1908×10^{-4}
$\partial P_f / \partial \mu_t$	-2.2000×10^{-3}	-2.0500×10^{-3}
$\partial P_f / \partial \sigma_t$	1.1000×10^{-3}	1.4000×10^{-3}
$\partial P_f / \partial \mu_b$	-4.4905×10^{-4}	-5.8711×10^{-4}
$\partial P_f / \partial \sigma_b$	3.2129×10^{-4}	1.8541×10^{-4}
$\partial P_f / \partial \mu_h$	-5.5812×10^{-4}	-6.4358×10^{-4}
$\partial P_f / \partial \sigma_h$	1.1000×10^{-3}	1.5000×10^{-3}
$\partial P_f / \partial \mu_l$	9.3914×10^{-5}	8.9505×10^{-5}
$\partial P_f / \partial \sigma_l$	1.9045×10^{-4}	1.9436×10^{-4}
$\partial P_f / \partial \mu_F$	5.5696×10^{-6}	1.5504×10^{-6}
$\partial P_f / \partial \sigma_F$	8.4887×10^{-5}	8.8620×10^{-7}
$\partial P_f / \partial \mu_d$	-1.9000×10^{-3}	-2.1000×10^{-3}
$\partial P_f / \partial \sigma_d$	1.6000×10^{-3}	1.5000×10^{-3}
$\partial P_f / \partial \mu_E$	-2.2514×10^{-7}	-3.6546×10^{-7}
$\partial P_f / \partial \sigma_E$	1.8949×10^{-7}	2.5002×10^{-7}

5. Conclusions

The univariate dimension-reduction method, reliability analysis based on Edgeworth series, and reliability sensitivity analysis are employed to present a computational procedure for estimating failure probability sensitivity of random engineering structures. Three numerical examples are employed to illustrate the performance of the proposed method. To illustrate the feasibility of the proposed method, failure probability and failure probability sensitivity results obtained by the new method are compared with those obtained by direct MCS. It is concluded as follows: The proposed method is suitable for models of which the input random variables have small coefficient of variation (≤ 0.1). The nonnormal input variables need not to be transformed into equivalent normal ones, so the new method can be conveniently applied to solve problems with nonnormal input variables.

Data Availability

The data and models used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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