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Retraction

Retracted: Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Z. Ullah, H. Bashir, R. Anjum, S. A. AlQahtani, S. Al-Hadhrami, and A. Ghaffar, "Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm," *Mathematical Problems in Engineering*, vol. 2021, Article ID 7946936, 15 pages, 2021.

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Research Article

Analysis of the Shortest Path in Spherical Fuzzy Networks Using the Novel Dijkstra Algorithm

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The concept of fuzzy graph (FG) and its generalized forms has been developed to cope with several real-life problems having some sort of imprecision like networking problems, decision making, shortest path problems, and so on. This paper is based on some developments in generalization of FG theory to deal with situation where imprecision is characterized by four types of membership grades. A novel concept of T-spherical fuzzy graph (TSFG) is proposed as a common generalization of FG, intuitionistic fuzzy graph (IFG), and picture fuzzy graph (PFG) based on the recently introduced concept of T-spherical fuzzy set (TSFS). The significance and novelty of proposed concept is elaborated with the help of some examples, graphical analysis, and results. Some graph theoretic terms are defined and their properties are studied. Specially, the famous Dijkstra algorithm is proposed in the environment of TSFGs and is applied to solve a shortest path problem. The comparative analysis of the proposed concept and existing theory is made. In addition, the advantages of the proposed work are discussed over the existing tools.

1. Introduction

In the past decades, the development of graph theory, specifically the fuzzy graph (FG) theory, and its applications in numerous scientific subjects indicates its significance. The addition of FGs in graph theory is of worth as it increases the viability of graph theory. From application point of view, FGs have been widely utilized in practical problems, for example, reference [1] provided a list of possible regions handled by FGs and fuzzy hypergraphs, reference [2] modelled some traffic problems using FGs, reference [3] utilized FGs in optimization of networks, reference [4] is based on application of telecommunication system in FGs, and reference [5] applied FGs in fuzzy neural networks. The theory of FGs has been initiated in [6] but briefly elaborated in [7] by Rosenfield after the remarkable work of Zadeh [8] on fuzzy sets (FSs). For some works on FGs, one may refer to [9–16].

An FS only described the membership grade of an event/object while the non-membership grade is obtained by subtracting the membership grade from 1, i.e., the non-membership grade could not be chosen independently. Therefore, Atanassov [17] developed the theory of intuitionistic fuzzy set (IFS) as an advanced form of FSs and provided an opening for the theory of IFGs which was proposed in [18]. Atanassov's tool of IFSs gave strength to Zadeh's FSs, and in the same way, theory of IFG generalizes FGs and makes it more valuable. For some quality work on

IFGs and its applications, one may refer to [19–23]. IFSs could not model human opinion properly as described in [24, 25], and hence a new tool of picture fuzzy set (PFS) was introduced describing not only yes or no type situations but also situations having some abstinence or refusal grade involved like in voting situation. PFS strongly generalizes FSs and IFSs, and some useful work in this direction could be found in [26–30]. The idea of PFGs was developed in [31] generalizing the FGs and IFGs.

If we observe the structure of PFSs, it is clear that they generalize the FSs and IFSs. They know how to handle the situations or data that FSs or IFSs might not. But the structure of PFS has some limitations. Its constraint on the membership, abstinence, and non-membership grades states that their sum must be less than or equal to one. Due to this formation of PFSs, one is unable to assign the values to these membership, abstinence, and non-membership functions by their own choice. Keeping this issue in mind, Mahmood et al. [32] proposed the concept of spherical fuzzy sets (SFSs) and consequently T-spherical fuzzy sets (TSFSs), which improves the construction of PFS and does not have limitations at all. Such type of framework of TSFSs not only models human opinion other than yes or no but also can deal with any form of data without any limitations. For example, if we look at the constraint of PFSs and TSFSs, then it becomes very much clear that the framework of TSFSs has no limitations. The constraints of IFSs, PFSs, and TSFSs are as follows:

(i) For IFSs $A = \{x_i, (s(x_i), d(x_i))\}$, we have $0 \leq s(x_i) + d(x_i) \leq 1$.

(ii) For PFSs $A = \{x_i, (s(x_i), i(x_i), d(x_i))\}$, we have $0 \leq s(x_i) + i(x_i) + d(x_i) \leq 1$.

(iii) For TSFSs $A = \{x_i, (s(x_i), i(x_i), d(x_i))\}$, we have $0 \leq s^n(x_i) + i^n(x_i) + d^n(x_i) \leq 1$ for some $n \in \mathbb{Z}^+$.

The diverse structure and novelty of TSFSs is clear from its constraints and comparison with existing structures. Further diversity of proposed structure is discussed in Section 2 with the aid of some pictorial representations in Figures 1–5 .

The problem of the shortest path is one of the well-known problems that has been discussed prominently in various extended structures of FSs. Okada and Soper [33] worked out the shortest path problems utilizing fuzzy arcs, and Deng et al. [34] presented the Dijkstra algorithm that is the technique for finding out the shortest path. References [35–37] provide some good work on fuzzy shortest path problems. Gani and Jabarulla [38] also studied the shortest paths in the environment of IFSs, and for details on finding out the shortest path in an IFG using Dijkstra algorithm, see [39]. Plenty of works have been carried out on the topic of shortest path problems (one may refer to [40–43]).

As discussed, the framework of TSFS is more generalized than FS, IFS, and PFS. Therefore, the graph of TSFS could be more useful in dealing with uncertain situations. Keeping in view the developments in FG, IFG, and PFG and their several real-life applications, the aim of this study is to propose the graphs of TSFSs named as TSFGs. The TSFG

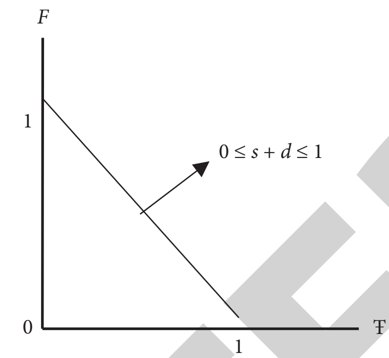

Figure 1: Intuitionistic fuzzy space.

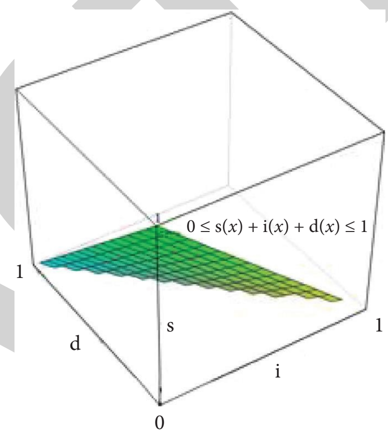

Figure 2: Pictorial fuzzy space.

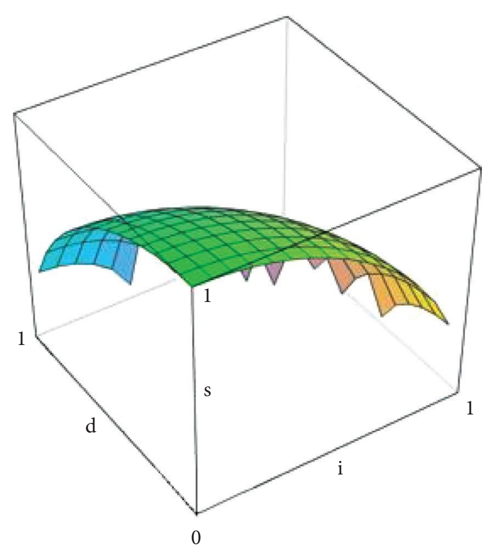

Figure 3: Space of spherical fuzzy sets [32].

generalizes the FG, IFG, and PFG. It discusses the membership, abstinence, and non-membership grades of an entity. Moreover, there is complete freedom for a decision maker; they can assign any fuzzy number as the membership, abstinence, or non-membership grades. Unlike PFGs, there are no limitations in the structure of TSFGs. The TSFG

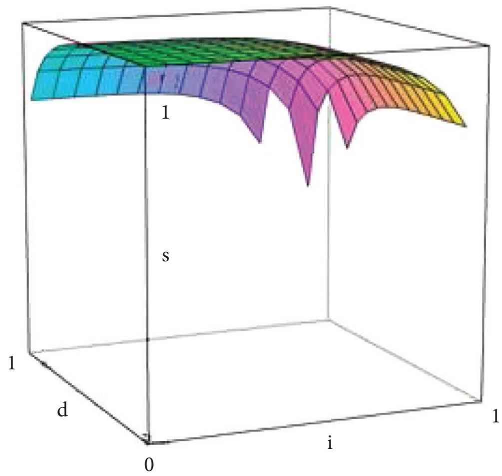

Figure 4: TSFS for $n = 5$.

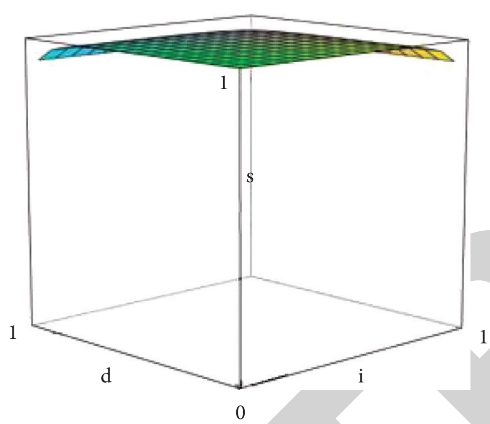

Figure 5: TSFS for $n = 10$.

can handle any problem that its predecessors could handle. It is the most powerful modelling tool among all the existing tools. The new structure of TSFG is investigated and some related terms are defined. The terms subgraph, complement, degree, and strength are defined for TSFGs and supported with examples. Several operations are also defined for TSFG, and some examples are discussed to support the defined concepts. To discuss the diversity and significance of TSFGs, a shortest path problem in the environment of TSFSs and TSFGs is also studied.

This paper is organized as follows. Section 1 is based on some history and motivation for proposing TSFGs. In Section 2, the basic definitions of IFS, PFS, SFS, TSFS, IFG, PFG, and the novelty of the proposed idea are discussed with the help of geometrical shapes. Also, in this section, TSFGs are proposed along with some basic graph theoretic terms like complement of TSFGs, order, degree and size of TSFGs and subgraphs of TSFGs, and the results of proposed notions are studied. In Section 3, operations of join and union are defined for TSFGs along with Cartesian product and composition of TSFGs. In Section 4, we propose a modified Dijkstra algorithm for the TSF shortest path that is then applied to find out the shortest path in a network. Furthermore, a comparative study is provided. In Section 5, we discussed the summary of our work along with its advantages and some future directions.

2. Preliminaries

In this section, the basic definitions of IFSs, PFSs, and TSFSs are reviewed and their spaces are geometrically described. Some elementary definitions of graphs of IFS and PFS are also discussed and explained with the help of some examples.

Definition 1 (see [17]). Let X be a universal set. An IFS on X is characterized by two mappings Ŧ and F on $[0,1]$ given that $0 \le s(x) + d(x) \le 1$. The values of s and d in the unit interval described the grade of membership and grade of non-membership of an element x in X. Also, $1 - (s(x) + d(x))$ denotes the hesitancy of $x \in X$. Moreover, the duplet (s, d) is said to be an intuitionistic fuzzy number (IFN). The range of the IFNs is portrayed in Figure 1.

In the voting situations, we might end up with four types of statuses, i.e., vote against, vote in favour, refusal, and abstain (nor in favour nor against). IFSs cannot cope with issues like this. Realizing this, a novel concept of PFS was developed by B. C. Cuong in 2013.

Definition 2 (see [25]). Let X be a universal set. A PFS on X is characterized by three mappings s, i, and d on $[0,1]$ provided that $0 \le s(x) + i(x) + d(x) \le 1$. The value of s, i, and d in the interval [0, 1] describes the membership, abstinence, and non-membership grades of x in$X$. Also, $1 - (s(x) + i(x) + d(x))$ denoted the refusal grade of $x \in X$. The triplet (s, i, d) is called the picture fuzzy number (PFN). The space of PFNs is depicted in Figure 2.

The problem with the framework of PFS is its check on the grade mappings, as depicted in Figure 2. Realizing this concern, Tahir et al. [32] proposed SFSs and consequently TSFSs. The definition of TSFSs is described below. Moreover, in order to make the point clear that TSFSs generalize IFSs and PFSs, a pictorial representation is given.

Definition 3 (see [32]). An SFS on X (a universal set) consists of three mappings s, i, and d on $[0,1]$ provided that $0 \le s^2(x) + i^2(x) + d^2(x) \le 1$. The values of s, i, and d in the interval [0, 1] describe the membership, abstinence, and non-membership grades of x in X. Also, the refusal grade of $x \in X$ is denoted by $r(x) = \sqrt{1 - (s^2(x) + i^2(x) + d^2(x))}$. The triplet (s, i, d) is called a spherical fuzzy number (SFN).

Definition 4 (see [32]). A TSFS on X (a universal set) consists of three mappings s, i, and d on $[0,1]$ provided that $0 \le s^n(x) + i^n(x) + d^n(x) \le 1$ for some $n \in \mathbb{Z}$. The values of s, i, and d in the interval [0, 1] describe the grade of membership, grade of abstinence, and grade of non-membership of x in X. Also, the refusal grade of $x \in X$ is denoted by $r(x) = \sqrt[n]{1 - (s^n(x) + i^n(x) + d^n(x))}$. The triplet (s, i, d) is called a T-spherical fuzzy number (TSFN).

The following figures described SFSs and TSFSs geometrically presenting their innovation and diverse structure. Figures 3–5 also show that TSFSs have no limitation.

From all the observations in this section, we conclude that the concept of TSFSs is the generalization of FSs, IFSs, PFSs, and SFSs and their structure does not have any limitations. Now, some elementary definitions associated with the graphs of IFS and PFS are discussed providing a base for the proposed work.

Definition 5 (see [25]). A pair $\mathcal{G} = (N, Ę)$ is known as IFG if

(i) $N = \{x_1, x_2, x_3, x_4, \ldots, x_n\}$ is the collection of vertices such that $s_1\colon N \longrightarrow [0,1]$ and $d_1\colon N \longrightarrow [0,1]$ denote the grade of membership and grade of non-membership of the element $x_i \in N$, respectively, with the condition that $0 \le s_1 + d_1 \le 1$ for all $x_i \in N$, $(i \in I)$.

(ii) $\check{E} \subseteq N \times N$ where $s_2\colon N \times N \longrightarrow [0,1]$ and $d_2\colon N \times N \longrightarrow [0,1]$ denote the grade of membership and grade of non-membership of the element $(x_i, x_j) \in \check{E}$ such that $s_2(x_i, x_j) \le \min\{s_1(x_i), s_1(x_j)\}$ and $d_2(x_i, x_j) \le \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \le s_2(x_i, x_j) + d_2(x_i, x_j) \le 1$ for all $(x_i, x_j) \in Ę$, $(i \in I)$.

Example 1. Figure 6 is an example of IFG.

Definition 6 (see [31]). A pair $\mathcal{G} = (N, Ę)$ is said to be a PFG if

(i) $N = \{x_1, x_2, x_3, x_4, \ldots, x_n\}$ is the set of vertices such that $s_1\colon N \longrightarrow [0,1]$ describes the grade of membership, $i_1\colon N \longrightarrow [0,1]$ describes the grade of abstinence, and $d_1\colon N \longrightarrow [0,1]$ describes the grade of non-membership of the element $x_i \in N$ on the condition that $0 \le s_1(x_i) + i_1(x_i) + d_1(x_i) \le 1$ for all $x_i \in N$, $(i \in I)$, and $1 - (s_{1i} + i_{1i} + d_{1i})$ is known as refusal grade of x in N.

(ii) $\check{E} \subseteq N \times N$ where $s_2\colon N \times N \longrightarrow [0,1]$ describes the grade of membership, $i\colon N \times N \longrightarrow [0,1]$ describes the grade of abstinence, and $d_2\colon N \times N \longrightarrow [0,1]$ describes the grade of non-membership of the element $(x_i, x_j) \in \check{E}$ such that $s_2(x_i, x_j) \le \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) \le \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i, x_j) \le \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \le s_2(x_i, x_j) + i_2(x_i, x_j) + d_2(x_i, x_j) \le 1$ for all $(x_i, x_j) \in Ę$, $(i \in I)$, and $1 - s_2(x_i, x_j)$ $+ i_2(x_i, x_j) + d_2(x_i, x_j)$ is known as refusal grade of (x_i, x_j) in Ę.

Example 2. Let $\mathcal{G} = (N, Ę)$ represent a graph with the collection of vertices N and the collection of edges Ę. Figure 7 is an example of PFG.

2.1. T-Spherical Fuzzy Graphs

Definition 7 (see [43]). A pair $\mathcal{G} = (N, Ę)$ is said to be TSFG if

(i) $N = \{x_1, x_2, x_3, \ldots, x_n\}$ is the set of vertices such that $s_1\colon N \longrightarrow [0,1]$ describes the grade of membership, $i_1\colon N \longrightarrow [0,1]$ describes the grade of abstinence, and $d_1\colon N \longrightarrow [0,1]$ describes the grade of non-membership of the element $x_i \in N$ on the condition that for some positive integers n $0 \le s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i) \le 1$ for all $x_i \in N (i \in I)$, and $\sqrt[n]{1 - (s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i))}$ is known as refusal grade of x in N.

(ii) $Ę \subseteq N \times N$ where $s_2\colon N \times N \longrightarrow [0,1], i_2\colon N \times N \longrightarrow [0,1]$ and $d_2\colon N \times N \longrightarrow [0,1]$ describes the grades of membership, abstinence, and non-membership of the element $(x_i, x_j) \in Ę$ such that $s_2(x_i, x_j) \le \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) \le \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i, x_j) \le \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \le s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j) \le 1$ for all $(x_i, x_j) \in Ę$, and $\sqrt[n]{1 - (s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j))}$ is known as refusal grade of (x_i, x_j) in Ę.

Example 3. Let $\breve{G} = (N, Ę)$ represent a graph with the collection of vertices N and the collection of edges Ę.

The vertices shown in Figures 8 and 9 are purely T-spherical fuzzy numbers (TSFNs) for $n = 5$.

Remark 1. PFG and SFG are TSFGs, but generally, the converse is not true.

Example 4. The graph in Figure 8 is clearly TSFG, but it is neither PFG nor SFG. Consider $(0.8, 0.9, 0.8)$; then, $0.8 + 0.9 + 0.8 = 2.5 \nleq 1$ and $0.8^2 + 0.9^2 + 0.8^2 = 2.09 \nleq 1$.

Definition 8 (see [43]). For TSFNs $A = \{s_A, \imath_A, d_A\}$ and $B = \{s_B, \imath_B, d_B\}$, we define

$$
\begin{aligned}
A \oplus B &= \left\{ \left\{ t, \begin{pmatrix} \sqrt[n]{s_A^n(x) + s_B^n(x) - s_A^n(x) \cdot s_B^n(x)}, \sqrt[n]{\imath_A^{\mathrm{n}}(\mathrm{x}) + \imath_B^{\mathrm{n}}(\mathrm{x}) - \imath_A^{\mathrm{n}}(\mathrm{x}) \cdot \imath_B^{\mathrm{n}}(\mathrm{x})} \\ ii d_A \cdot d_B \end{pmatrix} i \right\} \right\}, \\
A \otimes B &= \left\{ \left\{ \varkappa, \left((s_A(x) \cdot s_B(x)), i(\imath_A(x) \cdot \imath_B(x)), i \sqrt[n]{d_A^n(x) + d_B^n(x) - d_A^n(x) \cdot d_B^n(x)} \right) i \right\} \right\}.
\end{aligned} \tag{1}
$$

In FS theory, the rules of comparison have always been a challenge. For IFSs, several score functions have been established regularly. These score functions fall under the title of comparison rules. A better score function (SF) for

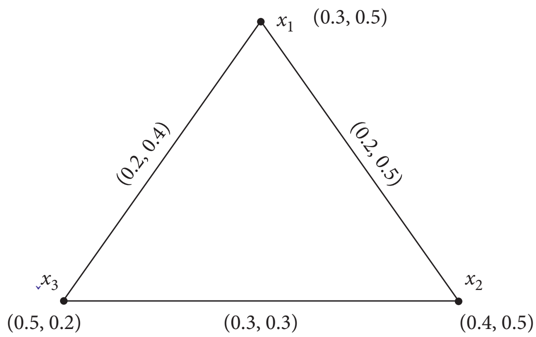

Figure 6: Intuitionistic fuzzy graph.

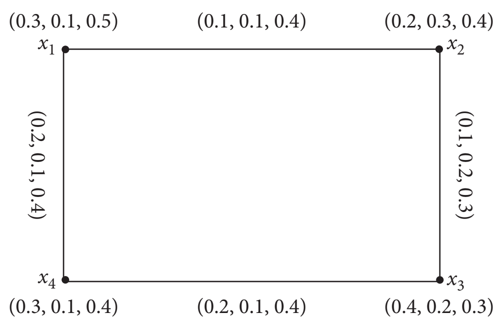

Figure 7: Picture fuzzy graph.

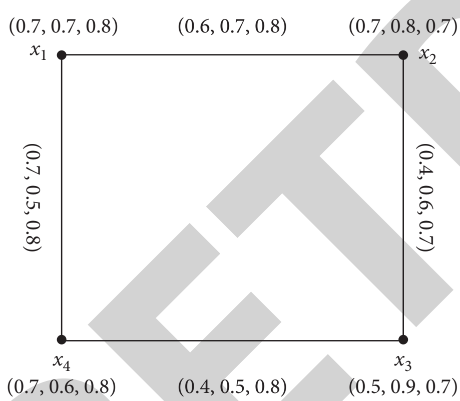

Figure 8: T-spherical fuzzy graph.

IFSs is established in [39] and it discusses the limitations of existing score functions which is demonstrated using examples. Further, work done on PFSs is significantly less; hence, in literature, there does not exist any SFs. Therefore, this article establishes a novel SF as a generalized SF proposed in [39]. In Section 4, this SF shall be utilized in the problems of the shortest path.

Definition 9 (see [43]). The SF for a TSFN $A = (s, i, d)$ is defined as

$$\mathrm{SC}(A) = \frac{(s)^n \left(1 - (i)^n - (d)^n\right)}{3}, \quad \mathrm{SC}(A) \in [0, 1]. \tag{2}$$

Remark 2. Replacing $i = 0$ and $n = 1$ reduces the defined score function in the environment of IFSs.

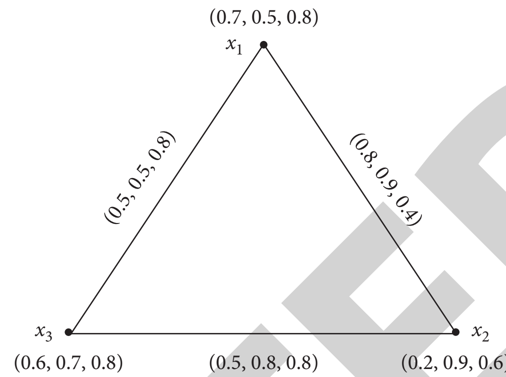

Figure 9: An example of not T-spherical fuzzy graph.

Definition 10 (see [43]). A pair $\mathrm{H} = (N^*, Ȩ^*)$ is said to be T-spherical fuzzy subgraph (TSFSG) of TSFG $\mathcal{G} = (N, Ȩ)$ if $N^* \subseteq N$ and $Ȩ^* \subseteq Ȩ$, that is, $s^*_{1i} \leq s_{1i}$, $i^*_{1i} \leq i_{1i}$, $d^*_{1i} \geq d_{1i}$ and $s^*_{2ij} \leq s_{2ij}$, $i^*_{2ij} \leq i_{2ij}$, $d^*_{2ij} \geq d_{2i} j$ for all $i, j = 1, 2, \ldots, n$.

Definition 11 (see [43]). The complement of a TSFG $\mathcal{G} = (N, Ȩ)$ is defined as

(i) $\overline{N} = N$.

(ii) $\overline{s_i} = s_i$, $\overline{i_i} = i_i$ and $\overline{d_i} = d_i$ for $i = 1, 2, \ldots, n$.

(iii) $\overline{s_{2ij}} = \min(s_i, is_j) - s_{2ij}$, $\overline{i_{2ij}} = \min(i_i, ii_j) - s_{2ij}$ and $\overline{d_{2ij}} = \max(d_i, id_j) - d_{2ij}$ for any $i, j = 1, 2, \ldots, n$.

Example 5. Figures 10 and 11 are examples of complement of TSFG.

The vertices are purely TSFNs for $n = 3$ in Figures 10 and 11.

Definition 12 (see [43]). The degree of a TSFG $\mathcal{G} = (N, Ȩ)$ is denoted and is defined by $\widetilde{d}(x) = (\widetilde{d}_s(x), \widetilde{d}_i(x), \widetilde{d}_d(x))$, where $\widetilde{d}_s(x) = \sum_{y \neq x} s_2(x, y) i$, $\widetilde{d}_i(x) = \sum_{y \neq x} i_2(x, y) i$ and $\widetilde{d}_d(x) = \sum_{y \neq x} \widetilde{d}_2(x, y) i$ for $i\ x, y \in N$.

Example 6. Let $\mathcal{G} = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges Ȩ.

The vertices are purely TSFNs for $n = 4$ in Figure 12.

The degree of vertices shown in Figure 12 is

$$\begin{aligned} \widetilde{d}(x_1) &= (0.9, 1.1, 1.3), \\ \widetilde{d}(x_2) &= (1, 1, 1.1), \\ \widetilde{d}(x_3) &= (0.8, 0.8, 1.4), \\ \widetilde{d}(x_4) &= (0.7, 0.9, 1.6). \end{aligned} \tag{3}$$

Definition 13 (see [43]). A pair $\mathcal{G} = (N, Ȩ)$ is said to be strong TSFG if

(i) $N = \{x_1, x_2, x_3, \ldots, x_n\}$ is the set of vertices such that s: $N \longrightarrow [0, 1]$ denotes the grade of membership, i: $N \longrightarrow [0, 1]$ denotes the grade of abstinence, and d: $N \longrightarrow [0, 1]$ represents the grade of nonmembership of the element $x_i \in N$ with the

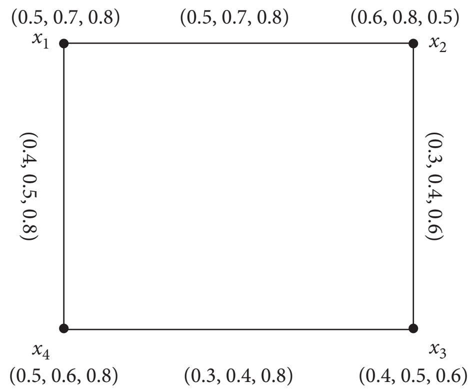

FIGURE 10: T-spherical fuzzy graph for $n = 3$.

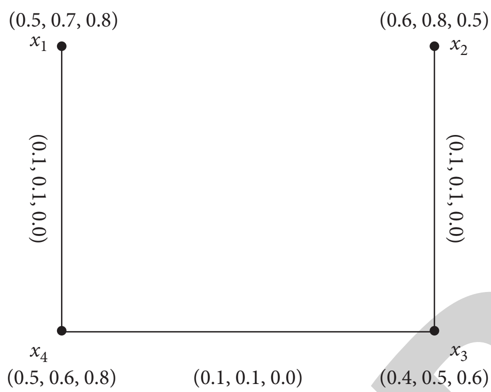

FIGURE 11: Complement of Figure 10.

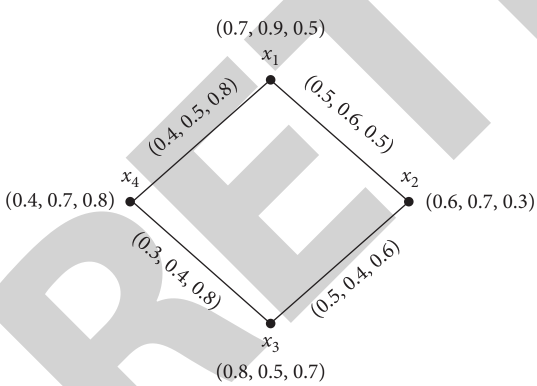

FIGURE 12: For the degree of TSFG.

condition that for some positive integers $n0 \leq s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i) \leq 1$ for all $x_i \in N (i \in I)$, and $\sqrt[n]{1 - (s_1^n(x_i) + i_1^n(x_i) + d_1^n(x_i))}$ is known as refusal grade of x in N.

(ii) $Ȩ \subseteq N \times N$ where $s_2\colon N \times N \longrightarrow [0,1]$ denotes the grade of membership, $i_2\colon N \times N \longrightarrow [0,1]$ describes the grade of abstinence, and $d_2\colon N \times N \longrightarrow [0,1]$ represents the grade of non-membership of the element $(x_i, x_j) \in Ȩ$ such that $s_2(x_i, x_j) = \min\{s_1(x_i), s_1(x_j)\}$, $i_2(x_i, x_j) = \min\{i_1(x_i), i_1(x_j)\}$ and $d_2(x_i, x_j) = \max\{d_1(x_i), d_1(x_j)\}$ with the condition $0 \leq s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j) \leq 1$ for all $(x_i, x_j) \in Ȩ$, and $\sqrt[n]{1 - (s_2^n(x_i, x_j) + i_2^n(x_i, x_j) + d_2^n(x_i, x_j))}$ is known as refusal grade of (x_i, ix_j) in Ȩ.

Example 7. Figure 13 is an example of strong TSFG.

The vertices are purely TSFNs for $n = 4$ in Figure 13.

Definition 14 (see [43]). An edge (x_i, x_j) in a TSFG $G = (N, Ȩ)$ is known to be a bridge, if by removal of that edge decreases the strength of the connectedness among any pair of vertices in G.

Example 8. Let $G = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges Ȩ.

Here, (x_1, x_4) is a bridge.

The vertices shown in Figure 14 are purely TSFNs for $n = 5$.

Definition 15 (see [43]). A vertex x_i in a TSFG $G = (N, Ȩ)$ is known to be cut vertex, if the removal of that vertex decreases the strength of the connectedness among any pair of vertices.

Example 9. Let $G = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges Ȩ.

Here, x_1 is a cut vertex.

The vertices are purely TSFNs for $n = 3$ in Figure 15.

3. Operations on T-Spherical Fuzzy Graphs

In this section, the operations on T-spherical fuzzy graph are defined and their results are studied.

Definition 18. The union of a TSFG $G_1 = (N_1, Ȩ_1)$and $G_2 = (N_2, Ȩ_2)$with $N_1 \cap N_2 = \varnothing$ and $G = G_1 \cup G_2 = (N_1 \cup N_2, Ȩ_1 \cup Ȩ_2)$ is defined by

$$\begin{aligned}
(s_1 \cup s_1')(x) &= \left\{\begin{matrix} s_1(x) & \text{if } x \in N_1 - N_2 \\ s_1'(x) & \text{if } ix \in N_2 - N_1 \end{matrix}\right\},\\
(i_1 \cup i_1')(x) &= \left\{\begin{matrix} i_1(x) & \text{if } x \in N_1 - N_2 \\ i_1{}'(x) & \text{if } x \in N_2 - N_1 \end{matrix}\right\},\\
(d_1 \cup d_1')(x) &= \left\{\begin{matrix} d_1(x) & \text{if } x \in N_1 - N_2 \\ d_1'(x) & \text{if } x \in N_2 - N_1 i \end{matrix}\right\},\\
(s_2 \cup s_2')(x_i x_j) &= \left\{ s_{2ij} \text{ if } \widehat{e}_{ij} \in Ȩ_1 - Ȩ_2 s_{2ij}' \text{ if } \widehat{e}_{ij} \in Ȩ_2 - Ȩ_1 \right\},\\
(i_2 \cup i_2')(x_i x_j) &= \left\{ i_{2ij} \text{ if } \widehat{e}_{ij} \in Ȩ_1 - Ȩ_2 i_{2ij}' \text{ if } \widehat{e}_{ij} \in Ȩ_2 - Ȩ_1 \right\},\\
(d_2 \cup d_2')(x_i x_j) &= \left\{ d_{2ij} \text{ if } \widehat{e}_{ij} \in Ȩ_1 - Ȩ_2 d_{2ij}' \text{ if } \widehat{e}_{ij} \in Ȩ_2 - Ȩ_1 \right\},
\end{aligned} \tag{4}$$

where (s_1, i_1, d_1) and (s_1', i_1', d_1') represent the vertices of truth membership, abstinence membership, and false membership of G_1 and G_2, respectively, and (s_2, i_2, d_2) and (s_2', i_2', d_2') represent the edges of truth, abstinence, and false memberships G_1 and G_2, respectively.

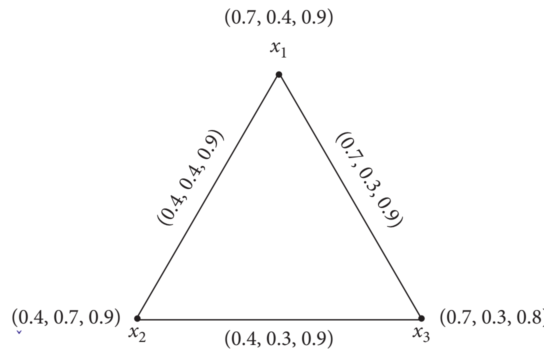

Figure 13: Strong T-spherical fuzzy graph.

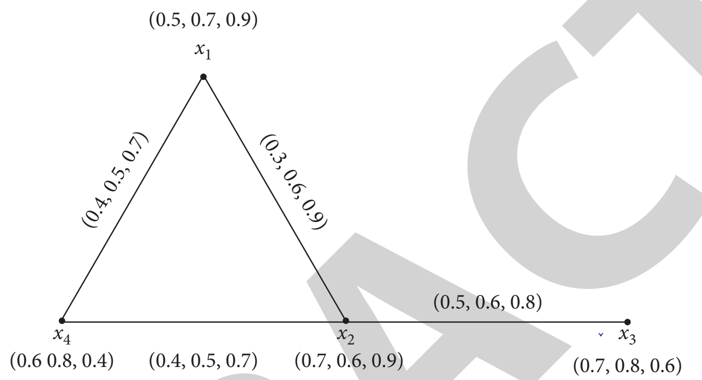

Figure 14: TSFG for a bridge.

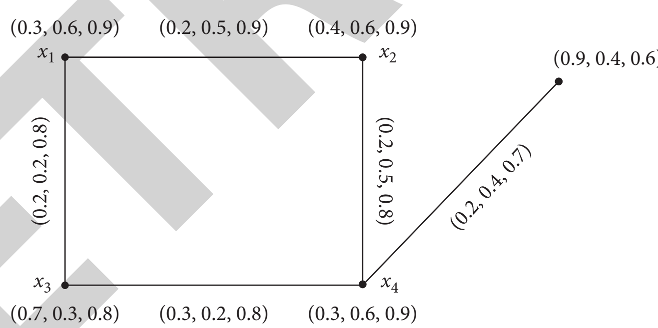

Figure 15: T-spherical fuzzy graph for cut vertex.

Example 10. Let $G = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges Ȩ. Figures 16–18 are examples of the union of two TSFGs.

The vertices are purely TSFNs for $n = 5$ in Figures 16–18.

Definition 19. Let G_1 and G_2 be two TSFGs. Then, the join of G_1 and G_2 is a TSFG, $G = G_1 + G_2 = (N_1 \cup N_2, Ȩ_1 \cup Ȩ_2 \cup Ȩ')$ defined by: $(s_1 + s_1')(x) = (s_1 \cup s_1')(x)$if $x \in N_1 \cup N_2$, $(i_1 + i_1')(x) = (i_1 \cup i_1')(x)$if $x \in N_1 \cup N_2$, $(d_1 + d_1')(x) = (d_1 \cup d_1')(x)$if $x \in N_1 \cup N_2$ and $(s_2 + s_2')(x_i x_j) = (s_1 \cup s_1')(x_i x_j)$ if $x \in Ȩ_1 \cup Ȩ_2 = \min(s_1(x_i), s_1'(x_j)$ if $x_i x_j \in Ȩ'$, and $(i_2 + i_2')(x_i x_j) = (i_1 \cup i_1')(x_i x_j)$if $x \in Ȩ_1 \cup Ȩ_2 = \min\ (i_1(x_i), i_1'(x_j)$ if $x_i x_j \in Ȩ'$ and $(d_2 + d_2')\ (x_i x_j) = (d_1 \cup d_1')(x_i x_j)$ if $x \in Ȩ_1 \cup Ȩ_2 = \max\ (d_1(x_i), {d_1}'(x_j)$ if $x_i x_j \in Ȩ'$.

Theorem 4. *If $G_1 = (N_1, Ȩ_1)$ and $G_2 = (N_2, Ȩ_2)$ are two TSFGs, then*

(i) $\overline{G_1 + G_2} \cong \overline{G_1} \cup \overline{G_2}$

(ii) $\overline{G_1 \cup G_2} \cong \overline{G_1} + \overline{G_2}$

Proof. Let I: $N_1 \cup N_2 \longrightarrow N_1 \cup N_2$ be the identity map. The following steps are calculated to prove (i)

(a) $\overline{(s_1 + s_1')}(x_i) = \overline{s_1} \cup \overline{s_1'}(x_i)$, $\overline{(i_1 + i_1')}(x_i) = \overline{i_1} i \cup \overline{i_1'}(x_i)$, $\overline{(d_1 + d_1')}(x_i) = \overline{d_1} i \cup \overline{d_1'}(x_i)$.

(b) $\overline{(s_2 + s_2')}(x_i, x_j) = \overline{s_2} i \cup \overline{s_2'}(x_i, x_j)$, $i\,\overline{(i_2 + i_2')\ }(x_i, x_j) = \overline{i_2} i \cup \overline{i_2'}(x_i, x_j)$, $\overline{(d_2 + d_2')}(x_i, x_j) = \overline{d_2} i \cup \overline{d_2'} i (x_i, x_j)$. Now to prove (a).

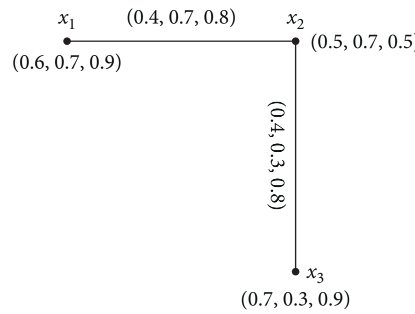

Figure 16: TSFG-A for union.

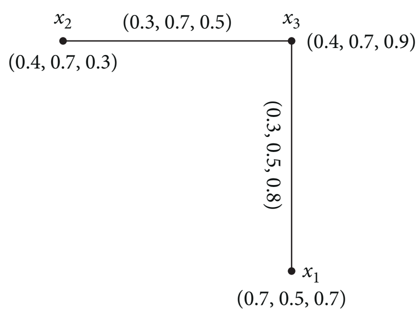

Figure 17: TSFG-B for union.

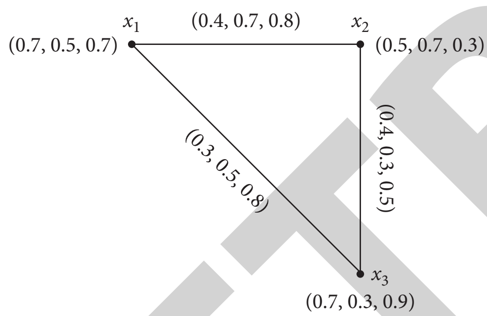

Figure 18: Union of two TSFGs.

(i) $\overline{(s_1+s_1')}\ (x_i) = (s_1+s_1')\ (x_i)$, by definition $= \{s_1\ (x_i)$ if $(x_i) \in N_1 s_1'(x_i)$ if $(x_i) \in N_2 i\} i = \{\ \overline{s_1}(x_i)$ if $(x_i) \in N_1 \overline{s_1'}(x_i)$if $i(x_i) \in N_2\} = (\overline{s_1} i \cup \overline{s_1'})(x_i)$ $= i\overline{(i_1+i_1')}, (x_i) = (i_1+i_1')\quad (x_i) = \{i_1(x_i)$ if $(x_i) \in N_1 i_1'(x_i)$ if $(x_i) \in N_2 i\} = \begin{Bmatrix} \overline{i_1}(\mathcal{V}_i) & \text{if } (x_i) \in N_1 \\ \overline{i_1'} i(\mathcal{V}_i) & \text{iif } (x_i) \in N_2 \end{Bmatrix}$ $= (\overline{i_1} \cup \overline{i_1'}) i(x_i)$, $\overline{(d_2+d_2')}(x_i) = (d_2+d_2')(x_i) = \{d_1(\mathcal{V}_i)$ if $(x_i) \in N_1 d_1'(\mathcal{V}_i)$ if $(x_i) \in N_2 i\}\ i = \{\overline{d_1}\ (x_i)$if $(x_i) \in N_1 \overline{d_1'}\ ii(x_i)$if $(x_i) \in N_2\} = (\overline{d_1} \cup \overline{d_1'})(x_i)$.

(ii) $\overline{i(s_2+s_2')}\ i(x_i,x_j) = \min i\ ((s_1+s_1')(x_i), i(s_1+s_1')(x_j)i) - i\quad (s_1+s_1')(x_i,x_j) = \{\min((s_1+s_1')(x_i), (s_1+s_1')(x_j)) - ((s_2 \cup s_2')(x_i,\ x_j)$ if $(x_i,x_j) \in Ȩ_1 \cup Ȩ_2 \min((s_1 \cup s_1')(x_i),\ i(s_1 \cup s_1')(x_j)) - \min(s_1(x_i), s_1'(x_j))$if $(x_i,x_j) \in Ȩ'\} = \{\min(s_1(x_i), s_1'\ (x_j) - s_2(x_i,x_j)$if $(x_i,x_j) \in Ȩ_1 \min(s_1(x_i), s_1'\ (x_j) - s_2(x_i,x_j)$if $(x_i,x_j) \in Ȩ_2 \min(s_1(ix_i), s_1'(x_j)) - \min(s_1(ix_i),\ s_1'(x_j)$ if $i(x_i,\ x_j) \in Ȩ'\} = \{\overline{s_2}(x_i,\ x_j)$ if $(x_i,x_j) \in Ȩ_1 \overline{s_2'}\ (x_i,y_i)$if $(x_i,x_j) \in Ȩ_2 0$if $i\ (x_i,x_j) \in Ȩ\} = \overline{s_2} \cup \overline{s_2'}(x_i,x_j)$, $\overline{(i_2+i_2')}\ (x_i,x_j) = \min((i_1+i_1')\ (x_i), (i_1+i_1')(x_j) - (i_1+i_1')(x_i,x_j) = \{\min((i_1+i_1')\ (x_i), (i_1+i_1')(x_j) - ((i_2 \cup i_2')\ (x_i,x_j)$ if $(x_i,x_j) \in Ȩ_1 \cup Ȩ_2 \min((i_1 \cup i_1')(x_i),\ (i_1 \cup i_1')(x_j)) - \min(i_1(x_i), i_1'(x_j))$if $(x_i,x_j) \in Ȩ'\} = \{\min(i_1(x_i), i_1'\ (x_j) - i_2(x_i,x_j)$if $(x_i,x_j) \in Ȩ_1 \min\ (i_1(x_i),\ i_1'(x_j) - i_2(x_i,x_j)$if $(x_i,x_j) \in Ȩ_2 \min(i_1(ix_i), i_1'(x_j)) - \min\ (i_1(ix_i), i_1'(x_j)$if $i(x_i,x_j) \in Ȩ'\} = \{\overline{i_2}(x_i,x_j)$ if $(x_i,x_j) \in Ȩ_1 \overline{i_2'}(x_i,x_j)$if $(x_i,x_j) i \in Ȩ_1 0$if $(x_i,x_j) \in Ȩ\} = \overline{i_2} \cup \overline{i_2'}(x_i,\ x_j)$, $\overline{(d_2+d_2')}(x_i,x_j) = \max((d_1+d_1')(x_i), (d_1+d_1')\quad (x_j)) - (d_1+d_1')(x_i,x_j)\{\max((d_1+d_1')(x_i), (d_1+d_1')(x_j)) - ((d_2 \cup d_2')(x_i,x_j)$ if $(x_i,x_j) \in Ȩ_1 \cup Ȩ_2 \max((d_1 \cup d_1')(x_i),\ (d_1 \cup d_1')(x_j)) - \max(d_1(x_i), d_1'\ (x_j)) i$ if $(x_i,x_j) \in Ȩ_1\} = \{\max(d_1(x_i), d_1'(x_j) - d_2(x_i,x_j)$ if $(x_i,x_j) \in Ȩ_1 \max(d_1(x_i), d_1'(x_j) - d_2(x_i,x_j)$ if $(x_i,x_j) \in Ȩ_2 \max(d_1(x_i),\ d_1'(x_j)) - \max(d_1(x_i), d_2'(x_j)$if $(x_i,x_j) \in Ȩ'\} = \{\overline{d_2}(x_i,x_j)$ if $(x_i,\ x_j) \in Ȩ_1 \overline{d_2'}(x_i,x_j)$if $(x_i,x_j) \in Ȩ_2\ 0$if $(x_i,\ x_j) \in Ȩ\} = \overline{d_2} \cup \overline{d_2'}(x_i,x_j)$.

Now to prove (ii), we have to show that

(a) $\overline{(s_1 \cup s_1')}(x_i) = \overline{s_1} + \overline{s_1'}(x_i)$, $\overline{(i_1 \cup i_1')}(x_i) = \overline{i_1} + \overline{i_1'}\ (x_i)$, $i\overline{(d_1 \cup d_1')}(x_i) = \overline{d_1} + \overline{d_1'}\ (x_i)$.

(b) $\overline{(s_2 \cup s_2')}(x_i,x_j) = \overline{s_2} + \overline{s_2'}(x_i,x_j)$, $\overline{(i_2 \cup i_2')}(x_i,\ x_j) = \overline{i_2} + \overline{i_2'}(x_i,x_j)$, $i\overline{(d_2 \cup d_2')}\ (x_i,x_j) = \overline{d_2} + \overline{d_2'}(x_i,\ x_j)$.

Let I: $N_1 \cup N_2 \longrightarrow N_1 \cup N_2$ be the identity map.

(a) $\overline{(s_1 \cup\quad s_1')}(x_i) = (s_1 \cup s_1')(x_i)$, by definition $= \begin{Bmatrix} s_1(x_i) \text{ if } i(x_i) \in N_1 \\ s_1'(x_i) \text{ if } i(x_i) \in N_2 \end{Bmatrix}$ $i = \begin{Bmatrix} \overline{s_1}(x_i) \text{ if } i(x_i) \in N_1 \\ \overline{s_1'} i(x_i) \text{ iif } i(x_i) \in N_2 \end{Bmatrix}$ $i = (\overline{s_1} + \overline{s_1'})(x_i)$, $i\overline{(i_1 \cup i_1')}(x_i) = (i_1 \cup i_1')(x_i) = \begin{Bmatrix} i_1(x_i) \text{ if } (x_i) \in N_1 \\ i_1'(x_i) \text{ if } (x_i) \in N_2 i \end{Bmatrix} i = \begin{Bmatrix} \overline{i_1}(\mathcal{V}_i) \text{ if } (x_i) \in N_1 \\ \overline{i_1'} i(\mathcal{V}_i) \text{ if } (x_i) \in N_2 \end{Bmatrix} =$ $(\overline{i_1} + \overline{i_1'})(x_i)$, $\overline{(d_2 \cup d_2')}\ (x_i) = (d_2 \cup d_2')\ (x_i) =\ i\{d_1\ (\mathcal{V}_i)$ if $i(x_i) \in N_1 d_1'(\mathcal{V}_i)$ if $i(x_i) \in N_2 i\} i = \{\overline{d_1}\quad (x_i)$ if $(x_i) \in\ N_1 \overline{d_1'}\ (x_i\ )$if $(x_i) \in N_2\}\ (\overline{d_1} + \overline{d_1'})\ (x_i)$.

(b) $\overline{(s_2 \cup s_2')}\ (x_i, x_j) = \min((s_1 \cup s_1')(x_i), (s_1 \cup s_1')(x_j)) - (s_1 \cup s_1')\ (x_i, x_j)\{\min(s_1(x_i), s_1'\ (x_j) - s_2(x_i, x_j)$ if $(x_i, x_j) \in Ȩ_1 \min(s_1\ (x_i), s_1'(x_j) - s_2\ (x_i, x_j)$ if $(x_i, x_j) \in Ȩ_2 \min(s_1(x_i), s_1'\ (x_j)) - 0$ if $(x_i) \in N_1,\ x_j \in N_2\} = iii\{\overline{s_2}\ (x_i, x_j)$ if $(x_i, x_j) \in Ȩ_1 s_2'\ (x_i, x_j)$ if $(x_i, x_j) \in Ȩ_2 \min(s_1(x_i), s_1'\ (x_j))$ if $(x_i) \in N_1,\ x_j \in N_2\} = \{\overline{s_2} \cup \overline{s_2'}$ if $(x_i, x_j) \in \underline{Ȩ_1}$ or $Ȩ_1 \min(s_1(x_i), s_1'\ (x_j))$ if $(x_i, x_j) \in Ȩ' = \overline{s_2} i \cup s_2'\ (x_i.\ x_j)\overline{(i_2 \cup}\ i_2')(x_i, x_j) = \min((i_1 \cup i_1')(x_i), i(i_1 \cup i_1')\ (x_j)) - i(i_1 \cup i_1')\ (x_i, x_j) = iiii\{\min(i_1\ (x_i), i_1'(x_j)\ - i_2(x_i, x_j)$ if $(x_i, x_j) \in Ȩ_1 \min(i_1(x_i), i_1'\ (x_j) - i_2(x_i, x_j)$ if $(x_i, x_j) \in Ȩ_2 \min(i_1\ (x_i), i_1'(x_j)) - 0$ if $i(x_i) \in N_1, x_j \in N_2\} = iii\{\overline{i_2}\ (x_i, x_j)$ if $(x_i, x_j) \in Ȩ_1 i_2'(x_i,\ x_j)$ if $(x_i, x_j) \in \underline{Ȩ_2} \min(i_1(x_i), i_1'\ (x_j))$ if $i\ (x_i) \in N_1, x_j \in\ N_2 = \{\overline{i_2} \cup i_2' i(x_i, x_j)$ if $(x_i, x_j) \underline{\in}\ Ȩ_1$ or $Ȩ_1 \min(i_1(x_i),\ i_1'(x_j))$ if $(x_i, x_j) \in Ȩ' = \overline{i_2} \cup i_2'(x_i, x_j)\overline{(d_2 \cup}\ d_2')(x_i, x_j) = \max\ ((d_1 \cup d_1')(x_i), (d_1\ \cup d_1')(x_j)) - (d_1 \cup d_1')\ (x_i, x_j) = \{\max(d_1\ (x_i), d_1'(x_j) - d_2\ (x_i, x_j)$ if $(x_i,\ x_j) \in Ȩ_1 \max(d_1(x_i), d_1'(x_j) - d_2(x_i, x_j)$ if $(x_i,\ x_j) \in Ȩ_2 \max(d_1(x_i), d_1'\ (x_j)) - 0$ if $i(x_i) \in N_1, x_j \in N_2\} = \{\overline{d_2}\ (x_i, x_j)$ if $(x_i, x_j) \in Ȩ_1 d_2'\ (x_i, x_j)$ if $(x_i,\ x_j) \in Ȩ_1 \max(d_1(x_i),\ d_1'(x_j)) - 0$ if $(x_i) \in\ N_1, x_j \in\ N_2\} = \{\overline{d_2}\ \cup d_2'(x_i, x_j)$ if $(x_i, x_j) \in \underline{Ȩ_1}$ or $\underline{Ȩ_1} \min(d_1(x_i),\ d_1'(x_j))$ if $(x_i, x_j) \in Ȩ'\} = \overline{d_2} \cup d_2'(x_i, x_j)\}$ □

Definition 20. The Cartesian product of two TSFGs G_1 and G_2 is denoted and defined by $G = G_1 \times G_2 = (N_1 \times N_2, Ȩ_1 \times Ȩ_2 i)$ where

(a) $(s_1 \times\ s_1')(u_1, u_2) = \min(s_1(u_1), s_1'\ (u_2))$, for every $u_1, u_2 \in N$, $(i_1 \times i')(u_1, u_2) = \min(i_1(u_1), i_1'(u_2))$ for every $u_1, u_2 \in N$, and $(d_1 \times d_1')\ (u_1, u_2) = \max(d_1(u_1), d_1'(u_2))$ for every $u_1, u_2 \in N$.

(b) $(s_1 \times s_1')(u, u_2)(u, x_2) = \min(s_1(u), s_2(u_2, x_2) \forall\ u \in N_1$, and $u_2 x_2 \in Ȩ_2\ (i_1 \times i_1')(u, u_2)\ (u, x_2) = \min(i_1(u), i_2(u_2, x_2)\ \forall\ ui \in N_1$, and $u_2 x_2 \in Ȩ_2, (d_1 \times d_1')(u, u_2)(u, x_2) = \max(d_1(u), d_2\ (u_2, ix_2)\ \forall\ u \in N_1$, and $u_2 x_2 \in Ȩ_2$. And $(s_2 \times s_2')(u_1, w)(x_1, w) = \min(s_1(w), s_2(u_1 x_1)\ \forall\ w \in N_2,\ u_1 x_1 \in Ȩ_1$, $(i_2 \times i_2')(u_1, w)(x_1, w) = \min(i_1(w), i_2(u_1 x_1)\ \forall\ w \in N_2,\ u_1 x_1 \in Ȩ_1$, $(d_2 \times d_2')(u_1, w)(x_1, w) = \max(d_1(w), d_2(u_1 x_1), \forall\ w \in N_2,\ u_1 x_1 \in Ȩ_1$.

Example 11. Let $G = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges $Ȩ$. Figures 19–21 present an example of Cartesian product of two TSFGs.

The vertices are purely TSFNs for $n = 5$ in Figures 19–21.

Definition 21. If $G = G_1 \circ G_2 = (N_1 \times N_2, Ȩ)$ is the composition between two graphs iG_1 and iG_2, where

$$
\begin{aligned}
Ȩ = (&\{(u, u_2)(u, x_2): ui \in N_1, \text{and}\, u_2 x_2 \in Ȩ_2\} \cup \{(u_1, w)(x_1, w): w \in N_2, u_1 x_1 \in Ȩ_1\} \\
&\cup \{(u_1, u_2)(x_1, x_2): u_1 x_1 \in Ȩ_1, u_2 \neq x_2\}),
\end{aligned}
\tag{5}
$$

then the composition of TSFGs G_1 and $G_2 G = G_1 \circ G_2$ is defined by

(a) $(s_1 \circ s_1')(u_1, u_2) = \min(s_1(u_1), s_1'(u_2))$ for every $u_1, u_2 \in N_1 \times N_2$, $(i_1 \circ i_1')(u_1, u_2) = \min(i_1\ (u_1), i_1'\ (u_2))$ for every $u_1, u_2 \in N_1 \times N_2$, and $(d_1 \circ d_1')(u_1, u_2) = \min(d_1(u_1), d_1'(u_2))$ for every $u_1, u_2 \in N_1 \times N_2$.

(b) $(s_2 \circ s_2')(u, u_2)(u, x_2) = \min(s_1(u), s_2(u, x_2)$ for every $ui \in N_1$, and $u_2, x_2 \in Ȩ_2$, $(i_2 i_2')\ (u, u_2)(u, x_2) = \min(i_1(u), i_2(u, x_2)$ for every $ui \in N_1$, and $u_2, x_2 \in Ȩ_2$, $(d_2 \circ d_2')(u, u_2)(u, x_2) = \max(d_1(u), d_2\ (u, x_2)$ for every $ui \in N_1$, and $u_2,\ x_2 \in Ȩ_2$ And $(s_2 \circ s_2')(u_1, w)(x_1, w) = \min\ (s_1(w),\ s_2(u_1 \mathscr{V}_1)$ for every $w \in N_2,\ u_1 x_1 \in Ȩ_1$, $(i_2 \circ i_2')\ (u_1, w)\ (x_1, w) = \min(i_1(w), i_2(u_1 x_1)$ for every $w \in\ N_2,\ u_1 x_1 \in Ȩ_1$, $(d_2 \circ d_2')(u_1, w)(x_1, w) = \max(d_1\ (w), d_2(u_1 x_1)$ for every $w \in \widetilde{V}_2,\ u_1 x_1 \in Ȩ_1$. $(s_2 \circ s_2')(u_1, u_2)(x_1, x_2) = \min(s_1'(u_2), s_1'(x_2), s_2(u_1,\ x_1))$ for every $(u_1, u_2)(x_1, x_2) \in \widetilde{E} -\ \widetilde{E}''$, $(i_2 \circ i_2')\ (u_1, u_2)(x_1, x_2) = \min(i_1'(u_2),\ i_1'(x_2), i_2(u_1, x_1))$ for every $(u_1, u_2)(x_1, x_2) \in Ȩ - Ȩ''$, $(d_2 \circ d_2')(u_1, u_2)(x_1,\ x_2) = \max(d_1'(u_2),\ d_1'(x_2), d_2(u_1, x_1))$ for every $(u_1, u_2)(x_1, x_2) \in Ȩ - Ȩ''$, where $Ȩ'' = \{(u, u_2)(u, x_2):\ u \in N_1$ for every $iu_2 x_2 \in Ȩ_2\} \cup \{(u_1, w)(x_1, w):\ w \in N_2$ for every $u_1 x_1 \in Ȩ_1\}$.

Theorem 5. *If $G_1 = (N_1, Ȩ_1)$ and $G_2 = (N_2, Ȩ_2)$ are two strong TSFGs, then $G_1 \circ G_2$ is a TSFG.*

Proof. Straight forward. □

Example 12. Let $G = (N, Ȩ)$ represent a graph with the collection of vertices N and the collection of edges $Ȩ$. Figures 22–24 present an example of composition of two TSFGs.

The vertices shown in Figures 22–24 are purely strong TSFNs for $n = 4$.

Remark 3. All the definitions and results discussed in Sections 3 and 4 for TSFGs are valid for spherical fuzzy graphs (SFGs) if we assume that the vertices and edges are purely SFNs for $n = 2$.

4. Shortest Path Problem

There are several problems that have been solved using the concept of graphs in which the shortest path problem is the one which got great attention in recent decades. Finding the shortest path has been always a challenge in different sciences, and several algorithms have been developed so far in which Dijkstra algorithm is of great interest. As demonstrated in introduction section of this paper, this type of

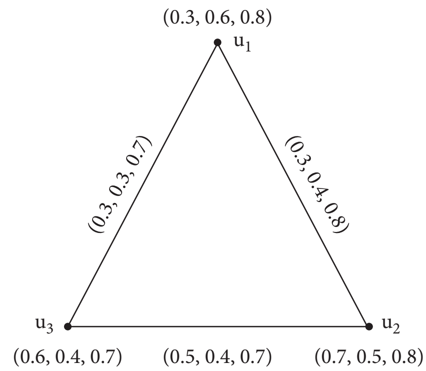

Figure 19: TSFG-A for product.

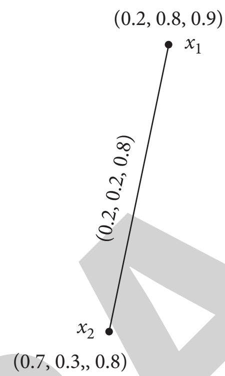

Figure 20: TSFG-B for product.

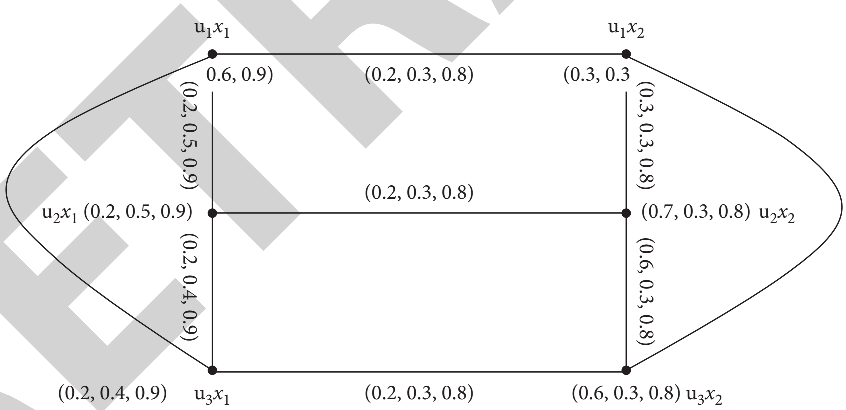

Figure 21: Product of two TSFGs.

problems has been given great importance in various fuzzy algebraic structures, which results in the development of several new approaches. Our aim is to follow the Dijkstra algorithm and apply it to a network of nodes where the path information has been provided in the form of TSFNs.

In this section, we assume a network where the shortest path must be computed from source node (SN) to destination node (DN) and the information about path between every two nodes is provided in the form of TSFNs. Usually, the shortest path is the one which is less costly or requires less time or the one on which one must travel less distance between SN and DN. The Dijkstra algorithm in T-spherical fuzzy environment is demonstrated briefly in the following.

4.1. T-Spherical Fuzzy Dijkstra Algorithm. The most reasonable approach to find shortest path in a network is to follow Dijkstra algorithm which is the successful algorithm used by many researchers such as [39–42]. The detailed steps of Dijkstra algorithm is for T-spherical fuzzy network are stated as follows.

(i) The source node is marked as permanent node (P). Moreover, it is labelled as $((0, 0, 1), -)$. Therefore, this node is involved in shortest path by default and distance travelled is zero at this stage.

(ii) Compute the label $[v_i \oplus d_{ij}, i]$ if j is not a permanent node where j is a node whose path is

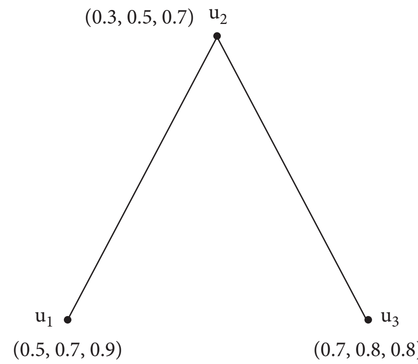

Figure 22: TSFG for composition-A.

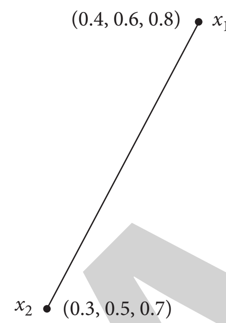

Figure 23: TSFG for composition-B.

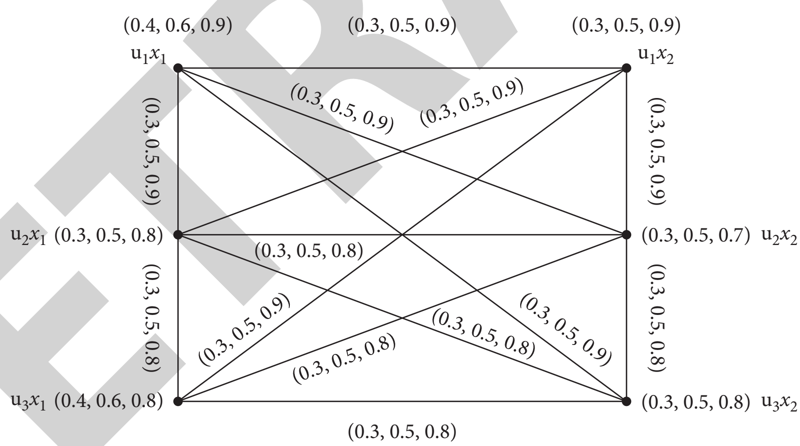

Figure 24: Composition of TSFG.

from node i. Furthermore, if j is labelled as $[v_j, ik]$ through some other node, then replace $[v_j, ik]$ by $[v_i \oplus d_{ij}, i]$ only if $\mathrm{SC}(v_i \oplus d_{ij})$ is less than $\mathrm{SC}(v_j)$.

(iii) If all of the nodes are labelled permanently, then the algorithm terminates. Otherwise, choose $[v_r, is]$ having shortest distance v_r and repeat Step 2 by setting $i = r$.

(iv) Using the information of the label, find the shortest path from SN to DN.

The flowchart the algorithm is shown in Figure 25.

Remark 4. $[v_i \oplus d_{ij}, i]$ is a label which states that the current location is node i and we travelled a distance $v_i \oplus d_{ij}$. Further, it is to be noted that the process cannot be continued to a permanent node but can be reversed. For two directly connected adjacent nodes i and j, node i is considered as the predecessor of node j if the path connecting them is directed from i to j.

Example 13. In Figure 26, a network is portrayed which is composed of 6 nodes and 8 edges. The aim is to find out the shortest path from SN (N_1) to DN (N_2) using modified Dijkstra algorithm.

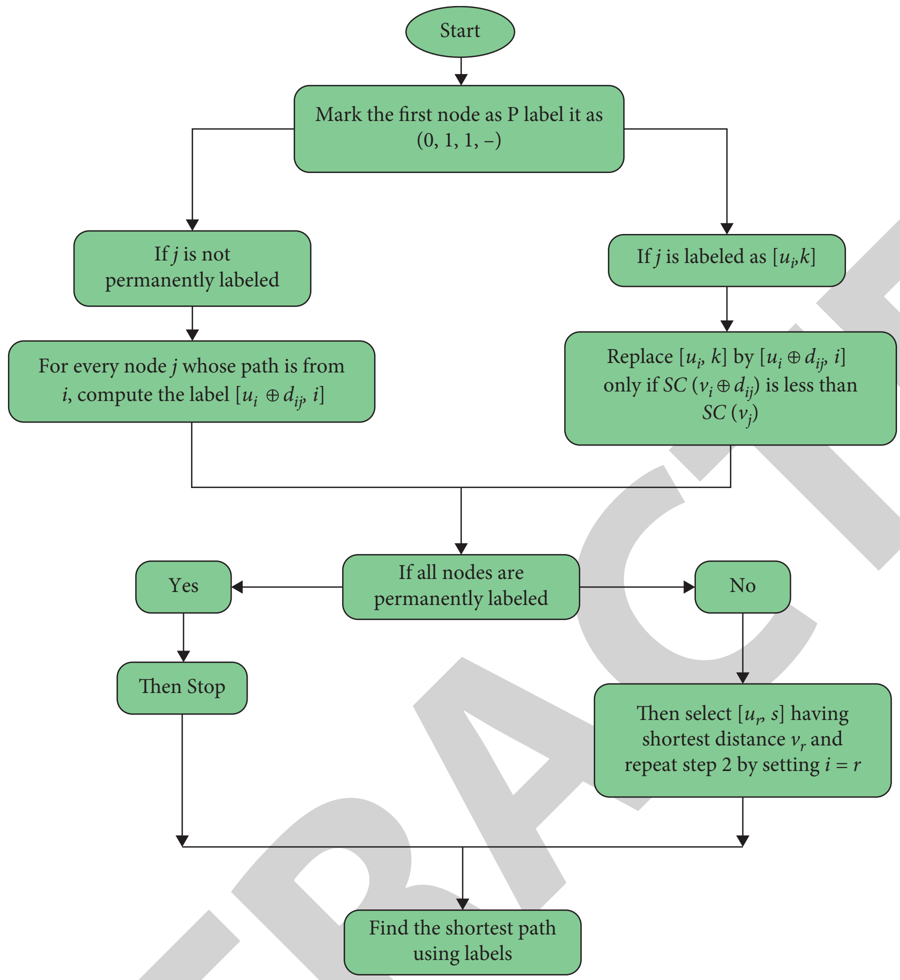

Figure 25: Flowchart of modified Dijkstra algorithm for computing the shortest path.

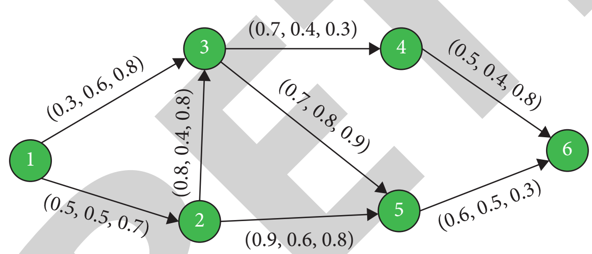

Figure 26: T-spherical fuzzy network.

The list of edges involved in this network is given in Table 1.

Now, we apply the modified Dijkstra algorithm and carry out the step-wise computations.

Step 1. Node 1 is in the shortest path by default, and thus we mark it as a permanent node.

Step 2. Node 1 is connected to two other nodes, and thus there are two ways, i.e., we might move from node 1 to node 3 or from node 1 to node 2. Hence, the list of nodes is given in Table 2.

Now we compute the scores by Definition 9 of $(0.3, 0.6, 0.8)$ and $(0.5, 0.5, 0.7)$.

$$\begin{aligned} SC(0.3, 0.6, 0.8) &= 0.00245, \\ SC(0.5, 0.5, 0.7) &= 0.022. \end{aligned} \tag{6}$$

As the score of $(0.3, 0.6, 0.8)$ is less than $(0.5, 0.5, 0.7)$, we mark node 3 as $((0.3, 0.6, 0.8), N_1)$ and label it as permanent.

Step 3. Again there are two ways to initiate from node 3, i.e., we might move from node 3 to node 5 or from node 3 to node 4. Hence, the list of nodes is given in Table 3.

Now we compute the scores of $(0.54, 0.24, 0.24)$ and $(0.54, 0.48, 0.72)$ as follows:

$$\begin{aligned} SC(0.54, 0.24, 0.24) &= 0.086, \\ SC(0.54, 0.48, 0.72) &= 0.028. \end{aligned} \tag{7}$$

As the score of $(0.54, 0.48, 0.72)$ is less than $(0.54, 0.24, 0.24)$, we mark node 5 as $((0.54, 0.48, 0.72), N_3)$ and label it as permanent.

Step 4. The only way out of node 5 leads to node 6. Hence, the list of nodes is given in Table 4.

Since there is a single way from node 5 to 6, node 6 is marked as $((0.37, 0.24, 0.23), N_5)$ and labelled as permanent.

Table 1: Weights of edges.

Edges	T-spherical distances
$(\mathbf{N}_1, \mathbf{N}_2)$	$(0.5, 0.5, 0.7)$
$(\mathbf{N}_1, \mathbf{N}_3)$	$(0.3, 0.6, 0.8)$
$(\mathbf{N}_2, \mathbf{N}_3)$	$(0.8, 0.4, 0.8)$
$(\mathbf{N}_2, \mathbf{N}_5)$	$(0.9, 0.6, 0.8)$
$(\mathbf{N}_3, \mathbf{N}_4)$	$(0.7, 0.4, 0.3)$
$(\mathbf{N}_3, \mathbf{N}_5)$	$(0.7, 0.8, 0.9)$
$(\mathbf{N}_4, \mathbf{N}_6)$	$(0.5, 0.4, 0.8)$
$(\mathbf{N}_5, \mathbf{N}_6)$	$(0.6, 0.5, 0.3)$

Table 2: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$((0, 0, 1), -)$	Permanent
$\mathbf{N}_2$	$((0.5, 0.5, 0.7), N_1)$	Temporary
$\mathbf{N}_3$	$((0.3, 0.6, 0.8), N_1)$	Temporary

Table 3: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$((0, 0, 1), -)$	Permanent
$\mathbf{N}_2$	$((0.5, 0.5, 0.7), N_1)$	Temporary
$\mathbf{N}_3$	$((0.3, 0.6, 0.8), N_1)$	Permanent
$\mathbf{N}_4$	$((0.54, 0.24, 0.24), N_3)$	Temporary
$\mathbf{N}_5$	$((0.54, 0.48, 0.72), N_3)$	Temporary

Step 5. Nodes 4 and 2 are the temporary nodes left over; therefore, their status is altered to permanent and the following list of nodes is obtained (Table 5).

Step 6. Table 6 implies the following sequence of shortest path from SN to DN, i.e., from node 1 to node 6.

Hence, according to modified Dijkstra algorithm, the shortest path is

$$N_1 \longrightarrow N_3 \longrightarrow N_5 \longrightarrow N_6. \tag{8}$$

4.1.1. Comparative Study. In this section, our aim is to analyse and compare the networks of TSFGs with existing concepts and prove the superiority of T-spherical fuzzy Dijkstra algorithm over existing approaches.

We take a network in the environment of IFSs where the information of paths is provided in IFNs. The network presented in Figure 27 is based on IFNs as all the values of paths are in IFNs and such information could be very easily converted into TSFNs if we assume the value of $i = 0$. Hence, we can determine the shortest path using the proposed approach.

Similarly, a network where information is in the form of FNs can also be transformed to a network of TSFNs by assuming the values of $i = d = 0$. For example, the network in Figure 28 is based on fuzzy information, and hence using the proposed approach of T-spherical fuzzy Dijkstra algorithm, we can easily compute the shortest path from source node to destination node.

Table 4: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$((0, 0, 1), -)$	Permanent
$\mathbf{N}_2$	$((0.5, 0.5, 0.7), N_1)$	Temporary
$\mathbf{N}_3$	$((0.3, 0.6, 0.8), N_1)$	Permanent
$\mathbf{N}_4$	$((0.54, 0.24, 0.24), N_3)$	Temporary
$\mathbf{N}_5$	$((0.54, 0.48, 0.72), N_3)$	Permanent
$\mathbf{N}_6$	$((0.37, 0.24, 0.23), N_5)$	Permanent

Table 5: List of nodes.

Nodes	Label	Status
$\mathbf{N}_1$	$((0, 0, 1), -)$	Permanent
$\mathbf{N}_2$	$((0.5, 0.5, 0.7), N_1)$	Permanent
$\mathbf{N}_3$	$((0.3, 0.6, 0.8), N_1)$	Permanent
$\mathbf{N}_4$	$((0.54, 0.24, 0.24), N_3)$	Permanent
$\mathbf{N}_5$	$((0.54, 0.48, 0.72), N_3)$	Permanent
$\mathbf{N}_6$	$((0.37, 0.24, 0.23), N_5)$	Permanent

Table 6: List of nodes.

$\mathbf{N}_6$	$((0.37, 0.24, 0.23), \mathbf{N}_5)$
$\mathbf{N}_5$	$((0.54, 0.48, 0.72), N_3)$
$\mathbf{N}_3$	$((0.3, 0.6, 0.8), N_1)$

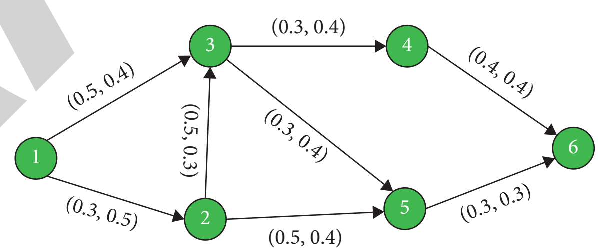

Figure 27: Intuitionistic fuzzy network.

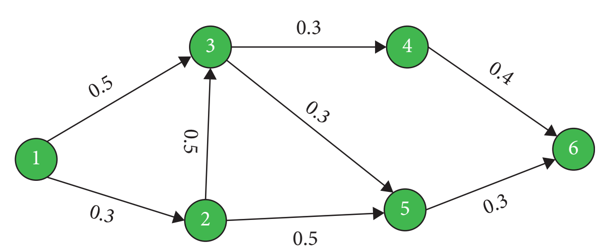

Figure 28: Fuzzy network.

5. Conclusion

In this paper, the concept of TSFG is introduced based on the novel theory of TSFSs. In view of the novelty of TSFSs, the importance of TSFGs is elaborated and it is discussed that TSFGs are generalizations of IFGs and PFGs and can be applicable in those situations where the frameworks of IFG and PFG failed to be applied. Some very basic graph theoretic terms like complement of TSFGs, T-spherical fuzzy subgraph, degree of vertices in TSFGs, strength of TSFGs, and bridges in TSFGs are defined. A study of operations of TSFGs is also established and related results are studied. The famous Dijkstra algorithm for TSFGs has been developed and the shortest path in a network of TSFGs has been solved. The main benefit of the proposed work is that it could be applied in the conditions that are handled by using the concepts of IFG or PFG, but these structures are incapable of handling the information given in the T-spherical fuzzy environment. In the near future, the framework of TSFGs could prove to be very useful tool that can be applied in the traffic signal problems, optimization in networks, and other problems of computer sciences and engineering. Additionally, the proposed work can be extended to interval-valued and cubic-valued frameworks that will give rise to much stronger and interesting structures with extended range of applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] J. N. Mordeson and P. S. Nair, "Applications of fuzzy graphs," in *Fuzzy Graphs and Fuzzy Hypergraphs*, pp. 83–133, Springer, Berlin, Germany, 2000.

[2] A. Kishore and M. Sunitha, "Strong chromatic number of fuzzy graphs," *Annals of Pure and Applied mathematics*, vol. 7, no. 2, pp. 52–60, 2014.

[3] L. Kóczy, "Fuzzy graphs in the evaluation and optimization of networks," *Fuzzy Sets and Systems*, vol. 46, no. 3, pp. 307–319, 1992.

[4] S. Samanta and M. Pal, "Telecommunication system based on fuzzy graphs," *Journal of Telecommunications System & Management*, vol. 3, no. 1, pp. 1–6, 2013.

[5] K. Sameena and M. S Sunitha, "Fuzzy graphs in fuzzy neural networks," *Proyecciones (Antofagasta)*, vol. 28, no. 3, pp. 239–252, 2009.

[6] A. Kaufmann, *Introduction à la Théorie Des Sous-Ensembles Flous à L'usage Des Ingénieurs: Éléments Théoriques De Base*, Vol. 1, Masson, Paris, France, 1973.

[7] A. Rosenfeld, "Fuzzy graphs," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pp. 77–95, Elsevier, Amsterdam, Netherlands, 1975.

[8] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[9] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987.

[10] K. R. Bhutani, "On automorphisms of fuzzy graphs," *Pattern Recognition Letters*, vol. 9, no. 3, pp. 159–162, 1989.

[11] K. R. Bhutani and A. Rosenfeld, "Strong arcs in fuzzy graphs," *Information Sciences*, vol. 152, pp. 319–322, 2003.

[12] A. N. Gani and S. Latha, "On irregular fuzzy graphs," *Applied Mathematical Sciences*, vol. 6, no. 11, pp. 517–523, 2012.

[13] A. N. Gani and K. Radha, "On regular fuzzy graphs," *Journal of Physical Sciences*, vol. 12, 2008.

[14] C. M. Klein, "Fuzzy shortest paths," *Fuzzy Sets and Systems*, vol. 39, no. 1, pp. 27–41, 1991.

[15] J. N. Mordeson, "Fuzzy line graphs," *Pattern Recognition Letters*, vol. 14, no. 5, pp. 381–384, 1993.

[16] J. N. Mordeson and P. Chang-Shyh, "Operations on fuzzy graphs," *Information Sciences*, vol. 79, no. 3-4, pp. 159–170, 1994.

[17] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[18] R. Parvathi and M. Karunambigai, "Intuitionistic fuzzy graphs," in *Computational Intelligence, Theory and Applications*, pp. 139–150, Springer, Berlin, Germany, 2006.

[19] A. N. Gani and S. S. Begum, "Degree, order and size in intuitionistic fuzzy graphs," *International Journal of Algorithms, Computing and Mathematics*, vol. 3, no. 3, pp. 11–16, 2010.

[20] M. Akram and B. Davvaz, "Strong intuitionistic fuzzy graphs," *Filomat*, vol. 26, no. 1, pp. 177–196, 2012.

[21] G. Pasi, R. Yager, and K. Atanassov, "Intuitionistic fuzzy graph interpretations of multi-person multi-criteria decision making: generalized net approach," in *Proceedings of the, 2004 2nd International IEEE Conference Intelligent Systems. Proceedings*, IEEE, Varna, Bulgaria, June 2004.

[22] R. Parvathi, M. Karunambigai, and K. T. Atanassov, "Operations on intuitionistic fuzzy graphs," in *Proceedings of the FUZZ-IEEE 2009 IEEE International Conference on Fuzzy Systems, 2009*, August 2009.

[23] S. Mishra and A. Pal, "Product of interval valued intuitionistic fuzzy graph," *Annals of pure and applied mathematics*, vol. 5, no. 1, pp. 37–46, 2013.

[24] B. Cuong, "Picture fuzzy sets-first results. part 1," *Journal of Computer Science and Cybernetics*, vol. 30, 2014.

[25] B. C. Cường, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, 409 pages, 2014.

[26] B. C. Cuong, V. Kreinovitch, and R. T. Ngan, "A classification of representable t-norm operators for picture fuzzy sets," in *Proceedings of the 2016 Eighth International Conference on Knowledge and Systems Engineering (KSE)*, October 2016.

[27] B. C. Cuong and P. Van Hai, "Some fuzzy logic operators for picture fuzzy sets," in *Proceedings of the 2015 Seventh International Conference on Knowledge and Systems Engineering (KSE)*, October 2015.

[28] H. Garg, "Some picture fuzzy aggregation operators and their applications to multicriteria decision-making," *Arabian Journal for Science and Engineering*, vol. 42, no. 12, pp. 5275–5290, 2017.

[29] P. H. Phong, D. T. Hieu, H. T. R. Ngan, and T. P. Them, "Some compositions of picture fuzzy relations," in *Proceedings of the 7th National Conference on Fundamental and Applied Information Technology Research (FAIR'7)*, Thai Nguyen, Vietnam, June 2014.

[30] C. Wang, H. Zhou, H. Tu, and S. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making," *Italian Journal of Pure and Applied Mathematics*, vol. 37, pp. 477–492, 2017.

[31] T. Al Hawary, T. Mahmood, N. Jan, K. Ullah, and A. Hussain, "On intuitionistic fuzzy graphs and some operations on picture fuzzy graphs," *Italian Journal of Pure and Applied Mathematics*, vol. 32, 2018.
[32] T. Mahmood, K. Ullah, Q. khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, 2018.
[33] S. Okada and T. Soper, "A shortest path problem on a network with fuzzy arc lengths," *Fuzzy Sets and Systems*, vol. 109, no. 1, pp. 129–140, 2000.
[34] Y. Deng, Y. Chen, Y. Zhang, and S. Mahadevan, "Fuzzy Dijkstra algorithm for shortest path problem under uncertain environment," *Applied Soft Computing*, vol. 12, no. 3, pp. 1231–1237, 2012.
[35] K.-C. Lin and M.-S. Chern, "The fuzzy shortest path problem and its most vital arcs," *Fuzzy Sets and Systems*, vol. 58, no. 3, pp. 343–353, 1993.
[36] S. Okada, "Fuzzy shortest path problems incorporating interactivity among paths," *Fuzzy Sets and Systems*, vol. 142, no. 3, pp. 335–357, 2004.
[37] F. Hernandes, M. T. Lamata, J. L. Verdegay, and A. Yamakami, "The shortest path problem on networks with fuzzy parameters," *Fuzzy Sets and Systems*, vol. 158, no. 14, pp. 1561–1570, 2007.
[38] A. N. Gani and M. M. Jabarulla, "On searching intuitionistic fuzzy shortest path in a network," *Applied Mathematical Sciences*, vol. 4, no. 69, pp. 3447–3454, 2010.
[39] S. Mukherjee, "Dijkstra's algorithm for solving the shortest path problem on networks under intuitionistic fuzzy environment," *Journal of Mathematical Modelling and Algorithms*, vol. 11, no. 4, pp. 345–359, 2012.
[40] J. Y. Kung and T. N. Chuang, "The shortest path problem with discrete fuzzy arc lengths," *Computers & Mathematics with Applications*, vol. 49, no. 2-3, pp. 263–270, 2005.
[41] T.-N. Chuang and J.-Y. Kung, "A new algorithm for the discrete fuzzy shortest path problem in a network," *Applied Mathematics and Computation*, vol. 174, no. 1, pp. 660–668, 2006.
[42] M. G. Karunambigai, P. Rangasamy, K. Atanassov, and N. Palaniappan, "An intuitionistic fuzzy graph method for finding the shortest paths in networks," in *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing*, pp. 3–10, Springer, Berlin, Germany, 2007.
[43] L. Zedam, N. Jan, E. Rak, T. Mahmood, and K. Ullah, "An approach towards decision-making and shortest path problems based on T-spherical fuzzy information," *International Journal of Fuzzy Systems*, vol. 22, no. 5, pp. 1521–1534, 2020.