Research Article

On the Spectrum of Laplacian Matrix

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Let G be a simple graph of order n . The matrix $\mathcal{L}(G) = D(G) - A(G)$ is called the Laplacian matrix of G , where $D(G)$ and $A(G)$ denote the diagonal matrix of vertex degrees and the adjacency matrix of G , respectively. Let $l_1(G)$, $l_{n-1}(G)$ be the largest eigenvalue, the second smallest eigenvalue of $\mathcal{L}(G)$ respectively, and $\lambda_1(G)$ be the largest eigenvalue of $A(G)$. In this paper, we will present sharp upper and lower bounds for $l_1(G)$ and $l_{n-1}(G)$. Moreover, we investigate the relation between $l_1(G)$ and $\lambda_1(G)$.

1. Introduction

We begin with the preliminaries which are required throughout this paper. Let G be a simple graph with vertex set $V = V(G)$ and edge set $E(G)$. The integers $n = n(G) = |V(G)|$ and $\varepsilon = \varepsilon(G) = |E(G)|$ are the order and the size of the graph G , respectively. The open neighborhood of vertex v_i is $N_G(v_i) = N(v_i) = \{v_j \in V(G) \mid v_i v_j \in E(G)\}$, and the degree of v_i is $d_G(v_i) = d_i = |N(v_i)|$. Let K_n be the complete graph of order n and $\bar{G}$ be the complement of the graph G . Let Δ and δ be the maximum degree and the minimum degree of the vertices of G , respectively. The eigenvalues of the adjacency matrix $A(G)$, are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$. The matrix $\mathcal{L}(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees, is called the Laplacian matrix of G and rarely appears in the literature. The eigenvalues of Laplacian matrix G are denoted as $l_1(G) \geq l_2(G) \geq \dots \geq l_n(G) = 0$. The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix. Many related physical quantities have the same relation to $\mathcal{L}(G)$; also, there are many problems in physics and chemistry where the Laplacian matrices of graphs and their spectra play the central role. Recently, its applications to several difficult problems in graph theory were discovered (see [1–7]).

Merris [8] discussed the Laplacian matrices of graphs. In [9], some bounds are established for Laplacian eigenvalues of graphs. Taheri et al. [10] presented some bounds for the largest Laplacian eigenvalue of graphs. Patra et al. [11] obtained bounds for the Laplacian spectral radius of graphs. In [12], the authors investigated some bounds for the Laplacian spectral radius of an oriented hypergraph. Chen [13] established some bounds for $\lambda_1(G)$.

In this paper, we first present sharp upper and lower bounds for $l_1(G)$ and $l_{n-1}(G)$, and then we investigate the relation between $l_1(G)$ and $\lambda_1(G)$.

2. Preliminaries

In this section, some fundamental results that are used in this paper are recalled. We begin with the following result, which plays a key role in this section.

Lemma 1 (see [14]). Let G be a graph of order n and size ε . Then,

$$\begin{aligned} \sum_{i=1}^{n-1} l_i &= \sum_{i=1}^n l_i = \text{tr} \mathcal{L}(G) = 2\varepsilon, \\ \sum_{i=1}^{n-1} l_i^2 &= \sum_{i=1}^n l_i^2 = \text{tr} \mathcal{L}^2(G) = 2\varepsilon + M_1(G), \end{aligned} \quad (1)$$

where $M_1(G) = \sum_{i=1}^n d_i^2$ is the well-known graph invariant called the first Zagreb index [15].

Favaron and Mahéo [16] proved the following result:

Lemma 2 (see [16]). Let G be a graph of order n . Then,

$$\lambda_1 \geq \sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}. \quad (2)$$

The proof of the next result can be found in [17].

Lemma 3. Let G be a graph of order n . Then, $l_1 = l_2 = \dots = l_{n-1}$ if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

Das in [18] proved the following lemma.

Lemma 4. Let G be a connected graph of order $n \geq 3$. Then, $l_2 = l_3 = \dots = l_{n-1}$ if and only if $G \cong K_n$.

In [14], a class of real polynomials $P_n(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + b_3 x^{n-3} + \dots + b_n$, denoted as $\mathcal{P}_n(a_1, a_2)$, where a_1 and a_2 are fixed real numbers, was considered.

Theorem 1. For the roots $y_1 \geq y_2 \geq \dots \geq y_n$ of an arbitrary polynomial $\varphi_n(y)$ from this class, the following values were introduced:

$$\begin{aligned} \bar{y} &= \frac{1}{n} \sum_{i=1}^n y_i, \\ \Gamma &= n \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n y_i \right)^2. \end{aligned} \quad (3)$$

Then upper and lower bounds for the polynomial roots, $y_i, i = 1, 2, \dots, n$, were determined in terms of the introduced values

$$\begin{aligned} \bar{y} + \frac{1}{n} \sqrt{\frac{\Gamma}{n-1}} &\leq y_1 \leq \bar{y} + \frac{1}{n} \sqrt{(n-1)\Gamma}, \\ \bar{y} - \frac{1}{n} \sqrt{\frac{(i-1)}{n-i+1} \Gamma} &\leq y_i \leq \bar{y} + \frac{1}{n} \sqrt{\frac{(n-i)}{i} \Gamma}, \quad \text{for } i = 2, 3, \dots, n-1. \end{aligned} \quad (4)$$

3. Main Results

In this section, we will obtain some sharp upper and lower bounds for $l_1(G)$ and $l_{n-1}(G)$ involving the first Zagreb index and order and size of graphs. Moreover, we investigate the relation between $l_1(G)$ and $\lambda_1(G)$. The first result is an immediate consequence of Theorem 1 and Lemma 1.

Lemma 5. Let G be a graph of order $n \geq 2$ and size ε . Then,

$$\frac{2\varepsilon + \sqrt{(n-1)(2\varepsilon + M_1(G)) - 4\varepsilon^2/n - 2}}{n-1} \leq l_1 \quad (5)$$

$$\leq \frac{2\varepsilon + \sqrt{(n-2)((n-1)(2\varepsilon + M_1(G)) - 4\varepsilon^2)}}{n-1},$$

$$\frac{2\varepsilon - \sqrt{(n-2)((n-1)(2\varepsilon + M_1(G)) - 4\varepsilon^2)/2}}{n-1} \leq l_{n-1} \leq \frac{2\varepsilon}{n-1}. \quad (6)$$

Here, we will obtain a lower and an upper bound for the largest Laplacian eigenvalue l_1 and the second smallest Laplacian eigenvalue l_{n-1} , respectively.

Theorem 2. Let G be a graph of order $n \geq 3$ and size ε . Then,

$$l_1 \geq \frac{2\varepsilon}{n-1} + \sqrt{\frac{1}{(n-1)(n-2)} \left(\frac{2\varepsilon(n-1+2\varepsilon)}{n-1} + \sum_{i=1}^n d_i^2 \right)}, \quad (7)$$

$$l_{n-1} \leq \frac{2\varepsilon}{n-1} - \sqrt{\frac{1}{(n-1)(n-2)} \left(\frac{2\varepsilon(n-1+2\varepsilon)}{n-1} + \sum_{i=1}^n d_i^2 \right)}, \quad (8)$$

and the equalities hold if and only if $G \cong K_n$ or $G \cong \overline{K}_n$.

Proof. For every fixed number t , we can write that

$$\begin{aligned} \left(\sum_{i=1}^{n-1} l_i - (n-1)l_t \right)^2 &= \left(\sum_{i=1}^{n-1} (l_i - l_t) \right)^2 \\ &= \sum_{i=1}^{n-1} (l_i - l_t)^2 + 2 \sum_{1 \leq i < j \leq n-1} (l_i - l_t)(l_j - l_t). \end{aligned} \quad (9)$$

It is not hard to see that when $t = 1$ or $t = n-1$, we get

$$\sum_{1 \leq i < j \leq n-1} (l_i - l_t)(l_j - l_t) \geq 0. \quad (10)$$

Hence, we have

$$\left(\sum_{i=1}^{n-1} l_i - (n-1)l_t \right)^2 \geq \sum_{i=1}^{n-1} (l_i - l_t)^2. \quad (11)$$

So, we can write

$$\begin{aligned} & \left(\sum_{i=1}^{n-1} l_i \right)^2 - 2(n-1)l_t \sum_{i=1}^{n-1} l_i + (n-1)^2 l_t^2 \\ & \geq \sum_{i=1}^{n-1} l_i^2 - 2l_t \sum_{i=1}^{n-1} l_i + (n-1)l_t. \end{aligned} \quad (12)$$

This is equivalent to

$$l_t^2 - \frac{2l_t \sum_{i=1}^{n-1} l_i}{n-1} + \frac{\left(\sum_{i=1}^{n-1} l_i \right)^2}{(n-1)^2 - (n-1)} \geq \frac{\sum_{i=1}^{n-1} l_i^2}{(n-1)^2 - (n-1)}, \quad (13)$$

or

$$\left(l_t - \frac{\sum_{i=1}^{n-1} l_i}{n-1} \right)^2 - \frac{\left(\sum_{i=1}^{n-1} l_i \right)^2}{(n-1)^2} + \frac{\left(\sum_{i=1}^{n-1} l_i \right)^2}{(n-1)(n-2)} \geq \frac{\sum_{i=1}^{n-1} l_i^2}{(n-1)(n-2)}. \quad (14)$$

Therefore, we have

$$\left(l_t - \frac{\sum_{i=1}^{n-1} l_i}{n-1} \right)^2 \geq \frac{1}{(n-1)(n-2)} \left(\sum_{i=1}^{n-1} l_i^2 - \frac{1}{n-1} \left(\sum_{i=1}^{n-1} l_i \right)^2 \right). \quad (15)$$

Hence, by using Lemma 1, we have

$$\sum_{i=1}^{n-1} l_i = \sum_{i=1}^n l_i = \text{tr} \mathcal{L}(G) = 2\varepsilon, \quad (16)$$

$$\sum_{i=1}^{n-1} l_i^2 = \sum_{i=1}^n l_i^2 = \text{tr} \mathcal{L}^2(G) = 2\varepsilon + M_1(G). \quad (17)$$

By combining inequalities (15)–(17), we get the following inequality:

$$\left(l_1 - \frac{2\varepsilon}{n-1} \right)^2 \geq \frac{1}{(n-1)(n-2)} \left(2\varepsilon - \frac{4\varepsilon^2}{n-1} + \sum_{i=1}^n d_i^2 \right). \quad (18)$$

By inequalities (5) and (6), we have

$$l_1 - \frac{2\varepsilon}{n-1} \geq 0, l_{n-1} - \frac{2\varepsilon}{n-1} \leq 0. \quad (19)$$

Therefore, we have

$$\begin{aligned} l_1 & \geq \frac{2\varepsilon}{n-1} + \sqrt{\frac{1}{(n-1)(n-2)} \left(\frac{2\varepsilon(n-1+2\varepsilon)}{n-1} + \sum_{i=1}^n d_i^2 \right)} \\ l_{n-1} & \leq \frac{2\varepsilon}{n-1} - \sqrt{\frac{1}{(n-1)(n-2)} \left(\frac{2\varepsilon(n-1+2\varepsilon)}{n-1} + \sum_{i=1}^n d_i^2 \right)}. \end{aligned} \quad (20)$$

If the equality in (7) holds, then the inequality in (10) must hold, and hence we have $l_1 = l_2 = \dots = l_{n-1} = 2\varepsilon/n-1$; thus, by Lemma 3, we have $G \cong K_n$ or $G \cong \overline{K}_n$. Conversely, if

$G \cong K_n$ or $G \cong \overline{K}_n$, then it is not difficult to see that the equalities in (7) and (8) hold.

Next, we present an upper bound for spectral radius of the Laplacian matrix. $\square$

Theorem 3. Let G be a connected graph of order $n \geq 2$ and size ε . Then,

$$l_1 \leq \frac{\sqrt{(16n-16)((2\varepsilon + M_1(G))(n-2) - 4\varepsilon^2)} + 8\varepsilon}{8n-8}. \quad (21)$$

Proof. Applying Lemma 1, we can write

$$\beta := \sum_{i=1}^{n-1} l_i^2 = 2\varepsilon + \sum_{i=1}^{n-1} d_i^2 = 2\varepsilon + M_1(G), \quad (22)$$

or

$$l_1^2 = \beta - \sum_{i=2}^{n-1} l_i^2 \leq \beta - \frac{1}{n-2} \left(\sum_{i=2}^{n-1} l_i \right)^2 = \beta - \frac{(2\varepsilon - l_1)^2}{n-2}. \quad (23)$$

By inequality (23), we have

$$l_1^2 \leq \beta - \frac{4\varepsilon^2 + l_1^2 - 4\varepsilon l_1}{n-2}. \quad (24)$$

Using inequality (24), we get

$$l_1^2 \left(1 + \frac{1}{n-2} \right) + \frac{4\varepsilon^2}{n-2} - \frac{4\varepsilon l_1}{n-2} - \beta \leq 0, \quad (25)$$

or

$$l_1^2(n-1) + 4\varepsilon^2 - 4\varepsilon l_1 - (n-2)\beta \leq 0. \quad (26)$$

By inequality (26), we can write

$$l_1^2(n-1) + 4\varepsilon^2 - 4\varepsilon l_1 - 2\varepsilon(n-2) - M_1(G)(n-2) \leq 0. \quad (27)$$

Solving this inequality leads to

$$l_1 \leq \frac{\sqrt{(16n-16)((2\varepsilon + M_1(G))(n-2) - 4\varepsilon^2)} + 8\varepsilon}{8n-8}. \quad (28)$$

Finally, we will describe a relationship between spectral radius (l_1) of the Laplacian matrix and the spectral radius (λ_1) of the adjacency matrix. $\square$

Theorem 4. Let G be a connected graph of order $n \geq 3$ and size ε . Then,

$$\lambda_1 \geq \sqrt{\frac{l_1^2(n-1)}{n(n-2)} + \frac{4\varepsilon^2}{n(n-2)} - \frac{4\varepsilon l_1}{n(n-2)} - \frac{2\varepsilon}{n}}, \quad (29)$$

and the equality holds if and only if $G \cong K_n$.

Proof. By inequality (26) and Lemma 2, we have

$$l_1^2(n-1) + 4\varepsilon^2 - 4\varepsilon l_1 \leq (n-2)\beta = (n-2)\left(2\varepsilon + \sum_{i=1}^n d_i^2\right), \quad (30)$$

$$\leq (n-2)(2\varepsilon + n\lambda_1^2) = 2\varepsilon(n-2) + n(n-2)\lambda_1^2. \quad (31)$$

Now suppose that the equality holds in (29). Then, all the inequalities in the proof must be equalities.

If the equality holds in (30), then inequality (23) must be equality; in other words,

$$\beta - \sum_{i=2}^{n-1} l_i^2 = \beta - \frac{1}{n-2} \left(\sum_{i=2}^{n-1} l_i \right)^2, \quad (32)$$

or

$$\sum_{i=2}^{n-1} l_i^2 = \frac{1}{n-2} \left(\sum_{i=2}^{n-1} l_i \right)^2. \quad (33)$$

Therefore, by equality (33), we get

$$l_2 = l_3 = \dots = l_{n-1}. \quad (34)$$

Hence, by Lemma 4, we get $G \cong K_n$. Conversely, one can easily see that equality holds in (29) when $G \cong K_n$. $\square$

4. Conclusion

In this paper, we established some sharp upper and lower bounds for the largest eigenvalue and the second smallest eigenvalues of Laplacian matrix involving the first Zagreb index and order and size of graphs. Moreover, we investigate a relation between the largest eigenvalues of Laplacian matrix and the adjacency matrix.

There are still open and challenging problems for researchers. For example, the problem of ABC matrix, GA matrix, and so on remains open for further investigation.

Data Availability

The data involved in the examples of our manuscript are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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