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Research Article

Some New Kinds of Fractional Integral Inequalities via Refined $(\alpha, h-m)$-Convex Function

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In this article, we present new integral inequalities for refined $(\alpha, h-m)$-convex functions using unified integral operators (12) and (13). The established results provide the refinements of several well-known integral and fractional integral inequalities.

1. Introduction

Convex functions are important in diverse fields of mathematics, statistics, engineering, and optimization. Especially in the formation of inequalities, they play a very vital role. In the subject of mathematical analysis, inequalities provide a significant contribution in developing classical concepts and notions. For example, inequalities well known as Cauchy–Schwarz inequality, Chebyshev inequality, Minkowski inequality, Hadamard inequality, and Jensen inequality are utilized frequently in pure and applied mathematics. It is always a challenge to extend, generalize, and refine such inequalities by considering new classes of functions. In this era, researchers are working on classical inequalities concerning fractional integral and derivative operators. It can be observed that the Hadamard inequality is studied more for many kinds of fractional integral and derivative operators than any other classical inequality, see [1–7] for more details.

The aim of this paper is to study the refinements of Hadamard and other integral inequalities recently studied in [8–11]. The consequences of these inequalities also provide refinements of fractional integral inequalities connected with the integral inequalities studied in the recent past.

The article is organized as follows. In Section 2, we suggest some preliminaries. In Section 3, the bounds of unified integral operators are given using refined $(\alpha, h-m)$-convex functions. These are the refinements of bounds already obtained in the literature. In Section 4, some applications of the main results are given in the form of fractional integral inequalities and their refinements.

2. Preliminaries

In this section, we give definitions of different kinds of convex functions and integral operators which will be useful in formulating the results of this paper. Throughout the paper, all the functions are assumed to be real-valued functions until specified.

Definition 1 (see [12]). A function Ω is called convex if

$$\Omega\left(tx_1' + (1-t)y_1'\right) \le t\Omega\left(x_1'\right) + (1-t)\Omega\left(y_1'\right), \qquad (1)$$

holds for all $x_1', y_1' \in I \subseteq \mathbb{R}$ and $t \in [0,1]$.

Definition 2 (see [1]). A function Ω is called (s,m)-convex if for each $x_1', y_1' \in [0,v] \subseteq \mathbb{R}$, we have

$$\Omega\left(tx_1' + m(1-t)y_1'\right) \le t^s\Omega\left(x_1'\right) + m(1-t)^s\Omega\left(y_1'\right), \qquad (2)$$

where $t \in [0,1]$ and $(s,m) \in [0,1]^2$.

Definition 3 (see [13]). A function Ω is called (α, m)-convex if for each $x_1', y_1' \in [0, v] \subseteq \mathbb{R}$, we have

$$\Omega(tx_1' + m(1-t)y_1') \leq t^{\alpha}\Omega(x_1') + m(1-t^{\alpha})\Omega(y_1'), \quad (3)$$

where $(\alpha, m) \in [0,1]^2$ and $t \in [0,1]$.

Definition 4 (see [4]). Let h: $J \longrightarrow \mathbb{R}$ is a function with $h \equiv 0$ and $(0,1) \subseteq J$. A function Ω is said to be $(h-m)$-convex, if $\Omega, h \geq 0$ and for each $x_1', y_1' \in [0, v] \subseteq \mathbb{R}$, we have

$$\Omega(tx_1' + m(1-t)y_1') \leq h(t)\Omega(x_1') + mh(1-t)\Omega(y_1'), \quad (4)$$

where $m \in [0,1]$ and $t \in (0,1)$.

Definition 5 (see [4]). Let h: $J \longrightarrow \mathbb{R}$ is a function with $h \equiv 0$ and $(0,1) \subseteq J$. A function Ω is said to be $(\alpha, h-m)$-convex, if $\Omega, h \geq 0$ and for each $x_1', y_1' \in [0, v] \subseteq \mathbb{R}$, we have

$$\Omega(tx_1' + m(1-t)y_1') \leq h(t^{\alpha})\Omega(x_1') + mh(1-t^{\alpha})\Omega(y_1'), \quad (5)$$

where $(\alpha, m) \in [0,1]^2$ and $t \in (0,1)$.

Definition 6 (see [14]). Let h: $J \longrightarrow \mathbb{R}$ be a function with $h \equiv 0$ and $(0,1) \subseteq J$. A function Ω is called refined $(\alpha, h-m)$-convex function, if $\Omega, h \geq 0$ and for each $x_1', y_1' \in [0, v] \subseteq \mathbb{R}$, we have

$$\Omega(tx_1' + m(1-t)y_1') \leq h(t^{\alpha})h(1-t^{\alpha})(\Omega(x_1') + m\Omega(y_1')), \quad (6)$$

where $(\alpha, m) \in (0,1]^2$ and $t \in (0,1)$.

Inequality (6) gives refinements of several types of convexities when $0 < h(t) < 1$, see [14].

The need for integral operators in the study of fractional derivatives is of immense importance. In the recent era, integral operators are being used extensively for producing new results in the literature. For references, see [2, 4–6]. Next, we give some fundamental integral operators which are used in this paper.

Definition 7 (see [15]). Let $\Omega \in L_1[x_1', y_1']$ and Δ be positive and increasing function having a continuous derivative on (x_1', y_1'). The left and right fractional integrals of Ω with respect to Δ on $[x_1', y_1']$ of order κ are given by

$$\begin{aligned} {}_{\Delta}^{\kappa}I_{y_1'^{+}}\Omega(x) &= \frac{1}{\Gamma(\kappa)}\int_{x_1'}^{x}(\Delta(x)-\Delta(t))^{\kappa-1}\Delta'(t)\Omega(t)\mathrm{d}t, \quad x > x_1', \\ {}_{\Delta}^{\kappa}I_{y_1'^{-}}\Omega(x) &= \frac{1}{\Gamma(\kappa)}\int_{x}^{y_1'}(\Delta(t)-\Delta(x))^{\kappa-1}\Delta'(t)\Omega(t)\mathrm{d}t, \quad x < y_1', \end{aligned} \quad (7)$$

where $\Gamma(.)$ is the gamma function and $\mathfrak{R}(\kappa) > 0$.

Definition 8 (see [16]). Let $\Omega \in L_1[x_1', y_1']$ and Δ be positive and increasing function having a continuous derivative on (x_1', y_1'). The left and right k-fractional integrals of Ω with respect to Δ on $[x_1', y_1']$ of order κ are given by

$${}_{\Delta}^{\kappa}I_{x_1'^{-}}\Omega(x) = \frac{1}{k\Gamma_k(\kappa)}\int_{x_1'}^{x}(\Delta(x)-\Delta(t))^{(\kappa/k)-1}\Delta'(t)\Omega(t)\mathrm{d}t, \quad x > x_1', \quad (8)$$

$${}_{\Delta}^{\kappa}I_{y_1'^{-}}\Omega(x) = \frac{1}{k\Gamma_k(\kappa)}\int_{x}^{y_1'}(\Delta(t)-\Delta(x))^{(\kappa/k)-1}\Delta'(t)\Omega(t)\mathrm{d}t, \quad x < y_1', \quad (9)$$

where $\Gamma_k(.)$ is the k-gamma function and $\mathfrak{R}(\kappa), k > 0$.

Definition 9 (see [17]). Let $\Omega \in L_1[x_1', y_1']$ and $x \in [x_1', y_1']$, also let $\sigma, \kappa, \alpha, \xi, \gamma, \iota \in \mathbb{C}$, $\mathfrak{R}(\kappa), \mathfrak{R}(\alpha), \mathfrak{R}(\xi) > 0, \mathfrak{R}(\iota) > \mathfrak{R}(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \mathfrak{R}(\kappa)$, then the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,x_1'^{+}}\Omega$ and $\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,y_1'^{-}}\Omega$ are defined by

$$\begin{aligned} \left(\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,x_1'^{+}}\Omega\right)(x;p) &= \int_{x_1'}^{x}(x-t)^{\alpha-1}E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\sigma(x-t)^{\kappa};p)\Omega(t)\mathrm{d}t, \\ \left(\epsilon^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,\sigma,y_1'^{-}}\Omega\right)(x;p) &= \int_{x}^{y_1'}(t-x)^{\alpha-1}E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(\sigma(t-x)^{\kappa};p)\Omega(t)\mathrm{d}t, \end{aligned} \quad (10)$$

where $E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(t;p)$ is the extended generalized Mittag–Leffler function defined as

$$E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}(t;p) = \sum_{n=0}^{\infty} \frac{\rho_p(\gamma+nk,\iota-\gamma)}{\rho(\gamma,\iota-\gamma)} \frac{(\iota)_{nk}}{\Gamma(\kappa n+\alpha)} \frac{t^n}{(\xi)_{n\delta}}. \quad (11)$$

Definition 10 (see [16]). Let Ω, Δ be real-valued functions defined over $[x_1', y_1']$ with $0 < x_1' < y_1'$, where Ω is positive and integrable and Δ is differentiable and strictly increasing. Also, let Υ/x be an increasing function on $[x_1', \infty)$ and $\alpha, \xi, \gamma, \iota \in \mathbb{C}$, $p, \kappa, \delta \geq 0$, and $0 < k \leq \delta + \kappa$. Then, for $x \in [x_1', y_1']$, the left and right integral operators are defined as

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi x_1'^{+}}\Omega\right)(x,\sigma;p) = \int_{x_1'}^{x} J_x^y\left(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\Delta;\Upsilon\right)\Delta'(y)\Omega(y)\mathrm{d}y, \quad (12)$$

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi y_1'^{-}}\Omega\right)(x,\sigma;p) = \int_{x}^{y_1'} J_y^x\left(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\Delta;\Upsilon\right)\Delta'(y)\Omega(y)\mathrm{d}y, \quad (13)$$

where

$$J_x^y\left(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\Delta;\Upsilon\right) = \frac{\Upsilon(\Delta(x)-\Delta(y))}{\Delta(x)-\Delta(y)} E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi}\left(\sigma(\Delta(x)-\Delta(y))^{\kappa};p\right). \quad (14)$$

Mittag–Leffler functions give several fractional integrals by assigning particular choice to the parameters involved in it, see Remarks 6 and 7 in [16].

3. Main Results

Throughout the paper, we use the following notation:

$$\int_0^1 h(u^{\alpha})h(1-u^{\alpha})\Delta'(x-u(x-x_1'))\mathrm{d}u = H_x^{x_1'}(u^{\alpha};h,\Delta). \quad (15)$$

Theorem 1. *Let Ω be a positive, refined $(\alpha, h-m)$-convex and integrable function defined over $[x_1', y_1']$. Also, let Υ/x be an increasing function defined on $[x_1', y_1']$ and Δ be strictly increasing and differentiable function on (x_1', y_1'). Then, for $\beta, \xi, \gamma, \iota \in \mathbb{R}$, $p, \kappa, \vartheta, \delta \geq 0$, $0 < k \leq \delta + \kappa$, and $0 < k \leq \delta + \vartheta$, the following result holds:*

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{k,\beta,\xi,y_1'^{+}}\Omega\right)(x,\sigma;p) + \left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^{-}}\Omega\right)(x,\sigma;p)\\ &\leq J_x^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(x_1') + m\Omega\left(\frac{x}{m}\right)\right)(x-x_1')H_x^{x_1'}(u^{\alpha};h,\Delta)\\ &\quad + J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1') + m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x)H_{y_1'}^{x}(v^{\alpha};h,\Delta).\end{aligned} \quad (16)$$

Proof. For the functions Υ/x and Δ, the following inequality holds:

$$J_x^t\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(t) \leq J_x^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi},\Delta;\Upsilon\right)\Delta'(t). \quad (17)$$

Using refined $(\alpha, h-m)$-convexity of Ω, one can have

$$\Omega(t) \leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(\Omega(x_1') + m\Omega\left(\frac{x}{m}\right)\right). \quad (18)$$

From (17) and (18), we have the following integral inequality:

$$\int_{x_1'}^{x} J_x^t\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\Delta'(t)\Omega(t)dt \le J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right) \times \int_{x_1'}^{x} h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\Delta'(t)\mathrm{d}t. \tag{19}$$

Using (12) of Definition 10 on the left side of inequality (19) and making change of the variable by setting $u = x - t/x - x_1'$ on the right-hand side of the above inequality, we obtain

$$\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,y_1'^{+}}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(x,\sigma;p) \le J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)(x-x_1')\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right) \times \int_0^1 h(u^{\alpha})h(1-u^{\alpha})\Delta'(x-u(x-x_1'))\mathrm{d}u. \tag{20}$$

Thus, we obtain

$$\left({}_{\Delta}\mathbb{F}_{k,\beta,\xi,y_1'^{+}}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(x,\sigma;p) \le J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)(x-x_1')\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)(x-x_1')H_x^{x_1'}(u^{\alpha};h,\Delta). \tag{21}$$

Also, for $t \in (x, y_1']$ and $x \in (x_1', y_1')$, we can write

$$J_t^x\left(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\Delta'(t) \le J_{y_1'}^x\left(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\Delta'(t). \tag{22}$$

and

$$\Omega(t) \le h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right). \tag{23}$$

From (22) and (23), we have the following integral inequality:

$$\int_x^{y_1'} J_t^x\left(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\Delta'(t)\Omega(t)dt \le J_{y_1'}^x\left(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right) \times \int_x^{y_1'} h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\Delta'(t)\mathrm{d}t. \tag{24}$$

Using (13) of Definition 10 on the left-hand side and making change of the variable by setting $v = t - x/y_1' - x$ on the right-hand side of the above inequality, we obtain

$$\left({}_{\Delta}\mathbb{F}_{\vartheta,\beta,\xi,y_1'^{-}}^{\Upsilon,\gamma,\delta,k,\iota}\Omega\right)(x,\sigma;p) \le J_{y_1'}^x\left(E_{\vartheta,\beta,\xi}^{\gamma,\delta,k,\iota},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x) \times \int_0^1 h(v^{\alpha})h(1-v^{\alpha})\Delta'(x+v(y_1'-x)\mathrm{d}v \tag{25}$$

Therefore,

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega\right)(x,\sigma;p) \leq J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x)H^{x}_{y_1'}(v^{\alpha};h,\Delta). \tag{26}$$

Combining (21) and (26), the required inequality (16) is obtained. Hence, the proof is completed.

Next, we give the refinement of Theorem 1. □

Theorem 2. *Under the assumptions of Theorem 1, if* $0<h(t)<1$, *then the following result holds:*

$$\begin{aligned}
&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^+}\Omega\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega\right)(x,\sigma;p)\\
&\leq J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)(x-x_1')H^{x_1'}_{x}(u^{\alpha};h,\Delta)\\
&\quad+J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x)H^{x}_{y_1'}(v^{\alpha};h,\Delta)\\
&\leq J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(x-x_1')\left(\Omega(x_1')H^{x_1'}_{x}(u^{\alpha};h,\Delta)+m\Omega\left(\frac{x}{m}\right)H^{x_1'}_{x}(1-u^{\alpha};h,\Delta)\right)\\
&\quad+J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x)\left(\Omega(y_1')H^{x}_{y_1'}(v^{\alpha};h,\Delta)+m\Omega\left(\frac{x}{m}\right)H^{x}_{y_1'}(1-v^{\alpha};h,\Delta)\right).
\end{aligned} \tag{27}$$

Proof. From (18) and (23), one can see that, for $0<h(t)<1$,

$$\begin{aligned}
\Omega(t)&\leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)\\
&\leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\Omega(x_1')+mh\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\Omega\left(\frac{x}{m}\right),\\
\Omega(t)&\leq h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)\\
&\leq h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\Omega(y_1')+mh\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\Omega\left(\frac{x}{m}\right).
\end{aligned} \tag{28}$$

Hence, by following the proof of Theorem 1, one can obtain (27). Hence, the proof is completed. □

Corollary 1. *Under the assumptions of Theorem 1, (16) gives the following result:*

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{k,\beta,\xi,y_1'^{+}}\Omega\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{k,\beta,\xi,y_1'^{-}}\Omega\right)(x,\sigma;p)$$
$$\leq J_x^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(x-x_1')\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)H_x^{x_1'}(u^{\alpha};h,\Delta)$$
$$+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)H_{y_1'}^{x}(v^{\alpha};h,\Delta). \tag{29}$$

Now, we give the refinement of Theorem 5 in [9] in the following corollary.

Corollary 2. *The following inequality for refined $(h-m)$-convex function holds:*

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{k,\beta,\xi,y_1'^{+}}\Omega\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^{-}}\Omega\right)(x,\sigma;p)$$
$$\leq\left(J_x^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(x_1)+m\Omega\left(\frac{x}{m}\right)\right)(\Delta(x)-\Delta(x_1'))\right.$$
$$\left.+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(\Delta(y_1')-\Delta(x))\right)\|h\|_{\infty}^{2}$$
$$\leq\left(J_x^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)(\Delta(x)-\Delta(x_1'))\right)$$
$$+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(\Delta(y_1')-\Delta(x))\Big)\|h\|_{\infty}. \tag{30}$$

Proof. Using $\alpha=1$ and $h\in L_{\infty}[0,1]$ in (27), we obtain inequality (30). □

Remark 1

(i) For $\Upsilon(x)=x^{\alpha\imath/k}\Gamma(\alpha\imath)/k\Gamma_k(\alpha\imath)$, $\alpha\imath>k>0$ with $p=\omega=0$, inequality (16) coincides with Theorem 10 in [18]

(ii) For $k=1$ along with the conditions of (i), inequality (29) coincides with Theorem 6 in [18]

(iii) For Δ as identity function along with the conditions of (i), inequality (29) coincides with Theorem 5 in [14]

(iv) For Δ as identity function and $k=1$ along with the conditions of (i), inequality (29) coincides with Theorem 1 in [14]

(v) For $h(t)=t$ and $m=1=\alpha$, inequality (16) coincides with Theorem 4 in [19]

(vi) For $h(t)=t$ and $m=1=\alpha$, inequality (29) coincides with Corollary 1 in [19]

(vii) For $h(t)=t$ and $m=1=\alpha$ along with the conditions of (i), inequality (29) coincides with Theorem 3.1 in [20]

(viii) For $h(t)\leq 1/\sqrt{2}$ along with the conditions of (iv), inequality (29) coincides with Theorem 2 in [14]

(ix) For $h(t)=t$ and $m=1=\alpha$ along with the conditions of (iii), inequality (29) coincides with Corollary 8 in [14]

(x) For $\alpha=1$ and $h(t)=t$ along with the conditions of (iii), inequality (29) coincides with Corollary 14 in [14]

(xi) For $h(t)=t^{s}$ and $\alpha=1$ along with the conditions of (iii), inequality (29) coincides with Corollary 15 in [14]

(xii) For $h(t)=t$ and $\alpha=1$ along with the conditions of (iii), inequality (29) coincides with Corollary 16 in [14]

(xiii) For $h(t)=t$ and $m=1=\alpha$ along with the conditions of (iv), inequality (29) coincides with Corollary 1 in [14]

(xiv) For $\alpha=1$ and $h(t)=t$ along with the conditions of (iv), inequality (29) coincides with Corollary 2 in [14]

(xv) For $h(t)=t^{s}$ and $\alpha=1$ along with the conditions of (iv), inequality (29) coincides with Corollary 4 in [14]

(xvi) For $h(t)=t$ and $\alpha=1$ along with the conditions of (iv), inequality (29) coincides with Corollary 5 in [14]

By using $0<h(t)<1$ and making different choices of functions h and Δ and the parameters in (16), one can get the refinements of many well-known inequalities for different classes of convex functions which are mentioned in Remark 3 in [9].

Next, we give a lemma which we will use in the proof of upcoming Theorem 3.

Lemma 1. *Let* $\Omega\colon [0,\infty) \longrightarrow \mathbb{R}$ *be a refined* $(\alpha, h-m)$*-convex function. If* $\Omega(x) = \Omega(x_1' + y_1' - x/m)$, $x \in [x_1', y_1']$, *and* $m \in (0,1]$, *then the following inequality holds:*

$$\Omega\left(\frac{x_1'+y_1'}{2}\right) \le h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)\Omega(x). \quad (31)$$

Proof. Since Ω is refined $(\alpha, h-m)$-convex, then following inequality holds:

$$\begin{aligned}\Omega\left(\frac{x_1'+y_1'}{2}\right) &\le h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\\ &\times\left[\Omega\left(\frac{x-x_1'}{y_1'-x_1'}y_1' + \frac{y_1'-x}{y_1'-x_1'}x_1'\right) + m\Omega\left(\frac{(x-x_1'/y_1'-x_1')x_1' + (y_1'-x/y_1'-x_1')y_1'}{m}\right)\right]\\ &\le h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(\Omega(x) + m\Omega\left(\frac{x_1'+y_1'-x}{m}\right)\right).\end{aligned} \quad (32)$$

Using $\Omega(x) = \Omega(x_1' + y_1' - x/m)$ in the above inequality, we obtain (31). This completes the proof. □

Remark 2.

(i) For $h(t) = t$ and $m = \alpha = 1$, (31) coincides with Lemma 1 in [19]

(ii) For $0 < h(t) < 1$, (31) gives refinement of Lemma 1 in [9]

Theorem 3. *Under the assumptions of Theorem 1, the following result holds for* $\Omega(x) = \Omega(x_1' + y_1' - x/m)$:

$$\begin{aligned}&\frac{1}{h(1/2^\alpha)h(2^\alpha-1/2^\alpha)(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\\ &\times\left(\left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}1\right)(x_1',\sigma;p) + \left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^+}1\right)(y_1',\sigma;p)\right)\\ &\le \left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p) + \left({}_\Delta\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^+}\Omega\right)(y_1',\sigma;p)\\ &\le (y_1'-x_1')\left(\Omega(y_1') + m\Omega\left(\frac{x_1'}{m}\right)\right)\left[J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)H^{x_1'}_{y_1'}(\nu^\alpha;h,\Delta)\right.\\ &\left.+J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)H^{x_1'}_{y_1'}(\nu^\alpha;h,\Delta)\right].\end{aligned} \quad (33)$$

Proof. For the kernel defined in (14) and function Δ, the following inequality holds:

$$J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x) \le J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x), \quad x \in (x_1', y_1'). \quad (34)$$

Using refined $(\alpha, h-m)$-convexity of Ω, we have

$$\Omega(x) \le h\left(\left(\frac{x-x_1'}{y_1'-x_1'}\right)^\alpha\right)h\left(1-\left(\frac{x-x_1'}{y_1'-x_1'}\right)^\alpha\right)\left(\Omega(y_1') + m\Omega\left(\frac{x_1'}{m}\right)\right). \quad (35)$$

From (34) and (35), we have the following integral inequality:

$$\begin{aligned}\int_{x_1'}^{y_1'} J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Omega(x)\Delta'(x)\mathrm{d}x &\le J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1') + m\Omega\left(\frac{x_1'}{m}\right)\right)\\ &\times\int_{x_1}^{y_1'} h\left(\left(\frac{x-x_1'}{y_1'-x_1'}\right)^\alpha\right)h\left(1-\left(\frac{x-x_1'}{y_1'-x_1'}\right)^\alpha\right)\Delta'(x)\mathrm{d}x.\end{aligned} \quad (36)$$

Using (13) of Definition 10 on the right-hand side and making change of the variable by setting $v = x - x_1'/y_1' - x_1'$ on the right-hand side of the above inequality, we obtain

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}1\Omega\right)(x_1',\sigma;p)\\&\leq J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x_1')\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta).\end{aligned}\tag{37}$$

The following inequality also holds true for $x \in (x_1', y_1')$:

$$J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\leq J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x).\tag{38}$$

From (35) and (38), the following integral inequality is obtained:

$$\begin{aligned}\int_{x_1'}^{y_1'}J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\Omega(x)\mathrm{d}x&\leq J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(\Omega(y_1')+m\Omega\left(\frac{x_1}{m}\right)\right)\\&\quad\times\int_{x_1'}^{y_1'}h\left(\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)\Delta'(x)\mathrm{d}x.\end{aligned}\tag{39}$$

Using (12) of Definition 10 on the left-hand side and making change of the variable on the right-hand side of the above inequality, we obtain

$$\begin{aligned}&\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{k,\beta,\xi,y_1'^+}1\Omega\right)(y_1',\sigma;p)\\&\leq J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x_1')\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta).\end{aligned}\tag{40}$$

Now, using Lemma 1, we can write

$$\begin{aligned}&\int_{x_1'}^{y_1'}\Omega\left(\frac{x_1'+y_1'}{2}\right)J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\mathrm{d}x\\&\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\int_{x_1'}^{y_1'}J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\Omega(x)\mathrm{d}x,\end{aligned}\tag{41}$$

which by using (13) of Definition 10 gives the following integral inequality:

$$\begin{aligned}&\frac{1}{h(1/2^{\alpha})h(2^{\alpha}-1/2^{\alpha})(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}1\right)(x_1',\sigma;p)\\&\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p).\end{aligned}\tag{42}$$

Again, using Lemma 1, we can write

$$\begin{aligned}&\Omega\left(\frac{x_1'+y_1'}{2}\right)J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\mathrm{d}x\\&\leq h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)\int_{x_1'}^{y_1'}J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(x)\Omega(x)\mathrm{d}x,\end{aligned}\tag{43}$$

which by using (12) of Definition 10 gives the following fractional integral inequality:

$$\begin{aligned}&\frac{1}{h(1/2^{\alpha})h(2^{\alpha}-1/2^{\alpha})(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^+}1\right)(y_1',\sigma;p)\\&\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^+}\Omega\right)(y_1',\sigma;p).\end{aligned}\tag{44}$$

Inequality (33) will be obtained by using (37), (40), (42), and (44).

The following theorem is the refinement of Theorem 3. □

Theorem 4. *Under the assumptions of Theorem 3, if* $0 < h(t) < 1$, *then the following refinement holds:*

$$\begin{aligned}
&\frac{1}{h(1/2^{\alpha})h(2^{\alpha}-1/2^{\alpha})(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\\
&\quad\times\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}1\right)(x_1',\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}1\right)(y_1',\sigma;p)\right)\\
&\leq\frac{1}{h(1/2^{\alpha})+mh(2^{\alpha}-1/2^{\alpha})}\Omega\left(\frac{x_1'+y_1'}{2}\right)\\
&\quad\times\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}1\right)(x_1',\sigma;p)+\left({}_{\Delta}F^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,x_1'^+}1\right)(y_1',\sigma;p)\right)\\
&\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,x_1'^+}\Omega\right)(y_1',\sigma;p)\\
&\leq(y_1'-x_1')\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)\left[J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta)\right.\\
&\quad\left.+J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta)\right]\\
&\leq(y_1'-x_1')\left(J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)+J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\right)\\
&\quad\times\left(\Omega(y_1')H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta)+m\Omega\left(\frac{x_1'}{m}\right)H^{x_1'}_{y_1'}(1-v^{\alpha};h,\Delta)\right).
\end{aligned}\tag{45}$$

Proof. From (35), one can see that, for $0 < h(t) < 1$,

$$\begin{aligned}
\Omega(x)&\leq h\left(\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)\\
&\leq h\left(\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)\Omega(y_1')+mh\left(1-\left(\frac{x-x_1'}{y_1'-x_1'}\right)^{\alpha}\right)\Omega\left(\frac{x_1'}{m}\right).
\end{aligned}\tag{46}$$

Hence, by following the proof of Theorem 3, one can obtain (45). This completes the proof. □

Corollary 3. *Under the assumptions of Theorem 3, (33) gives the following result:*

$$\begin{aligned}
&\frac{1}{h(1/2^{\alpha})h(2^{\alpha}-1/2^{\alpha})(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^-}1\right)(x_1',\sigma;p)\right.\\
&\left.+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}1\right)(y_1',\sigma;p)\right)\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}\Omega\right)(y_1',\sigma;p)\\
&\leq 2(y_1'-x_1')\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)H^{x_1'}_{y_1'}(v^{\alpha};h,\Delta).
\end{aligned}\tag{47}$$

Now, we give the refinement of Theorem 6 in [9] in the following corollary.

Corollary 4. *The following inequality for refined $(h-m)$-convex function holds:*

$$\begin{aligned}
&\frac{1}{h^2(1/2)(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,y_1'^-}1\right)(x_1',\sigma;p)+\left({}_{\Delta}F^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,x_1'^+}1\right)(y_1',\sigma;p)\right)\\
&\leq\frac{1}{h(1/2)(m+1)}\Omega\left(\frac{x_1'+y_1'}{2}\right)\left(\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p)+\left({}_{\Delta}F^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\alpha,\xi,x_1'^+}1\right)(y_1',\sigma;p)\right)\\
&\leq\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^-}\Omega\right)(x_1',\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}\Omega\right)(y_1',\sigma;p)\\
&\leq 2\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(\Delta(y_1')-\Delta(x_1'))\|h\|^2_{\infty}\\
&\leq 2\left(\Omega(y_1')+m\Omega\left(\frac{x_1'}{m}\right)\right)J^{x_1'}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(\Delta(y_1')-\Delta(x_1'))\|h\|_{\infty}.
\end{aligned}\tag{48}$$

Proof. For $h\in L_{\infty}[0,1]$ and $\alpha=1$ in (45), one can obtain (48). □

Remark 3

(i) For $h(t)=t$ and $m=1=\alpha$, inequality (33) coincides with Theorem 5 in [19]

(ii) For $h(t)=t$ and $m=1=\alpha$, inequality (47) coincides with Corollary 2 in [19]

By using $0<h(t)<1$ and making different choices of functions h and Δ and the parameters in (33), one can get the refinements of many well-known inequalities for different classes of convex functions which are mentioned in Remark 5 of [9].

Theorem 5. *Let Ω,Δ be differentiable functions such that $|\Omega'|$ is refined $(\alpha,h-m)$-convex and Δ be strictly increasing over $[x_1',y_1']$ and differentiable over $[x_1',y_1']$. Also, Υ/x be an increasing function on $[x_1',y_1']$ and $\beta,\xi,\gamma,\iota\in\mathbb{R}$, $p,\kappa,\vartheta,\delta\geq 0$, $0<k\leq\delta+\kappa$, and $0<k\leq\delta+\vartheta$. Then, for $x\in(x_1',y_1')$, we have*

$$\begin{aligned}
&\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}\Omega*\Delta\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega*\Delta\right)(x,\sigma;p)\right|\\
&\leq J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(x-x_1')\left(|\Omega'(x_1')|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x_1'}_{x}(u^{\alpha};h,\Delta)\\
&\quad+J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x)\left(|\Omega'(y_1')|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x}_{y_1'}(v^{\alpha};h,\Delta),
\end{aligned}\tag{49}$$

where

$$\begin{aligned}
\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+}\Omega*\Delta\right)(x,\sigma;p)&=\int_{x_1'}^{x}J^{t}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\Delta'(t)\Omega'(t)\mathrm{d}t,\\
\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega*\Delta\right)(x,\sigma;p)&=\int_{x}^{y_1'}J^{x}_{t}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\Delta'(t)\Omega'(t)\mathrm{d}t.
\end{aligned}\tag{50}$$

Proof. Using refined $(\alpha,h-m)$-convexity of $|\Omega\prime|$ over $[x_1',y_1']$ implies

$$|\Omega'(t)| \leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right) h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(|\Omega'(x_1')| + m\left|\Omega'\left(\frac{x}{m}\right)\right|\right). \tag{51}$$

Absolute value property implies the following relation:

$$\begin{aligned} &-h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right) h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(|\Omega'(x_1')| + m\left|\Omega'\left(\frac{x}{m}\right)\right|\right) \leq \Omega'(t) \\ &\leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right) h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(|\Omega'(x_1')| + m\left|\Omega'\left(\frac{x}{m}\right)\right|\right). \end{aligned} \tag{52}$$

From (17) and the second inequality of (52), we have the following inequality:

$$\begin{aligned} &\int_{x_1}^{x} J_x^t\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon\right)\Delta'(t)\Omega'(t)\mathrm{d}t \leq J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon\right)\left(|\Omega'(x_1')| + m\left|\Omega'\left(\frac{x}{m}\right)\right|\right) \\ &\times \int_{x_1'}^{x} h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right) h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\Delta'(t)\mathrm{d}t, \end{aligned} \tag{53}$$

which leads to the following fractional integral inequality:

$$\begin{aligned} &\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,y_1'^{+}}^{\Upsilon,\gamma,\delta,k,\iota}\Omega * \Delta\right)(x,\sigma;p) \\ &\leq J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon\right)(x-x_1')\left(|\Omega'(x_1')| + m\left|\Omega\left(\frac{x}{m}\right)\right|\right) H_x^{x_1'}(u^{\alpha}; h, \Delta). \end{aligned} \tag{54}$$

Also, inequality (17) and the first inequality of (52) give the following fractional integral inequality:

$$\begin{aligned} &\left({}_{\Delta}\mathbb{F}_{\kappa,\beta,\xi,y_1'^{+}}^{\Upsilon,\gamma,\delta,k,\iota}\Omega * \Delta\right)(x,\sigma;p) \\ &\geq -J_x^{x_1'}\left(E_{\kappa,\beta,\xi}^{\gamma,\delta,k,\iota}, \Delta; \Upsilon\right)(x-x_1')\left(|\Omega'(x_1')| + m\left|\Omega'\left(\frac{x}{m}\right)\right|\right) H_x^{x_1'}(u^{\alpha}; h, \Delta). \end{aligned} \tag{55}$$

Again, using refined $(\alpha, h-m)$-convexity of $|\Omega'|$ over $[x_1', y_1']$, we can write

$$|\Omega'(t)| \leq h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right) h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\left(m\left|\Omega'\left(\frac{x}{m}\right)\right| + |\Omega'(y_1')|\right). \tag{56}$$

and

$$h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\left(m\left|\Omega'\left(\frac{x}{m}\right)\right|+\left|\Omega'\left(y_1'\right)\right|\right)\leq\Omega'(t)$$
$$\leq h\left(\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)h\left(1-\left(\frac{t-x}{y_1'-x}\right)^{\alpha}\right)\left(m\left|\Omega'\left(\frac{x}{m}\right)\right|+\left|\Omega'\left(y_1'\right)\right|\right). \tag{57}$$

From (22) and the second inequality of (57), the following fractional integral inequality is obtained:

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega*\Delta\right)(x,\sigma;p)$$
$$\leq J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(y_1'-x\right)\left(\left|\Omega'\left(y_1'\right)\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x}_{y_1'}\left(v^{\alpha};h,\Delta\right), \tag{58}$$

and (22) and the first inequality of (57) give the following fractional integral inequality:

$$\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-}\Omega*\Delta\right)(x,\sigma;p)$$
$$\geq -J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(y_1'-x\right)\left(\left|\Omega'\left(y_1'\right)\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x}_{y_1'}\left(v^{\alpha};h,\Delta\right). \tag{59}$$

Inequality (49) will be obtained by using (54), (55), (58), and (59). Hence, the proof is completed.

Next, we give refinement of Theorem 5 in the following theorem. □

Theorem 6. *Under the assumptions of Theorem 5, if* $0<h(t)<1$*, then the following refinement holds:*

$$\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^+},\Omega*\Delta\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\Omega*\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^-},\Delta\right)(x,\sigma;p)\right|$$
$$\leq J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(x-x_1'\right)\left(\left|\Omega'\left(x_1'\right)\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x_1'}_{x}\left(u^{\alpha};h,\Delta\right)$$
$$+J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(y_1'-x\right)\left(\left|\Omega'\left(y_1'\right)\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H^{x}_{y_1'}\left(v^{\alpha};h,\Delta\right)$$
$$\leq J^{x_1'}_{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)\left(x-x_1'\right)\left(\left|\Omega'\left(x_1'\right)\right|H^{x_1'}_{x}\left(u^{\alpha};h,\Delta\right)\right.$$
$$\left.+m\left|\Omega'\left(\frac{x}{m}\right)\right|H^{x_1'}_{x}\left(1-u^{\alpha};h,\Delta\right)\right)+J^{x}_{y_1'}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)\left(y_1'-x\right)$$
$$\times\left(\left|\Omega'\left(y_1'\right)\right|H^{x}_{y_1'}\left(v^{\alpha};h,\Delta\right)+m\left|\Omega'\left(\frac{x}{m}\right)\right|H^{x}_{y_1'}\left(1-v^{\alpha};h,\Delta\right)\right). \tag{60}$$

Proof. From (51), one can see that, for $0<h(t)<1$,

$$\begin{aligned}\left|\Omega'(t)\right| &\leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)h\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left(\left|\Omega'(x_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)\\ &\leq h\left(\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left|\Omega'(x_1')\right|+mh\left(1-\left(\frac{x-t}{x-x_1'}\right)^{\alpha}\right)\left|\Omega'\left(\frac{x}{m}\right)\right|.\end{aligned} \tag{61}$$

Hence, by following the proof of Theorem 5, one can obtain (60). This completes the proof. □

Corollary 5. *Under the assumptions of Theorem 5, (49) gives the following result:*

$$\begin{aligned}&\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^{+}}\Omega*\Delta\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^{-}}\Omega*\Delta\right)(x,\sigma;p)\right|\\ &\leq J_{x}^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(x-x_1')\left(\left|\Omega'(x_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H_{x}^{x_1'}(u^{\alpha};h,\Delta)\\ &\quad+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(y_1'-x)\left(\left|\Omega'(y_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)H_{y_1'}^{x}(v^{\alpha};h,\Delta).\end{aligned} \tag{62}$$

The following corollary presents the refinement of Theorem 7 in [9].

Corollary 6. *The following inequality for refined $(h-m)$-convex function holds:*

$$\begin{aligned}&\left|\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\kappa,\beta,\xi,y_1'^{+}}\Omega*\Delta\right)(x,\sigma;p)+\left({}_{\Delta}\mathbb{F}^{\Upsilon,\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi,y_1'^{-}}1\Omega*g\right)(x,\sigma;p)\right|\\ &\leq\left[J_{x}^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(\Delta(x)-\Delta(x_1'))\left(\left|\Omega'(x_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)\right.\\ &\quad\left.+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)(\Delta(y_1')-\Delta(x))\left(\left|\Omega'(y_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)\right]\|h\|_{\infty}^{2}\\ &\leq\left[J_{x}^{x_1'}\left(E^{\gamma,\delta,k,\iota}_{\kappa,\beta,\xi},\Delta;\Upsilon\right)(\Delta(x)-\Delta(x_1'))\left(\left|\Omega'(x_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)\right.\\ &\quad\left.+J_{y_1'}^{x}\left(E^{\gamma,\delta,k,\iota}_{\vartheta,\beta,\xi},\Delta;\Upsilon\right)(\Delta(y_1')-\Delta(x))\left(\left|\Omega'(y_1')\right|+m\left|\Omega'\left(\frac{x}{m}\right)\right|\right)\right]\|h\|_{\infty}.\end{aligned} \tag{63}$$

Proof. For $h\in L_{\infty}[0,1]$ and $\alpha=1$ in (60), we obtain (63). □

Remark 4

(i) For $h(t)=t$ and $m=1=\alpha$, inequality (49) coincides with Theorem 6 in [19]

(ii) For $h(t)=t$ and $m=1=\alpha$, inequality (62) coincides with Corollary 3 in [19]

By using $0<h(t)<1$ and making different choices of functions h and Δ and the parameters in (49), one can get the refinements of many well-known inequalities for different classes of convex functions which are mentioned in Remark 6 of [9].

4. Inequalities for Fractional Integral Operators

In this section, we present the bounds of some of the fractional integral operators which will be deduced from the results of Section 3.

Proposition 1. *Under the assumptions of Theorem 1, the following result holds:*

$$\Gamma(\beta)\left(\left({}^{\beta}_{\Delta}I_{x_1'^{+}}\Omega\right)(x)+\left({}^{\beta}_{\Delta}I_{y_1'^{-}}\Omega\right)(x)\right)$$
$$\leq\left(\Delta(x)-\Delta(x_1')\right)^{\beta-1}\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)(x-x_1')H_x^{x_1'}(u^{\alpha};h,\Delta)$$
$$+\left(\Delta(y_1')-\Delta(x)\right)^{\beta-1}\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x)H_{y_1'}^{x}(v^{\alpha};h,\Delta). \tag{64}$$

Proof. For $\Upsilon(t)=t^{\beta},\beta>0$, and $p=\sigma=0$ in the proof of Theorem 1, we obtain (64). □

Proposition 2. *Under the assumptions of Theorem 1, the following inequality holds:*

$$\Gamma(\beta)\left(\left({}_{x_1'^{+}}I_{\Upsilon}\Omega\right)(x)+\left({}_{y_1'^{+}}I_{\Upsilon}\Omega\right)(x)\right)$$
$$\leq\Upsilon(x-x_1')\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)\int_0^1 h(u^{\alpha})h(1-u^{\alpha})\mathrm{d}u$$
$$+\Upsilon(y_1'-x)\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)\int_0^1 h(v^{\alpha})h(1-v^{\alpha})\mathrm{d}v. \tag{65}$$

Proof. Using Δ as identity function with $\sigma=p=0$ in the proof of Theorem 1, we obtain the required result. □

Corollary 7. *For $\Upsilon(t)=\Gamma(\beta)t^{\beta/k}/k\Gamma_k(\beta)$ with $\beta>k$ and $p=\sigma=0$, (12) and (13) reduce to the fractional integral operators (8) and (9), which satisfy the following upper bound:*

$$k\Gamma_k(\beta)\left[\left({}^{\beta}_{\Delta}I^{\kappa}_{x_1'^{+}}\Omega\right)(x)+\left({}^{\beta}_{\Delta}I^{\kappa}_{y_1'^{+}}\Omega\right)(x)\right]$$
$$\leq\left(\Delta(x)-\Delta(x_1')\right)^{(\beta/k)-1}\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)(x-x_1')H_x^{x_1'}(u^{\alpha};h,\Delta) \tag{66}$$
$$+\left(\Delta(y_1')-\Delta(x)\right)^{(\beta/k)-1}\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)(y_1'-x)H_{y_1'}^{x}(v^{\alpha};h,\Delta).$$

Remark 5. For $0<h(t)<1$, (66) gives refinement of Corollary 8 in [9].

Corollary 8. *Using $\Upsilon(t)=t^{\beta}$ and Δ as identity function for $\beta\geq 1$ along with $p=\sigma=0$, (12) and (13) give fractional integral ${}^{\beta}_{x_1'^{+}}I\Omega(x)$ and ${}^{\beta}_{y_1'^{-}}I\Omega(x)$ defined in [15], which satisfy the following upper bound:*

$$\Gamma(\beta)\left(\left({}^{\beta}_{\Delta}I_{x_1'^{+}}\Omega(x)\right)(x)+\left({}^{\beta}_{\Delta}I_{y_1'^{+}}\Omega\right)(x)\right)$$
$$\leq(x-x_1')^{\beta}\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)\int_0^1 h(u^{\alpha})h(1-u^{\alpha})\mathrm{d}u$$
$$+(y_1'-x)^{\beta}\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)\int_0^1 h(v^{\alpha})h(1-v^{\alpha})\mathrm{d}v. \tag{67}$$

Corollary 9. *Using $\Upsilon(t)=\Gamma(\beta)t^{\beta/k}/k\Gamma_k(\beta)$ and Δ as identity function along with $p=\sigma=0$, (12) and (13) reduce to the fractional integral operators ${}^{\beta}I^{\kappa}_{x_1'^{+}}\Omega(x)$ and ${}^{\beta}I^{\kappa}_{y_1'^{+}}\Omega(x)$ given in [21], which satisfy the following upper bound:*

$$\left({}^{\beta}I^{\kappa}_{x_1'^{+}}\Omega\right)(x)+\left({}^{\beta}I^{\kappa}_{y_1'^{+}}\Omega\right)(x)$$
$$\leq\frac{1}{k\Gamma_k(\beta)}\left[(x-x_1')^{\beta/k}\left(\Omega(x_1')+m\Omega\left(\frac{x}{m}\right)\right)\right.$$
$$\times\int_0^1 h(u^{\alpha})h(1-u^{\alpha})\mathrm{d}u+(y_1'-x)^{\beta/k}\left(\Omega(y_1')+m\Omega\left(\frac{x}{m}\right)\right)$$
$$\left.\times\int_0^1 h(v^{\alpha})h(1-v^{\alpha})\mathrm{d}v\right]. \tag{68}$$

Remark 6. For $0<h(t)<1$, (68) gives refinement of Corollary 10 in [9].

Corollary 10. *For $k = 1$ in Corollary 9, the following upper bound for Riemann–Liouville fractional integral is satisfied:*

$$\left({}^{\beta}I_{x_1'^+}\Omega\right)(x) + \left({}^{\beta}I_{y_1'^+}\Omega\right)(x) \leq \frac{1}{\Gamma(\beta)}\Bigg[(x - x_1')^{\beta}\left(\Omega(x_1') + m\Omega\left(\frac{x}{m}\right)\right) \times \int_0^1 h(u)^{\alpha} h(1 - u^{\alpha})\mathrm{d}u + (y_1' - x)^{\beta}\left(\Omega(y_1') + m\Omega\left(\frac{x}{m}\right)\right) \times \int_0^1 h(v^{\alpha}) h(1 - v^{\alpha})\mathrm{d}v\Bigg]. \tag{69}$$

Remark 7. For $0 < h(t) < 1$, (69) gives refinement of Corollary 11 in [9].

Similar bounds can be obtained for Theorems 3 and 5, which we leave for the reader.

5. Conclusions

This article is about the bounds of unified integral operators via refined $(\alpha, h - m)$-convexity. The obtained results are the refinements of some already published results. Moreover, some deducible fractional integral operators and their related bounds are also given.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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