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Research Article

Solutions of Integral Equations via Fixed-Point Results on Orthogonal Gauge Structure

Mazhar Mehmood,[1] Hassen Aydi,[2,3] Muhammad Usman Ali,[4] Fahimuddin,[5] Abdullah Shoaib,[1] and Manuel De La Sen[6]

[1] *Department of Mathematics and Statistics, Riphah International University, Islamabad 44000, Pakistan*
[2] *Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, Sousse 4000, Tunisia*
[3] *China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[4] *Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan*
[5] *Department of Mathematics, International Islamic University Islamabad, Islamabad, Pakistan*
[6] *Institute of Research and Development of Processes IIDP, University of the Basque Country, Campus of Leioa, P.O. Box 48940, Leioa 48940, Bizkaia, Spain*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

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The main outcome of this paper is to introduce the notion of orthogonal gauge spaces and to present some related fixed-point results. As an application of our results, we obtain existence theorems for integral equations.

1. Introduction

Fixed-point theory is a very important tool for proving the existence and uniqueness of the solutions to various mathematical models, such as integral and partial differential equations, optimization, variational inequalities, and approximation theory. Fixed-point theory has also gained considerable importance in the recent past after the famous Banach contraction theorem [1]. Since then, there have been many results related to mappings satisfying various types of contractive inequalities [2–5]. Recently, Gordji et al. [6] introduced an exciting notion of the orthogonal sets after which, orthogonal metric spaces were introduced. The concept of a sequence, continuity, and completeness has been redefined in this space. Further, they gave an extension of the Banach fixed-point theorem on this newly described shape and also applied their theorem to show the existence of a solution for a differential equation, which cannot be applied by the Banach fixed-point theorem.

On the other hand, many definitions and theorems in the literature do not require that all of the properties of a metric hold true. In the last decades, many concepts of generalized metrics (as controlled and double controlled metrics) have been introduced (see [7, 8]).

Gauge spaces are characterized by the fact that the distance between two points may be zero even if the two points are distinct. For instance, Frigon [9] and Chis and Precup [10] gave a generalization of the Banach contraction principle on gauge spaces. In the same direction, many interesting results have been raised obtained by different authors in [11–17]. In 2013, Ali et al. [18] ensured the existence of fixed points for an integral operator via a fixed-point theorem on complete gauge spaces. In 2012, Wardowski [19] gave a new type of contractions, named as F-contractions, and established new related fixed-point results. This contraction type nicely generalizes the most famous Banach contraction condition. Later on, many researchers worldwide generalized this result (see [20–24]).

To give the ongoing research a new direction, we have combined the above statements in two directions of research. For this, we apply the concept of orthogonality in gauge spaces and investigate the existence of solutions of integral equations through the fixed-point theorem on orthogonal complete gauge spaces.

2. Preliminaries and Basic Definitions

First, we include some basic definitions and theorems which are useful to understand the results presented in this paper.

Wardowski [19] introduced the family $\mathcal{F}$ of all functions $F\colon (0,\infty) \longrightarrow \mathbb{R}$ so that

(F_1): for all $\eta_1, \eta_2 \in (0,\infty)$ with $\eta_1 < \eta_2$, we have $F(\eta_1) < F(\eta_2)$

(F_2): for each positive sequence $\{\mathfrak{K}_n\}$, $\lim_{n\longrightarrow\infty} \mathfrak{K}_n = 0$ iff $\lim_{n\longrightarrow\infty} F(\mathfrak{K}_n) = -\infty$

(F_3): there is $\mathfrak{k} \in (0,1)$ so that $\lim_{\mathfrak{K}\longrightarrow 0^+} F(\mathfrak{K}) = 0$

The following are elements in $\mathcal{F}$:

(i) $F_\alpha(v) = \ln(v)$ for $v > 0$

(ii) $F_\beta(v) = v + \ln(v)$ for $v > 0$

(iii) $F_\gamma(v) = -1/\sqrt{v}$ for $v > 0$

Further, Wardowski [19] introduced F-contractions and the related fixed-point theorem in the following way.

Theorem 1 (see [19]). *Let (X,d) be a complete metric space and let $T\colon X \longrightarrow X$ be an F-contraction mapping, that is, there are $F \in \mathcal{F}$ and $\tau > 0$ so that for all $\varsigma, v \in X$, $d(T\varsigma, Tv) > 0$ implies*

$$\tau + F(d(T\varsigma, Tv)) \leq F(d(\varsigma, v)). \tag{1}$$

Then T admits a unique fixed point in X.

Minak et al. [25] generalized this result as follows.

Theorem 2 (see [25]). *Let (X,d) be a complete metric space and $T\colon X \longrightarrow X$ be a self-mapping. Suppose that there are $F \in \mathcal{F}$ and $\tau > 0$ so that*

$$\tau + F(d(T\varsigma, Tv)) \leq F\left(\max\left\{d(\varsigma, v), d(\varsigma, T\varsigma), d(v, Tv), \frac{d(\varsigma, Tv) + d(v, T\varsigma)}{2}\right\}\right), \tag{2}$$

for all $\varsigma, v \in X$, with $d(T\varsigma, Tv) > 0$. If F or T is continuous, then T admits a unique fixed point.

Now, we explain the notion of a pseudometric.

Definition 1 (see [26]). Let X be a nonempty set. A function $d\colon X \times X \longrightarrow [0,\infty)$ is said to be a pseudometric on X if for all $\varrho, \rho, \sigma \in X$:

(i) $d(\varrho, \varrho) = 0$

(ii) $d(\varrho, \rho) = d(\rho, \varrho)$

(iii) $d(\varrho, \sigma) \leq d(\varrho, \rho) + d(\rho, \sigma)$

Example 1. Denote by $X = C[0,\infty), \mathbb{R})$ the set of continuous real-valued functions $f\colon X \longrightarrow \mathbb{R}$ with $t_0 \in [0,\infty)$. This point t_0 then induces a pseudometric on X defined as $d(f,g) = |f(t_0) - g(t_0)|$ for $f, g \in X$.

In 2017, Gordji et al. [6] initiated the notion of an orthogonal set (or o-set).

Definition 2 (see [6]). Let $X \neq \varnothing$ and $\perp$ be a binary relation defined on $X \times X$. The pair $(X, \perp)$ is called an orthogonal set (or an o-set), if

$$\begin{gathered} (\forall y \in X \quad y \perp x_0) \\ \text{or } (\forall y \in X \quad x_0 \perp y). \end{gathered} \tag{3}$$

The element x_0 is said to be orthogonal. An orthogonal set may have more than one orthogonal element.

Example 2 (see [6]). Let $X = \mathbb{Z}$. We write that $\varrho \perp \rho$ if there is $k \in \mathbb{Z}$ so that $\rho = k\varrho$. Note that $0 \perp \varrho$ for each $\varrho \in \mathbb{Z}$. Hence, $(X, \perp)$ is an o-set.

Definition 3 (see [6]). Let $(X, \perp)$ be an o-set. Any two elements $\varrho, \rho \in X$ are said to be orthogonally related iff $\varrho \perp \rho$.

Definition 4 (see [6]). Let $(X, \perp)$ be an o-set. A mapping $\Gamma\colon X \longrightarrow X$ is said to be $\perp$-preserving if for all orthogonally related elements ϱ, ρ, we have $\Gamma(\varrho) \perp \Gamma(\rho)$.

Definition 5 (see [6]). Let $(X, \perp)$ be an orthogonal set. $\{\varrho_n\}_{n\in\mathbb{N}}$ is said to be an orthogonal sequence (briefly, an o-sequence) if

$$\begin{gathered} (\forall n, \quad \varrho_n \perp \varrho_{n+1}) \\ \text{or } (\forall n, \quad \varrho_{n+1} \perp \varrho_n). \end{gathered} \tag{4}$$

3. Orthogonal Gauge Space

The simplest way of defining an orthogonal gauge space is that the gauge space defined on an o-set is called an orthogonal gauge space. The precise discussion is given below.

Let $(X, \perp)$ be an o-set and $d\colon X \times X \longrightarrow [0,\infty)$ be a pseudometric on X, then $(X, d, \perp)$ is said to be an orthogonal pseudometric space (or an o-pseudometric space).

Example 3. Let $X = C([0,\infty), \mathbb{R})$ be the set of continuous real-valued functions $f\colon X \longrightarrow \mathbb{R}$ with $t_0 \in [0,\infty)$. This element induces an orthogonal pseudometric on X defined by $d(\varrho(t), \rho(t)) = |\varrho(t_0) - \rho(t_0)|$ for $\varrho(t), \rho(t) \in X$ and the orthogonality on X is given as $\varrho \perp \rho$ iff $\varrho(t)\rho(t) \geq \varrho(t)$ or $\varrho(t)\rho(t) \geq \rho(t)$.

Definition 6. Let $(X, d, \perp)$ be an orthogonal pseudometric space. Then, the orthogonal d-ball (or d-ball) of radius $\mu > 0$

and centered at $\sigma \in X$ is the set $\mathscr{B}(\sigma, d, \mu) = \{\xi \in X\colon d(\sigma, \xi) < \mu\}$.

Note that when we say that d is an o-pseudometric on X, it means that $(X, \perp)$ is an o-set and d is a pseudometric on X.

Definition 7. A family $\mathscr{M} = \{d_\lambda\colon \lambda \in \mathscr{U}\}$ of o-pseudometrics on X is said to be separating if for each pair (ϱ, ρ) such that $\varrho \neq \rho$, there is $d_\lambda \in \mathscr{M}$ with $d_\lambda(\varrho, \rho) \neq 0$.

Definition 8. Let $\mathscr{M} = \{d_\lambda\colon \lambda \in \mathscr{U}\}$ be the family of o-pseudometrics on X. Then, the topology $\mathscr{T}(\mathscr{M})$ having as a subbasis the set of balls

$$\beta(\mathscr{M}) = \{\mathscr{B}(\varrho, d_\lambda, \varepsilon)\colon \quad \varrho \in X, d_\lambda \in \mathscr{M}, \varepsilon > 0\} \tag{5}$$

is said to be the topology induced by the set $\mathscr{M}$ of o-pseudometrics. The pair $(X, \mathscr{T}(\mathscr{M}), \perp)$ is called an orthogonal gauge space.

Definition 9. Let $(X, \mathscr{T}(\mathscr{M}), \perp)$ be an orthogonal gauge space with respect to the family $\mathscr{M} = \{d_\delta\colon \delta \in \mathscr{U}\}$ of o-pseudometrics on X.

(i) An orthogonal sequence $\{\upsilon_n\}$ converges to $\upsilon \in X$ if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ so that } d_\delta(\upsilon_n, \upsilon) < \varepsilon, \quad \forall n \geq n_0, \forall \delta \in \mathscr{U}. \tag{6}$$

(ii) An orthogonal sequence $\{\upsilon_n\}$ is Cauchy if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ so that } d_\delta(\upsilon_m, \upsilon_n) < \varepsilon, \quad \forall m, n \geq n_0, \forall \delta \in \mathscr{U}. \tag{7}$$

(iii) $(X, \mathscr{T}(\mathscr{F}), \perp)$ is an o-complete gauge space iff each Cauchy o-sequence in $(X, \mathscr{T}(\mathscr{F}), \perp)$ converges to an element of X.

(iv) A subset of X is called closed if it contains the limit of each convergent o-sequence.

4. Fixed-Point Results on Orthogonal Gauge Structure

In this section, we study the existence of fixed points for a mapping satisfying certain conditions on an orthogonal gauge structure. Throughout this article, $\mathscr{U}$ is a directed set and X is a nonempty o-set with an orthogonal element (say ω_a) and also endowed with a separating o-complete gauge structure $\{d_\lambda\colon \lambda \in \mathscr{U}\}$ of o-pseudometrics.

Theorem 3. *Let X be a nonempty o-set endowed with a separating o-complete gauge structure $\{d_\lambda\colon \lambda \in \mathscr{U}\}$ of o-pseudometrics. Let $T\colon X \longrightarrow X$ be a mapping with $F \in \mathscr{F}$ and $\tau > 0$ so that*

$$\begin{aligned} \alpha(x, y) \geq 1 \Rightarrow \tau + F(d_\lambda(Tx, Ty)) \leq F(a_\lambda d_\lambda(x, y) &+ b_\lambda d_\lambda(x, Tx) + c_\lambda d_\lambda(y, Ty) + e_\lambda d_\lambda(x, Ty) \\ &+ L_\lambda d_\lambda(y, Tx)), \end{aligned} \tag{8}$$

for all $x, y \in X$ with $x \perp y$ and for each $\lambda \in \mathscr{U}$, whenever $d_\lambda(Tx, Ty) \neq 0$ for $\lambda \in \mathscr{U}$, where

$a_\lambda, b_\lambda, c_\lambda, e_\lambda, L_\lambda \geq 0$ and $a_\lambda + b_\lambda + c_\lambda + 2e_\lambda = 1$ for all $\lambda \in \mathscr{U}$. Further, assume that

(i) T is $\perp$-preserving

(ii) There is an element $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $\alpha(\omega_0, T\omega_0) \geq 1$

(iii) For each $x, y \in X$ with $x \perp y$ and $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$

(iv) For any o-sequence $\{\rho_n\}$ in X such that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for each $n \geq 1$ and $\rho_n \longrightarrow \rho$ as $n \longrightarrow \infty$, we have $\alpha(\rho_n, \rho) \geq 1$ and $\rho_n \perp \rho$ for each $n \geq 1$

Then, T possesses a fixed point.

Proof. Due to (ii), there is $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $\alpha(\omega_0, T\omega_0) \geq 1$, and by considering (iii), we get $\alpha(T\omega_0, T^2\omega_0) \geq 1$. Moreover, we have $T\omega_0 \perp T^2\omega_0$, since T is $\perp$-preserving. Repetition of the same arguments implies $\alpha(T^n\omega_0, T^{n+1}\omega_0) \geq 1$ and $T^n\omega_0 \perp T^{n+1}\omega_0$ for each $n \in \mathbb{N}$. Consider $\omega_n = T^n\omega_0$ for each $n \geq 1$. Then we say that $\{\omega_n\}$ is an o-sequence with $\alpha(\omega_n, \omega_{n+1}) \geq 1$ for each $n \geq 0$. Also, note that if there is some $m_0 \in \mathbb{N}$ so that $\omega_{m_0} = \omega_{m_0+1}$, then ω_{m_0} is a fixed point of T. Thus, we assume there does not exist any such a natural number. As $\omega_0 \in X$ with $\omega_0 \perp \omega_1$ and $\alpha(\omega_0, \omega_1) \geq 1$, then from (8), we have

$$\begin{aligned} \tau + F(d_\lambda(\omega_1, \omega_2)) &= \tau + F(d_\lambda(T\omega_0, T\omega_1)) \\ &\leq F(a_\lambda d_\lambda(\omega_0, \omega_1) + b_\lambda d_\lambda(\omega_0, T\omega_0) + c_\lambda d_\lambda(\omega_1, T\omega_1) + e_\lambda d_\lambda(\omega_0, T\omega_1) + L_\lambda d_\lambda(\omega_1, T\omega_0)) \\ &= F(a_\lambda d_\lambda(\omega_0, \omega_1) + b_\lambda d_\lambda(\omega_0, \omega_1) + c_\lambda d_\lambda(\omega_1, \omega_2) + e_\lambda d_\lambda(\omega_0, \omega_2) + L_\lambda.(0)) \\ &\leq F(a_\lambda d_\lambda(\omega_0, \omega_1) + b_\lambda d_\lambda(\omega_0, \omega_1) + c_\lambda d_\lambda(\omega_1, \omega_2) + e_\lambda)(d_\lambda(\omega_0, \omega_1) + d_\lambda(\omega_1, \omega_2)) \\ &= F((a_\lambda + b_\lambda + e_\lambda) d_\lambda(\omega_0, \omega_1) + (c_\lambda + e_\lambda) d_\lambda(\omega_1, \omega_2)), \quad \text{forall } \lambda \in \mathscr{U}. \end{aligned} \tag{9}$$

Since F is strictly increasing, from above we get

$$d_\lambda(\omega_1, \omega_2) < (a_\lambda + b_\lambda + e_\lambda) d_\lambda(\omega_0, \omega_1) + (c_\lambda + e_\lambda) d_\lambda(\omega_1, \omega_2), \tag{10}$$

that is,

$$\begin{aligned} &(1 - c_\lambda - e_\lambda) d_\lambda(\omega_1, \omega_2) \\ &\quad < (a_\lambda + b_\lambda + e_\lambda) d_\lambda(\omega_0, \omega_1), \quad \text{for all } \lambda \in \mathscr{U}. \end{aligned} \tag{11}$$

Since $a_\lambda + b_\lambda + c_\lambda + 2e_\lambda = 1$, we have

$$d_\lambda(\omega_1,\omega_2) < d_\lambda(\omega_0,\omega_1), \quad \text{for all } \lambda \in \mathcal{U}. \tag{12}$$

Now, from (9), we have

$$\tau + F(d_\lambda(\omega_1,\omega_2)) \le F(d_\lambda(\omega_0,\omega_1)), \quad \text{for all } \lambda \in \mathcal{U}. \tag{13}$$

Also, we know that $\omega_1 \perp \omega_2$ and $\alpha(\omega_1,\omega_2) \ge 1$. From (8), we get

$$\begin{aligned}\tau + F(d_\lambda(\omega_2,\omega_3)) &= \tau + F(d_\lambda(T\omega_1,T\omega_2))\\ &\le F(a_\lambda d_\lambda(\omega_1,\omega_2) + b_\lambda d_\lambda(\omega_1,T\omega_1) + c_\lambda d_\lambda(\omega_2,T\omega_2) + e_\lambda d_\lambda(\omega_1,T\omega_2) + L_\lambda d_\lambda(\omega_2,T\omega_1))\\ &= F(a_\lambda d_\lambda(\omega_1,\omega_2) + b_\lambda d_\lambda(\omega_1,\omega_2) + c_\lambda d_\lambda(\omega_2,\omega_3) + e_\lambda d_\lambda(\omega_1,\omega_3) + L_\lambda.(0))\\ &\le F(a_\lambda d_\lambda(\omega_1,\omega_2) + b_\lambda d_\lambda(\omega_1,\omega_2) + c_\lambda d_\lambda(\omega_2,\omega_3) + e_\lambda)(d_\lambda(\omega_1,\omega_2) + d_\lambda(\omega_2,\omega_3))\\ &= F((a_\lambda + b_\lambda + e_\lambda)d_\lambda(\omega_1,\omega_2) + (c_\lambda + e_\lambda)d_\lambda(\omega_2,\omega_3)), \quad \text{for all } \lambda \in \mathcal{U}.\end{aligned} \tag{14}$$

Since F is strictly increasing, again from above we get

$$d_\lambda(\omega_2,\omega_3) < (a_\lambda + b_\lambda + e_\lambda)d_\lambda(\omega_1,\omega_2) + (c_\lambda + e_\lambda)d_\lambda(\omega_2,\omega_3), \tag{15}$$

that is,

$$(1 - c_\lambda - e_\lambda)d_\lambda(\omega_2,\omega_3) < (a_\lambda + b_\lambda + e_\lambda) d_\lambda(\omega_1,\omega_2), \quad \text{for all } \lambda \in \mathcal{U}. \tag{16}$$

As $a_\lambda + b_\lambda + c_\lambda + 2e_\lambda = 1$, thus we have

$$d_\lambda(\omega_2,\omega_3) < d_\lambda(\omega_1,\omega_2). \tag{17}$$

Now, from (14), we have

$$\tau + F(d_\lambda(\omega_2,\omega_3)) \le F(d_\lambda(\omega_1,\omega_2)). \tag{18}$$

By the obtained inequalities, we get

$$\begin{aligned}F(d_\lambda(\omega_2,\omega_3)) &\le F(d_\lambda(\omega_1,\omega_2)) - \tau\\ &\le F(d_\lambda(\omega_0,\omega_1)) - 2\tau, \quad \text{for all } \lambda \in \mathcal{U}.\end{aligned} \tag{19}$$

Working on the same steps, we conclude that

$$\begin{aligned}&F(d_\lambda(\omega_n,\omega_{n+1})) \le F(d_\lambda(\omega_0,\omega_1)) - n\tau,\\ &\quad \text{for each } n \in \mathbb{N} \text{ and for all } \lambda \in \mathcal{U}.\end{aligned} \tag{20}$$

Letting $n \longrightarrow \infty$ in (20), we get $\lim_{n\longrightarrow\infty} F(d_\lambda(\omega_n,\omega_{n+1})) = -\infty$ for all $\lambda \in \mathcal{U}$. Thus, by property (F_2), we have $\lim_{n\longrightarrow\infty} d_\lambda(\omega_n,\omega_{n+1}) = 0\ \forall\, \lambda \in \mathcal{U}$. Let $(d_\lambda)_n = d_\lambda(\omega_n,\omega_{n+1})$ for all $\lambda \in \mathcal{U}$ and for each $n \in \mathbb{N}$. Using (F_3), there is $\eta \in (0,1)$ so that

$$\lim_{n\longrightarrow\infty} (d_\lambda)_n^\eta F((d_\lambda)_n) = 0, \quad \text{for all } \lambda \in \mathcal{U}. \tag{21}$$

From (20), we have

$$\begin{aligned}&(d_\lambda)_n^\eta F((d_\lambda)_n) - (d_\lambda)_n^\eta F((d_\lambda)_0) \le -(d_\lambda)_n^\eta n\tau\\ &\le 0, \quad \text{for each } n \in \mathbb{N} \text{ and } \lambda \in \mathcal{U}.\end{aligned} \tag{22}$$

Letting $n \longrightarrow \infty$ in (22), we get

$$\lim_{n\longrightarrow\infty} n(d_\lambda)_n^\eta = 0, \quad \text{for all } \lambda \in \mathcal{U}. \tag{23}$$

This implies that there is $n_1 \in \mathbb{N}$ with $n(d_\lambda)_n^\eta \le 1$ for each $n \ge n_1$ and for all $\lambda \in \mathcal{U}$. Thus, we have

$$(d_\lambda)_n \le \frac{1}{n^{1/n}}, \quad \text{for each } n \ge n_1 \text{ and } \lambda \in \mathcal{U}. \tag{24}$$

To ensure that $\{\omega_n\}$ is a Cauchy o-sequence, take $m, n \in \mathbb{N}$ with $m > n > n_1$. Using (24) and triangular inequality, we have

$$\begin{aligned}d_\lambda(\omega_n,\omega_m) &\le d_\lambda(\omega_n,\omega_{n+1}) + d_\lambda(\omega_{n+1},\omega_{n+2}) + \cdots + d_\lambda(\omega_{m-1},\omega_m)\\ &= \sum_{i=n}^{m-1} (d_\lambda)_i \le \sum_{i=n}^{\infty} (d_\lambda)_i \le \sum_{i=n}^{\infty} \frac{1}{i^{/1\eta}}, \quad \text{for all } \lambda \in \mathcal{U}.\end{aligned} \tag{25}$$

The series $\sum_{i=n}^{\infty} 1/i^{1/n}$ is convergent, so $\lim_{n,m\longrightarrow\infty} d_\lambda(\omega_n,\omega_m) = 0$ for all $\lambda \in \mathcal{U}$. It yields that $\{\omega_n\}$ is a Cauchy o-sequence. Since X is o-complete, there is $\omega^* \in X$ so that $\omega_n \longrightarrow É^*$ as $n \longrightarrow \infty$. By (iv), we have $\alpha(\omega_n,\omega^*) \ge 1$ and $\omega_n \perp \omega^*$ for each $n \in \mathbb{N}$. We now claim that $d_\lambda(\omega^*, T\omega^*) = 0$ for all $\lambda \in \mathcal{U}$. On contrary, suppose that there is $\lambda_0 \in \mathcal{U}$ with $d_{\lambda_0}(\omega^*, T\omega^*) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $d_{\lambda_0}(\omega_{n+1}, T\omega^*) > 0$ for each $n \geq n_0$. Now, note that we have $\omega_n, \omega^* \in X$ with $\alpha(\omega_n, \omega^*) \geq 1$, $\omega_n \perp \omega^*$, and $d_{\lambda_0}(T\omega_n, T\omega^*) > 0$ for all $n \geq n_0$. Then, from (8), we get

$$\begin{aligned} \tau + F\left(d_{\lambda_0}(T\omega_n, T\omega^*)\right) \leq F\Big(& a_{\lambda_0} d_{\lambda_0}(\omega_n, \omega^*) \\ &+ b_{\lambda_0} d_{\lambda_0}(\omega_n, T\omega_n) + c_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega^*) + e_{\lambda_0} d_{\lambda_0}(\omega_n, T\omega^*) + L_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega_n)\Big), \quad \text{for all } n \geq n_0. \end{aligned} \tag{26}$$

This implies that

$$d_{\lambda_0}(T\omega_n, T\omega^*) < a_{\lambda_0} d_{\lambda_0}(\omega_n, \omega^*) + b_{\lambda_0} d_{\lambda_0}(\omega_n, T\omega_n) + c_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega^*) + e_{\lambda_0} d_{\lambda_0}(\omega_n, T\omega^*) + L_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega_n), \quad \text{for all } n \geq n_0. \tag{27}$$

Thus, by considering the triangular property and (27), we have for each $n \geq n_0$

$$\begin{aligned} d_{\lambda_0}(\omega^*, T\omega^*) &\leq d_{\lambda_0}(\omega^*, \omega_{n+1}) + d_{\lambda_0}(\omega_{n+1}, T\omega^*) \\ &= d_{\lambda_0}(\omega^*, \omega_{n+1}) + d_{\lambda_0}(T\omega_n, T\omega^*) \\ &< d_{\lambda_0}(\omega^*, \omega_{n+1}) + a_{\lambda_0} d_{\lambda_0}(\omega_n, \omega^*) + b_{\lambda_0} d_{\lambda_0}(\omega_n, \omega_{n+1}) + c_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega^*) \\ &\quad + e_{\lambda_0} d_{\lambda_0}(\omega_n, T\omega^*) + L_{\lambda_0} d_{\lambda_0}(\omega^*, \omega_{n+1}). \end{aligned} \tag{28}$$

At the limit $n \longrightarrow \infty$, one obtains

$$d_{\lambda_0}(\omega^*, T\omega^*) \leq \left(c_{\lambda_0} + e_{\lambda_0}\right) d_{\lambda_0}(\omega^*, T\omega^*) < d_{\lambda_0}(\omega^*, T\omega^*). \tag{29}$$

It is a contradiction, so $d_\lambda(\omega^*, T\omega^*) = 0$ for all $\lambda \in \mathcal{U}$. As X is separating, we obtain $\omega^* = T\omega^*$. □

Example 4. Let $X = D([0, 100], \mathbb{R})$ be the collection of all twice differentiable real-valued functions. Take the metric

$$d_n(x, y) = \max_{t \in [0,n]} |x(t) - y(t)| \tag{30}$$

for each $n \in \{1, 2, 3, \ldots, 100\}$. For $x, y \in X$, consider $x \perp y \Leftrightarrow xy = 0$. Let $T: X \longrightarrow X$ be defined by

$$Tx(t) = \begin{cases} \dfrac{x(t)}{3} + \dfrac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}, & x(t) \geq 0\ \forall t, \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}, & \text{otherwise.} \end{cases} \tag{31}$$

Given $\alpha: X \times X \longrightarrow [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1, & x, y \text{ both are constant or linear functions,} \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Suppose that $x, y \in X$ with $x \perp y$ and $\alpha(x, y) = 1$. Then x and y are both constant functions with at least one of them is the zero function. Say $x = 0$ and $y \neq 0$. Then, we have the following two cases:

Case 1: for $x = 0$ and $y > 0$, we have $Tx = 0$ and $Ty = y/3$. Thus, $d_n(Tx, Ty) = y/3$ and $d_n(x, y) = y$ for each $n \in \{1, 2, 3, \ldots, 100\}$.

Case 2: for $x = 0$ and $y < 0$, we have $Tx = 0$ and $Ty = 0$. Thus, $d_n(Tx, Ty) = 0$ and $d_n(x, y) = -y$ for each $n \in \{1, 2, 3, \ldots, 100\}$.

Take $a_\lambda = 1$ and $b_\lambda = c_\lambda = e_\lambda = L_\lambda = 0$ for all $\lambda \in \{1, 2, 3, \ldots, 100\}$. Then

$$\alpha(x, y) \geq 1 \Rightarrow d_\lambda(Tx, Ty) \leq \frac{a_\lambda}{3} d_\lambda(x, y) \leq \frac{a_\lambda}{\exp(1)} d_\lambda(x, y). \tag{33}$$

By taking $\tau = 1$ and $F(t) = \ln t$, one can conclude that (8) holds for all $x, y \in X$ with $x \perp y$ and for each $\lambda \in \mathcal{U} = \{1, 2, 3, \ldots, 100\}$.

Also, for $x \perp y$, we have $x(t)y(t) = 0$ for each t, then at least one of them is the zero function, and hence, $Tx(t)Ty(t) = 0$ for each t, that is, $Tx \perp Ty$. Thus, T is $\perp$-preserving. Further, for $w_0 = -1$, $Tw_0 = 0$, and so $w_0 \perp Tw_0$ and $\alpha(w_0, Tw_0) = 1$. Furthermore, for all $x, y \in X$ with $x \perp y$ and $\alpha(x, y) = 1$, we know that x and y are constant functions, then Tx and Ty are also constant functions, and so $\alpha(Tx, Ty) = 1$.

Moreover, for each o-sequence $\{\rho_n\}$ in X with $\alpha(\rho_n, \rho_{n+1}) = 1$ for each $n \geq 1$ and $\rho_n \longrightarrow \rho$ as $n \longrightarrow \infty$, we have $\rho = 0$. Therefore, $\alpha(\rho_n, \rho) \geq 1$ and $\rho_n \perp \rho$ for each $n \geq 1$.

Consequently, all the conditions in Theorem 3 are verified. One can conclude that T possesses a fixed point.

Remark 1. For the functions defined in the above example, note that (8) does not hold for every $x, y \in X$. It suffices to take $x(t) = -5t$ and $y(t) = -15t$, then $Tx(t) = -5$ and $Ty(t) = -15$. Also note that $d_1(x, y) = 10$ and $d_1(Tx, Ty) = 10$.

Theorem 4. *Let X be a nonempty o-set endowed with a separating o-complete gauge structure $\{d_\lambda: \lambda \in \mathcal{U}\}$ of o-pseudometrics. Let $T: X \longrightarrow X$ be a self-mapping with $F \in \mathcal{F}$ and $\tau > 0$ so that*

$$\alpha(x, y) \geq 1 \Rightarrow \tau + F(d_\lambda(Tx, Ty)) \leq F\left(a_\lambda \max\left\{d_\lambda(x, y), d_\lambda(x, Tx), d_\lambda(y, Ty), \frac{d_\lambda(x, Ty) + d_\lambda(y, Tx)}{2}\right\} + b_\lambda d_\lambda(y, Tx)\right), \tag{34}$$

for all $x, y \in X$ with $x \perp y$ and for each $\lambda \in \mathcal{U}$, whenever $d_\lambda(Tx, Ty) \neq 0$ for $\lambda \in \mathcal{U}$, where a_λ, b_λ are positive real numbers with $a_\lambda + b_\lambda = 1$ for all $\lambda \in \mathcal{U}$. Further, assume that

(i) T is $\perp$-preserving

(ii) There is $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $\alpha(\omega_0, T\omega_0) \geq 1$

(iii) For each $x, y \in X$ with $x \perp y$ and $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$

(iv) For any o-sequence $\{\rho_n\}$ in X so that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $\rho_n \longrightarrow \rho$ as $n \longrightarrow \infty$, we have $\alpha(\rho_n, \rho) \geq 1$ and $\rho_n \perp \rho$ for each $n \in \mathbb{N}$

Then, T admits a fixed point.

Proof. Using (ii), there is $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $\alpha(\omega_0, T\omega_0) \geq 1$, and by considering (iii), we get $\alpha(T\omega_0, T^2\omega_0) \geq 1$. Moreover, we have $T\omega_0 \perp T^2\omega_0$, since T is $\perp$-preserving. Repetition of the same arguments implies that $\alpha(T^n\omega_0, T^{n+1}\omega_0) \geq 1$ and $T^n\omega_0 \perp T^{n+1}\omega_0$ for each $n \geq 1$. Consider $\omega_n = T^n\omega_0$. Then $\{\omega_n\}$ is an o-sequence with $\alpha(\omega_n, \omega_{n+1}) \geq 1$ for each $n \geq 0$. Also, note that if there is some $m_0 \geq 1$ such that $\omega_{m_0} = \omega_{m_0+1}$, then ω_{m_0} is a fixed point of T. Thus, we assume that such a natural number does not exist. As $\omega_0 \in X$ with $\omega_0 \perp \omega_1$ and $\alpha(\omega_0, \omega_1) \geq 1$, then from (34), we get

$$\begin{aligned}\tau + F(d_\lambda(\omega_1, \omega_2)) &= \tau + F(d_\lambda(T\omega_0, T\omega_1)) \\ &\leq F\left(a_\lambda \max\left\{d_\lambda(\omega_0, \omega_1), d_\lambda(\omega_0, T\omega_0), d_\lambda(\omega_1, T\omega_1), \frac{d_\lambda(\omega_1, T\omega_0) + d_\lambda(\omega_0, T\omega_1)}{2}\right\} + b_\lambda d_\lambda(\omega_1, T\omega_0)\right) \\ &< F(\max\{d_\lambda(\omega_0, \omega_1), d_\lambda(\omega_1, \omega_2)\}), \quad \text{for all } \lambda \in \mathcal{U}.\end{aligned} \tag{35}$$

If we assume that $\max\{d_\lambda(\omega_0, \omega_1), d_\lambda(\omega_1, \omega_2)\} = d_\lambda(\omega_1, \omega_2)$, then we have a contradiction with respect to (35). Thus, $\max\{d_\lambda(\omega_0, \omega_1), d_\lambda(\omega_1, \omega_2)\} = d_\lambda(\omega_0, \omega_1)$ for all $\lambda \in \mathcal{U}$. Using (35), we have

$$\tau + F(d_\lambda(\omega_1, \omega_2)) < F(d_\lambda(\omega_0, \omega_1)), \quad \text{for all } \lambda \in \mathcal{U}. \tag{36}$$

Again, we know that $\omega_1 \perp \omega_2$ and $\alpha(\omega_1, \omega_2) \geq 1$; then, from (34), we have

$$\begin{aligned}\tau + F(d_\lambda(\omega_2, \omega_3)) &= \tau + F(d_\lambda(T\omega_1, T\omega_2)) \\ &\leq F\left(a_\lambda \max\left\{d_\lambda(\omega_1, \omega_2), d_\lambda(\omega_1, T\omega_1), d_\lambda(\omega_2, T\omega_2), \frac{d_\lambda(\omega_2, T\omega_1) + d_\lambda(\omega_1, T\omega_2)}{2}\right\} + b_\lambda d_\lambda(\omega_2, T\omega_1)\right) \\ &< F(\max\{d_\lambda(\omega_1, \omega_2), d_\lambda(\omega_2, \omega_3)\}), \quad \text{for all } \lambda \in \mathcal{U}.\end{aligned} \tag{37}$$

If we assume that $\max\{d_\lambda(\omega_1,\omega_2), d_\lambda(\omega_2,\omega_3)\} = d_\lambda(\omega_2,\omega_3)$, then we have a contradiction to (37). Thus, $\max\{d_\lambda(\omega_1,\omega_2), d_\lambda(\omega_2,\omega_3)\} = d_\lambda(\omega_1,\omega_2)\ \forall\lambda\in\mathscr{U}$. Thus, from (37), we have

$$\tau + F(d_\lambda(\omega_2,\omega_3)) < F(d_\lambda(\omega_1,\omega_2)), \quad \text{for all } \lambda\in\mathscr{U}. \tag{38}$$

From (36) and (38), we have

$$F(d_\lambda(\omega_2,\omega_3)) < F(d_\lambda(\omega_0,\omega_1)) - 2\tau, \quad \text{for all } \lambda\in\mathscr{U}. \tag{39}$$

Working with the same steps, we obtain

$$F(d_\lambda(\omega_n,\omega_{n+1})) < F(d_\lambda(\omega_0,\omega_1)) - n\tau, \quad \text{for each } n\in\mathbb{N} \text{ for all } \lambda\in\mathscr{U}. \tag{40}$$

Letting $n\longrightarrow\infty$ in (40), we get $\lim_{n\longrightarrow\infty}F(d_\lambda(\omega_n,\omega_{n+1})) = -\infty$ for all $\lambda\in\mathscr{U}$. Thus, by property (F_2), we have $\lim_{n\longrightarrow\infty}d_\lambda(\omega_n,\omega_{n+1}) = 0$. Let $(d_\lambda)_n = d_\lambda(\omega_n,\omega_{n+1})$ for all $\lambda\in\mathscr{U}$ and for each $n\in\mathbb{N}$. From (F_3), there is $\eta\in(0,1)$ so that

$$\lim_{n\longrightarrow\infty}(d_\lambda)_n^\eta F((d_\lambda)_n) = 0, \quad \text{for all } \lambda\in\mathscr{U}. \tag{41}$$

From (40), we have

$$(d_\lambda)_n^\eta F((d_\lambda)_n) - (d_\lambda)_n^\eta F((d_\lambda)_0) < -(d_\lambda)_n^\eta n\tau \le 0 \quad \text{for each } n\in\mathbb{N} \text{ and for all } \lambda\in\mathscr{U}. \tag{42}$$

Letting $n\longrightarrow\infty$ in (42), we get

$$\lim_{n\longrightarrow\infty} n(d_\lambda)_n^\eta = 0, \quad \forall\lambda\in\mathscr{U}. \tag{43}$$

This implies that there is $n_1\in\mathbb{N}$ so that $n(d_\lambda)_n^\eta \le 1$ for each $n\ge n_1$ and for all $\lambda\in\mathscr{U}$. Thus, we have

$$(d_\lambda)_n \le \frac{1}{n^{1/\eta}}, \quad \text{for each } n\ge n_1 \text{ and for all } \lambda\in\mathscr{U}. \tag{44}$$

We claim that $\{\omega_n\}$ is a Cauchy o-sequence. Take the integers m,n with $m>n>n_1$. Using (44) and the triangular inequality, one writes

$$\begin{aligned} d_\lambda(\omega_n,\omega_m) &\le d_\lambda(\omega_n,\omega_{n+1}) + d_\lambda(\omega_{n+1},\omega_{n+2}) + \cdots + d_\lambda(\omega_{m-1},\omega_m) \\ &= \sum_{i=n}^{m-1}(d_\lambda)_i \le \sum_{i=n}^{\infty}(d_\lambda)_i \le \sum_{i=n}^{\infty}\frac{1}{i^{1/\eta}}, \quad \text{for all } \lambda\in\mathscr{U}. \end{aligned} \tag{45}$$

The series $\sum_{i=n}^{\infty}1/i^{1/\eta}$ converges, so $\lim_{n,m\longrightarrow\infty}d_\lambda(\omega_n,\omega_m) = 0$ for all $\lambda\in\mathscr{U}$. That is, $\{\omega_n\}$ is a Cauchy o-sequence. Since X is o-complete, there is $\omega^*\in X$ so that $\omega_n\longrightarrow\omega^*$ as $n\longrightarrow\infty$. By (iv), we have $\alpha(\omega_n,\omega^*)\ge 1$ and $\omega_n\perp\omega^*$ for each $n\in\mathbb{N}$. We now claim that $d_\lambda(\omega^*,T\omega^*) = 0$ for all $\lambda\in\mathscr{U}$. On contrary, suppose that there is $\lambda_0\in\mathscr{U}$ with $d_{\lambda_0}(\omega^*,T\omega^*)>0$. Then, there exists $n_0\in\mathbb{N}$ such that $d_{\lambda_0}(\omega_{n+1},T\omega^*)>0$ for each $n\ge n_0$. Now, note that $\omega_n,\omega^*\in X$ with $\alpha(\omega_n,\omega^*)\ge 1$, $\omega_n\perp\omega^*$ and $d_{\lambda_0}(T\omega_n,T\omega^*)>0$ for all $n\ge n_0$. Then, from (34), we get

$$\begin{aligned} \tau + F\left(d_{\lambda_0}(T\omega_n,T\omega^*)\right) \le F\Bigg(& a_{\lambda_0}\max\left\{d_{\lambda_0}(\omega_n,\omega^*), d_{\lambda_0}(\omega_n,T\omega_n), d_{\lambda_0}(\omega^*,T\omega^*), \frac{d_{\lambda_0}(\omega^*,T\omega_n) + d_{\lambda_0}(\omega_n,T\omega^*)}{2}\right\} \\ & + b_{\lambda_0}d_{\lambda_0}(\omega^*,T\omega_n)\Bigg). \end{aligned} \tag{46}$$

This implies that

$$d_{\lambda_0}(T\omega_n,T\omega^*) < a_{\lambda_0}\max\left\{d_{\lambda_0}(\omega_n,\omega^*), d_{\lambda_0}(\omega_n,T\omega_n), d_{\lambda_0}(\omega^*,T\omega^*), \frac{d_{\lambda_0}(\omega^*,T\omega_n) + d_{\lambda_0}(\omega_n,T\omega^*)}{2}\right\} + b_{\lambda_0}d_{\lambda_0}(\omega^*,T\omega_n). \tag{47}$$

Thus, for each $n \geq n_0$, by considering the triangular property and (47), we have

$$\begin{aligned}
d_{\lambda_0}(\omega^*, T\omega^*) &\leq d_{\lambda_0}(\omega^*, \omega_{n+1}) + d_{\lambda_0}(\omega_{n+1}, T\omega^*) \\
&= d_{\lambda_0}(\omega^*, \omega_{n+1}) + d_{\lambda_0}(T\omega_n, T\omega^*) \\
&< d_{\lambda_0}(\omega^*, \omega_{n+1}) + a_{\lambda_0} \max\left\{d_{\lambda_0}(\omega_n, \omega^*), d_{\lambda_0}(\omega_n, T\omega_n), d_{\lambda_0}(\omega^*, T\omega^*), \frac{d_{\lambda_0}(\omega^*, T\omega_n) + d_{\lambda_0}(\omega_n, T\omega^*)}{2}\right\} \\
&\quad + b_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega_n) \\
&\leq d_{\lambda_0}(\omega^*, \omega_{n+1}) + a_{\lambda_0} \max\Bigg\{d_{\lambda_0}(\omega_n, \omega^*), d_{\lambda_0}(\omega_n, T\omega_n), d_{\lambda_0}(\omega^*, T\omega^*), \\
&\quad \frac{d_{\lambda_0}(\omega^*, \omega_n) + d_{\lambda_0}(\omega_n, T\omega_n) + d_{\lambda_0}(\omega_n, \omega^*) + d_{\lambda_0}(\omega^*, T\omega^*)}{2}\Bigg\} + b_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega_n) \\
&\leq d_{\lambda_0}(\omega^*, \omega_{n+1}) + a_{\lambda_0}\left(d_{\lambda_0}(\omega_n, \omega^*) + d_{\lambda_0}(\omega_n, T\omega_n) + d_{\lambda_0}(\omega^*, T\omega^*)\right) + b_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega_n).
\end{aligned} \tag{48}$$

Letting $n \longrightarrow \infty$ in the above inequality, we have

$$d_{\lambda_0}(\omega^*, T\omega^*) \leq a_{\lambda_0} d_{\lambda_0}(\omega^*, T\omega^*) < d_{\lambda_0}(\omega^*, T\omega^*). \tag{49}$$

It is a contradiction, so $d_\lambda(\omega^*, T\omega^*) = 0$ for all $\lambda \in \mathcal{U}$. As X is separating, hence we obtain $\omega^* = T\omega^*$.

In the following corollaries, we assume that X is a nonempty o-set with an orthogonal element (say ω_a) and endowed with a separating o-complete gauge structure $\{d_\lambda: \lambda \in \mathcal{U}\}$. Further, assume that a directed graph $G = (V, E)$ is defined on X so that the set of its vertices V coincides with X (i.e., $V = X$) and the set of edges E is so that $\Delta \subset E$, where $\Delta = \{(\sigma, \sigma): \sigma \in X\}$. Moreover, G is supposed that it has no parallel edges.

The following corollaries can be obtained from our results by defining $\alpha: X \times X \longrightarrow [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

□

Corollary 1. *Let $T: X \longrightarrow X$ be a mapping with $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$(x, y) \in E \Rightarrow \tau + F(d_\lambda(Tx, Ty)) \leq F(a_\lambda d_\lambda(x, y) + b_\lambda d_\lambda(x, Tx) + c_\lambda d_\lambda(y, Ty) + e_\lambda d_\lambda(x, Ty) + L_\lambda d_\lambda(y, Tx)), \tag{51}$$

for each $x, y \in X$ with $x \perp y$ and for each $\lambda \in \mathcal{U}$, whenever $d_\lambda(Tx, Ty) \neq 0$ for $\lambda \in \mathcal{U}$, where $a_\lambda, b_\lambda, c_\lambda, e_\lambda, L_\lambda \geq 0$ and $a_\lambda + b_\lambda + c_\lambda + 2e_\lambda = 1$ for all $\lambda \in \mathcal{U}$. Further, assume that

(i) T is $\perp$-preserving

(ii) There is an element $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $(\omega_0, T\omega_0) \in E$

(iii) For each $x, y \in X$ with $x \perp y$ and $(x, y) \in E$, we have $(Tx, Ty) \in E$

(iv) For any o-sequence $\{\omega_n\}$ in X with $(\omega_n, \omega_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $\omega_n \longrightarrow x$ as $n \longrightarrow \infty$, we have $(\omega_n, x) \in E$ and $\omega_n \perp x$ for each $n \in \mathbb{N}$

Then, T admits a fixed point.

Corollary 2. *Let $T: X \longrightarrow X$ be a mapping with $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\begin{aligned}
(x, y) \in E \Rightarrow \tau + F(d_\lambda(Tx, Ty)) \\
\leq F\left(a_\lambda \max\left\{d_\lambda(x, y), d_\lambda(x, Tx), d_\lambda(y, Ty), \frac{d_\lambda(x, Ty) + d_\lambda(y, Tx)}{2}\right\} + b_\lambda d_\lambda(y, Tx)\right),
\end{aligned} \tag{52}$$

for each $x, y \in X$ with $x \perp y$ and for each $\lambda \in \mathcal{U}$, whenever $d_\lambda(Tx, Ty) \neq 0$ for $\lambda \in \mathcal{U}$, where a_λ, b_λ are positive real numbers with $a_\lambda + b_\lambda = 1$ for all $\lambda \in \mathcal{U}$. Further, assume that

(i) T is $\perp$-preserving

(ii) There exists an element $\omega_0 \in X$ with $\omega_0 \perp T\omega_0$ and $(\omega_0, T\omega_0) \in E$

(iii) For each $x, y \in X$ with $x \perp y$ and $(x, y) \in E$, we have $(Tx, Ty) \in E$

(iv) For any o-sequence $\{\omega_n\}$ in X such that $(\omega_n, \omega_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $\omega_n \longrightarrow x$ as $n \longrightarrow \infty$, we have $(\omega_n, x) \in E$ and $\omega_n \perp x$ for each $n \in \mathbb{N}$

Then, T admits a fixed point.

5. Application to Integral Equations

Consider the following Volterra-type integral equation:

$$x(t) = g(t) + \int_0^{f(t)} K(t, s, x(s))\mathrm{d}s, \quad t \in I = [0, \infty), \tag{53}$$

where

(i) $g: I \longrightarrow [1, \infty)$ is continuous
(ii) $f: I \longrightarrow [0, \infty)$ is continuous
(iii) $K: I \times I \times \mathbb{R} \longrightarrow \mathbb{R}^+ = [0, \infty)$ is a continuous function

Let $X = C([0, \infty), \mathbb{R}^+)$ be the space of all real-valued continuous functions from $[0, \infty)$ into $[0, \infty)$. We can define orthogonality relation on X by

$$x \perp y \Leftrightarrow x(t)y(t) \geq y(t) \text{ or } x(t)y(t) \geq x(t). \tag{54}$$

Define the family of pseudometrics as $d_n(x, y) = \max_{t\in[0,n]}|x(t) - y(t)|e^{-|\tau t|}$, for each $n \in \mathbb{N}$, where τ is a positive real number. Clearly, $\mathcal{F} = \{d_n: n \in \mathbb{N}\}$ defines a gauge structure on X, which is separating and o-complete.

Theorem 6. *Take $X = C([0, \infty), \mathbb{R}^+)$. Define the operator $T: X \longrightarrow X$ by*

$$Tx(t) = g(t) + \int_0^{f(t)} K(t, s, x(s))ds, \quad t \in I = [0, \infty), \tag{55}$$

where $g: I \longrightarrow [1, \infty)$, $f: I \longrightarrow [0, \infty)$, and $K: I \times I \times \mathbb{R} \longrightarrow \mathbb{R}^+ = [0, \infty)$ are continuous functions. Also, assume that there are $\tau > 0$ and $\gamma: X \longrightarrow (0, \infty)$ so that for all $x, y \in X$ with $x(t)y(t) \geq y(t)$ or $x(t)y(t) \geq x(t)$ and $t, s \in [0, n]$, we have

$$|K(t, s, x) - K(t, s, y)| \leq \frac{e^{-\tau}}{\gamma(x + y)} d_n(x, y) \text{ for each } n \in \mathbb{N}. \tag{56}$$

Moreover,

$$\int_0^{f(t)} \frac{1}{\gamma(x(s) + y(s))} \mathrm{d}s \leq e^{|\tau t|}, \tag{57}$$

for each $t \in I$. Then, (53) admits at least one solution.

Proof. For $x \in X$, take

$$Tx(t) = g(t) + \int_0^{f(t)} K(t, s, x(s))\mathrm{d}s \geq 1, \tag{58}$$

for every $t \in I = [0, \infty)$. Note that $Tx(t)Ty(t) \geq Ty(t)$ for every $t \in I$. Hence, we say that if $x \perp y$, then $Tx \perp Ty$. Also, note that for each $\omega_0 \in X$, we have $\omega_0(t)T\omega_0(t) \geq \omega_0(t)$ for every $t \in I$, that is, $\omega_0 \perp T\omega_0$.

Now, for all $x, y \in X$ with $x(t)y(t) \geq y(t)$ or $x(t)y(t) \geq x(t)$ and $t \in [0, n]$ for each $n \geq 1$, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^{f(t)} |K(t, s, x(s)) - K(t, s, y(s))|\mathrm{d}s \\
&\leq \int_0^{f(t)} \frac{e^{-\tau}}{\gamma(x(s) + y(s))} d_n(x, y)\mathrm{d}s \\
&= e^{-\tau} d_n(x, y) \int_0^{f(t)} \frac{1}{\gamma(x(s) + y(s))} \mathrm{d}s \\
&\leq e^{|\tau t|} e^{-\tau} d_n(x, y).
\end{aligned} \tag{59}$$

Thus, we have

$$|Tx(t) - Ty(t)|e^{-|\tau t|} \leq e^{-\tau} d_n(x, y). \tag{60}$$

This implies that

$$d_n(Tx, Ty) \leq e^{-\tau} d_n(x, y). \tag{61}$$

One writes

$$\ln d_n(Tx, Ty) \leq \ln(e^{-\tau} d_n(x, y)). \tag{62}$$

That is,

$$\tau + \ln d_n(Tx, Ty) \leq \ln d_n(x, y) \quad \text{for each } n \in \mathbb{N}. \tag{63}$$

Therefore, it can be concluded that Theorem 4 is applied for the operator T with the choice of $\alpha(\varsigma, \upsilon) = 1$ for all $\varsigma, \upsilon \in X$, $a_\lambda = 1$ and $b_\lambda = c_\lambda = e_\lambda = L_\lambda = 0$ for each $\lambda \in \mathbb{N}$ and $F(x) = \ln(x)$. Hence, T possesses a fixed point, i.e., (53) admits at least one solution.

Take

$$x(t) = g(t) + \int_a^b K(t, s, x(s))\mathrm{d}s, \quad t \in I = [0, \infty), \tag{64}$$

where

(i) $g: I \longrightarrow \mathbb{R}$ is continuous
(ii) $K: I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

The above equation is a Fredholm-type integral equation. □

Theorem 7. *Let $X = C([0, \infty), \mathbb{R})$ and let the operator $T: X \longrightarrow X$ be defined by*

$$Tx(t) = g(t) + \int_a^b K(t, s, x(s))\mathrm{d}s, \quad t \in I = [0, \infty), \tag{65}$$

where $g: I \longrightarrow$ and $K: I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions ($a < b$). Assume that

(i) *If*

$$x(t)y(t) \geq y(t) \text{ or } x(t)y(t) \geq x(t), \tag{66}$$

then we have

$$Tx(t)Ty(t) \geq Ty(t) \text{ or } Tx(t)Ty(t) \geq Tx(t). \tag{67}$$

(ii) *There are $\tau > 0$ and $\gamma: X \longrightarrow (0, \infty)$ so that for each $x, y \in X$ with $x(t)y(t) \geq y(t)$ or $x(t)y(t) \geq x(t)$ and $t, s \in [0, n]$, we have*

$$|K(t,s,x) - K(t,s,y)| \leq \frac{e^{-\tau}}{\gamma(x+y)} d_n(x,y) \text{ for each } n \in \mathbb{N}. \tag{68}$$

Moreover,

$$\int_a^b \frac{1}{\gamma(x(s) + y(s))} \mathrm{d}s \leq 1. \tag{69}$$

(iii) *There is $\omega_0 \in X$ so that $\omega_0(t)T\omega_0(t) \geq T\omega_0(t)$ or $\omega_0(t)T\omega_0(t) \geq \omega_0(t)$.*

(iv) *For any sequence $\{\omega_n\}$ in X with $\omega_n\omega_{n+1} \geq \omega_n$ or $\omega_n\omega_{n+1} \geq \omega_{n+1}$ for each $n \in \mathbb{N}$ and $\omega_n \longrightarrow x$, we have $\omega_n x \geq \omega_n$ or $\omega_n x \geq x$.*

Then, (64) admits at least one solution.

Let $X = (C[0,\infty), \mathbb{R})$ be the set of all real-valued continuous functions. Again, define orthogonality relation on X by

$$x \perp y \Leftrightarrow x(t)y(t) \geq y(t) \text{ or } x(t)y(t) \geq x(t) \tag{70}$$

and family of pseudometrics given as $d_n(x,y) = \max_{t\in[0,n]}|x(t) - y(t)|e^{-|\tau t|}$, for each $n \in \mathbb{N}$, where τ is a positive real number. Note that $\mathcal{F} = \{d_n: n \in \mathbb{N}\}$ defines a gauge structure on X, which is separating and o-complete, so the conclusion of this theorem can be obtained from Theorem 4 by taking $\alpha(\varsigma, \upsilon) = 1$ for all $\varsigma, \upsilon \in X$, $a_\lambda = 1$ and $b_\lambda = c_\lambda = e_\lambda = L_\lambda = 0$ for each $\lambda \in \mathbb{N}$ and $F(x) = \ln(x)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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