Research Article

Incoherence Analysis of RD-AIC-Based Observation Matrix and Its Application in Power Quality Disturbance Signal

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In the theory of compressed sensing, restricted isometry property (RIP) decides the universality and reconstruction robustness of an observation matrix. At present, an observation matrix based on RD-AIC (RD-AIC-based observation matrix) can compress sparse continuous signals with a simple structure, but RIP analysis of this matrix is lack and challenging to prove. In this paper, this problem is relaxed. Instead, we demonstrate the incoherence analysis process, derive the orthogonality and nonsingularity of the matrix, propose the generalized relevance calculation steps of the matrix, and propose the hardware parameter design principles to improve the incoherence of the matrix. Moreover, compression and reconstruction experiments used in power quality disturbance signals are developed for testing the incoherence. The results show that the RD-AIC-based observation matrix has substantial incoherence under suitable hardware parameters, equivalent to the Gaussian random matrix and the Bernoulli random matrix.

1. Introduction

Compressed sensing (CS) is an efficient theory for signal compression [1], widely applied in medical imaging, radar imaging, and wireless sensors [2–4]. The classical CS theory is distributed CS, 1-bit CS, and blind CS [5–7], and the recent CS theory has developed with advanced control theory and deep learning technology. For example, Kyriiadis [8] proposes an adaptive CS method to improve tracking performance, Palangi et al. [9] design a distributed CS with deep learning approach, and Wang et al. [10] introduce a method of combining convolutional neural network and CS. However, no matter what kind of CS theory, the observation matrix is an important part of obtaining sparse and compressible signal information. The restricted isometry property (RIP) is a standard tool for studying the universality and reconstruction robustness of an observation matrix [11]. Although Gaussian random matrix, Bernoulli random matrix, Toeplitz matrix, and Chaotic matrix [12–15] have been proved, which satisfied RIP, and are optimal for sparse recovery, they have limited use in practice because of complicated hardware structure, high complexity of calculation, and extensive expense on storage. Therefore, they are still stuck in theory and are only used to compress discrete signals:

\[ y(m) = \Phi x(n), \]

where \( \Phi \) is an observation matrix mentioned above, \( x(n) \) is a discrete signal with sparsity, and \( y(m) \) is a compressed sampling data.

An analog-to-information converter (AIC) is a vital under-sampling technique based on CS theory and samples the analog signals at the sub-Nyquist rate [16]. Recently, many AIC architectures have been designed in the literature, such as random demodulation AIC (RD-AIC) [17], modulated wideband converters (MWC) [18], and random modulation preintegrator (RMPI) [19]. Since RD-AIC has a broad application prospect with simple structure and low cost, it is usually used in practice to compress analog signals:

\[ y(m) = \Phi_{RD-AIC} x(t), \]
where $\Phi_{\text{RD-AIC}}$ is an observation matrix based on RD-AIC (RD-AIC-based observation matrix), $x(t)$ is an analog signal with sparsity, and $y(m)$ is a compressed sampling data.

Comparing equation (2) to equation (1), the RD-AIC-based observation matrix is significantly better than the traditional observation matrixes in engineering. Unfortunately, although many works discuss the structure and hardware implementation of an RD-AIC [20–22], few works discuss RIP of an RD-AIC-based observation matrix [23]. However, it is well known that RIP is an NP-hard problem to prove. Recently, some discriminating theorems instead of RIP property were proposed by some researchers [24, 25], which was to judge the incoherence of an observation.

This paper proposes to analyze the incoherence of RD-AIC-based observation matrix and apply it in power quality disturbance data. The main innovations and contributions of this work are as follows:

1. A noncoherent analysis method based on the RD-AIC-based observation matrix is demonstrated to avoid an NP-hard problem for RIP proving
2. An orthogonality and nonsingularity of RD-AIC-based observation matrix are deduced for the first time, solving the premise of applying discriminant theorems
3. A design rule of hardware parameters is proposed, which can be used to enhance the incoherence of RD-AIC-based observation matrix
4. Compression and reconstruction experiments used in power quality disturbance signals are developed for testing the incoherence

This paper is structured into five sections. The basic concepts and the mathematical model of the RD-AIC-based observation matrix are introduced in Section 2. In Section 3, we analyze the orthogonality and the nonsingularity of the RD-AIC-based observation matrix. We also propose the generalized relevance calculation step and the design principle to the RD-AIC-based observation matrix. The simulation experiment studies are exposed in Section 4. The conclusions are finally drawn in Section 5.

### 2. The Mathematical Model of RD-AIC-Based Observation Matrix

The structure of RD-AIC and RD-AIC-based observation matrix is shown in Figure 1. RD-AIC consists of signal modulation, low-pass filter (LPF), and uniform sampling at low speed, and the construction processing of $\Phi_{\text{RD-AIC}}$ is as follows.

It is assumed that an analog signal $x(t)$ is $K$ sparse under the transform function $\varphi_n(t)$:

$$x(t) = \sum_{n=0}^{N-1} \alpha_n \varphi_n(t), \quad (3)$$

where $\alpha_n$ is the transforming factor, which has $K$ nonzero values.

1. $x(t)$ passes the signal modulation:

$$x_1(t) = x(t)p_c(t), \quad (4)$$

where $p_c(t)$ is a binary random sequence consisted of [-1, 1] and $x_1(t)$ is the output.

2. $x_1(t)$ passes the LPF:

$$y_1(t) = \int_{-\infty}^{\infty} x(t)p_c(t)h(t - \tau)d\tau, \quad (5)$$

where $h(t)$ is the unit impulse response of the LPF and $y_1(t)$ is the output. When equation (3) is substituted with (1), the following can be gained:

$$y_1(t) = \sum_{n=0}^{N-1} \alpha_n \int_{-\infty}^{\infty} \varphi_n(t)p_c(t)h(t - \tau)d\tau. \quad (6)$$

3. $y_1(t)$ passes the uniform sampling at low speed:

$$y(m) = \sum_{n=0}^{N-1} \theta_{m,n} \alpha_n = \Theta \alpha, \quad (9)$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T \in R^{N \times 1}$ and $\Theta \in R^{M \times N}$ is the integral result of the transformed function, the binary random sequence, and the unit impulse response. If the discretization is applied onto $\Theta$, we obtain

$$\Theta = BHP\Psi, \quad (10)$$

$$\Psi = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \ldots & \varphi_{N-1}(0) \\ \varphi_1(1) & \varphi_2(1) & \ldots & \varphi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(N-1) & \varphi_2(N-1) & \ldots & \varphi_{N-1}(N-1) \end{bmatrix}, \quad (11)$$

$$P = \begin{bmatrix} p_c(0) & 0 & \ldots & 0 \\ 0 & p_c(1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_c(N-1) \end{bmatrix}, \quad (12)$$

$$H = \begin{bmatrix} h(0) & 0 & \ldots & 0 \\ h(1) & h(0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h(N-1) & \ldots & h(1) & h(0) \end{bmatrix}. \quad (13)$$
where $\Psi \in R^{N \times N}$ is the transform function matrix that is consisted of the discretized $\phi_{n}(n)$, $P \in R^{N \times N}$ is the binary random sequence matrix that is consisted of the discretized $p_{c}(n)$, $H \in R^{N \times N}$ is the unit impulse response matrix that is consisted of the circular movement of the discretized $h(n)$, and $B \in R^{M \times N}$ is the uniform sampling matrix consisted of the direct sum of $M$ same $\kappa$ vectors.

According to equation (10) and CS principle [1], $\Phi_{RD-AIC}$ is

$$\Phi_{RD-AIC} = BHP,$$  \hfill (15)

where $\Phi_{RD-AIC} \in R^{M \times N}$ and $c = (M/N) \times 100\%$ is the compression ratio. The smaller the value of $c$, the less the number of samples.

### 3. Analysis Method of Incoherence

#### 3.1. Relevance Discriminating Theorem

**Theorem 1.** If an observation matrix $\Phi$ is constructed by $M$ rows which are drawn randomly from a nonsingular matrix $\Phi_{Z} = [\phi_{Z1}, \phi_{Z2}, \ldots, \phi_{ZN}] \in R^{N \times N}$, the sufficient condition which allows solving reconstruction by a norm optimization method is

$$M \geq lu^2(\Phi_{Z}, \Psi) \frac{K}{\log(N)},$$  \hfill (16)

where $l$ is the constant, $u \in [1, \sqrt{N}]$ is the relevance between the orthogonal matrix $\Phi_{Z}$, and the transform function matrix $\Psi$:

$$u(\Phi_{Z}, \Psi) = \sqrt{N} \max_{i,j} \langle \phi_{zi}, \psi_{j} \rangle.$$  \hfill (17)

#### 3.2. Analysis of the Orthogonality and Nonsingularity

Let $\Phi_{HP} = HP$,

$$\Phi_{HP} \in R^{N \times N}$$

where $\Phi_{HP}$ is the part of the RD-AIC-based observation matrix $\Phi_{RD-AIC}$.

According to equation (15), $\Phi_{RD-AIC}$ is constructed by $M$ rows which are drawn randomly from the square matrix $\Phi_{HP}$. Therefore, to use the above theorems, the orthogonality and nonsingularity of $\Phi_{HP}$ need to be analyzed.

According to equations (12) and (13), $\Phi_{HP}$ can be drawn:
where $\Phi_{HP}$ is the triangular matrix without the orthogonality.

The nonsingularity of $\Phi_{HP}$ depending on the diagonal element is zero. Because $p_c(m)$ is not equal to zero, the nonsingularity of $\Phi_{HP}$ relies on the first value of $h(n)$.

According to the LPF design theory, many prototype filters have different structures, for example, Butterworth, Chebyshev, and ellipse. Recently, the fractional-order idea is used in the LPF design [26–28]. The order and the parameter number of the filters mentioned above are not the same, which undoubtedly increases the difficulty of deriving the first value of $h(n)$.

As it is well known, the ideal LPF is nonrealizable. However, the results of its research can be used to guide the design of actual filters. Therefore, in order to facilitate the analysis, we replaced the actual filter with the ideal filter and assumed that $\Phi_{HP}$ is consisted of a binary random sequence matrix and an ideal unit impulse response matrix.

Then, let

$$\begin{bmatrix}
    p_c(0)h(0) & 0 & 0 & \ldots & \ldots & 0 \\
p_c(0)h(1) & p_c(1)h(0) & 0 & \ldots & \ldots & 0 \\
p_c(0)h(2) & p_c(1)h(1) & p_c(2)h(0) & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
p_c(0)h(N-1) & \ldots & \ldots & \ldots & p_c(N-2)h(0) & 0 \\
\end{bmatrix},$$

which is shown in Table 1, and the calculation results imply the first trend of how to increase the incoherence of $\Phi_{HP}$. If $\beta$ is small, it shows less fluctuation and better stability. Calculation of the statistical results under different parameters is shown in Table 2.

Above the calculation and statistical results imply the prerequisite of the hardware cost of the guaranteed RD-AIC-based observation matrix. The details and parameter design principles are as follows:

(1) There is a relationship that exists between $\tilde{u}$ and $T_p$. Both $\bar{u}_{\text{mean}}$ and $\beta$ will increase along with the dropping of $T_p$ at the same $f_p$. Therefore, $T_p$ should be extended as long as possible at the prerequisite of the hardware cost of the guaranteed RD-AIC-based observation matrix.

(2) There is a relationship that exists between $\tilde{u}$ and $f_p$. Both $\bar{u}_{\text{mean}}$ and $\beta$ will augment along with the dropping of $f_p$ at the same $T_p$. Therefore, $f_p$ increase appropriately can enhance the performance of the RD-AIC-based observation matrix. However, $f_p$ can be constricted without demolishing the stronger incoherence when $T_p$ is bigger.

(3) There is no relation existed between $\tilde{u}$ and $T$. $T$ is only relative to the compression rate $c$. Therefore, the
The time-frequency comparison between the reconstructed signal and the original signal is shown in Figure 5. It can be spotted a good reconstruction performance. Also, the experiment is implemented under different $T_p$, $f_p$, and white Gaussian noise. Furthermore, the results are shown in Table 3, where each index statistic is the mean of 100 parallel experiments.

4.2. Harmonic Experiment. The harmonic is commonly generated by the nonlinear loadings in the power users [30]. According to IEEE Electromagnetic Compatibility Standard, the mathematical expression of the normalized amplitudes of harmonic is

$$x(t) = \cos(2\pi ft) + \sum a_i \cos(2\pi f_i t),$$

(26)

where $a_i$ is the amplitude of harmonic and $f_i$ is the frequency of harmonic.

Let $f_j = 85$ Hz, $150$ Hz, $350$ Hz and the other parameter setting of $T_p$, $f_p$, $f_x$, $M$, and $c$ be the same as the voltage sag experiment. The harmonic signal passing through each part of the RD-AIC-based observation matrix is shown in Figure 6.

The time-frequency comparison between the reconstructed signal and the original signal is shown in Figure 7. It also can be spotted as a good reconstruction performance. The results of the comparison experiment under different parameters are shown in Table 4.

4.3. Data Analysis. According to Tables 3 and 4, the decreasing tendency of MAE and MSE is denoted along with the increase of SNR, yet the reconstruction performance is different evidently under different $T_p$ and $f_p$. To simplify, in this section, only MSE is compared and analyzed, and the different MSE curves are shown in Figures 8 and 9.

The analysis results of Figures 8 and 9 are as follows:

1. When $T_p = 1000$, MSE is all little under different $f_p$ and SNR and the maximum MSE is not more than $1e-02$. It indicates that even if $f_p$ is significantly reduced, the RD-AIC-based observation matrix still has outstanding reconstruction performance and robustness against noise with a bigger $T_p$.

2. When $T_p = 31$, MSE drops along with the ascending $f_p$ and the minimum MSE is $4.35e-0.4$. It indicates that there is still excellent reconstruction performance and robustness against noise with a smaller $T_p$ and a bigger $f_p$.

3. When $T_p = 7$, MSE is all big under any $f_p$ and SNR and the minimum MSE is $0.1977$. It indicates that even if $f_p$ is significantly increased, the RD-AIC-based observation matrix still has worse reconstruction performance and robustness against noise with a very small $T_p$. 

 augmentation of $c$ is required, and $f_p$ can be restricted appropriately with the prerequisites of (a) and (b) which are satisfied.

<table>
<thead>
<tr>
<th>Table 1: The generalized relevance calculation step based on Theorem 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The calculation step</td>
</tr>
<tr>
<td>Step 1: set period of the binary random sequence $T_p$ and the</td>
</tr>
<tr>
<td>passband cut-off frequency $f_p$</td>
</tr>
<tr>
<td>Step 2: construct the binary random sequence $p_c(n)$ and the</td>
</tr>
<tr>
<td>ideal unit impulse response $h'(n)$</td>
</tr>
<tr>
<td>Step 3: construct $\Phi_{HP}$ based on equation (20)</td>
</tr>
<tr>
<td>Step 4: conduct Schmid orthogonalization on column vectors in</td>
</tr>
<tr>
<td>$\Phi_{UL}$ to get standard orthogonal bases $\phi_{ULP1}, \phi_{ULP2}, \ldots, \phi_{ULPN}'$</td>
</tr>
<tr>
<td>Step 5: calculate the generalized relevance $\bar{u}$ based on</td>
</tr>
<tr>
<td>equation (17)</td>
</tr>
</tbody>
</table>

4. Experiment

In this section, to verify the compression and reconstruction performance, the RD-AIC-based observation matrix with the design principle mentioned above is applied to the power quality disturbance signal. Meanwhile, the mean absolute error (MAE), the mean squared error (MSE), and the compression rate $c$ have been adopted.

4.1. Voltage Sag Experiment. The voltage sag refers to a sudden dropping of mean squared values of power frequency voltage at a certain point in the power grid and returning to normal after a short duration [29]. According to IEEE Electromagnetic Compatibility Standard, the normalized mathematical expression of the voltage sag is

$$x(t) = \cos(2\pi ft)(1 - a_i u(t - t_1) - u(t - t_2)),$$

(25)

where $f$ is the frequency of the base wave in the power grid, $a_i \in [0.1, 0.9]$ is the depth of the voltage sag, $t_1$ is the initiation time, and $t_2$ is the ending time.

Let $a_i = 0.3$, $t_1 = 41.2$ ms, $t_2 = 123$ ms, $T = 0.5$ ms, $N = 2000$, $T_p = 1000$, $f_p = 1$ KHz, and $M = 400$. The都没压缩采样 processing that the voltage sag signal passes through each part of RD-AIC-based observation matrix is shown in Figure 4.
The above analysis results are consistent with the statistics results on incoherence in Section 3.3, verifying this paper’s content.

4.4. Comparison with the Random Observation Matrix.

The RD-AIC-based observation matrix has been compared with Gaussian random matrixes and Bernoulli random matrixes.
<table>
<thead>
<tr>
<th>Statistical results</th>
<th>$T_p = 1000$</th>
<th>$T_p = 31$</th>
<th>$T_p = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 kHz$</td>
<td>$0.5 kHz$</td>
<td>$0.1 kHz$</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>1.845</td>
<td>2.592</td>
<td>6.028</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.280</td>
<td>0.286</td>
<td>0.329</td>
</tr>
</tbody>
</table>

Figure 4: The compression sampling process of the voltage sag signal passing through the RD-AIC-based observation matrix. (a) The original signal, (b) the modulated signal, (c) the filtered signal, and (d) the compressed sampling signal.

Figure 5: The time-frequency comparison between the reconstruction voltage sag signal and the original voltage sag signals. (a) The time domain. (b) The frequency domain.
Table 3: The reconstruction performance comparison under different parameters.

<table>
<thead>
<tr>
<th>$f_p$ (kHz)</th>
<th>SNR (dB)</th>
<th>$T_p = 1000$</th>
<th>$T_p = 31$</th>
<th>$T_p = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MAE</td>
<td>MSE</td>
<td>MAE</td>
<td>MSE</td>
</tr>
<tr>
<td>20</td>
<td>0.059</td>
<td>0.0055</td>
<td>0.051</td>
<td>0.050</td>
</tr>
<tr>
<td>25</td>
<td>0.028</td>
<td>0.0014</td>
<td>0.027</td>
<td>4.2e-03</td>
</tr>
<tr>
<td>30</td>
<td>0.018</td>
<td>5.94e-04</td>
<td>0.080</td>
<td>2.4e-03</td>
</tr>
<tr>
<td>35</td>
<td>0.012</td>
<td>5.10e-04</td>
<td>0.038</td>
<td>1.3e-03</td>
</tr>
<tr>
<td>40</td>
<td>0.013</td>
<td>4.89e-04</td>
<td>0.013</td>
<td>4.35e-04</td>
</tr>
<tr>
<td>20</td>
<td>0.053</td>
<td>4.6e-03</td>
<td>0.266</td>
<td>0.2046</td>
</tr>
<tr>
<td>25</td>
<td>0.039</td>
<td>2.8e-03</td>
<td>0.167</td>
<td>0.1165</td>
</tr>
<tr>
<td>30</td>
<td>0.022</td>
<td>1.0e-03</td>
<td>0.148</td>
<td>0.1082</td>
</tr>
<tr>
<td>35</td>
<td>0.017</td>
<td>6.47e-04</td>
<td>0.133</td>
<td>0.0981</td>
</tr>
<tr>
<td>40</td>
<td>0.015</td>
<td>5.95e-04</td>
<td>0.131</td>
<td>0.0992</td>
</tr>
<tr>
<td>20</td>
<td>0.081</td>
<td>0.0106</td>
<td>0.388</td>
<td>0.3325</td>
</tr>
<tr>
<td>25</td>
<td>0.057</td>
<td>5.3e-03</td>
<td>0.385</td>
<td>0.3046</td>
</tr>
<tr>
<td>30</td>
<td>0.042</td>
<td>3.1e-03</td>
<td>0.378</td>
<td>0.2952</td>
</tr>
<tr>
<td>35</td>
<td>0.039</td>
<td>2.8e-03</td>
<td>0.347</td>
<td>0.2828</td>
</tr>
<tr>
<td>40</td>
<td>0.033</td>
<td>2.3e-03</td>
<td>0.266</td>
<td>0.2075</td>
</tr>
</tbody>
</table>

Figure 6: The compression sampling process of the harmonic passing through the RD-AIC-based observation matrix. (a) The original signal, (b) the modulated signal, (c) the filtered signal, and (d) the compressed sampling signal.
matrixes at different $c$ (2%, 4%, 5%, 10%, 20%, 25%, 30%, 35%, 40%, and 50%). The comparison results are shown in Figure 10. An obvious reconstruction error only occurs under $c$ at 2%. However, under other $c$, the reconstruction performance is similar to the Gaussian random matrix and Bernoulli random matrix. In summary, an RD-AIC-based observation matrix following the design principle is sufficient to realize the highly precious reconstruction at high compression ratios. It is equivalent to that of the classical Gaussian random matrix and Bernoulli random matrixes.
Figure 8: The reconstruction performance comparison of the voltage sag experiment at the parameters $f_p$, $T_p$, and SNR. (a) Different MSE curve under $f_p = 1$ KHz. (b) Different MSE curve under $f_p = 0.5$ KHz. (c) Different MSE curve under $f_p = 0.1$ KHz.

Figure 9: The reconstruction performance comparison of the harmonic experiment at the parameters $f_p$, $T_p$, and SNR. (a) The different MSE curve under $f_p = 1$ KHz. (b) The different MSE curve under $f_p = 0.5$ KHz. (c) The different MSE curve under $f_p = 0.1$ KHz.
5. Conclusions

The work in this paper provides a theoretical and technical reference for incoherence analysis of the RD-AIC-based observation matrix. Some conclusions are shown as follows:

(1) The square matrix of the RD-AIC-based observation matrix based on the binary random sequence and the ideal low-pass filter unit impulse response is not orthogonal but nonsingular.

(2) The incoherence of the RD-AIC-based observation matrix is related to the binary random sequence period and the passband cut-off frequency of LPF.

(3) The RD-AIC-based observation matrix with substantial incoherence is suitable for power quality disturbance data and is great as the Gaussian random matrix and Bernoulli random matrix on reconstruction performance.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References


Figure 10: The reconstruction performance comparison of using the different observation matrices. (a) The reconstruction comparison of the voltage sag. (b) The reconstruction comparison of the harmonic.


