Research Article

A B-Spline Quasi Interpolation Crank–Nicolson Scheme for Solving the Coupled Burgers Equations with the Caputo–Fabrizio Derivative

M. Taghipour¹ and H. Aminikhah ^{1,2}

¹Department of Applied Mathematics and Computer Science, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 1914, Rasht 41938, Iran

²Center of Excellence for Mathematical Modelling, Optimization and Combinatorial Computing (MMOCC), University of Guilan, P. O. Box 1914, Rasht 41938, Iran

Correspondence should be addressed to H. Aminikhah; aminikhah@guilan.ac.ir

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In this paper, a Crank–Nicolson finite difference scheme based on cubic B-spline quasi-interpolation has been derived for the solution of the coupled Burgers equations with the Caputo–Fabrizio derivative. The first- and second-order spatial derivatives have been approximated by first and second derivatives of the cubic B-spline quasi-interpolation. The discrete scheme obtained in this way constitutes a system of algebraic equations associated with a bi-pentadiagonal matrix. We show that the proposed scheme is unconditionally stable. Numerical examples are provided to verify the efficiency of the method.

1. Introduction

The coupled Burgers equations are coupled partial differential equations which are capable of describing realitic polydispersive suspensions. The coupled Burgers equation predicts an interesting phenomenon, which is called phase shifts [1]. This equation is one of the fundamental models in fluid mechanics and arises in gas dynamics, chromatography, and flood waves in rivers [2]. The coupled viscous Burgers equation is given by

$$\begin{aligned} u_t - u_{xx} + \eta uu_x + \alpha_1 (uv)_x &= 0, & x \in [a, b], t \in [0, T], \\ v_t - v_{xx} + \eta vv_x + \alpha_2 (uv)_x &= 0, & x \in [a, b], t \in [0, T], \end{aligned} \quad (1)$$

with the initial conditions

$$\begin{aligned} u(x, 0) &= \phi_1(x), \\ v(x, 0) &= \phi_2(x), \end{aligned} \quad (2)$$

and the boundary conditions

$$\begin{aligned} u(a, t) &= f_1(a, t), \\ u(b, t) &= f_2(b, t), \\ v(a, t) &= g_1(a, t), \\ v(b, t) &= g_2(b, t), \end{aligned} \quad (3)$$

where η is a real constant and α and β are arbitrary constants depending on the system parameters such as Peclet number, Stokes velocity of particles due to gravity, and Brownian diffusivity [3].

Spline is a special function defined piecewise by polynomials. The spline approximation first appeared in a paper by Schoenberg [4]. Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called a spline. Applications of spline function in fractional partial differential equations can be found in [5–15].

In this article, we consider the following coupled Burgers equation with time fractional derivative:

$$\frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} + \eta u \frac{\partial u}{\partial x} + \alpha_1 \frac{\partial(uv)}{\partial x} = q_1, \quad (4)$$

$$x \in [a, b], t \in [0, T], 0 < \gamma < 1,$$

$$\frac{\partial^\gamma v}{\partial t^\gamma} - \frac{\partial^2 v}{\partial x^2} + \eta v \frac{\partial v}{\partial x} + \alpha_2 \frac{\partial(uv)}{\partial x} = q_2, \quad (5)$$

$$x \in [a, b], t \in [0, T], 0 < \gamma < 1,$$

with initial

$$\begin{aligned} u(x, 0) &= \phi_1(x), \\ v(x, 0) &= \phi_2(x), \end{aligned} \quad (6)$$

and the boundary conditions

$$\begin{aligned} u(a, t) &= f_1(a, t), \\ u(b, t) &= f_2(b, t), \\ v(a, t) &= g_1(a, t), \\ v(b, t) &= g_2(b, t), \end{aligned} \quad (7)$$

where γ is order of time fractional derivative. Also, η , α_1 , and α_2 are those ones we said before.

${}_0^{\text{CF}}D_t^\gamma u(x, t) = ((\partial^\gamma u(x, t))/\partial t^\gamma)$ denotes the Caputo–Fabrizio derivative of the function $u(x, t)$ defined as [16]

$${}_0^{\text{CF}}D_t^\gamma u(x, t) = \frac{M(\gamma)}{1-\gamma} \int_0^t u'(x, s) e^{-\sigma(t-s)} ds, \quad (8)$$

where $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$ and $\sigma = (\gamma/1 - \gamma)$.

Recently, the Caputo–Fabrizio derivative has received more attention from researchers due to their description of some physical phenomenon [17–26].

For the description of memory and some physical properties of various materials and processes, modeling with fractional derivatives is very appropriate. This is the main benefit of fractional derivatives in comparison with classical integer order models, in which such effects are missed. In recent years, the coupled system of Burgers equations with fractional derivatives has been the focus of attention. For example, in [27], the Adomian decomposition method is directly extended to study the coupled Burgers equations with time and space fractional derivatives. Khan et al. [28] proposed the generalized differential transform method (GDTM) and homotopy perturbation method (HPM) for time fractional Burgers and coupled Burgers equations. The fractional variational iteration method (FVIM) to solve a time and space fractional coupled burgers equations is given by Prakash et al. [29]. In [30], a q-homotopy analysis transform method (q-HATM) for time and space fractional coupled Burgers equations is introduced. Aminikhah and Malekzadeh [31] introduced a new homotopy perturbation method for system of variable coefficient coupled Burgers equations with time fractional derivative. In [32], the Laplace-Adomian decomposition method (LADM), the

Laplace-variational iteration method (LVIM), and the reduced differential transform method (RDTM) are proposed to solve the one- and two-dimensional fractional coupled Burgers equations. Albuohimad and Adibi derived a hybrid spectral exponential Chebyshev method (HSECM) for time fractional coupled Burgers equations [33]. In [34], authors investigate the fractional coupled viscous Burgers equation involving Mittag–Leffler kernel. In [35], the generalized two-dimensional differential transform method (DTM) was applied to solve the coupled Burgers equations with space and time fractional derivatives. Ozdemir et al. used the Gegenbauer wavelets-based computational methods to find the approximate solutions of the coupled system of Burgers equations with time fractional derivative [36].

Our aim is to propose a Crank–Nicolson finite difference scheme using cubic B-spline quasi-interpolation to solve time fractional coupled viscous Burgers equations. The first- and second-order spatial derivatives have been approximated by first and second derivatives of the cubic B-spline quasi-interpolation. This approximations have not been used for the fractional coupled Burgers equations before.

The paper is organized as follows. In Section 2, we present some basic definitions and concepts of quasi-interpolants. In Section 3, using the quasi-interpolant and Crank–Nicolson finite difference method, we obtain a numerical scheme. The stability of this method is studied in Section 4. In Section 5, some numerical examples are proposed. Finally, conclusions are given in Section 6.

2. Univariate Spline Quasi-Interpolants

In this section, we introduce the basic concepts about B-spline and univariate B-spline quasi-interpolants that we will use in Section 3.

According to [37], let

$$P_d^1 := \text{space of univariate polynomials of degree at most } d, \quad (9)$$

and $\Omega = [a, b]$ be an interval that has been partitioned into subintervals via a set of points $\Delta = \{x_i\}_{i=0}^{k+1}$ with

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b. \quad (10)$$

We define the space of univariate polynomial splines of smoothness r and degree d with knots Δ as

$$\mathcal{S}_d^r(\Delta) := \left\{ s \in C^r(\Omega) : s|_{(x_i, x_{i+1})} \in P_d^1, \quad i = 0, \dots, k \right\}, \quad (11)$$

where $0 \leq r < d$ are given integers. We have

$$n := \dim \mathcal{S}_d^r(\Delta) = k(d - r) + r + 1. \quad (12)$$

For a formal proof of this fact, see Theorem 4.4 of [38].

Given $0 \leq r < d$ and $\Delta = \{x_i\}_{i=0}^{k+1}$, the associated extended partition Δ_e is defined to be $\{y_i\}_{i=0}^{n+d+1}$, where n is the dimension of $\mathcal{S}_d^r(\Delta)$ given in (11):

$$\begin{aligned}
a &= y_1 = \dots = y_{d+1}, \\
y_{n+1} &= \dots = y_{n+d+1} = b, \\
y_{d+2} &\leq \dots \leq y_n = \frown x_1, \dots, x_1, \dots, \frown x_k, \dots, x_k.
\end{aligned} \tag{13}$$

Given an extended partition Δ_e , let

$$Q_i^1(t) := \begin{cases} \frac{1}{y_{i+1} - y_i}, & y_i \leq t < y_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

for $i = 1, \dots, n+d$, and let

$$Q_i^m(t) := \begin{cases} \frac{(t - y_i)Q_i^{m-1}(t) + (y_{i+m} - t)Q_{i+1}^{m-1}(t)}{y_{i+m} - y_i}, & y_i \leq t < y_{i+m}, \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

for $2 \leq m \leq d+1$ and $i = 1, \dots, n+d-m+1$. Let

$$N_i^m(t) := (y_{i+m} - y_i)Q_i^m(t), \quad i = 1, \dots, n+d-m+1. \tag{16}$$

We call these the normalized B-splines of order m (or degree $m-1$) associated with the extended partition Δ_e .

In [37], univariate B-spline quasi-interpolants can be defined as a formula of the form

$$Q_d f(x) = \sum_{i=1}^{n+d} (\lambda_i f) N_i(x), \tag{17}$$

where $\{N_i\}_{i=1}^{n+d}$ are the B-splines forming a basis of $\mathcal{S}_d^r(\Delta)$.

Quasi-interpolants have been heavily studied in the literature. Some basic ideas and sources for further information can be found in [38]. For a good approximations, we need to make sure it reproduces polynomials, i.e., $Qp = p$ for all $p \in P_d^1$. For each $i = 1, \dots, n$, we assume that the coefficient λ_i is a linear functional defined on $C[a, b]$ that can be computed from samples of f at some set of points $\sigma(\lambda_i)$ in $[a, b]$.

According to [39], the error of a quasi-interpolation satisfies

$$|f(x) - (Q_d f)(x)| \leq \frac{\|Q_d\|}{(d+1)!} \|f^{(d+1)}\|_{\infty, D_x} h(x)^{d+1}, \quad x \in D_y^d, \tag{18}$$

where $D_y^d = [y_{d+1}, y_{n+1}]$, D_x is the union of the supports of all B-splines N_i , $i \sim x$ and $\|f^{(d+1)}\|_{\infty, D_x}$ denote the maximum norm of $f^{(d+1)}$ on D_x and $h(x) = \max_{y \in D_x} |y - x|$ that $\sim$ is used to indicate proportionality. If the local mesh ratio is bounded, i.e., if the quotients of the lengths of adjacent knot intervals are $\leq r_y$, then the error of the derivatives on the knot intervals (y_l, y_{l+1}) can be estimated by

$$|f^{(j)}(x) - (Q_d f)^{(j)}(x)| \leq c(d, r_y) \|Q_d\| \|f^{(d+1)}\|_{\infty, D_x} h(x)^{d+1-j}, \tag{19}$$

for $j \leq d$.

Suppose $a = t_0 < \dots < t_n = b$ are equally spaced points in the interval $[a, b]$. Let

$$\lambda_i f := \begin{cases} f(t_0), & i = 1, \\ \frac{1}{18} (7f(t_0) + 18f(t_1) - 9f(t_2) + 2f(t_3)), & i = 2, \\ \frac{1}{6} (-f(t_{i-3}) + 8f(t_{i-2}) - f(t_{i-1})), & 3 \leq i \leq n+1, \\ \frac{1}{18} (2f(t_{n-3}) - 9f(t_{n-2}) + 18f(t_{n-1}) + 7f(t_n)), & i = n+2, \\ f(t_n), & i = n+3. \end{cases} \tag{20}$$

Then, (17) defines a linear operator mapping $C[a, b]$ into $\mathcal{S}_3^2(\Delta)$ with $Qp = p$ for all cubic polynomials p . For approximate derivatives of f by derivatives of $Q_3 f$ up to the order h^3 , we can evaluate the value of f' and f'' at x_i by $(Q_3 f)'(x) = \sum_{i=1}^{n+3} (\lambda_i f) N_i'(x)$ and $(Q_3 f)''(x) = \sum_{i=1}^{n+3}$

$(\lambda_i f) N_i''(x)$. We set $Y = (f_0, f_1, \dots, f_n)^T$, $Y' = (f_0', f_1', \dots, f_n')^T$, and $Y'' = (f_0'', f_1'', \dots, f_n'')^T$, where $f_j' = (Q_3 f)'(x_j)$, $j = 1, \dots, n$, and $f_j'' = (Q_3 f)''(x_j)$, $j = 1, \dots, n$. By solution of the linear systems,

$$\begin{aligned} f'_i &= \sum_{i=1}^{n+3} (\lambda_i f) N'_i(x), \quad i = 0, 1, \dots, n, \\ f''_i &= \sum_{i=1}^{n+3} (\lambda_i f) N''_i(x), \quad i = 0, 1, \dots, n, \end{aligned} \quad (21)$$

we obtain

$$\begin{aligned} Y' &= \frac{1}{h} D^1 Y, \\ Y'' &= \frac{1}{h^2} D^2 Y, \end{aligned} \quad (22)$$

where $D^1, D^2 \in \mathbb{R}^{(n+1) \times (n+1)}$ and is obtained as follows:

$$D^1 = \begin{pmatrix} -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 & -\frac{1}{6} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{6} & -1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \dots & 0 & 0 & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{pmatrix},$$

$$D^2 = \begin{pmatrix} 2 & -5 & 4 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{6} & \frac{5}{3} & -3 & \frac{5}{3} & -\frac{1}{6} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{6} & \frac{5}{3} & -3 & \frac{5}{3} & -\frac{1}{6} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{6} & \frac{5}{3} & -3 & \frac{5}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & \dots & 0 & -\frac{1}{6} & \frac{5}{3} & 0 & \frac{5}{3} & -\frac{1}{6} \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 4 & -5 & 2 \end{pmatrix}. \quad (23)$$

3. Numerical Scheme

We consider a grid $x_i = a + ih, i = 0, 1, \dots, M$, with $h = x_{i+1} - x_i$. The step length in time is denoted by τ and $t_k = k\tau, \tau = (T/N), 0 \leq k \leq N$. So, the domain $[a, b] \times [0, T]$ is divided into a uniform grid of mesh points (x_j, t_k) . The values of the function u at the grid points are denoted $u_j^k = u(x_j, t_k)$ and U_j^k is the approximate solution at the point (x_j, t_k) .

A discrete approximation to the ${}_0^{\text{CF}} D_t^\gamma u(x, t)$ at (x_j, t_k) can be obtained by the following approximation [40]:

$$\begin{aligned} {}_0^{\text{CF}} D_t^\gamma u(x_j, t_k) &= \frac{M(\gamma)}{1-\gamma} \int_0^{t_k} \frac{\partial u(x_j, s)}{\partial s} e^{-\sigma(t_k-s)} ds \\ &= \frac{M(\gamma)}{1-\gamma} \sum_{l=1}^k \int_{t_{k-1}}^{t_k} \frac{\partial u(x_j, s)}{\partial s} e^{-\sigma(t_k-s)} ds \\ &= \frac{M(\gamma)}{1-\gamma} \sum_{l=1}^k \int_{t_{k-1}}^{t_k} \frac{u_j^l - u_j^{l-1}}{\tau} e^{-\sigma(t_k-s)} ds + R \\ &= \frac{M(\gamma)}{\gamma\tau} \sum_{l=1}^k (u_j^l - u_j^{l-1}) w_k + R \\ &= \frac{M(\gamma)}{\gamma\tau} \left(w_0 u_j^k - \sum_{l=1}^{k-1} (w_{k-l-1} - w_{k-l}) u_j^l - w_{k-1} u_j^0 \right) + R, \end{aligned} \quad (24)$$

where

$$w_j = e^{-\sigma\tau j} - e^{-\sigma\tau(j+1)}. \quad (25)$$

Theorem 1. Suppose $v(t) \in C^2[0, t_k]$. Let

$$A = \sum_{l=1}^k \int_{t_{l-1}}^{t_l} \left[v'(\tau) - \frac{v(t_l) - v(t_{l-1})}{\Delta t} \right] e^{-\sigma(t_k - \tau)} d\tau. \quad (26)$$

Then,

$$|A| \leq \frac{(\Delta t)^2}{\sigma} \max_{0 \leq t \leq t_k} |v''(t)| \left| \frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \dots \right| t_k. \quad (27)$$

Proof. Using the Taylor series expansion with integral remainder, we have

$$\begin{aligned} A &= \frac{1}{\Delta t} \sum_{l=1}^k \int_{t_{l-1}}^{t_l} \left[\int_{t_{l-1}}^{\tau} v''(s)(s - t_{l-1}) ds \right. \\ &\quad \left. - \int_{\tau}^{t_l} v''(s)(t_l - s) ds \right] e^{-\sigma(t_k - \tau)} d\tau, \\ &= \frac{1}{\sigma \Delta t} \sum_{l=1}^k \int_{t_{l-1}}^{t_l} \left[(s - t_{l-1}) (e^{-\sigma(t_k - t_l)} - e^{-\sigma(t_k - s)}) \right. \\ &\quad \left. - (t_l - s) (e^{-\sigma(t_k - s)} - e^{-\sigma(t_k - t_{l-1})}) \right] v''(s) ds. \end{aligned} \quad (28)$$

Since,

$$\begin{aligned} &\int_{t_{l-1}}^{t_l} \left[(s - t_{l-1}) (e^{-\sigma(t_k - t_l)} - e^{-\sigma(t_k - s)}) \right. \\ &\quad \left. - (t_l - s) (e^{-\sigma(t_k - s)} - e^{-\sigma(t_k - t_{l-1})}) \right] ds, \\ &= (\Delta t)^4 e^{-\sigma \Delta t(k-l)} \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \dots \right) \\ &\leq (\Delta t)^4 \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \Delta t}{24} + \dots \right), \end{aligned} \quad (29)$$

and $e^{-\sigma \Delta t(k-l)} < 1$ for $l = 1, 2, \dots, k$ and $t_k = k\Delta t$; hence, the result will be achieved.

Now, using Theorem 1, we obtain

$$\begin{aligned} \frac{M(\gamma)}{1-\gamma} \sum_{l=1}^k \int_{t_{l-1}}^{t_l} \left(\frac{\partial u}{\partial s} \Big|_{(x_j, s)} - \frac{u_j^l - u_j^{l-1}}{\tau} \right) e^{-\sigma(t_k - s)} ds, \\ \leq \frac{M(\gamma)}{1-\gamma} \frac{\tau^2}{\sigma} \max_{0 \leq t \leq t_k} |u''(t)| \left| \left(\frac{\sigma^2}{12} - \frac{\sigma^3 \tau}{24} + \dots \right) \right| \sum_{l=1}^k e^{-\sigma\tau(k-l)} = O(\tau^2). \end{aligned} \quad (30)$$

We introduce some lemmas which will be used in numerical scheme and stability analysis. $\square$

Lemma 1 (see [41]). Suppose $u(t) \in C^3[0, t_{k+1}]$; then, we have

$$\left| {}_0^{\text{CF}} D_t^\gamma u^{k+(1/2)} - T_t^\gamma u^{k+(1/2)} \right| = O(\tau^2), \quad 0 \leq k \leq N-1, \quad (31)$$

where

$$T_t^\gamma u^{k+(1/2)} = \frac{1}{\gamma \tau} \left(w_0 \frac{u_j^k + u_j^{k+1}}{2} - \sum_{l=1}^k (w_{k-l} - w_{k-l+1}) \frac{u_j^l + u_j^{l-1}}{2} - w_k u_j^0 \right), \quad 0 \leq k \leq N-1. \quad (32)$$

Lemma 2 (see [42]). For the definition M_j , we have $M_j > 0$ and $M_{j+1} < M_j, \forall j \leq k$.

Lemma 3 (see [42]). Suppose $u(t) \in C_{x,t}^{4,4}([0, L] \times [0, T])$; it holds that $0 \leq w_j \leq C\tau$, and $0 \leq w_j - w_{j+1} \leq C\tau w_j$.

Now, we present the numerical scheme for solving (4)–(7) based on the Crank–Nicolson method and cubic B-spline quasi-interpolant. We approximate equations (4) and (5) at the point $(x_j, t_{k+(1/2)})$ by the Crank–Nicolson finite difference approximation:

$$\begin{aligned} &\frac{\partial^2 u(x_j, t_{k+(1/2)})}{\partial t^2} - \frac{1}{2} [u_{xx}(x_j, t_{k+1}) + u_{xx}(x_j, t_k)] \\ &+ \frac{\eta}{2} [u(x_j, t_{k+1}) u_x(x_j, t_{k+1}) + u(x_j, t_k) u_x(x_j, t_k)] \\ &+ \frac{\alpha_1}{2} [((uv)_x)_j^{k+1} + ((uv)_x)_j^k] = q_1(x_j, t_{k+(1/2)}), \end{aligned} \quad (33)$$

$$\begin{aligned} &\frac{\partial^\alpha v(x_j, t_{k+(1/2)})}{\partial t^\alpha} - \frac{1}{2} [v_{xx}(x_j, t_{k+1}) + v_{xx}(x_j, t_k)] \\ &+ \frac{\eta}{2} [v(x_j, t_{k+1}) v_x(x_j, t_{k+1}) + v(x_j, t_k) v_x(x_j, t_k)] \\ &+ \frac{\alpha_2}{2} [((uv)_x)_j^{k+1} + ((uv)_x)_j^k] = q_2(x_j, t_{k+(1/2)}). \end{aligned} \quad (34)$$

The nonlinear terms in equations (33) and (34) are linearized using the following quasi-linearization [43]:

$$(uu_x)_j^{k+1} = u_j^{k+1} (u_x)_j^k + u_j^k (u_x)_j^{k+1} - (uu_x)_j^k + O(\tau^2), \quad (35)$$

$$(uv_x)_j^{k+1} = u_j^{k+1} (v_x)_j^k + u_j^k (v_x)_j^{k+1} - (uv_x)_j^k + O(\tau^2), \quad (36)$$

$$(u_x v)_j^{k+1} = (u_x)_j^{k+1} v_j^k + (u_x)_j^k v_j^{k+1} - (u_x v)_j^k + O(\tau^2), \quad (37)$$

$$(v v_x)_j^{k+1} = v_j^{k+1} (v_x)_j^k + v_j^k (v_x)_j^{k+1} - (v v_x)_j^k + O(\tau^2). \quad (38)$$

Now, using (22), (24), Lemma 1, and relations (36)–(38), we have the following difference scheme which is accurate of the order $O(\tau^2 + h^2)$:

$$\begin{aligned}
& \left[\frac{w_0}{2} + \frac{\gamma\tau\eta}{2h} \sum_{l=0}^M D_{jl}^1 U_l^k + \frac{\alpha_1\gamma\tau}{2h} \sum_{l=0}^M D_{jl}^1 V_l^k \right] U_j^{k+1} \\
& - \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 U_l^{k+1} + \frac{\gamma\tau\eta}{2h} U_j^k \sum_{l=0}^M D_{jl}^1 U_l^{k+1} \\
& + \frac{\alpha_1\gamma\tau}{2h} V_j^k \sum_{l=0}^M D_{jl}^1 U_l^{k+1} + \frac{\alpha_1\gamma\tau}{2h} V_j^{k+1} \sum_{l=0}^M D_{jl}^1 U_l^k \\
& + \frac{\alpha_1\gamma\tau}{2h} U_j^k \sum_{l=0}^M D_{jl}^1 V_l^{k+1} \\
& = \gamma\tau q_{1j}^{k+(1/2)} - \frac{w_0}{2} U_j^k + \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 U_l^k + w_k U_j^0 \\
& + \sum_{l=0}^M (w_{k-l} - w_{k-l+1}) \frac{U_j^l + U_j^{l-1}}{2},
\end{aligned} \tag{39}$$

$$\begin{aligned}
& \left[\frac{w_0}{2} + \frac{\gamma\tau\eta}{2h} \sum_{l=0}^M D_{jl}^1 V_l^k + \frac{\alpha_2\gamma\tau}{2h} \sum_{l=0}^M D_{jl}^1 U_l^k \right] V_j^{k+1} \\
& - \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 V_l^{k+1} + \frac{\gamma\tau\eta}{2h} V_j^k \sum_{l=0}^M D_{jl}^1 V_l^{k+1} \\
& + \frac{\alpha_2\gamma\tau}{2h} U_j^k \sum_{l=0}^M D_{jl}^1 V_l^{k+1} + \frac{\alpha_2\gamma\tau}{2h} U_j^{k+1} \sum_{l=0}^M D_{jl}^1 V_l^k \\
& + \frac{\alpha_2\gamma\tau}{2h} V_j^k \sum_{l=0}^M D_{jl}^1 U_l^{k+1} \\
& = \gamma\tau q_{2j}^{k+(1/2)} - \frac{w_0}{2} V_j^k + \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 V_l^k + w_k V_j^0 \\
& + \sum_{l=0}^M (w_{k-l} - w_{k-l+1}) \frac{V_j^l + V_j^{l-1}}{2}.
\end{aligned} \tag{40}$$

Therefore, in each time step we solve the following bi-pentadiagonal linear system of dimension $(2M-2) \times (2M-2)$:

$$AM^{k+1} = F_k, \tag{41}$$

where $M = [U_1^{k+1}, \dots, U_{M-1}^{k+1}, V_1^{k+1}, \dots, V_{M-1}^{k+1}]'$.

4. Stability Analysis

To study the stability analysis of the proposed scheme, we use the Fourier method. In applying the Fourier stability

method, the nonlinear terms are temporarily frozen, since the stability analysis is strictly only applicable to linear equations. Thus, we have linearized the nonlinear terms uu_x and $(uv)_x$ in equation (4) by freezing u and v as a local constants β_1 and β_2 , respectively. We have

$$\begin{aligned}
& {}_0^{\text{CF}} D_t^\alpha U^{k+(1/2)} - \frac{1}{2} [(U_{xx})_j^{k+1} + (U_{xx})_j^k] \\
& + \frac{\eta\beta_1}{2} [(U_x)_j^{k+1} + (U_x)_j^k] + \frac{\alpha_1\beta_2}{2} [(U_x)_j^{k+1} + (U_x)_j^k] \\
& + \frac{\alpha_1\beta_1}{2} [(V_x)_j^{k+1} + (V_x)_j^k] = q_1(x_j, t_{k+(1/2)}).
\end{aligned} \tag{42}$$

Substituting approximations (22) and (32) yield the following difference equation:

$$\begin{aligned}
& \frac{w_0}{2} U_j^{k+1} - \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 U_l^{k+1} + \frac{\gamma\tau}{2h} (\eta\beta_1 + \alpha_1\beta_2) \sum_{l=0}^M D_{jl}^1 U_l^{k+1} \\
& + \frac{\gamma\tau\alpha_1\beta_1}{2h} \sum_{l=0}^M D_{jl}^1 V_l^{k+1} \\
& = -\frac{w_0}{2} U_j^k + \sum_{l=0}^M (w_{l-k} - w_{l-k+1}) \frac{U_j^l + U_j^{l-1}}{2} \\
& + w_k U_j^0 + \frac{\gamma\tau}{2h^2} \sum_{l=0}^M D_{jl}^2 U_l^k \\
& - \frac{\gamma\tau}{2h} (\eta\beta_1 + \alpha_1\beta_2) \sum_{l=0}^M D_{jl}^1 U_l^k - \frac{\gamma\tau\alpha_1\beta_1}{2h} \sum_{l=0}^M D_{jl}^1 V_l^k + q_{1j}^{k+(1/2)}.
\end{aligned} \tag{43}$$

Let $\tilde{U}_j^k$ and $\tilde{V}_j^k$ be the approximate solutions of (43), and define $\xi_j^k = U_j^k - \tilde{U}_j^k$, $\xi_j^k = V_j^k - \tilde{V}_j^k$, $1 \leq j \leq M-1$, $0 \leq k \leq N-1$, with corresponding vectors:

$$\begin{aligned}
& \zeta^k = (\zeta_1^k, \zeta_2^k, \dots, \zeta_{M-1}^k)^T, \\
& \xi^k = (\xi_1^k, \xi_2^k, \dots, \xi_{M-1}^k)^T.
\end{aligned} \tag{44}$$

So, we have

$$\begin{aligned}
& \frac{w_0 \zeta_j^{k+1}}{2} - \frac{\gamma \tau}{2h^2} \left(-\frac{\zeta_{j-2}^{k+1}}{6} + \frac{5\zeta_{j-1}^{k+1}}{3} - 3\zeta_j^{k+1} + \frac{5\zeta_{j+1}^{k+1}}{3} - \frac{\zeta_{j+2}^{k+1}}{6} \right) + \frac{\gamma \tau \nu}{2} \left(\frac{\zeta_{j-2}^{k+1}}{12} - \frac{2\zeta_{j-1}^{k+1}}{3} + \frac{2\zeta_{j+1}^{k+1}}{3} - \frac{\zeta_{j+2}^{k+1}}{12} \right) \\
& + \frac{\gamma \tau \alpha_1 \beta_1}{2h} \left(\frac{1}{12} \zeta_{j-2}^{k+1} - \frac{2}{3} \zeta_{j-1}^{k+1} + \frac{2}{3} \zeta_{j+1}^{k+1} - \frac{1}{12} \zeta_{j+2}^{k+1} \right) = -\frac{w_0 \zeta_j^k}{2} + \sum_{l=0}^M (w_{l-k} - w_{l-k+1}) \frac{\zeta_j^l + \zeta_j^{l-1}}{2} \\
& + \frac{\gamma \tau}{2h^2} \left(-\frac{1}{6} \zeta_{j-2}^k + \frac{5}{3} \zeta_{j-1}^k - 3\zeta_j^k + \frac{5}{3} \zeta_{j+1}^k - \frac{1}{6} \zeta_{j+2}^k \right) - \frac{\gamma \tau \nu}{2} \left(\frac{1}{12} \zeta_{j-2}^k - \frac{2}{3} \zeta_{j-1}^k + \frac{2}{3} \zeta_{j+1}^k - \frac{1}{12} \zeta_{j+2}^k \right) \\
& - \frac{\gamma \tau \alpha_1 \beta_1}{2h} \left(\frac{1}{12} \zeta_{j-2}^k - \frac{2}{3} \zeta_{j-1}^k + \frac{2}{3} \zeta_{j+1}^k - \frac{1}{12} \zeta_{j+2}^k \right) + w_k \zeta_j^0,
\end{aligned} \tag{45}$$

where $\nu = (\eta\beta_1 + \alpha_1\beta_2)$.

Now, we define the grid functions as follows:

$$\begin{aligned}
\zeta^k(x) &= \begin{cases} \zeta_j^k, & x_j - \frac{h}{2} < x \leq x_j + \frac{h}{2}, \\ 0, & 0 \leq x \leq \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leq L, \end{cases} \\
\xi^k(x) &= \begin{cases} \xi_j^k, & x_j - \frac{h}{2} < x \leq x_j + \frac{h}{2}, \\ 0, & 0 \leq x \leq \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leq L. \end{cases}
\end{aligned} \tag{46}$$

We expand the $\zeta^k(x)$ and $\xi^k(x)$ into the following Fourier series expansions:

$$\begin{aligned}
\zeta^k(x) &= \sum_{l=-\infty}^{\infty} A_k(l) e^{i(2\pi l x/L)}, \\
\xi^k(x) &= \sum_{l=-\infty}^{\infty} B_k(l) e^{i(2\pi l x/L)},
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
A_k(l) &= \frac{1}{L} \int_0^L \zeta^k(x) e^{(-i2\pi l x/L)} dx, \\
B_k(l) &= \frac{1}{L} \int_0^L \xi^k(x) e^{(-i2\pi l x/L)} dx.
\end{aligned} \tag{48}$$

Applying the Parseval equality,

$$\begin{aligned}
\int_0^L \|\zeta^k(x)\|^2 dx &= \sum_{l=-\infty}^{\infty} \|A_k(l)\|^2, \\
\int_0^L \|\xi^k(x)\|^2 dx &= \sum_{l=-\infty}^{\infty} \|B_k(l)\|^2,
\end{aligned} \tag{49}$$

$$\begin{aligned}
\int_0^L \|\zeta^k(x)\|^2 dx &= \sum_{j=1}^{M-1} h \|\zeta_j^k\|^2, \\
\int_0^L \|\xi^k(x)\|^2 dx &= \sum_{j=1}^{M-1} h \|\xi_j^k\|^2,
\end{aligned} \tag{50}$$

we have

$$\begin{aligned}
\|\zeta^k\|_2^2 &= \sum_{l=-\infty}^{\infty} \|A_k(l)\|^2, \\
\|\xi^k\|_2^2 &= \sum_{l=-\infty}^{\infty} \|B_k(l)\|^2.
\end{aligned} \tag{51}$$

Now, we suppose that

$$\begin{aligned}
\zeta_j^k &= A_k e^{i\sigma_x jh}, \\
\xi_j^k &= B_k e^{i\sigma_x jh},
\end{aligned} \tag{52}$$

where $\sigma_x = (2l\pi/L)$. Substituting the above relations into (45) leads to

$$\begin{aligned}
& \left(\frac{w_0}{2} + \frac{\gamma \tau}{6h^2} \cos(2\sigma_x h) - \frac{5\gamma \tau}{3h^2} \cos(\sigma_x h) \right) A_{k+1} + i \left(-\frac{\gamma \tau \alpha_1 \beta_1}{12h} \sin(2\sigma_x h) + \frac{2\gamma \tau \alpha_1 \beta_1}{3h} \sin(\sigma_x h) \right) B_{k+1} \\
& + i \left(-\frac{\gamma}{6} \sin(2\sigma_x h) + \frac{4\gamma}{3} \sin(\sigma_x h) \right) A_{k+1} = \left(-\frac{w_0}{2} - \frac{\gamma \tau}{6h^2} \cos(2\sigma_x h) + \frac{5\gamma \tau}{3h^2} \cos(\sigma_x h) - \frac{3\gamma \tau}{2h^2} \right) A_k \\
& + w_k A_0 + i \left(\frac{\gamma}{6} \sin(2\sigma_x h) - \frac{4\gamma}{3} \sin(\sigma_x h) \right) A_k + i \left(\frac{\gamma \tau \alpha_1 \beta_1}{12h} \sin(2\sigma_x h) + \frac{2\gamma \tau \alpha_1 \beta_1}{3h} \sin(\sigma_x h) \right) B_k \\
& + \sum_{l=1}^k (w_{k-l} - w_{k-l+1}) \frac{A_l + A_{l-1}}{2}.
\end{aligned} \tag{53}$$

Set

$$\begin{aligned} X &= \frac{w_0}{2} + \frac{\gamma\tau}{6h^2} \cos(2\sigma_x h) - \frac{5\gamma\tau}{3h^2} \cos(\sigma_x h), \\ Y &= -\frac{\gamma}{6} \sin(2\sigma_x h) + \frac{4\gamma}{3} \sin(\sigma_x h), \\ Z &= -\frac{\gamma\tau\alpha_1\beta_1}{12h} \sin(2\sigma_x h) + \frac{2\gamma\tau\alpha_1\beta_1}{3h} \sin(\sigma_x h). \end{aligned} \quad (54)$$

We have

$$\begin{aligned} (X + iY)A_{k+1} + iZB_{k+1} &= -(X + iY)A_k - iZB_k \\ &\quad + \sum_{l=1}^k (w_{k-l} - w_{k-l+1}) \frac{A_l + A_{l-1}}{2} + w_k A_0 \end{aligned} \quad (55)$$

so that

$$\begin{aligned} |A_{k+1}| &\leq |A_k| + \frac{|iZ|}{|X + iY|} (|B_{k+1}| + |B_k|) \\ &\quad + \sum_{l=1}^k (w_{k-l} - w_{k-l+1}) \frac{|A_l + A_{l-1}|}{2} \left(\frac{1}{|X + iY|} \right) + \frac{w_k |A_0|}{|X + iY|}. \end{aligned} \quad (56)$$

Theorem 2. If A_k is the solution of equation (55), then there are positive constants C_k such that

$$|A_k| \leq C_k |A_0|, \quad k = 1, 2, \dots, N-1. \quad (57)$$

Proof. We use the mathematical induction for proof. For $k = 1$, we have

$$|A_1| \leq |A_0| + \frac{|iZ|}{|X + iY|} (|B_1| + |B_0|) + \frac{w_0 |A_0|}{|X + iY|}. \quad (58)$$

Using the convergence of the series on the right-hand side of equation (49), we know that there exists a positive constant P_2 such that

$$|B_k| \leq P_2 |A_0|, \quad k = 0, 1, \dots, N-1. \quad (59)$$

So,

$$|A_1| \leq |A_0| + D(|A_0|) + E|A_0| \leq C_1 |A_0|. \quad (60)$$

Now, suppose that

$$|A_k| \leq C_k |A_0|, \quad k = 1, 2, \dots, N-2. \quad (61)$$

By equation (56), we have

$$\begin{aligned} |A_{k+1}| &\leq C_k |A_0| + FC_2 |A_0| + G \sum_{l=1}^k (w_{k-l} - w_{k-l+1}) \\ &\quad \cdot \frac{C_l |A_0| + C_l |A_0|}{2} + H |A_0|. \end{aligned} \quad (62)$$

Now, assume that

$$C' = \max\{C_1, C_2, \dots, C_{N-2}\}. \quad (63)$$

So,

$$|A_{k+1}| \leq C_k |A_0| + FP_2 |A_0| + GC' |A_0| + H |A_0| \leq C_{k+1} |A_0|. \quad (64)$$

□

Remark 1. From (5), with similar way, there are positive constants Q_k such that

$$|B_k| \leq Q_k |B_0|, \quad k = 1, 2, \dots, N-1. \quad (65)$$

Theorem 3. The finite difference schemes (39) and (40) are unconditionally stable for $\gamma \in (0, 1)$.

Proof. According to Theorem 2 and Remark 1, using (51), we obtain

$$\begin{aligned} \|\zeta^k\|_2^2 &= \sum_{l=-\infty}^{\infty} \|A_k(l)\|^2 \leq \sum_{l=-\infty}^{\infty} C_k^2 \|A_0(l)\|^2 = C_k^2 \|\zeta^0\|_2^2, \\ \|\eta^k\|_2^2 &= \sum_{l=-\infty}^{\infty} \|B_k(l)\|^2 \leq \sum_{l=-\infty}^{\infty} Q_k^2 \|B_0(l)\|^2 = Q_k^2 \|\eta^0\|_2^2, \end{aligned} \quad (66)$$

so that

$$\begin{aligned} \|U^k - \tilde{U}^k\|_2^2 &\leq C_k \|U^0 - \tilde{U}^0\|_2^2, \\ \|V^k - \tilde{V}^k\|_2^2 &\leq Q_k \|V^0 - \tilde{V}^0\|_2^2, \end{aligned} \quad (67)$$

which shows that schemes (39) and (40) are unconditionally stable. □

5. Numerical Results

In this section, we provide two examples to illustrate efficiency of schemes (39) and (40). All experiments are performed on a Windows 10 (64 bit) Intel(R) Core(TM) i7-7500U CPU 2.70 GHz, 8.0 GB of RAM using MATLAB R2017b. In all examples, we use the error norm

$$\|e(\tau, h)\| = \|e^N\| = \left(h \sum_{j=1}^M (e_j^N)^2 \right)^{(1/2)}, \quad (68)$$

where $e_j^n = u_j^n - U_j^n$ and $e_j^n = v_j^n - V_j^n$. We evaluate the convergence order with the following formula:

$$r(\tau, h) = \log_2 \left(\frac{\|e(\tau, 2h)\|}{\|e(\tau, h)\|} \right). \quad (69)$$

Example 1. Consider the coupled Burgers equation with time fractional derivative with exact solutions $u(x, t) = v(x, t) = e^t x^2 (1 - x)^2$:

FIGURE 1: The plot of exact solution (a) and numerical solution (b) at $\tau = (1/1000)$ and $h = (1/160)$ with $\gamma = 0.9$ for Example 1.

$$\begin{cases}
 \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} + \eta u \frac{\partial u}{\partial x} + \alpha_1 \frac{\partial(uv)}{\partial x} = q_1, & x \in [0, 1], t \in [0, 1], 0 < \gamma < 1, \\
 \frac{\partial^\gamma v}{\partial t^\gamma} - \frac{\partial^2 v}{\partial x^2} + \eta v \frac{\partial v}{\partial x} + \alpha_2 \frac{\partial(uv)}{\partial x} = q_2, & x \in [0, 1], t \in [0, 1], 0 < \gamma < 1, \\
 u(0, t) = 0, v(0, t) = 0, \\
 u(1, t) = 0, v(1, t) = 0, \\
 u(x, 0) = x^2(1-x)^2, v(x, 0) = x^2(1-x)^2.
 \end{cases} \quad (70)$$

We derive the functions q_1 and q_2 with the help from exact solutions. Numerical solution and exact solution have been demonstrated in Figure 1. Tables 1–4 give the approximation errors and CPU times for the difference schemes. We choose different space step sizes to obtain the numerical results and order of convergence. Figure 2 shows the comparison of numerical solution and exact solution for $\gamma = 0.9$ at $t = 1$ and contour plot of numerical solution. Also, Figure 3 shows the pointwise errors for $u(x, t)$.

Example 2. Consider the following coupled Burgers equation with time fractional derivative with exact solutions $u(x, t) = v(x, t) = x(1-x)\sin(t)$:

$$\begin{cases}
 \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x} + \alpha_1 \frac{\partial(uv)}{\partial x} = q_1, & x \in [0, 1], t \in [0, T], 0 < \gamma < 1, \\
 \frac{\partial^\gamma v}{\partial t^\gamma} - \frac{\partial^2 v}{\partial x^2} + 2v \frac{\partial v}{\partial x} + \alpha_2 \frac{\partial(uv)}{\partial x} = q_2, & x \in [0, 1], t \in [0, T], 0 < \gamma < 1, \\
 u(0, t) = 0, v(0, t) = 0, \\
 u(1, t) = 0, v(1, t) = 0, \\
 u(x, 0) = 0, v(x, 0) = 0.
 \end{cases} \quad (71)$$

We solve this problem with the method developed in this article with several values of T and γ . Graphs of numerical solution and exact solution at different times have been demonstrated in Figure 4. Figure 4 shows that the proposed method is efficient. Table 5 gives the approximation errors for $t = 0.5, 1, 1.5, 2, 2.5, 3, 4$ with

different α_1, α_2 and CPU times. We choose $h = (1/160)$ for space step size to obtain the numerical results. Figure 5 shows the comparison of numerical solution and exact solution for $\gamma = 0.1$ at $t = 1$. Also, Figure 6 shows the pointwise errors and contour plot of numerical solution for $u(x, t)$.

TABLE 1: Take $\tau = 0.001$ and $\eta = 800$ and $\alpha_1 = 0.01, \alpha_2 = 0.01$ experiment order of convergence for $u(x, t)$ at $\gamma = 0.1, 0.3, 0.5$ for Example 1.

h	$\gamma = 0.1$			$\gamma = 0.3$			$\gamma = 0.5$		
	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU
(1/10)	$1.1161e-03$		1.6557 s	$1.1133e-03$		1.4701 s	$1.1094e-03$		1.5101 s
(1/20)	$3.8484e-04$	2.5941	2.8074 s	$1.8406e-04$	2.5966	2.7374 s	$1.8282e-04$	2.6013	2.8795 s
(1/40)	$4.7604e-05$	1.9571	5.4495 s	$4.7393e-05$	1.9574	5.2651 s	$4.7059e-05$	1.9579	5.6347 s
(1/80)	$1.1883e-05$	2.0022	10.6861 s	$1.1833e-05$	2.0018	10.6804 s	$1.1756e-05$	2.0011	12.0549 s
(1/160)	$2.7784e-06$	2.0966	22.2362 s	$2.7680e-06$	2.0018	23.9606	$2.7529e-06$	2.0944	24.3564 s

TABLE 2: Take $\tau = 0.001$ and $\eta = 800$ and $\alpha_1 = 0.01$ and $\alpha_2 = 0.01$ experiment order of convergence for $u(x, t)$ at $\gamma = 0.7, 0.9$ for Example 1.

h	$\gamma = 0.7$			$\gamma = 0.9$		
	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU
(1/10)	$1.1018e-03$		1.6541 s	$1.0653e-03$		1.7714 s
(1/20)	$1.8045e-04$	2.6102	3.1754 s	$1.7495e-04$	2.6062	3.4511 s
(1/40)	$4.6426e-05$	1.9586	6.2369 s	$4.5085e-05$	1.9562	6.7489 s
(1/80)	$1.1614e-05$	1.9991	13.0887 s	$1.1364e-05$	1.9882	15.1479 s
(1/160)	$2.7286e-06$	2.0896	25.7009 s	$2.7292e-06$	2.0579	31.5002 s

TABLE 3: Take $\tau = 0.001$ and $\eta = 200$ and $\alpha_1 = 0.03$ and $\alpha_2 = 0.03$ experiment order of convergence for $v(x, t)$ at $\gamma = 0.1, 0.3, 0.5$ for Example 1.

h	$\gamma = 0.1$			$\gamma = 0.3$			$\gamma = 0.5$		
	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU
(1/10)	$1.1129e-03$		1.6208 s	$1.1110e-03$		1.3725 s	$1.1061e-03$		1.5264 s
(1/20)	$1.8595e-04$	2.5813	1.5632 s	$1.8514e-04$	2.5852	2.6589 s	$1.8385e-04$	2.5889	2.5895 s
(1/40)	$4.7628e-05$	1.9650	5.2549 s	$4.7411e-05$	1.9653	5.2359 s	$1.1695e-05$	2.0088	5.3654 s
(1/80)	$1.1827e-05$	2.0097	10.6584 s	$1.1775e-05$	2.0095	10.2545 s	$1.1695e-05$	2.0088	12.0036 s
(1/160)	$2.7407e-06$	2.1095	22.12358 s	$2.7296e-06$	2.1090	23.8547 s	$2.7131e-06$	2.1079	24.1354 s

TABLE 4: Take $\tau = 0.001$ and $\eta = 200$ and $\alpha_1 = 0.03$ and $\alpha_2 = 0.03$, experiment order of convergence for $v(x, t)$ at $\gamma = 0.7, 0.9$ for Example 1.

h	$\gamma = 0.7$			$\gamma = 0.9$		
	$\ e^N\ $	Order	CPU	$\ e^N\ $	Order	CPU
(1/10)	$1.0983e-03$		1.6151 s	$1.0614e-03$		1.6646 s
(1/20)	$1.8136e-04$	2.5983	3.0172 s	$1.7506e-04$	2.6000	3.3234 s
(1/40)	$4.6406e-05$	1.9665	6.1189 s	$4.4958e-05$	1.9612	6.7189 s
(1/80)	$1.1544e-05$	2.0072	13.5179 s	$1.1262e-05$	1.9971	15.1345 s
(1/160)	$2.6855e-06$	2.1039	25.4015 s	$2.7119e-06$	2.0755	31.4012 s

FIGURE 2: The comparison (a) and absolute error (b) between numerical solution and exact solution with $\tau = (1/1000)$, $h = (1/160)$, and $\gamma = 0.9$ at $t = 1$ for Example 1.

FIGURE 3: Pointwise errors for $u(x, t)$ with $\tau = (1/1000)$, $h = (1/160)$, and $\gamma = 0.9$ (a) and contour plot of numerical solution $u(x, t)$ (b) for Example 1.

FIGURE 4: Graphs of numerical and exact solution at different times with $\tau = 0.001$ for Example 2.

TABLE 5: Errors for $u(x, t)$ and $v(x, t)$ at different times for Example 2.

T	α_1	α_2	$\gamma = 0.1$			$\gamma = 0.3$			$\gamma = 0.7$		
			$\ e_u^N\ $	$\ e_v^N\ $	CPU	$\ e_u^N\ $	$\ e_v^N\ $	CPU	$\ e_u^N\ $	$\ e_v^N\ $	CPU
0.5	0.01	0.01	$2.2750e-07$	$1.0966e-08$	19.7143s	$2.2269e-07$	$1.0692e-08$	19.6825s	$2.1196e-07$	$2.3957e-09$	22.1880s
1	0.03	0.02	$2.1872e-06$	$1.9266e-08$	19.7329s	$2.1574e-06$	$1.8640e-08$	19.9076s	$2.0966e-06$	$5.8427e-09$	24.0896s
1.5	0.001	0.05	$8.1009e-08$	$2.2678e-08$	19.6884s	$8.1626e-08$	$2.1703e-08$	20.8275s	$9.8131e-08$	$6.6570e-09$	25.0396s
2	0.001	0.001	$6.6249e-08$	$2.0562e-08$	19.8765s	$6.8923e-08$	$1.9294e-08$	21.4610s	$9.1599e-08$	$3.4789e-09$	25.4790s
2.5	0.002	0.002	$6.3910e-08$	$1.3390e-08$	19.9632s	$7.1535e-08$	$1.1951e-08$	22.1602s	$1.0437e-07$	$3.7207e-09$	25.9242s
3	0.01	0.01	$3.3597e-08$	$2.8784e-09$	19.9462s	$6.9805e-08$	$1.4200e-09$	22.4854s	$1.3465e-07$	$1.2871e-08$	26.0649s
4	0.01	0.01	$6.2662e-07$	$1.7635e-08$	20.2347s	$6.3810e-07$	$1.8725e-08$	23.4227s	$6.0534e-07$	$2.9270e-08$	26.4143s

FIGURE 5: The plot of exact solution (a) and numerical solution (b) at $\tau = 1/1000$ and $h = (1/160)$ with $\gamma = 0.1$ for Example 2. FIGURE 6: Pointwise errors for $v(x, t)$ with $\tau = (1/1000)$, $h = (1/160)$, and $\gamma = 0.1$ (a) and contour plot of numerical solution $u(x, t)$ (b) for Example 2.

6. Conclusion

In this article, we constructed a Crank–Nicolson finite difference scheme based on cubic B-spline quasi-interpolation to solve time fractional coupled Burgers equations. By the Fourier series method, we proved that this scheme is unconditionally stable. Numerical examples have been carried out to show the convergence orders and applicability of the scheme and error norms are calculated with respect to different space step sizes. From the error tables and graphs of exact and numerical solution, we can say that our method has a good

accuracy. For the numerical computations, we have used Matlab.

Data Availability

All results have been obtained by conducting the numerical procedure and the ideas can be shared for the researchers.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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