Research Article

A Numerical Method for Compressible Model of Contamination from Nuclear Waste in Porous Media

Zhifeng Wang

School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China

Correspondence should be addressed to Zhifeng Wang; wangzhifeng1997@163.com

Received 6 September 2020; Accepted 25 April 2021; Published 19 May 2021

Academic Editor: Yong Hong Wu

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This paper studies and analyzes a model describing the flow of contaminated brines through the porous media under severe thermal conditions caused by the radioactive contaminants. The problem is approximated based on combining the mixed finite element method with the modified method of characteristics. In order to solve the resulting algebraic nonlinear equations efficiently, a two-grid method is presented and discussed in this paper. This approach includes a small nonlinear system on a coarse grid with size H and a linear system on a fine grid with size h . It follows from error estimates that asymptotically optimal accuracy can be obtained as long as the mesh sizes satisfy $H = O(h^{1/3})$.

1. Introduction

A compressible nuclear waste disposal contamination problem in porous media is presented by the following coupled systems of partial differential equations. The physical processes can be concreted to be a high-level waste disposal buried in a salt dome, and next the salt dissolves to generate a brine, radioactive elements decay to generate heat, and finally the radionuclides are transported by the flow.

Fluid:

$$\phi_1 \frac{\partial p}{\partial t} + \nabla \cdot u = -q + R'_s(\hat{c}), \quad (1)$$

$$u = -\frac{k}{\mu(\hat{c})} \nabla p = -a(\hat{c})^{-1} \nabla p, \quad (2)$$

where p and u are the fluid pressure and Darcy velocity, respectively, $\phi_1 = \phi c_w$ and ϕ is the porosity. $q = q(x, t)$ is a production term, $R'_s(\hat{c}) = [c_s \phi K_s f_s / (1 + c_s)] (1 - \hat{c})$ is a salt dissolution term, $k(x)$ is the permeability of the rock, and $\mu(\hat{c})$, the viscosity of the fluid, is dependent upon $\hat{c}$, the concentration of the brine in the fluid.

Brine:

$$\phi \frac{\partial \hat{c}}{\partial t} + u \cdot \nabla \hat{c} - \nabla \cdot (E_c \nabla \hat{c}) = g(\hat{c}), \quad (3)$$

where E_c is the diffusion tensor including the molecular diffusion and mechanical diffusion and $E_c = \mathbf{D} + D_m \mathbf{I}$, $\mathbf{D} = (D)_{ij} = ((\alpha_T |u| \delta_{ij} + (\alpha_L - \alpha_T) u_i u_j) / |u|)$, and $g(\hat{c}) = -\hat{c} \{ [c_s \phi K_s f_s / (1 + c_s)] (1 - \hat{c}) \} - q_c + R_s$. Here, D_m is molecular diffusion. u_i and u_j are two direction cosines of Darcy velocity. $\mathbf{I}$ is an identity matrix.

Heat:

$$d_1(p) \frac{\partial p}{\partial t} + d_2 \frac{\partial T}{\partial t} + c_p u \cdot \nabla T - \nabla \cdot (\bar{E}_H \nabla T) = Q(u, T, \hat{c}, p), \quad (4)$$

where T is the temperature of the fluid, $d_1(p) = -\phi c_w [U_0 + (p/\rho)]$, $d_2 = \phi c_p + (1 - \phi) \rho_R c_{pR}$, $\bar{E}_H = \mathbf{D} c_{pw} + K'_m \mathbf{I}$, $K'_m = k_m / \rho_0$, and $Q(u, T, \hat{c}, p) = -\{ [\nabla U_0 - c_p \nabla T_0] \cdot u + [U_0 + c_p (T - T_0) + (p/\rho)] [-q + R'_s] \} - q_L - q^H - q_H$.

Radionuclide (component i):

$$\begin{aligned} & \phi K_i \frac{\partial c_i}{\partial t} + u \cdot \nabla c_i - \nabla \cdot (E_c \nabla c_i) + d_3(c_i) \frac{\partial p}{\partial t} \\ & = f_i(\hat{c}, c_1, \dots, c_N), \quad i = 1, \dots, N, \end{aligned} \quad (5)$$

where c_i is the trace concentration of the i th radionuclide, $d_3(c_i) = \phi c_w c_i (K_i - 1)$, and

$$f_i(\hat{c}, c_1, \dots, c_N) = c_i \left\{ q - \left[\frac{c_s \phi K_s f_s}{(1 + c_s)} \right] (1 - \hat{c}) \right\} - q c_i - q_{ci} + q_{0i} + \sum_{j=1}^N k_{ij} \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i. \quad (6)$$

We assume the following:

- (1) Zero Neumann boundary conditions for the equations
- (2) The initial conditions are assumed given
- (3) The medium Ω is vertically homogeneous and take $\Omega \in \mathbb{R}^2$
- (4) The solutions are smooth and Ω periodic
- (5) k, ϕ, ϕ_1, d_2 , and K_i are bounded below by positive constants, and $R'_s(\hat{c}), \mu(\hat{c}), g(\hat{c}), d_3(c_i), f_i(c_i)$, and $Q(T)$ are twice continuously differentiable with bounded partial derivatives about the variables in parentheses
- (6) $\mathbf{D} = 0$

Chou and Li [1], Ewing et al. [2], and Li et al. [3] have presented and studied several numerical methods for system (1)–(5) and its incompressible case. In this paper, we use the mixed finite element method to approximate the fluid problem and treat the brine, heat, and radionuclides by a modified method of characteristic finite element. It is well known that the full discrete approximation scheme is coupled and nonlinear. If simply lagging, the evaluation of the nonlinear items is used to obtain a linear system; it would be inevitable to introduce the constraint conditions about

the mesh grid due to the stability requirement. Moreover, it would take an expensive cost to choose the implicit scheme to nonlinear solutions. An efficient method motivated by Xu [4] is considered in this paper. The method is used by Bi et al. [5], Chen et al. [6–10], and Liu et al. [11, 12] for solving some nonlinear problem. We shall relegate all of nonlinear iterations on a coarse grid of size H much coarser than the original fine grid of size h . According to the error estimates in the context, it obtains the asymptotically optimal accuracy to take $H = O(h^{1/3})$.

The remainder of the paper is organized as follows. Notations and approximation assumptions are given in Section 2. A two-grid method is defined and the convergence error estimates are derived in Section 3. In Section 4, we give some conclusions and extensions.

2. Notations and Approximation Results

To analyze the temporal discretization on a time interval $(0, T)$, let M be a positive integer number, $\Delta t = T/M$, $t^n = n\Delta t$ ($0 \leq n \leq M$), and $\varphi^n = \varphi(\cdot, t^n)$. Let $L^p(J; W^{j,q}(\Omega))$ denote the usual set of functions with the norm

$$\|\psi\|_{L^p(J; W^{j,q}(\Omega))} = \left(\int_J \|\psi(\cdot, t)\|_{W^{j,q}(\Omega)}^p dt \right)^{1/p}, \quad (7)$$

where if $p = \infty$, the integral is replaced by the essential supremum. Let $l^p(J; W^{j,q}(\Omega))$ denote the time discrete analogue to the space $L^p(J; W^{j,q}(\Omega))$ with the norm

$$\|\psi\|_{l^p(J; W^{j,q}(\Omega))} = \left(\sum_{n=1}^N \|\psi^n\|_{W^{j,q}(\Omega)}^p \Delta t \right)^{1/p}. \quad (8)$$

Let $W = L^2(\Omega)$ and $V = \{v \in H(\text{div}; \Omega); v \cdot \gamma = 0\}$. The weak form is presented as follows:

$$\left(\phi_1 \frac{\partial p}{\partial t}, w \right) + (\nabla \cdot u, w) = (-q + R'_s(\hat{c}), w), \quad \forall w \in W, \quad (9)$$

$$(a(\hat{c})u, v) - (\nabla \cdot v, p) = 0, \quad \forall v \in V, \quad (10)$$

$$\left(\phi \frac{\partial \hat{c}}{\partial t}, z \right) + (u \cdot \nabla \hat{c}, z) + (E_c \nabla \hat{c}, \nabla z) = (g(\hat{c}), z), \quad (11)$$

$$\left(d_2 \frac{\partial T}{\partial t}, z \right) + c_p (u \cdot \nabla T, z) + (\tilde{E}_H \nabla T, \nabla z) + \left(d_1(p) \frac{\partial p}{\partial t}, z \right) = (Q(u, T, \hat{c}, p), z), \quad (12)$$

$$\left(\phi K_i \frac{\partial c_i}{\partial t}, z \right) + (u \cdot \nabla c_i, z) + (E_c \nabla c_i, \nabla z) + \left(d_3(c_i) \frac{\partial p}{\partial t}, z \right) = (f_i(\hat{c}, c_1, \dots, c_N), z), \quad (13)$$

for $z \in H^1(\Omega)$ and $i = 1, \dots, N$.

Assume that $V_h \times W_h$ is the Raviart–Thomas space of index at least k associated with a quasitriangulation of Ω such that the elements have diameters bounded by h_p . Let

$M_h = M_{h_c}$ and $R_h = R_{h_T}$ be finite-dimensional subspaces of $W^{1,\infty}(\Omega)$ for the approximation of concentrations and temperature, respectively, and we take M_h and R_h as the piecewise-polynomial space of degree at least l and r ,

respectively. As in [2, 11, 13], the approximation properties for $V_h \times W_h$ and M_h, R_h are given by

$$\inf_{v_h \in V_h} \|v - v_h\| \leq C \|v\|_{k+1} h_p^{k+1}, \quad (14)$$

$$\inf_{v_h \in V_h} \|\nabla \cdot (v - v_h)\| \leq C (\|v\|_{k+1} + \|\nabla \cdot v\|_{k+1}) h_p^{k+1}, \quad (15)$$

for $v \in V \cap H^{k+1}(\Omega)^2$ and $\nabla \cdot v \in H^{k+1}(\Omega)$, and

$$\inf_{w_h \in W_h} \|w - w_h\| \leq C \|w\|_{k+1} h_p^{k+1}, \quad w \in H^{k+1}(\Omega), \quad (16)$$

$$\inf_{z_h \in M_h} \|z - z_h\|_{1,q} \leq C \|z\|_{l+1,q} h_c^l, \quad z \in W^{l+1,q}(\Omega), 1 \leq q \leq \infty, \quad (17)$$

$$\inf_{z_h \in R_h} \|z - z_h\|_{1,q} \leq C \|z\|_{r+1,q} h_T^r, \quad z \in W^{r+1,q}(\Omega), 1 \leq q \leq \infty. \quad (18)$$

If the initial solutions $\{p_h^0, u_h^0, \tilde{c}_h^0, \tilde{T}_h^0, c_{ih}^0\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$, the characteristics-Galerkin and mixed finite element approximation schemes are to find $\{p_h^n, u_h^n, \tilde{c}_h^n, \tilde{T}_h^n, c_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ satisfying

$$\left(\phi_1 \frac{p_h^n - p_h^{n-1}}{\Delta t}, w \right) + (\nabla \cdot u_h^n, w) = (-q + R'_s(\tilde{c}_h^n), w), \quad \forall w \in W_h, \quad (19)$$

$$(a(\tilde{c}_h^n) u_h^n, v) - (\nabla \cdot v, p_h^n) = 0, \quad \forall v \in V_h, \quad (20)$$

$$\left(\phi \frac{\tilde{c}_h^n - \tilde{c}_h^{n-1}}{\Delta t}, z \right) + (E_c \nabla \tilde{c}_h^n, \nabla z) = (g(\tilde{c}_h^n), z), \quad \forall z \in M_h, \quad (21)$$

$$\left(d_2 \frac{\tilde{T}_h^n - \tilde{T}_h^{n-1}}{\Delta t}, z \right) + (\tilde{E}_h \nabla \tilde{T}_h^n, \nabla z) + \left(d_1 (p_h^n) \frac{p_h^n - p_h^{n-1}}{\Delta t}, z \right) = (Q(u_h^n, \tilde{T}_h^n, \tilde{c}_h^n, p_h^n), z), \quad \forall z \in R_h, \quad (22)$$

$$\left(\phi K_i \frac{c_{ih}^n - c_{ih}^{n-1}}{\Delta t}, z \right) + (E_c \nabla c_{ih}^n, \nabla z) + \left(d_3 (c_{ih}^n) \frac{p_h^n - p_h^{n-1}}{\Delta t}, z \right) = (f_i(\tilde{c}_h^n, c_{1h}^n, \dots, c_{Nh}^n), z), \quad \forall z \in M_h, \quad (23)$$

where $i = 1, \dots, N$ and

$$\begin{aligned} \bar{c}_h^{n-1} &= \tilde{c}_h^{n-1}(\bar{x}^{n-1}), \\ \bar{x}^{n-1} &= x - \frac{u_h^n \Delta t}{\phi}, \\ \bar{c}_{ih}^{n-1} &= c_{ih}^{n-1}(\bar{x}^{n-1}), \\ \bar{x}^{n-1} &= x - \frac{u_h^n \Delta t}{(\phi K_i)}, \\ \bar{T}_h^{n-1} &= \tilde{T}_h^{n-1}(\bar{x}^{n-1}), \\ \bar{x}^{n-1} &= x - \frac{c_p u_h^n \Delta t}{d_2}. \end{aligned} \quad (24)$$

Remark. If $\bar{x}^{n-1}$ is located outside Ω , we can join $\bar{x}^{n-1}$ with $Y \in \partial\Omega$ so that $(\bar{x}^{n-1} - Y)/\|\bar{x}^{n-1} - Y\|$ is the outer-normal direction to the boundary $\partial\Omega$ at Y . Take $x^* \in \Omega$ so that $Y - x^{*n-1} = \bar{x}^{n-1} - Y$, then we define $\bar{x}^{n-1} = x^{*n-1}$.

In order to deduce the error estimates, we employ the elliptic projections by labeling them with tildes.

$$(\nabla \cdot (u - \tilde{U}), w) + (p - \tilde{P}, w) = 0, \quad w \in W_h, \quad (25)$$

$$(a(\tilde{c})(u - \tilde{U}), v) + (\nabla \cdot v, p - \tilde{P}) = 0, \quad v \in V_h, \quad (26)$$

$$(E_c \nabla (\tilde{c} - \tilde{C}), \nabla z) + \lambda(\tilde{c} - t\tilde{C}, z) = 0, \quad z \in M_h, \quad (27)$$

$$(E_H \nabla (T - \tilde{T}), \nabla z) + \lambda(T - \tilde{T}, z) = 0, \quad z \in R_h, \quad (28)$$

$$(E_c \nabla (c_i - \tilde{C}_i), \nabla z) + \lambda(c_i - \tilde{C}_i, z) = 0, \quad z \in M_h, \quad (29)$$

where $\{\tilde{U}, \tilde{P}\}: J \longrightarrow V_h \times W_h, \tilde{C}: J \longrightarrow M_h, \tilde{T}: J \longrightarrow R_h$ and $\tilde{C}_i: J \longrightarrow M_h$ for $t \in J$ and introduce the following notations:

$$\begin{aligned} \eta &= p - \tilde{P}, \\ \alpha &= \tilde{P} - p_h, \\ \varrho &= u - \tilde{U}, \\ \beta &= \tilde{U} - u_h, \\ \tilde{\zeta} &= \tilde{c} - \tilde{C}, \\ \hat{\chi} &= \tilde{C} - \tilde{c}_h, \\ \zeta_i &= c_i - \tilde{C}_i, \\ \chi_i &= \tilde{C}_i - c_{ih}, \\ \theta &= T - \tilde{T}, \\ \psi &= \tilde{T} - \tilde{T}_h. \end{aligned} \quad (30)$$

Subtracting (19) from (9) and taking $w = d_i \alpha^{n-1}$, we get the error equation about the pressure function as follows:

$$\begin{aligned}
& (\phi_1 d_t \alpha^{n-1}, d_t \alpha^{n-1}) + (\nabla \cdot \beta^n, d_t \alpha^{n-1}) \\
&= - \left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \alpha^{n-1} \right) + \left(\phi_1 \left(d_t \bar{P}^{n-1} - \frac{\partial \bar{P}^n}{\partial t} \right), d_t \alpha^{n-1} \right) \\
&+ \left(\frac{\partial R'_s}{\partial \bar{c}} (\delta_1^n) (\bar{c}^n - \bar{c}_h^n), d_t \alpha^{n-1} \right),
\end{aligned} \tag{31}$$

where $d_t \alpha^{n-1} = (\alpha^n - \alpha^{n-1})/\Delta t$ and δ_1^n is between $\bar{c}^n$ and $\bar{c}_h^n$.

Next, combining (20) from (10) at $t = t^n$ with the test function β^n ,

$$(a(\bar{c}_h^n) \beta^n, \beta^n) - (\nabla \cdot \beta^n, \alpha^n) = - \left(\frac{\partial a}{\partial \bar{c}} (\delta_2^n) \bar{U}^n (\bar{c}^n - \bar{c}_h^n), \beta^n \right). \tag{32}$$

When $t = t^{n-1}$, we apply the Taylor expansion and obtain that

$$\begin{aligned}
& (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1}) - (\nabla \cdot \beta^n, \alpha^{n-1}) \\
&= (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1} - \beta^n) - \left(\frac{\partial a}{\partial \bar{c}} (\delta_2^{n-1}) \bar{U}^{n-1} (\bar{c}^{n-1} - \bar{c}_h^{n-1}), \beta^n \right),
\end{aligned} \tag{33}$$

where δ_2 is between $\bar{c}$ and $\bar{c}_h$.

Combining (32) with (33), we get

$$\begin{aligned}
& \frac{1}{2} d_t \{ (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1}) \} - (\nabla \cdot \beta^n, d_t \alpha^{n-1}) \\
&= - \frac{1}{2\Delta t} (a(\bar{c}_h^n) \beta^n, \beta^n) + \frac{1}{2\Delta t} (a(\bar{c}_h^{n-1}) \beta^n, \beta^n) - \frac{1}{2\Delta t} (a(\bar{c}_h^{n-1}) \beta^n, \beta^n) \\
&+ \frac{1}{\Delta t} (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^n) - \frac{1}{2\Delta t} (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1}) \\
&- \left(d_t \left\{ \frac{\partial a}{\partial \bar{c}} (\delta_2^{n-1}) \bar{U}^{n-1} (\bar{c}^{n-1} - \bar{c}_h^{n-1}), \beta^n \right\} \right).
\end{aligned} \tag{34}$$

By (31) and (34),

$$\begin{aligned}
& (\phi_1 d_t \alpha^{n-1}, d_t \alpha^{n-1}) + \frac{1}{2} d_t \{ (a(\bar{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1}) \} \\
&= - \left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \alpha^{n-1} \right) + \left(\phi_1 \left(d_t \bar{P}^{n-1} - \frac{\partial \bar{P}^n}{\partial t} \right), d_t \alpha^{n-1} \right) + \left(\frac{\partial R'_s}{\partial \bar{c}} (\delta_1^n) (\bar{c}^n - \bar{c}_h^n), d_t \alpha^{n-1} \right) \\
&- \frac{1}{2\Delta t} ([a(\bar{c}_h^n) - a(\bar{c}_h^{n-1})] \beta^n, \beta^n) - \frac{1}{2\Delta t} (a(\bar{c}_h^{n-1}) (\beta^n - \beta^{n-1}), (\beta^n - \beta^{n-1})) \\
&- \left(d_t \left\{ \frac{\partial a}{\partial \bar{c}} (\delta_2^{n-1}) \bar{U}^{n-1} (\bar{c}^{n-1} - \bar{c}_h^{n-1}), \beta^n \right\} \right) \\
&\leq - \left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \alpha^{n-1} \right) + \left(\phi_1 \left(d_t \bar{P}^{n-1} - \frac{\partial \bar{P}^n}{\partial t} \right), d_t \alpha^{n-1} \right) + \left(\frac{\partial R'_s}{\partial \bar{c}} (\delta_1^n) (\bar{c}^n - \bar{c}_h^n), d_t \alpha^{n-1} \right) \\
&- \frac{1}{2\Delta t} ([a(\bar{c}_h^n) - a(\bar{c}_h^{n-1})] \beta^n, \beta^n) - \left(d_t \left\{ \frac{\partial a}{\partial \bar{c}} (\delta_2^{n-1}) \bar{U}^{n-1} (\bar{c}^{n-1} - \bar{c}_h^{n-1}), \beta^n \right\} \right).
\end{aligned} \tag{35}$$

Using the deduction as [1, 2], we have

$$\begin{aligned} & \|d_t \alpha^{n-1}\|^2 + \frac{1}{2} d_t \{ (a(\hat{c}_h^{n-1}) \beta^{n-1}, \beta^{n-1}) \} \\ & \leq \left(\|\beta^n\|_\infty^2 + \varepsilon \right) \|d_t \hat{\chi}^{n-1}\|^2 + C \left(\|\hat{\chi}^{n-1}\|^2 + \|\beta^n\|^2 + h_p^{2k+2} + h_c^{2l+2} + \Delta t^2 \right). \end{aligned} \quad (36)$$

After making the induction hypothesis that $\sup \|\beta^n\|_\infty \rightarrow 0$, we multiply (36) by Δt and sum over $1 \leq n \leq M$ to get

$$\begin{aligned} & \sum_{n=1}^M \|d_t \alpha^{n-1}\|^2 \Delta t + (a(\hat{c}_h^{M-1}) \beta^M, \beta^M) \\ & \leq \left(\|\beta^n\|_\infty^2 + \varepsilon \right) \sum_{n=1}^M \|d_t \hat{\chi}^{n-1}\|^2 \Delta t \\ & \quad + C \left(\sum_{n=1}^M \left[\|\hat{\chi}^{n-1}\|_1^2 + \|\beta^n\|^2 \right] \Delta t + h_p^{2k+2} + h_c^{2l+2} + \Delta t^2 \right). \end{aligned} \quad (37)$$

Combining (11) and (21), we get the following equality, in which we choose the test function $z = \hat{\chi}^n - \hat{\chi}^{n-1} = d_t \hat{\chi}^{n-1} \Delta t$ and sum over $1 \leq n \leq M$:

$$\begin{aligned} & \sum_{n=1}^M (\phi d_t \hat{\chi}^{n-1}, d_t \hat{\chi}^{n-1}) \Delta t + \frac{1}{2} (E_c \nabla \hat{\chi}^M, \nabla \hat{\chi}^M) - \frac{1}{2} (E_c \nabla \hat{\chi}^0, \nabla \hat{\chi}^0) \\ & \leq \sum_{n=1}^M \left(\phi \frac{\partial \hat{c}^n}{\partial t} + u_h^n \cdot \nabla \hat{c}^n - \phi \frac{\hat{c}^n - \bar{\hat{c}}^{n-1}}{\Delta t}, d_t \hat{\chi}^{n-1} \right) \Delta t + \sum_{n=1}^M \left(\phi \frac{\bar{\hat{\chi}}^{n-1} - \hat{\chi}^{n-1}}{\Delta t}, d_t \hat{\chi}^{n-1} \right) \Delta t \\ & \quad + \sum_{n=1}^M \left(\phi d_t \hat{\zeta}^{n-1}, d_t \hat{\chi}^{n-1} \right) \Delta t + \sum_{n=1}^M \left(\phi \frac{\hat{\zeta}^{n-1} - \bar{\hat{\zeta}}^{n-1}}{\Delta t}, d_t \hat{\chi}^{n-1} \right) \Delta t + \sum_{n=1}^M \lambda(\hat{\zeta}^n, d_t \hat{\chi}^{n-1}) \Delta t \\ & \quad + \sum_{n=1}^M ((u^n - u_h^n) \cdot \nabla \hat{c}^n, d_t \hat{\chi}^{n-1}) \Delta t + \sum_{n=1}^M \left(\frac{\partial g}{\partial \hat{c}}(\hat{c}_h^n)(\hat{\zeta}^n + \hat{\chi}^n), d_t \hat{\chi}^{n-1} \right) \Delta t, \end{aligned} \quad (38)$$

where $\bar{\hat{\chi}}^{n-1} = \hat{\chi}^{n-1}(\bar{x}^{n-1})$, $\bar{\hat{\zeta}}^{n-1} = \hat{\zeta}^{n-1}(\bar{x}^{n-1})$, and $\bar{x}^{n-1} = x - (u_h^n / \phi(x)) \Delta t$.

Note that

$$\begin{aligned} & \left| \sum_{n=1}^M \left(\phi \frac{\hat{\zeta}^{n-1} - \bar{\hat{\zeta}}^{n-1}}{\Delta t}, d_t \hat{\chi}^{n-1} \right) \Delta t \right| \\ & = \left| \left[\left(\phi \left(\hat{\zeta}^{M-1} - \bar{\hat{\zeta}}^{M-1} \right), \hat{\chi}^M \right) - \left(\phi \left(\hat{\zeta}^0 - \bar{\hat{\zeta}}^0 \right), \hat{\chi}^0 \right) \right] - \sum_{n=1}^{M-1} \left(\phi d_t \left(\hat{\zeta}^n - \bar{\hat{\zeta}}^n \right), \hat{\chi}^n \right) \right| \\ & = C \Delta t \left\{ \|\hat{\zeta}^{M-1}\|^2 + \|\hat{\zeta}^0\|^2 + \sum_{n=1}^{M-1} \left(\|\hat{\zeta}^{n-1}\|^2 + \|d_t \hat{\zeta}^n\|^2 \right) \right\} + \varepsilon \Delta t \|\hat{\chi}^0\|_1^2. \end{aligned} \quad (39)$$

The reminder of the right side items in (38) is just as [2], that is,

$$\begin{aligned}
& \sum_{n=1}^M \|d_t \hat{\chi}^{n-1}\|^2 \Delta t + (E_c \nabla \hat{\chi}^M, \nabla \hat{\chi}^M) \\
& \leq C \left(\sum_{n=1}^M \left[\|\nabla \hat{\chi}^{n-1}\|_{\infty}^2 \|\hat{\chi}^{n-1}\|_1^2 + \|\beta^n\|^2 \right] \Delta t + h_p^{2k+2} + h_c^{2l+2} + \Delta t^2 \right).
\end{aligned} \tag{40}$$

It follows from the assumption $\sup \|\nabla \hat{\chi}^{n-1}\|_{\infty} \leq C$ and Gronwall lemma that

$$\begin{aligned}
& \|\hat{\chi}\|_{L^{\infty}(H^1)} + \|d_t \hat{\chi}\|_{L^2(L^2)} + \|\beta\|_{L^{\infty}(L^2)} + \|d_t \alpha\|_{L^2(L^2)} \\
& \leq C(\Delta t + h_c^{l+1} + h_p^{k+1}).
\end{aligned} \tag{41}$$

Then, by the inverse estimate and (40), we know that the induction hypotheses hold if

$$\begin{aligned}
h_p^{k+1} &= o(h_c), \\
h_c^{l+1} &= o(h_p), \\
\Delta t &= o(h_c), \\
\Delta t &= o(h_p).
\end{aligned} \tag{42}$$

Finally, from the approximation properties,

$$\begin{aligned}
& \|\hat{c} - \hat{c}_h\|_{L^{\infty}(H^1)} + \|d_t(\hat{c} - \hat{c}_h)\|_{L^2(L^2)} + \|u - u_h\|_{L^{\infty}(L^2)} + \|d_t(p - p_h)\|_{L^2(L^2)} \\
& \leq c(\Delta t + h_c^l + h_p^{k+1}).
\end{aligned} \tag{43}$$

Similar to the above analysis, we can obtain the error estimates for the radionuclide equation and heat equation as follows:

$$\begin{aligned}
& \sum_{i=1}^N \|\chi_i\|_{L^{\infty}(H^1)} + \sum_{i=1}^N \|d_t \chi_i\|_{L^2(L^2)} \leq C(\Delta t + h_c^{l+1} + h_p^{k+1}), \\
& \|\psi\|_{L^{\infty}(H^1)} + \|d_t \psi\|_{L^2(L^2)} \leq C(\Delta t + h_c^{l+1} + h_T^{r+1} + h_p^{k+1}),
\end{aligned} \tag{44}$$

where $h_p^{k+1} = o(h_T)$, $h_c^{l+1} = o(h_T)$, and $\Delta t = o(h_T)$ are satisfied.

Note that the time step is limited to be $o(h)$ due to the theoretical proof.

Theorem 1. Define $\{p_h^n, u_h^n, \hat{c}_h^n, \tilde{T}_h^n, c_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ for $n \geq 1$ by system (19)–(23) and assume that the approximation properties (25)–(29) hold. If

$$\begin{aligned}
h_p^{k+1} &= o(h_c), \\
h_c^{l+1} &= o(h_p), \\
\Delta t &= o(h_c), \\
\Delta t &= o(h_p), \\
h_p^{k+1} &= o(h_T), \\
h_c^{l+1} &= o(h_T), \\
\Delta t &= o(h_T),
\end{aligned} \tag{45}$$

then there exists a positive constant C independent of h and Δt , such that

$$\begin{aligned}
& \|\hat{c} - \hat{c}_h\|_{L^{\infty}(H^1)} + \|d_t(\hat{c} - \hat{c}_h)\|_{L^2(L^2)} + \|u - u_h\|_{L^{\infty}(L^2)} + \|d_t(p - p_h)\|_{L^2(L^2)} \\
& + \sum_{i=1}^N \|c_i - c_{ih}\|_{L^{\infty}(H^1)} + \sum_{i=1}^N \|d_t(c_i - c_{ih})\|_{L^2(L^2)} \\
& + \|T - \tilde{T}_h\|_{L^{\infty}(H^1)} + \|d_t(T - \tilde{T}_h)\|_{L^2(L^2)} \\
& \leq C(\Delta t + h_c^l + h_T^r + h_p^{k+1}).
\end{aligned} \tag{46}$$

Similarly, we can get the error estimates of fine grid scheme in L^2 norm.

Theorem 2. Define $\{p_h^n, u_h^n, \tilde{c}_h^n, \tilde{T}_h^n, c_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ for $n \geq 1$ by system (19)–(23) and assume that the approximation properties (25)–(29) hold. Then, there exists a positive constant C independent of h, H , and Δt , such that

$$\begin{aligned} & \|\tilde{c} - \tilde{c}_h\|_{L^\infty(L^2)} + \|u - u_h\|_{L^2(L^2)} + \|p - p_h\|_{L^\infty(L^2)} \\ & + \sum_{i=1}^N \|c_i - c_{ih}\|_{L^\infty(L^2)} + \|T - \tilde{T}_h\|_{L^\infty(L^2)} \\ & \leq C(\Delta t + h_c^{l+1} + h_T^{r+1} + h_p^{k+1}). \end{aligned} \quad (47)$$

3. An Efficient Method

We now use and analyze a two-grid method for iteratively solving the nonlinear problem. The method has two steps.

Stage 1. On the coarse grid $\mathcal{T}_H$ with a mesh size $H \in (0, 1)$, solve a small nonlinear system for $\{p_H^n, U_H^n, \tilde{C}_H^n, T_H^n, C_{iH}^n\} \in V_H \times W_H \times M_H \times R_H \times M_H^N$ given by (19)–(23).

Stage 2. On the fine grid $\mathcal{T}_h$ with a mesh size $h \in (0, 1)$ ($h \ll H$), solve the following linear system for $\{p_h^n, U_h^n, \tilde{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$:

$$\begin{aligned} & \left(\phi_1 \frac{P_h^n - P_h^{n-1}}{\Delta t}, w \right) + (\nabla \cdot U_h^n, w) - \left(\frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n) \tilde{C}_h^n, w \right) \\ & = \left(-q + R'_s(\tilde{C}_H^n) - \frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n) \tilde{C}_H^n, w \right), \quad \forall w \in W_h, \end{aligned} \quad (48)$$

$$\left(a(\tilde{C}_H^n) U_h^n + \frac{\partial a}{\partial \tilde{c}}(\tilde{C}_H^n) U_H^n \tilde{C}_h^n, v \right) - (\nabla \cdot v, P_h^n) = \left(\frac{\partial a}{\partial \tilde{c}}(\tilde{C}_H^n) U_H^n \tilde{C}_H^n, v \right), \quad \forall v \in V_h, \quad (49)$$

$$\begin{aligned} & \left(\phi \frac{\tilde{C}_h^n - \tilde{C}_h^{n-1}}{\Delta t}, z \right) + (U_h^n \cdot \nabla \tilde{C}_H^n, z) + (E_c \nabla \tilde{C}_h^n, \nabla z) - \left(\frac{\partial g}{\partial \tilde{c}}(\tilde{C}_H^n) \tilde{C}_h^n, z \right) \\ & = (U_H^n \cdot \nabla \tilde{C}_H^n, z) + \left(g(\tilde{C}_H^n) - \frac{\partial g}{\partial \tilde{c}}(\tilde{C}_H^n) \tilde{C}_H^n, z \right), \quad \forall z \in M_h, \end{aligned} \quad (50)$$

$$\begin{aligned} & \left(d_2 \frac{T_h^n - T_h^{n-1}}{\Delta t}, z \right) + (\tilde{E}_H \nabla T_h^n, \nabla z) - \left(\frac{\partial Q^H}{\partial T} T_h^n, z \right) \\ & = - \left(d_1 (P_h^n) \frac{P_h^n - P_h^{n-1}}{\Delta t}, z \right) + (Q^H, z) - \left(\frac{\partial Q^H}{\partial T} T_H^n, z \right), \quad \forall z \in R_h, \end{aligned} \quad (51)$$

$$\begin{aligned} & \left(\phi K_i \frac{C_{ih}^n - C_{ih}^{n-1}}{\Delta t}, z \right) + (E_c \nabla C_{iH}^n, \nabla z) + \left(\frac{\partial d_3}{\partial c_i}(C_{iH}^n) \frac{P_h^n - P_h^{n-1}}{\Delta t} C_{ih}^n, z \right) \\ & - \left(\frac{\partial f_i^H}{\partial c_1} C_{1h}^n + \dots + \frac{\partial f_i^H}{\partial c_N} C_{Nh}^n, z \right) = - \left(d_3(C_{iH}^n) \frac{P_h^n - P_h^{n-1}}{\Delta t}, z \right) \\ & + \left(\frac{\partial d_3}{\partial c_i}(C_{iH}^n) \frac{P_h^n - P_h^{n-1}}{\Delta t} C_{iH}^n + f_i^H - \frac{\partial f_i^H}{\partial c_1} C_{1H}^n - \dots - \frac{\partial f_i^H}{\partial c_N} C_{NH}^n, z \right), \quad \forall z \in M_h, \end{aligned} \quad (52)$$

where

$$\begin{aligned}
Q^H &= Q(U_h^n, T_H^n, \tilde{C}_h^n, P_h^n), \\
f_i^H &= f_i(\tilde{C}_h^n, C_{1H}^n, \dots, C_{NH}^n), \\
\bar{C}_h^{n-1} &= \tilde{C}_h^{n-1}(\bar{x}^{n-1}), \\
\bar{x}^{n-1} &= x - \left(\frac{U_H^n \Delta t}{\phi} \right), \\
\bar{C}_{ih}^{n-1} &= C_{ih}^{n-1}(\bar{x}^{n-1}), \\
\bar{x}^{n-1} &= x - \left(\frac{U_h^n \Delta t}{(\phi K_i)} \right), \\
\bar{T}_h^{n-1} &= T_h^{n-1}(\bar{x}^{n-1}), \\
\bar{x}^{n-1} &= x - \left(\frac{c_p U_h^n \Delta t}{d_2} \right),
\end{aligned} \tag{53}$$

and the projection on the fine grid is based on the numerical solutions on coarse grid.

The sequential solution processes are defined as follows. Firstly, we apply the Newton iteration to the coupled system on the coarse grid and obtain $\{P_H^n, U_H^n, \tilde{C}_H^n, T_H^n, C_{iH}^n\}$. Next, combining (48)–(50), we get P_h^n and U_h^n with RT_1 and $\tilde{C}_h^n$ with piecewise linear finite element using a coupled linear system. Finally, from (51) and (52), we can get C_{ih}^n and T_h^n by parallel computation.

In order to analyze the linear scheme on the fine grid, we define

$$\begin{aligned}
\pi &= \tilde{P} - P_h, \\
\sigma &= \tilde{U} - U_h, \\
\hat{\xi} &= \tilde{C} - \tilde{C}_h, \\
\xi &= \tilde{C}_i - C_{ih}, \\
\omega &= \tilde{T} - T_h.
\end{aligned} \tag{54}$$

According to Taylor expansion, there exists a positive δ_3 such that

$$\begin{aligned}
&R'_s(\tilde{c}^n) - R'_s(\tilde{C}_H^n) - \frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n)(\tilde{c}^n - \tilde{C}_H^n) \\
&= R'_s(\tilde{C}_H^n) + \frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n)(\tilde{c}^n - \tilde{C}_H^n) - R'_s(\tilde{C}_H^n) - \frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n)(\tilde{C}_H^n - \tilde{C}_H^n) \\
&\quad + \frac{\partial^2 R'_s}{2 \partial \tilde{c}^2}(\delta_3)(\tilde{c}^n - \tilde{C}_H^n)^2 \\
&= \frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n)(\tilde{c}^n - \tilde{C}_H^n) + \frac{\partial^2 R'_s}{2 \partial \tilde{c}^2}(\delta_3)(\tilde{c}^n - \tilde{C}_H^n)^2.
\end{aligned} \tag{55}$$

According to (48), (9), and (25),

$$\begin{aligned}
&(\phi_1 d_t \pi^{n-1}, d_t \pi^{n-1}) + (\nabla \cdot \sigma^n, d_t \pi^{n-1}) \\
&= -\left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \pi^{n-1} \right) + \left(\phi_1 \left(d_t \tilde{P}^{n-1} - \frac{\partial \tilde{P}^n}{\partial t} \right), d_t \pi^{n-1} \right) + (\eta^n, d_t \pi^{n-1}) \\
&\quad + \left(\frac{\partial R'_s}{\partial \tilde{c}}(\tilde{C}_H^n)(\tilde{c}^n - \tilde{C}_H^n) + \frac{\partial^2 R'_s}{2 \partial \tilde{c}^2}(\delta_1)(\tilde{c}^n - \tilde{C}_H^n)^2, d_t \pi^{n-1} \right).
\end{aligned} \tag{56}$$

Like the deduction of (34), we see that

$$\begin{aligned}
& \frac{1}{2} d_t \left\{ \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^{n-1} \right) \right\} - (\nabla \cdot \sigma^n, d_t \pi^{n-1}) \\
&= -\frac{1}{2\Delta t} \left(a \left(\widehat{C}_H^n \right) \sigma^n, \sigma^n \right) + \frac{1}{2\Delta t} \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^n, \sigma^n \right) - \frac{1}{2\Delta t} \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^n, \sigma^n \right) \\
&+ \frac{1}{\Delta t} \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^n \right) - \frac{1}{2\Delta t} \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^{n-1} \right) - \left(d_t \left\{ \frac{\partial a}{\partial \widehat{c}} \left(\widehat{C}_H^{n-1} \right) \widetilde{U}^{n-1} \right. \right. \\
&\quad \left. \left(\widehat{c}^{n-1} - \widehat{C}_h^{n-1} \right) + \frac{\partial a}{\partial \widehat{c}} \left(\widehat{C}_H^{n-1} \right) \left(\widetilde{U}^{n-1} - U_H^{n-1} \right) \left(\widehat{C}_h^{n-1} - \widehat{C}_H^{n-1} \right) \right\}, \sigma^n \right) \\
&\quad \left. - \frac{1}{2\Delta t} \left(\frac{\partial^2 a}{\partial \widehat{c}^2} (\delta_4) \widetilde{U}^n \left(\widehat{c}^n - \widehat{C}_H^n \right)^2 - \frac{\partial^2 a}{\partial \widehat{c}^2} (\delta_4) \widetilde{U}^{n-1} \left(\widehat{c}^{n-1} - \widehat{C}_H^{n-1} \right)^2, \sigma^n \right), \right.
\end{aligned} \tag{57}$$

where δ_4 is between $\widehat{c}$ and $\widehat{C}_H$.

Hence, (56) and (57) can give that

$$\begin{aligned}
& (\phi_1 d_t \pi^{n-1}, d_t \pi^{n-1}) + \frac{1}{2} d_t \left\{ \left(a \left(\widehat{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^{n-1} \right) \right\} \\
&= - \left(\phi_1 \frac{\partial \eta^n}{\partial t}, d_t \pi^{n-1} \right) + \left(\phi_1 \left(d_t \widetilde{P}^{n-1} - \frac{\partial \widetilde{P}^n}{\partial t} \right), d_t \pi^{n-1} \right) + (\eta^n, d_t \pi^{n-1}) \\
&+ \left(\frac{\partial R'_s}{\partial \widehat{c}} \left(\widehat{C}_H^n \right) (\widehat{c}^n - \widehat{C}_h^n) + \frac{\partial^2 R'_s}{2 \partial \widehat{c}^2} (\delta_3) (\widehat{c}^n - \widehat{C}_H^n)^2, d_t \pi^{n-1} \right) \\
&- \frac{1}{2\Delta t} \left(\left[a \left(\widehat{C}_H^n \right) - a \left(\widehat{C}_H^{n-1} \right) \right] \sigma^n, \sigma^n \right) - \frac{1}{2\Delta t} \left(a \left(\widehat{C}_H^{n-1} \right) (\sigma^n - \sigma^{n-1}), (\sigma^n - \sigma^{n-1}) \right), \\
&- \left(d_t \left\{ \frac{\partial a}{\partial \widehat{c}} \left(\widehat{C}_H^{n-1} \right) \widetilde{U}^{n-1} \left(\widehat{c}^{n-1} - \widehat{C}_h^{n-1} \right) + \frac{\partial a}{\partial \widehat{c}} \left(\widehat{C}_H^{n-1} \right) \left(\widetilde{U}^{n-1} - U_H^{n-1} \right) \left(\widehat{C}_h^{n-1} - \widehat{C}_H^{n-1} \right) \right\}, \sigma^n \right) \\
&- \frac{1}{2\Delta t} \left(\frac{\partial^2 a}{\partial \widehat{c}^2} (\delta_4) \widetilde{U}^n \left(\widehat{c}^n - \widehat{C}_H^n \right)^2 - \frac{\partial^2 a}{\partial \widehat{c}^2} (\delta_4) \widetilde{U}^{n-1} \left(\widehat{c}^{n-1} - \widehat{C}_H^{n-1} \right)^2, \sigma^n \right).
\end{aligned} \tag{58}$$

The error equation about $\widehat{c}$ shows that

$$\begin{aligned}
& \left(\phi \frac{\widehat{\xi}^n - \widehat{\xi}^{n-1}}{\Delta t}, z \right) + (E_c \nabla \widehat{\xi}^n, \nabla z) \\
&= - \left(\phi \frac{\partial \widehat{c}^n}{\partial t} + u_H^n \cdot \nabla \widehat{c}^n - \phi \frac{\widehat{c}^n - \widehat{c}^{n-1}}{\Delta t}, z \right) + \left(\phi \frac{\widehat{\xi}^{n-1} - \widehat{\xi}^{n-1}}{\Delta t}, z \right) \\
&- \left(\phi \frac{\widehat{\xi}^n - \widehat{\xi}^{n-1}}{\Delta t}, z \right) + \lambda (\widehat{\xi}^n, z) - ((u^n - U_H^n) \cdot (\nabla \widehat{c}^n - \nabla \widehat{C}_H^n) + (u^n - U_H^n) \cdot \nabla \widehat{C}_H^n, z) \\
&+ \left(\frac{\partial g}{\partial \widehat{c}} \left(\widehat{C}_H^n \right) (\widehat{\xi}^n + \widehat{\xi}^n), z \right) + \left(\frac{\partial^2 g}{2 \partial \widehat{c}^2} (\delta_5) (\widehat{c}^n - \widehat{C}_H^n)^2, z \right),
\end{aligned} \tag{59}$$

where $\bar{\xi}^{n-1} = \hat{\xi}^{n-1}(\bar{X}^{n-1})$, $\bar{\zeta}^{n-1} = \hat{\zeta}^{n-1}(\bar{X}^{n-1})$, and $\bar{X}^{n-1} = x - (U_H^n / \phi(x)) \Delta t$.

Taking the test function $z = \hat{\xi}^n - \hat{\xi}^{n-1} = d_t \hat{\xi}^{n-1} \Delta t$ and summing over $1 \leq n \leq M$, we have

$$\begin{aligned}
 & \sum_{n=1}^M \left(\phi d_t \hat{\xi}^{n-1}, d_t \hat{\xi}^{n-1} \right) \Delta t + \frac{1}{2} \left(E_c \nabla \hat{\xi}^M, \nabla \hat{\xi}^M \right) - \frac{1}{2} \left(E_c \nabla \hat{\xi}^0, \nabla \hat{\xi}^0 \right) \\
 & \leq \sum_{n=1}^M \left(\phi \frac{\partial \hat{c}^n}{\partial t} + u_H^n \cdot \nabla \hat{c}^n - \phi \frac{\hat{c}^n - \bar{\hat{c}}^{n-1}}{\Delta t}, d_t \hat{\xi}^{n-1} \right) \Delta t + \sum_{n=1}^M \left(\phi \frac{\bar{\xi}^{n-1} - \hat{\xi}^{n-1}}{\Delta t}, d_t \hat{\xi}^{n-1} \right) \Delta t \\
 & \quad + \sum_{n=1}^M \left(\phi \frac{\hat{\xi}^n - \bar{\xi}^{n-1}}{\Delta t}, d_t \hat{\xi}^{n-1} \right) \Delta t + \sum_{n=1}^M \lambda \left(\hat{\zeta}^n, d_t \hat{\xi}^{n-1} \right) \Delta t \\
 & \quad + \sum_{n=1}^M \left((u^n - U_H^n) \cdot (\nabla \hat{c}^n - \nabla \hat{C}_H^n) + (u^n - U_h^n) \cdot \nabla \hat{C}_H^n, d_t \hat{\xi}^{n-1} \right) \Delta t \\
 & \quad + \sum_{n=1}^M \left(\frac{\partial g}{\partial \hat{c}}(\hat{C}_H^n)(\hat{\zeta}^n + \hat{\xi}^n) + \frac{\partial^2 g}{2 \partial \hat{c}^2}(\delta_4)(\hat{c}^n - \hat{C}_H^n)^2, d_t \hat{\xi}^{n-1} \right) \Delta t.
 \end{aligned} \tag{60}$$

Since

$$\begin{aligned}
 & \|(\hat{c} - \hat{C}_H)^2\| \leq \|\hat{c} - \hat{C}_H\|_{L^\infty} \|\hat{c} - \hat{C}_H\|_{L^2} \\
 & \leq (\|\hat{c} - \bar{C}\|_{L^\infty} + \|\bar{C} - \hat{C}_H\|_{L^\infty}) \|\hat{c} - \hat{C}_H\|_{L^2} \\
 & \leq C(|\ln H_c| H_c^{l+1} + |\ln H_c| (H_c^{l+1} + H_p^{k+1})) \\
 & \quad \times (H_c^{l+1} + H_p^{k+1}) \\
 & \leq C(H_c^{2l+2} |\ln H_c| + H_p^{2k+2} |\ln H_c|),
 \end{aligned} \tag{61}$$

$$\begin{aligned}
 & \|(u - U_H)(\hat{c} - \hat{C}_H)\| \leq \|u - U_H\| \|\hat{c} - \hat{C}_H\|_{L^\infty} \\
 & \leq C(H_c^{l+1} + H_p^{k+1}) \\
 & \quad \times (|\ln H_c| (H_c^{l+1} + H_p^{k+1})) \\
 & \leq C(H_c^{2l+2} |\ln H_c| + H_p^{2k+2} |\ln H_c|),
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 & \|(u - U_H)(\nabla \hat{c} - \nabla \hat{C}_H)\| \leq \|u - U_H\| \|\hat{c} - \hat{C}_H\|_{1,\infty} \\
 & \leq C(H_c^{l+1} + H_p^{k+1}) \\
 & \quad \times (H_c^l + H_c^{-1} (H_c^{l+1} + H_p^{k+1})) \\
 & \leq C(H_c^{2l+1} + H_p^{2k+2} H_c^{-1}).
 \end{aligned} \tag{63}$$

It follows from (61)–(63) that

$$\begin{aligned}
 & \|\hat{c} - \hat{C}_h\|_{L^\infty(H^1)} + \|d_t(\hat{c} - \hat{C}_h)\|_{L^2(L^2)} + \|u - U_H\|_{L^\infty(L^2)} \\
 & \quad + \|d_t(p - P_h)\|_{L^2(L^2)} \\
 & \leq C(\Delta t + h_c^l + h_p^{k+1} + H_c^{2l+1} + H_p^{2k+2} H_c^{-1}).
 \end{aligned} \tag{64}$$

At last, we present the error results of the radionuclide equation and heat equation as follows:

$$\begin{aligned}
 & \sum_{i=1}^N \|c_i - C_{ih}\|_{L^\infty(H^1)} + \sum_{i=1}^N \|d_t(c_i - C_{ih})\|_{L^2(L^2)} \\
 & \leq C(\Delta t + h_c^l + h_p^{k+1} + H_c^{2l+2} |\ln H_c| + H_p^{2k+2} |\ln H_c|), \\
 & \|T - T_h\|_{L^\infty(H^1)} + \|d_t(T - T_h)\|_{L^2(L^2)} \\
 & \leq C(\Delta t + h_c^l + h_T^r + h_p^{k+1} + H_T^{2r+2} |\ln H_T| + H_c^{2l+2} |\ln H_T| + H_p^{2k+2} |\ln H_T|).
 \end{aligned} \tag{65}$$

If setting $l = 1, r = 1, k = 1$, $h_c = h_T = h_p$, and $H_c = H_T = H_p$, which accords with the assumption (45), we can get the following theorem of fine grid scheme.

Theorem 3. Define $\{P_h^n, U_h^n, \hat{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ for $n \geq 1$ by system (48)–(52) and assume that the approximation properties (14)–(18) hold. Then, there exists a positive constant C independent of h, H , and Δt , such that

$$\begin{aligned} & \|\hat{c} - \hat{C}_h\|_{L^\infty(H^1)} + \|d_t(\hat{c} - \hat{C}_h)\|_{L^2(L^2)} + \|u - U_h\|_{L^\infty(L^2)} + \|d_t(p - P_h)\|_{L^2(L^2)} \\ & + \sum_{i=1}^N \|c_i - C_{ih}\|_{L^\infty(H^1)} + \sum_{i=1}^N \|d_t(c_i - C_{ih})\|_{L^2(L^2)} \\ & + \|T - T_h\|_{L^\infty(H^1)} + \|d_t(T - T_h)\|_{L^2(L^2)} \\ & \leq C(\Delta t + h + H^3). \end{aligned} \quad (66)$$

Theorem 4. Define $\{P_h^n, U_h^n, \hat{C}_h^n, T_h^n, C_{ih}^n\} \in V_h \times W_h \times M_h \times R_h \times M_h^N$ for $n \geq 1$ by system (48)–(52) and assume that the approximation properties (14)–(18) hold. Then, there exists a positive constant C independent of h, H , and Δt , such that

$$\begin{aligned} & \|\hat{c} - \hat{C}_h\|_{L^\infty(L^2)} + \|u - U_h\|_{L^2(L^2)} + \|p - P_h\|_{L^\infty(L^2)} \\ & + \sum_{i=1}^N \|c_i - C_{ih}\|_{L^\infty(L^2)} + \|T - T_h\|_{L^\infty(L^2)} \\ & \leq C(\Delta t + h^2 + H^3). \end{aligned} \quad (67)$$

4. Conclusions and Extensions

The two-grid method presented in this paper reduces the complexity of problem. It involves a small nonlinear system on a coarse grid of size H and a linear system on a fine grid of size h . It is shown that the coarse space can be extremely coarse and still achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{1/3})$ in H^1 norm. Compared with the implicit scheme, the two-grid method reduces CPU time. Moreover, the method is suitable to make the large-scale computation and long time duration. The future work is to use the discontinuous finite volume element method [14], block-centered finite difference method [15], higher-order finite volume [16], and SAV method [17, 18] to consider this problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The author would like to thank Professor Luping Wang whose comments helped improve this paper.

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