

Research Article

An Approach of Interval-Valued Picture Fuzzy Uncertain Linguistic Aggregation Operator and Their Application on Supplier Selection Decision-Making in Logistics Service Value Concretion

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With respect to multiple criteria group decision-making (MCGDM) problems in which both the criteria weights and the expert weights take the form of crisp numbers and attribute values take the form of interval-valued picture fuzzy uncertain linguistic numbers, some new group decision-making analysis methods are developed. Firstly, some operational laws, expected value, and accuracy function of interval-valued picture fuzzy uncertain linguistic numbers are introduced. Then, an interval-valued picture fuzzy uncertain linguistic averaging and geometric aggregation operators are developed. Furthermore, some desirable properties of the developed operators, such as commutativity, idempotency, and monotonicity, have been studied. Based on these operators, an approach to multiple criteria group decision-making with interval-valued picture fuzzy uncertain linguistic information has been proposed. Finally, a practical example of 3PL supplier selection in logistics service value concretion is taken to test the defined method and to expose the effectiveness of the defined model.

1. Introduction

Decision-making (DM) has been influential in day-to-day activities such as education, economics, engineering, and medical. In DM, the problems contain a lot of information sources, giving the final result through the aggregating processes. Experts can take decisions on certain level due to the convolution of such decision-making (DM) problem and management information themselves, but they may have doubts about their interpretations. Specially, there may be a grade of hesitation, which is too necessary to focus on, while organizing completely beneficial models and problems. These degrees of hesitation are better defined by intuitionistic fuzzy set (IFS) values rather than objective numbers. The generalized form of Zadeh fuzzy sets (FSs) [1] is intuitionistic fuzzy sets [2]. The element of the IFS occurs in the ordered pair form, consisting of positive grade and negative grade, and the sum of

the two grades characterize is less than or equal to 1. Many researchers have made a significant contribution to the expansion of IFS generalization and its application to various fields, resulting in greater success of IFSs in theory and technology. The aggregation of IF information [3–6] is a big part of multicriteria decision-making (MCDM) with IFSs. It is why IFNs are too easy to reveal predilection details of a decision-maker over artifacts in the DM phase with unknown or firm chances. A significant step towards achieving the result of a decision problem is the aggregation of IFNs. The number of operators known as IFHA, IFOWA, and IFOWG operators has recently been introduced for this purpose to aggregate IFNs [7–12].

Wang and Li [13] proposed the Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in MADM. Verma and Sharma [14] proposed the exponential entropy on IFSs. Wan and Dong [15] discussed some

MADM based on triangular IFN Choquet integral operator. Wan [16] developed the power average operators of trapezoidal IFNs and application to MAGDM. Wan et al. [17] proposed the power geometric operators for trapezoidal IFNs and application to MAGDM. Verma [14] developed MAGDM approach based on intuitionistic fuzzy order- α divergence and entropy measures with the MABAC method. Wan et al. [18] defined trapezoidal IF prioritized AOs and application to MADM. Wan and Yi [19] defined power average of trapezoidal IFNs using strict t-norms and t-conorms. Xu et al. [20] developed aggregating decision information into Atanassov's intuitionistic fuzzy numbers for heterogeneous MAGDM. Wan et al. [21] defined some new generalized AOs for triangular IFNs and application to MAGDM. Dong and Wan [22] defined a new method for prioritized MCGDM with triangular IFNs. Dong et al. [23] developed some generalized Choquet integral operator of triangular Atanassov's IFNs and discussed their application to MAGDM. Verma and Sharma [24] studied some new measure of inaccuracy and its application to multicriteria MCDM under IF environment. Verma [25] discussed the generalized Bonferroni mean operator for IFNs and its application to MADM. Wan and Zhu [26] introduced the triangular IF triple Bonferroni harmonic mean operators and application to MAGDM. Wan and Dong [27] developed aggregating decision information into IVIFNs for heterogeneous MAGDM. Wan and Dong [28] give the DM theories and methods based on IVIF sets. Liu and Garg [29] defined the linguistic connection number of set pair analysis based on the TOPSIS method and numerical scale function. Verma [30] proposed some AOs for linguistic trapezoidal IFNs and their application to MAGDM. Verma and Merigo [31] defined the approach of MAGD based on 2-dimension linguistic intuitionistic fuzzy aggregation operators.

Batool et al. [32] defined the entropy-based Pythagorean probabilistic hesitant fuzzy decision-making technique and their application for Fog-Haze factor assessment problem. Khan et al. [33] proposed the Pythagorean fuzzy (PyF) Dombi AOs for the decision support system. Ashraf et al. [34] developed the fuzzy decision support modeling for Internet finance soft power evaluation using the sine trigonometric Pythagorean fuzzy information. Wan et al. [35] defined the Pythagorean fuzzy mathematical programming method for MAGDM with Pythagorean fuzzy truth degrees. Wan et al. [36] defined a new order relation for PyFNs and application to MAGDM. Garg [37] developed the linguistic interval-valued PyFNs and their application in MAGDM process. Wan et al. [38] introduced a three-phase method for PyF multiattribute group decision-making and application to haze management. Wang et al. [39] defined PyF interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight. Garg [29] introduced linguistic single-valued neutrosophic power AOs and their applications to group DM problems.

Since IFNs have two kinds of reports, i.e., yes and no, but in the case of election, there is some problem with the three styles of response, e.g., yes, no, and refused, where

the optimistic answer is a refusal. Cuong [40, 41] defined the principle of picture fuzzy set (PFS) to overcome this defect, dignifying positive, neutral, and negative grades in three separate functions. Cuong [42] discussed some PFS features and agreed with distance measurements as well. In the PF logic for fuzzy derivation forms, Cuong and Hai [43] defined fuzzy logic AOs and specified basic operational laws. The features of the fuzzy t-norm and t-conorm for PFS are analyzed by Cuong et al. [44]. Phong et al. [45] addressed a certain framework of PF relationships. Son et al. [46, 47] offer estimates of time and temperature based on information from the PF sets. Son [48, 49] defines picture fuzzy measures of isolation, distance, association, and often combined with the PFS condition. Wei et al. [50–52] have found several methods to measure the proximity between PFSs. Several researchers have currently created further models for PFSs: Singh [53] proposes the PFS coefficient of correlation and tested it to the clustering analysis. Son [54] defined a novel structure of the PFS fluid derivation and improved a classic method of fluid inference. Thong [55, 56] used the PF clustering approach to optimize the complex and particle problems. Wei [57] used the weighted cross-entropy theory of PFS to describe some simple leadership methods and utilized this approach to give ranking the alternatives. Yang et al. [58] used PFSs to define a versatile soft matrix of DM. Garg features an aggregation of MCDM problems with PFSs in [59]. The PFS solution was implemented by Peng et al. [60] and applied to DM. In addition, for the PFS, readers can also see [61, 62]. Shahzaib et al. [63] are expanding the PFS cubic set model. Thus, the study objective is divided into three parts under the IVPFULNs. Ashraf et al. [64] developed the cleaner production evaluation in gold mines using a novel distance measure method with cubic PFNs. Khan et al. [65] defined picture fuzzy aggregation information based on Einstein operations and their application in DM. Ashraf and Abdullah [66] proposed some novel aggregation operators for cubic picture fuzzy information and discussed their application for multiattribute decision support problem. Zeng et al. [67] defined the application of exponential Johnson picture fuzzy divergence measure in MCGD. Ashraf et al. [68] developed some aggregation operators of cubic picture fuzzy quantities and their application in decision support systems. Khalil [69] defined a new operation on interval-valued picture fuzzy set, interval-valued picture fuzzy soft set, and their applications. Akram et al. [70] proposed a DM model under complex picture fuzzy Hamacher AOs. Yang [71] proposed a group decision algorithm for aged health care product purchase under q-rung picture normal fuzzy environment using Heronian mean operator.

Moreover, in many multiple criteria group decision-making (MCGDM) problems, considering that the estimations of the criteria values are interval-valued picture fuzzy uncertain linguistic sets, it therefore is very necessary to give some aggregation techniques to aggregate the interval-valued picture fuzzy uncertain linguistic information. However, we are aware that the existing

aggregation techniques have difficulty in coping with group decision-making problems with interval-valued picture fuzzy uncertain linguistic information. Therefore, we in the current paper propose a series of aggregation operators for aggregating the interval-valued picture fuzzy uncertain linguistic information and investigate some properties of these operators. Then, based on the defined aggregation operators, we develop an approach to MCGDM with interval-valued picture fuzzy uncertain linguistic information. Moreover, we use a numerical example to show the application of the developed approach.

The remainder of the manuscript is arranged accordingly: in Section 2, first we discuss some fundamental ideas relating to the IVPFULS. Then, we described a number of AOs and discussed their basic properties, in Section 3. In Section 4, we discussed the supplier selection group decision model in logistics service value cocreation using the IVPFULG and IVPFULHG operators. An illustrative example of the selection of 3PL suppliers in the logistics service value cocreation information is given in Section 5, to explain the objective of the model. The article ends in Section 6.

2. Preliminaries

We defined some basic definitions relevant to the IVPFULSs in this section.

Definition 1 (see [1]). Let X be a nonempty set. Then, a fuzzy set is described as

$$\mathfrak{R} = \{(x, a_{\mathfrak{R}}(x)) | x \in X\}, \quad (1)$$

where $a_{\mathfrak{R}}: X \rightarrow [0, 1]$ is the positive membership function of \mathfrak{R} .

Definition 2 (see [2]). Let X be a nonempty set. Then, an intuitionistic fuzzy set is described as

$$\mathfrak{R} = \{(x, a_{\mathfrak{R}}(x), b_{\mathfrak{R}}(x)) | x \in X\}, \quad (2)$$

for an element $x \in X$, and the function $a_{\mathfrak{R}}(x), b_{\mathfrak{R}}(x): X \rightarrow [0, 1]$ represents the positive and negative grades, respectively, with $0 \leq a_{\mathfrak{R}}(x) + b_{\mathfrak{R}}(x) \leq 1$ for $x \in X$. And hesitation margin of x to \mathfrak{R} is obtained as $\pi_{\mathfrak{R}}(x) = 1 - a_{\mathfrak{R}}(x) - b_{\mathfrak{R}}(x)$.

Definition 3 (see [72]). Let X be a nonempty set. A picture fuzzy set \mathfrak{R} of X is defined as

$$\mathfrak{R} = \{\langle (x, a_{\mathfrak{R}}(x), b_{\mathfrak{R}}(x), c_{\mathfrak{R}}(x)) \rangle | x \in X\}, \quad (3)$$

where the function $a_{\mathfrak{R}}(x): X \rightarrow [0, 1]$ represents the function of positive and $c_{\mathfrak{R}}(x), b_{\mathfrak{R}}(x): X \rightarrow [0, 1]$ represents the function of neutral and negative membership, respectively, with the condition $0 \leq a_{\mathfrak{R}}(x) + b_{\mathfrak{R}}(x) + c_{\mathfrak{R}}(x) \leq 1$ for $x \in X$. The picture fuzzy hesitation margin of x to \mathfrak{R} is given by $\pi_{\mathfrak{R}}(x) = 1 - a_{\mathfrak{R}}(x) - b_{\mathfrak{R}}(x) - c_{\mathfrak{R}}(x)$,

$\pi_{\mathfrak{R}}(x)$ is called the indeterminacy grade of $x \in X$ to the PFS \mathfrak{R} .

Definition 4 (see [73]). Let X be a nonempty set. Then, the picture fuzzy linguistic set \mathfrak{R} in X is as

$$\mathfrak{R} = \{\langle (x, \langle s_{\theta(x)}, a_{\mathfrak{R}}(x), b_{\mathfrak{R}}(x), c_{\mathfrak{R}}(x) \rangle) \rangle | x \in X\}, \quad (4)$$

where $s_{\theta(x)} \in S$ is the linguistic number, $a_{\mathfrak{R}}(x) \in [0, 1]$ is a positive grade, $b_{\mathfrak{R}}(x) \in [0, 1]$ is a neutral grade, and $c_{\mathfrak{R}}(x) \in [0, 1]$ a negative grade of the element x to $s_{\theta}(x)$ under the condition $a_{\mathfrak{R}}(x) + b_{\mathfrak{R}}(x) + c_{\mathfrak{R}}(x) \leq 1, \forall x \in X$, and the refusal grade of \mathfrak{R} to $s_{\theta}(x)$ for all $x \in X$ is represented as

$$\pi_{\mathfrak{R}}(x) = 1 - a_{\mathfrak{R}}(x) - b_{\mathfrak{R}}(x) - c_{\mathfrak{R}}(x). \quad (5)$$

If $b_{\mathfrak{R}}(x) = 0, \forall x \in X$, then PFLS becomes to IFLS.

Definition 5 (see [74]). Let $[0, 1]$ be the closed intervals set and $X \neq \emptyset$ are the given set. Then, interval-valued picture fuzzy set (IVPFS) is described as

$$\Lambda = \{\langle (x, \tilde{a}_{\Lambda}(x), \tilde{b}_{\Lambda}(x), \tilde{c}_{\Lambda}(x)) \rangle | x \in X\}, \quad (6)$$

where $\tilde{a}_{\Lambda}, \tilde{b}_{\Lambda}, \tilde{c}_{\Lambda}: X \rightarrow [0, 1]$, and $0 \leq \sup(\tilde{a}_{\Lambda}(x)) + \sup(\tilde{b}_{\Lambda}(x)) + \sup(\tilde{c}_{\Lambda}(x)) \leq 1, \forall x \in X$. The intervals $\tilde{a}_{\Lambda}(x), \tilde{b}_{\Lambda}(x)$ and $\tilde{c}_{\Lambda}(x)$ represent the positive, neutral, and negative grades of the elements $x \in X$, respectively. Thus, for every $x \in X$, $\tilde{a}_{\Lambda}(x), \tilde{b}_{\Lambda}(x)$, and $\tilde{c}_{\Lambda}(x)$ are closed intervals, and their lower and upper end points are symbolized as, $a_{\Lambda}^{-}(x), a_{\Lambda}^{+}(x), b_{\Lambda}^{-}(x), b_{\Lambda}^{+}(x), c_{\Lambda}^{-}(x)$, and $c_{\Lambda}^{+}(x)$. We can write as

$$\Lambda = \{(x, [a_{\Lambda}^{-}(x), a_{\Lambda}^{+}(x)], [b_{\Lambda}^{-}(x), b_{\Lambda}^{+}(x)], [c_{\Lambda}^{-}(x), c_{\Lambda}^{+}(x)]) | x \in X\}, \quad (7)$$

where $0 \leq a_{\Lambda}^{+}(x) + b_{\Lambda}^{+}(x) + c_{\Lambda}^{+}(x) \leq 1, a_{\Lambda}^{-}(x) \geq 0, b_{\Lambda}^{-}(x) \geq 0$, and $c_{\Lambda}^{-}(x) \geq 0$ hesitation interval relative to Λ , for every element x , is computed as

$$\begin{aligned} \tilde{\pi}_{\Lambda}(x) &= [\pi_{\Lambda}^{-}(x), \pi_{\Lambda}^{+}(x)] \\ &= \{[1 - a_{\Lambda}^{+}(x) - b_{\Lambda}^{+}(x) - c_{\Lambda}^{+}(x), 1 - a_{\Lambda}^{-}(x) \\ &\quad - b_{\Lambda}^{-}(x) - c_{\Lambda}^{-}(x)]\}. \end{aligned} \quad (8)$$

For any element x , the triple $\{\tilde{a}_{\Lambda}(x), \tilde{b}_{\Lambda}(x), \tilde{c}_{\Lambda}(x)\}$ is known as interval-valued picture fuzzy numbers (IVPFNs). For convenience, the triple $\{\tilde{a}_{\Lambda}(x), \tilde{b}_{\Lambda}(x), \tilde{c}_{\Lambda}(x)\}$ is often represented by $([a, b], [c, d], [e, f])$, where $[a, b] \in e[0, 1], [c, d] \in e[0, 1], [e, f] \in e[0, 1]$ and $b + d + f \leq 1$.

Definition 6 (see [74]). Let $\Lambda_1 = \{\langle (x, [a_{\Lambda_1}^{-}(x), a_{\Lambda_1}^{+}(x)], [b_{\Lambda_1}^{-}(x), b_{\Lambda_1}^{+}(x)], [c_{\Lambda_1}^{-}(x), c_{\Lambda_1}^{+}(x)]) \rangle | x \in X\}$ and $\Lambda_2 = \{\langle (x, [a_{\Lambda_2}^{-}(x), a_{\Lambda_2}^{+}(x)], [b_{\Lambda_2}^{-}(x), b_{\Lambda_2}^{+}(x)], [c_{\Lambda_2}^{-}(x), c_{\Lambda_2}^{+}(x)]) \rangle | x \in X\}$ are the two IVPFSs in the set x and $n \geq 0$. Then, the following operational laws of IVPFN are developed:

$$\begin{aligned}
 \Lambda_1 \times \Lambda_2 &= \left\{ \left(\begin{array}{l} x, [a_{\Lambda_1}^-(x)a_{\Lambda_2}^-(x), a_{\Lambda_1}^+(x)a_{\Lambda_2}^+(x)] \\ [1 - (1 - b_{\Lambda_1}^-(x))(1 - b_{\Lambda_2}^-(x)), \\ 1 - (1 - b_{\Lambda_1}^+(x))(1 - b_{\Lambda_2}^+(x))] \\ [1 - (1 - c_{\Lambda_1}^-(x))(1 - c_{\Lambda_2}^-(x)), \\ 1 - (1 - c_{\Lambda_1}^+(x))(1 - c_{\Lambda_2}^+(x))] \end{array} \right) \middle| x \in X \right\}, \\
 \Lambda_1 + \Lambda_2 &= \left\{ \left(\begin{array}{l} x, [1 - (1 - a_{\Lambda_1}^-(x))(1 - a_{\Lambda_2}^-(x)), \\ 1 - (1 - a_{\Lambda_1}^+(x))(1 - a_{\Lambda_2}^+(x))] \\ [b_{\Lambda_1}^-(x)b_{\Lambda_2}^-(x), b_{\Lambda_1}^+(x)b_{\Lambda_2}^+(x)], \\ [c_{\Lambda_1}^-(x)c_{\Lambda_2}^-(x), c_{\Lambda_1}^+(x)c_{\Lambda_2}^+(x)] \end{array} \right) \middle| x \in X \right\}, \\
 \lambda \Lambda_1 &= \left\{ \left(\begin{array}{l} x, [1 - (1 - a_{\Lambda_1}^-(x))^\lambda, 1 - (1 - a_{\Lambda_1}^+(x))^\lambda], \\ [(b_{\Lambda_1}^-(x))^\lambda, (b_{\Lambda_1}^+(x))^\lambda], \\ [(c_{\Lambda_1}^-(x))^\lambda, (c_{\Lambda_1}^+(x))^\lambda] \end{array} \right) \middle| x \in X \right\}, \\
 \Lambda_1^\lambda &= \left\{ \left(\begin{array}{l} x, [(a_{\Lambda_1}^-(x))^\lambda, (a_{\Lambda_1}^+(x))^\lambda], \\ [1 - (1 - b_{\Lambda_1}^-(x))^\lambda, 1 - (1 - b_{\Lambda_1}^+(x))^\lambda], \\ [1 - (1 - c_{\Lambda_1}^-(x))^\lambda, 1 - (1 - c_{\Lambda_1}^+(x))^\lambda] \end{array} \right) \middle| x \in X \right\}.
 \end{aligned} \tag{9}$$

Definition 7 (see [75, 76]). Suppose that $S = (s_0, \dots, s_{l-1})$ be a discrete linguistic term set, where l is the odd number, and $l \geq 0$. For example, $l = 7$, and then the linguistic term set is defined as $S = (s_0, s_1, s_2, s_3, s_4) = \{\text{poor, slightly poor, fair, slightly good, good}\}$. If $e < f$, then the following properties must be satisfied by the linguistic term set:

- (1) Negation $\text{Neg}(s_N) = s_f, f = l - e$
- (2) Maximum $(s_e, s_f) = s_e$, if $s_e \geq s_f$
- (3) Minimum $(s_e, s_f) = s_e$, if $s_e \leq s_f$

Suppose that $\tilde{s} = [s_e, s_f], s_e, s_f \in \tilde{s}$ and $e \leq f, s_e, s_f$ are the lower limit and upper limit of \tilde{s} , correspondingly. Then, \tilde{s} is said to be an uncertain linguistic variable.

Definition 8 (see [77]). Let \tilde{S} denote the family of uncertain linguistic variables. Then, the following operation is defined for $\tilde{s}_1 = [s_{e_1}, s_{f_1}]$ and $\tilde{s}_2 = [s_{e_2}, s_{f_2}]$:

- (1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{e_1}, s_{f_1}] \oplus [s_{e_2}, s_{f_2}] = [s_{e_1} \oplus s_{e_2}, s_{f_1} \oplus s_{f_2}] = [s_{e_1+e_2}, s_{f_1+f_2}]$
- (2) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{e_1}, s_{f_1}] \otimes [s_{e_2}, s_{f_2}] = [s_{e_1} \otimes s_{e_2}, s_{f_1} \otimes s_{f_2}] = [s_{e_1 \times e_2}, s_{f_1 \times f_2}]$
- (3) $\lambda \tilde{s}_1 = \lambda [s_{e_1}, s_{f_1}] = [s_{\lambda e_1}, s_{\lambda f_1}]$
- (4) $\lambda (\tilde{s}_1 \oplus \tilde{s}_2) = \lambda \tilde{s}_1 \oplus \lambda \tilde{s}_2$, if $\lambda \geq 0$

Definition 9. Let $\tilde{s}_x = [s_{\theta(x)}, s_{\tau(x)}], \tilde{s}_x \in S$, and X is a non-empty set. Then,

$$\Lambda = \{x, (\tilde{s}_x, \tilde{a}_\Lambda(x), \tilde{b}_\Lambda(x), \tilde{c}_\Lambda(x)) \mid x \in X\} \tag{10}$$

is called IVPFULSs, where $\tilde{a}_\Lambda: X \rightarrow [0, 1], \tilde{b}_\Lambda: X \rightarrow [0, 1]$ and $\tilde{c}_\Lambda: X \rightarrow [0, 1]$ satisfy condition $0 \leq \sup(\tilde{a}_\Lambda(x)) + \sup(\tilde{b}_\Lambda(x)) + \sup(\tilde{c}_\Lambda(x)) \leq 1, \forall x \in X$. The intervals $\tilde{a}_\Lambda(x), \tilde{b}_\Lambda(x)$ and $\tilde{c}_\Lambda(x)$ represent the positive, neutral, and negative membership grades of the elements x to the uncertain linguistic variable $\tilde{s}_x = [s_{\theta(x)}, s_{\tau(x)}]$, respectively. Thus, for every $x \in X, \tilde{a}_\Lambda(x), \tilde{b}_\Lambda(x)$, and $\tilde{c}_\Lambda(x)$ are closed intervals, and their lower and upper endpoints are represented by $a_\Lambda^-(x), a_\Lambda^+(x), b_\Lambda^-(x), b_\Lambda^+(x), c_\Lambda^-(x)$, and $c_\Lambda^+(x)$. We can write as

$$\Lambda = \left\{ \left\langle x, \left(\begin{array}{l} [s_{\theta(x)}, s_{\tau(x)}], [a_\Lambda^-(x), a_\Lambda^+(x)], [b_\Lambda^-(x), b_\Lambda^+(x)], \\ [c_\Lambda^-(x), c_\Lambda^+(x)] \end{array} \right) \right\rangle \middle| x \in X \right\}, \tag{11}$$

where $s_{\theta(x)}, s_{\tau(x)} \in \tilde{S}, 0 \leq a_\Lambda^+(x) + b_\Lambda^+(x) + c_\Lambda^+(x) \leq 1, a_\Lambda^-(x) \geq 0, b_\Lambda^-(x) \geq 0$ and $c_\Lambda^-(x) \geq 0$.

Hesitation interval of x to the uncertain linguistic variable $\tilde{s}_x = [s_{\theta(x)}, s_{\tau(x)}]$ for every element x is as follows:

$$\tilde{\pi}_\Lambda(x) = [\pi_\Lambda^-(x), \pi_\Lambda^+(x)] = \left[\begin{array}{l} 1 - a_\Lambda^+(x) - b_\Lambda^+(x) - c_\Lambda^+(x), \\ 1 - a_\Lambda^-(x) - b_\Lambda^-(x) - c_\Lambda^-(x) \end{array} \right]. \quad (12)$$

Definition 10. Let $\Lambda = \{ \langle x, ([s_{\theta(x)}, s_{\tau(x)}], [a_\Lambda^-(x), a_\Lambda^+(x)], [b_\Lambda^-(x), b_\Lambda^+(x)], [c_\Lambda^-(x), c_\Lambda^+(x)]) \rangle | x \in X \}$ be IVPFULS, an 8-tuple $\langle \langle ([s_{\theta(x)}, s_{\tau(x)}], [a_\Lambda^-(x), a_\Lambda^+(x)], [b_\Lambda^-(x), b_\Lambda^+(x)], [c_\Lambda^-(x), c_\Lambda^+(x)]) \rangle \rangle$ is said to be IVPFULN, and unknown linguistic variables can also be viewed as a set of interval-valued numbers. Therefore, it can be expressed as

$$\Lambda = \{ \langle [s_{\theta(x)}, s_{\tau(x)}] [a_\Lambda^-(x), a_\Lambda^+(x)], [b_\Lambda^-(x), b_\Lambda^+(x)], [c_\Lambda^-(x), c_\Lambda^+(x)] \rangle | x \in X \}. \quad (13)$$

Definition 11. Let $\Lambda_1 = \langle [s_{\theta(\Lambda_1)}, s_{\tau(\Lambda_1)}], [a^-(\Lambda_1), a^+(\Lambda_1)], [b^-(\Lambda_1), b^+(\Lambda_1)], [c^-(\Lambda_1), c^+(\Lambda_1)] \rangle$ and $\Lambda_2 = \langle [s_{\theta(\Lambda_2)}, s_{\tau(\Lambda_2)}], [a^-(\Lambda_2), a^+(\Lambda_2)], [b^-(\Lambda_2), b^+(\Lambda_2)], [c^-(\Lambda_2), c^+(\Lambda_2)] \rangle$ are the two IVPFULNs and $\lambda \geq 0$. Then, we have defined the following operation for IVPFULNs:

- (1) $\Lambda_1 + \Lambda_2 = \left(\begin{array}{l} [s_{\theta(\Lambda_1)+\theta(\Lambda_2)}, s_{\tau(\Lambda_1)+\tau(\Lambda_2)}]; \\ [1 - (1 - a^-(\Lambda_1))(1 - a^-(\Lambda_2)), 1 - (1 - a^+(\Lambda_1))(1 - a^+(\Lambda_2))], \\ [b^-(\Lambda_1)b^-(\Lambda_2), b^+(\Lambda_1)b^+(\Lambda_2)], [c^-(\Lambda_1)c^-(\Lambda_2), c^+(\Lambda_1)c^+(\Lambda_2)] \end{array} \right)$
- (2) $\Lambda_1 \times \Lambda_2 = \left(\begin{array}{l} [s_{\theta(\Lambda_1)+\theta(\Lambda_2)}, s_{\tau(\Lambda_1)+\tau(\Lambda_2)}]; [a^-(\Lambda_1)a^-(\Lambda_2), a^+(\Lambda_1)a^+(\Lambda_2)] \\ [1 - (1 - b^-(\Lambda_1))(1 - b^-(\Lambda_2)), 1 - (1 - b^+(\Lambda_1))(1 - b^+(\Lambda_2))], \\ [1 - (1 - c^-(\Lambda_1))(1 - c^-(\Lambda_2)), 1 - (1 - c^+(\Lambda_1))(1 - c^+(\Lambda_2))] \end{array} \right)$

$$Sco^*(\Lambda_1) = \frac{1}{3} \times \left(\begin{array}{l} \frac{a^-(\Lambda_1) + a^+(\Lambda_1)}{2} + 1 - \frac{b^-(\Lambda_1) + b^+(\Lambda_1)}{2} \\ + 1 - \frac{c^-(\Lambda_1) + c^+(\Lambda_1)}{2} \end{array} \right) \times \frac{s_{\theta(\Lambda_1)+\tau(\Lambda_1)}}{2}. \quad (14)$$

Definition 13. Let $\Lambda_1 = \langle [s_{\theta(\Lambda_1)}, s_{\tau(\Lambda_1)}], [a^-(\Lambda_1), a^+(\Lambda_1)], [b^-(\Lambda_1), b^+(\Lambda_1)], [c^-(\Lambda_1), c^+(\Lambda_1)] \rangle$ be an IVPFULN, and an accuracy function is defined as

$$Ho^*(\Lambda_1) = \frac{s_{\theta(\Lambda_1)+\tau(\Lambda_1)}}{2} \times \left(\begin{array}{l} \frac{a^-(\Lambda_1) + a^+(\Lambda_1)}{2} + \frac{b^-(\Lambda_1) + b^+(\Lambda_1)}{2} \\ + \frac{c^-(\Lambda_1) + c^+(\Lambda_1)}{2} \end{array} \right). \quad (15)$$

Definition 14. Let $\Lambda_1 = \langle [s_{\theta(\Lambda_1)}, s_{\tau(\Lambda_1)}], [a^-(\Lambda_1), a^+(\Lambda_1)], [b^-(\Lambda_1), b^+(\Lambda_1)], [c^-(\Lambda_1), c^+(\Lambda_1)] \rangle$ and $\Lambda_2 = \langle [s_{\theta(\Lambda_2)}, s_{\tau(\Lambda_2)}], [a^-(\Lambda_2), a^+(\Lambda_2)], [b^-(\Lambda_2), b^+(\Lambda_2)], [c^-(\Lambda_2), c^+(\Lambda_2)] \rangle$ be an IVPFULN. Then,

- (1) if $Sco^*(\Lambda_1) > Sco^*(\Lambda_2)$, then, $\Lambda_1 > \Lambda_2$
- (2) if $Sco^*(\Lambda_1) = Sco^*(\Lambda_2)$, then
 - (a) if $Ho^*(\Lambda_1) > Ho^*(\Lambda_2)$, then $\Lambda_1 > \Lambda_2$

$$(3) \lambda \Lambda_1 = \left(\begin{array}{l} [s_{\lambda \times \theta(\Lambda_1)}, s_{\lambda \times \tau(\Lambda_1)}]; [1 - (1 - a^-(\Lambda_1))^\lambda, 1 - (1 - a^+(\Lambda_1))^\lambda], \\ [(b^-(\Lambda_1))^\lambda, (b^+(\Lambda_1))^\lambda], [(c^-(\Lambda_1))^\lambda, (c^+(\Lambda_1))^\lambda] \end{array} \right)$$

$$(4) \Lambda_1^\lambda = \left(\begin{array}{l} [s_{\theta(\Lambda_1)^\lambda}, s_{\tau(\Lambda_1)^\lambda}]; [(a^-(\Lambda_1))^\lambda, (a^+(\Lambda_1))^\lambda], \\ [1 - (1 - b^-(\Lambda_1))^\lambda, 1 - (1 - b^+(\Lambda_1))^\lambda], \\ [1 - (1 - c^-(\Lambda_1))^\lambda, 1 - (1 - c^+(\Lambda_1))^\lambda] \end{array} \right)$$

Theorem 1. Let $\Lambda_1 = \langle [s_{\theta(\Lambda_1)}, s_{\tau(\Lambda_1)}], [a^-(\Lambda_1), a^+(\Lambda_1)], [b^-(\Lambda_1), b^+(\Lambda_1)], [c^-(\Lambda_1), c^+(\Lambda_1)] \rangle$ and $\Lambda_2 = \langle [s_{\theta(\Lambda_2)}, s_{\tau(\Lambda_2)}], [a^-(\Lambda_2), a^+(\Lambda_2)], [b^-(\Lambda_2), b^+(\Lambda_2)], [c^-(\Lambda_2), c^+(\Lambda_2)] \rangle$ be the two IVPFULNs. Then, the following rules must be satisfied:

- (1) $\Lambda_1 + \Lambda_2 = \Lambda_2 + \Lambda_1$
- (2) $\Lambda_1 \times \Lambda_2 = \Lambda_2 \times \Lambda_1$
- (3) $\lambda(\Lambda_1 + \Lambda_2) = \lambda\Lambda_1 + \lambda\Lambda_2, \lambda \geq 0$
- (4) $\lambda_1\Lambda_1 + \lambda_2\Lambda_1 = (\lambda_1 + \lambda_2)\Lambda_1, \lambda_1, \lambda_2 \geq 0$
- (5) $\Lambda_1^{\lambda_1} \times \Lambda_1^{\lambda_2} = (\Lambda_1)^{\lambda_1+\lambda_2}, \lambda_1, \lambda_2 \geq 0$
- (6) $\Lambda_1^{\lambda_1} \times \Lambda_2^{\lambda_2} = (\Lambda_1 \times \Lambda_2)^{\lambda_1}, \lambda_1 \geq 0$

Proof. See for proof Appendix B. □

Definition 12. Let $\Lambda_1 = \langle [s_{\theta(\Lambda_1)}, s_{\tau(\Lambda_1)}], [a^-(\Lambda_1), a^+(\Lambda_1)], [b^-(\Lambda_1), b^+(\Lambda_1)], [c^-(\Lambda_1), c^+(\Lambda_1)] \rangle$ be an IVPFULN, and a score function is defined as

(b) if $Ho^*(\Lambda_1) = Ho^*(\Lambda_2)$, then $\Lambda_1 = \Lambda_2$

3. The Interval-Valued Picture Fuzzy Uncertain Linguistic Geometric Operators

Definition 15. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of IVPFULNs, and IVPFULWG: $\Psi^m \rightarrow \Psi$, if

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) = \prod_{j=1}^n (\Lambda_j)^{\omega_j}, \quad (16)$$

where Ψ are the set of all IVPFULNs, and $\omega = (\omega_1, \dots, \omega_n)^T$ is the weighting vector of Λ_j , $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$; then, IVPFULWG is said to be the IVPFUL-weighted geometric operator. Especially, if $\omega = ((1/n), \dots, (1/n))$, then the IVPFULWG operator is the IVPFUL geometric operator.

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) = \left(\begin{array}{c} \left[s \prod_{j=1}^n (\theta(\Lambda_j))^{\omega_j}, s \prod_{j=1}^n (\tau(\Lambda_j))^{\omega_j} \right], \left[\prod_{j=1}^n (a^-(\Lambda_j))^{\omega_j}, \prod_{j=1}^n (a^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^n (1 - b^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - c^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^n (1 - c^+(\Lambda_j))^{\omega_j} \right] \end{array} \right). \quad (17)$$

Proof. See for proof Appendix C. \square

Theorem 3. *Monotonicity:* let $\Lambda_j^* = (\Lambda_1^*, \dots, \Lambda_n^*)$ and $\Lambda_j = (\Lambda_1, \dots, \Lambda_n)$ be the two collections of interval-valued picture fuzzy uncertain linguistic numbers, if $\Lambda_j^* \leq \Lambda_j, \forall j = 1, \dots, n$. Then,

$$\text{IVPFULWG}_\omega(\Lambda_1^*, \dots, \Lambda_n^*) \leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n). \quad (18)$$

Proof. Let

$$\begin{aligned} \text{IVPFULWG}_\omega(\Lambda_1^*, \dots, \Lambda_n^*) &= \prod_{j=1}^n (\Lambda_j^*)^{\omega_j}, \\ \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) &= \prod_{j=1}^n (\Lambda_j)^{\omega_j}. \end{aligned} \quad (19)$$

Since $\Lambda_j^* \leq \Lambda_j, \forall j = 1, \dots, n$, we have

$$\text{IVPFULWG}_\omega(\Lambda_1^*, \dots, \Lambda_n^*) \leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n). \quad (20)$$

\square

Theorem 4. *Idempotency:* let $\Lambda_j = \Lambda, j = 1, \dots, n$. Then,

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) = \Lambda. \quad (21)$$

Proof. Science: $\Lambda_j = \Lambda$, for all j , and then we have

$$\begin{aligned} \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) &= \prod_{j=1}^n (\Lambda_j)^{\omega_j} \\ &= \prod_{j=1}^n (\Lambda)^{\omega_j} = (\Lambda)^{\sum_{j=1}^n \omega_j} = \Lambda. \end{aligned} \quad (22)$$

\square

Theorem 2. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of IVPFULNs. Then, the result obtained from Definition 15 is also an IVPFULN:

Theorem 5. *Boundedness:* the IVPFULWG operator lies between the maximum and minimum operators:

$$\begin{aligned} \min(\Lambda_1, \dots, \Lambda_n) &\leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) \\ &\leq \max(\Lambda_1, \dots, \Lambda_n). \end{aligned} \quad (23)$$

Proof. Let $\Lambda = \min(\Lambda_1, \dots, \Lambda_n)$, $\Psi = \max(\Lambda_1, \dots, \Lambda_n)$, and according to Theorem 3, we have

$$\begin{aligned} \text{IVPFULWG}_\omega(\Lambda, \dots, \Lambda) &\leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) \\ &\leq \text{IVPFULWG}_\omega(\Psi, \dots, \Psi). \end{aligned} \quad (24)$$

Furthermore,

$$\text{IVPFULWG}_\omega(\Lambda, \dots, \Lambda) = \Lambda, \text{IVPFULWG}_\omega(\Psi, \dots, \Psi) = \Psi. \quad (25)$$

So,

$$\Lambda \leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) \leq \Psi, \quad (26)$$

i.e.,

$$\begin{aligned} \min(\Lambda_1, \dots, \Lambda_n) &\leq \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) \\ &\leq \max(\Lambda_1, \dots, \Lambda_n). \end{aligned} \quad (27)$$

\square

Definition 16. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of the IVPFULNs, and IVPFULWG: $\Psi^n \rightarrow \Psi$, if

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) = \prod_{j=1}^n (\Lambda_{\delta_j})^{\omega_j}, \quad (28)$$

where Ψ are the family of all IVPFULNs, and $\omega = (\omega_1, \dots, \omega_n)^T$ is an associated weight with IVPFULWG; $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, $(\delta_1, \dots, \delta_n)$ are the permutation of $(1, \dots, n)$, such as $\Lambda_{\delta_{j+1}} > \Lambda_{\delta_j}, \forall j = 1, \dots, n$;

then, IVPFULOWG is said to be IVPFUL ordered weighted geometric operator, and w_j denoted the j^{th} position in the aggregation process. So, we called w is the location weight vector.

Theorem 6. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of IVPFULNs. Then, the result aggregated from Definition 15 is also an IVPFULN, and

$$\text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n) = \left(\begin{array}{c} \left[\prod_{j=1}^n (\theta(\Lambda_{\delta_j}))^{w_j}, \prod_{j=1}^n (\tau(\Lambda_{\delta_j}))^{w_j} \right], \\ \left[\prod_{j=1}^n (a^-(\Lambda_{\delta_j}))^{w_j}, \prod_{j=1}^n (a^+(\Lambda_{\delta_j}))^{w_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - b^-(\Lambda_{\delta_j}))^{w_j}, 1 - \prod_{j=1}^n (1 - b^+(\Lambda_{\delta_j}))^{w_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - c^-(\Lambda_{\delta_j}))^{w_j}, 1 - \prod_{j=1}^n (1 - c^+(\Lambda_{\delta_j}))^{w_j} \right] \end{array} \right), \quad (29)$$

where $w = (w_1, \dots, w_n)^T$ be the associated weighting vector of IVPFULOWG, and $w_j \in [0, 1], \sum_{j=1}^n w_j = 1, (\delta_1, \dots, \delta_n)$ are the permutation of $(1, \dots, n)$, such as $\Lambda_{\delta_{j+1}} > \Lambda_{\delta_j}$ for all $j = 1, \dots, n$.

Theorem 9. Idempotency: let $\Lambda_j = \Lambda, j = 1, \dots, n$. Then,
$$\text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n) = \Lambda. \quad (34)$$

Proof. Proof is the same as proof of Theorem 2. \square

Proof. Proof is the same as Theorem 4 \square

Theorem 7. Commutativity: let $(\Lambda_1^*, \dots, \Lambda_n^*)$ be any permutation of $(\Lambda_1, \dots, \Lambda_n)$. Then,

$$\text{IVPFULOWG}_w(\Lambda_1^*, \dots, \Lambda_n^*) = \text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n). \quad (30)$$

Theorem 10. Boundedness: the IVPFULOWG operator lies between the maximum and minimum operators:

$$\min(\Lambda_1, \dots, \Lambda_n) \leq \text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n) \leq \max(\Lambda_1, \dots, \Lambda_n). \quad (35)$$

Proof. As we know that

$$\begin{aligned} \text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n) &= \prod_{j=1}^n (\Lambda_{\delta_j})^{w_j}, \\ \text{IVPFULOWG}_w(\Lambda_1^*, \dots, \Lambda_n^*) &= \prod_{j=1}^n (\Lambda_{\delta_j}^*)^{w_j}. \end{aligned} \quad (31)$$

Since $(\Lambda_1^*, \dots, \Lambda_n^*)$ are the any permutation of $(\Lambda_1, \dots, \Lambda_n)$, we have $\Lambda_{\delta_j} = \Lambda_{\delta_j}^* (j = 1, \dots, n)$; then,

$$\text{IVPFULOWG}_w(\Lambda_1^*, \dots, \Lambda_n^*) = \text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n). \quad (32)$$

Theorem 8. Monotonicity: let $(\Lambda_1^*, \dots, \Lambda_n^*)$ and $(\Lambda_1, \dots, \Lambda_n)$ be the two families of IVPFULNs, if $\Lambda_j^* \leq \Lambda_j, \forall j = 1, \dots, n$. Then,

$$\text{IVPFULOWG}_w(\Lambda_1^*, \dots, \Lambda_n^*) \leq \text{IVPFULOWG}_w(\Lambda_1, \dots, \Lambda_n). \quad (33)$$

Proof. Proof is the same as Theorem 3. \square

Proof. Proof is the same as Theorem 5. \square

Definition 17. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of IVPFULNs, and IVPULHG: $\Psi^n \rightarrow \Psi$, if

$$\text{IVPFULHG}_w(\Lambda_1, \dots, \Lambda_n) = \prod_{j=1}^n (\Upsilon_{\delta_j})^{w_j}, \quad (36)$$

where Ψ are the set of all IVPFULNs, and $w = (w_1, \dots, w_n)^T$ be the associated weighting vector with IVPFULOWG; $w_j \in [0, 1], \sum_{j=1}^n w_j = 1, \Upsilon_{\delta_j}$ is j^{th} largest of the IVPFUL weighted arguments $\Upsilon_{\kappa}(\Upsilon_{\kappa}) = (\Lambda_{\kappa})^{n\omega_{\kappa}} (\kappa = 1, \dots, n)$, $\omega = (\omega_1, \dots, \omega_n)^T$ is the weights of $\Lambda_l (l = 1, \dots, n), \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$, and n denoted the balancing coefficient.

Theorem 11. Let $\Lambda_l = \langle [s_{\theta(\Lambda_l)}, s_{\tau(\Lambda_l)}], [a^-(\Lambda_l), a^+(\Lambda_l)], [b^-(\Lambda_l), b^+(\Lambda_l)], [c^-(\Lambda_l), c^+(\Lambda_l)] \rangle (l = 1, \dots, n)$ be a family of IVPFULNs. Then, the result aggregated from Definition 16 is also an IVPFULN, as

$$\text{IVPFULHG}_w(\Lambda_1, \dots, \Lambda_n) = \left(\begin{array}{c} \left[S \prod_{j=1}^n (\theta(\Upsilon_{\delta_j}))^{w_j}, S \prod_{j=1}^n (\tau(\Upsilon_{\delta_j}))^{w_j} \right], \\ \left[\prod_{j=1}^n (a^-(\Upsilon_{\delta_j}))^{w_j}, \prod_{j=1}^n (a^+(\Upsilon_{\delta_j}))^{w_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - b^-(\Upsilon_{\delta_j}))^{w_j}, 1 - \prod_{j=1}^n (1 - b^+(\Upsilon_{\delta_j}))^{w_j} \right], \\ \left[1 - \prod_{j=1}^n (1 - c^-(\Upsilon_{\delta_j}))^{w_j}, 1 - \prod_{j=1}^n (1 - c^+(\Upsilon_{\delta_j}))^{w_j} \right] \end{array} \right), \quad (37)$$

where $w = (w_1, \dots, w_n)^T$ be the associated weight vector with IVPULHG, $w_j \in [0, 1], \sum_{j=1}^n w_j = 1$, Υ_{δ_j} is j^{th} biggest of the IVPFUL weighted arguments $\Upsilon_{\kappa}(\Upsilon_{\kappa}) = (\Lambda_{\kappa})^{w_{\kappa}}$ ($\kappa = 1, \dots, n$), $\omega = (\omega_1, \dots, \omega_n)^T$ is the weights of Λ_l ($l = 1, \dots, n$), $\omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$, and n shows the balancing coefficient.

Proof. Proof is the same as the proof of Theorem 2. \square

4. Supplier Selection Group Decision Model in Logistics Service Value Cocreation Based on Interval-Valued Picture Fuzzy Sets

4.1. Supplier Selection Group Decision Model in Concretion Value of the Logistics Service. Logistics provider selection is a multicriteria concern which requires a wide variety of criteria. In their studies, Spencer et al. [78] reported 23 possible selection criteria and 35 selection factors were identified by Govindan et al. [79] which revealed eleven key 3PL selection criteria with a review of sixty-seven 3PL selection papers published in the period 1994–2013, each of which is defined by a list of attributes; the study revealed that cost was the commonly adopted criterion, followed by relationship, service, and quality [80]. While the above selection attribute is commonly used in the selection of 3PL, the selected attribute is operationally driven, whereas previous studies seldom considered the strategic supply chain and value creation variables when selecting logistics suppliers. It is important to review the selection criteria in the logistics service value concretion scenario that the development of value is the key premise of establishing and retaining the customer relationship and is the key goal and the central economic exchange mechanism [81].

More and more businesses are recognizing the importance of value cocreation for logistics services with partners in the supply chain management environment. Wan et al. [82] find the innovative way to attain competitive advantage and more personalized product and service offering for customers. Supplier selection is the most critical problems for logistics sector performance cocreation in supply chain management (SCM) setting. In the selection of 3PL suppliers, the emerging trend is the convergence of traditional selection characteristics such as cost, quality, response time, and location with new factors in the cocreation of service value, such as new value growth,

knowledge management, and service innovation. In order to create full selection criteria for supplier selection in the value cocreation scenario of logistics services, we combine traditional operational selection criteria and value-oriented SCM strategic selection criteria. The supplier selection attributes for the cocreation of the logistics service value are listed in Table 1.

4.2. Algorithm for Group Decision-Making with Interval-Valued Picture Fuzzy Uncertain Linguistic Information. Using these two operators IVPFULWG and IVPFULHG, we present a group DM problem, under the IVPFUL information. Let we have $A = \{A_1, \dots, A_n\}$ be the collection of alternative and $C = \{C_1, \dots, C_n\}$ be the set of criteria with the weight vector $\omega = (\omega_1, \dots, \omega_n)^T$. Let $D = \{d_1, \dots, d_p\}$ be the set of experts and $\vartheta = (\vartheta_1, \dots, \vartheta_p)^T$ be the weighting vector of experts. Let $R^k = [\psi_{ij}^k]_{m \times n}$ be the decision matrix, where $R_{ij}^k = \langle [s_{l_{jk}}^-, s_{l_{jk}}^+], [a_{l_{jk}}^-, a_{l_{jk}}^+], [b_{l_{jk}}^-, b_{l_{jk}}^+], [c_{l_{jk}}^-, c_{l_{jk}}^+] \rangle$ takes the form of the IVPULNs, given by the experts E_{κ} , for the alternative A_l with respect to the criteria C_j , and $a_{l_{jk}}^- \leq a_{l_{jk}}^+, b_{l_{jk}}^- \leq b_{l_{jk}}^+, c_{l_{jk}}^- \leq c_{l_{jk}}^+, a_{l_{jk}}^- + b_{l_{jk}}^- + c_{l_{jk}}^- \leq 1, s_{l_{jk}}^-, s_{l_{jk}}^+ \in S$.

The model for solving the abovementioned logistics service value MCGDM problem includes the following steps:

- Step 1: select the criteria for the selection of logistics suppliers using the model of logistics service value cocreation DM.
- Step 2: use the weights $\omega = (\omega_1, \dots, \omega_m)^T$ for the selection parameter of logistics suppliers and use the IVPFUL decision-matrix $R^k = [\psi_{ij}^k]_{m \times n}$ with weights $\vartheta = (\vartheta_1, \dots, \vartheta_p)^T$ for the experts.
- Step 3: then, use the IVPFULWG operator

$$\begin{aligned} \psi_{ij} &= \text{IVPFULWG}(\psi_{ij}^1, \dots, \psi_{ij}^p) \\ &= \prod_{\kappa=1}^p (\psi_{ij}^{\kappa})^{\vartheta_{\kappa}}, \quad (l = 1, \dots, m; j = 1, \dots, n), \end{aligned} \quad (38)$$

to summarize all the decision matrices R^k in a collective decision matrix $R = [\psi_{ij}]_{m \times n}$

Step 4: use the information of the matrix R and the IVPFULHG operator:

TABLE 1: Supplier selection criteria in the cocreation of logistic service performance.

Criteria	Criteria elements	Description of the criteria element
Collaborative value	Strategic alliance collaboration with the company goals business cooperation	Value cooperation is the foundation of value cocreation, to be viewed from the political, financial, and consumer aspects.
Knowledge matching	Heterogeneous experience corresponding information establishment of knowledge sharing and transfer of knowledge	Knowledge shows great role in cocreation and the value of logistics services; knowledge management is a part of the value cocreation process.
Capacity to innovate business	Innovation in service notion innovation to the service process innovation in service technology innovation service delivery	Service creativity denotes the capacity of 3PL to provide concrete service value in logistics service solutions.
Service quality	Time of response: service price reliability of business network logistics infrastructure knowledge sharing	Quality of service is the classical 3PL selection measurement criteria that explain the concrete importance of quality of service.

$$\begin{aligned} \psi &= \text{IVPFULHG}_{\omega,w}(\psi_{11}, \dots, \psi_{ln}) \\ &= \prod_{j=1}^n \left(Y_{l\delta_j} \right)^{\omega_j}, \quad (j = 1, \dots, m). \end{aligned} \quad (39)$$

Step 5: find the score values $\text{Sco}^*(\psi_l) (l = 1, \dots, n)$.

Step 6: according to the value of $\text{Sco}^*(\psi_l)$, rank the alternative A_l and choose the best.

5. Example

We study a group of the DM problem under the logistics supply chain information that involves a 4PL solution provider searching for the best 3PL service value cocreation supplier with its customer (the International Enterprise Manufacturing Group). Now, we have four global 3PL suppliers $A_l (l = 1, \dots, 4)$ and three experts (their weight vector is $\vartheta = (0.40, 0.30, 0.30)^T$ from various professional fields involved in the DM phase. Now, we have been using the following steps to find out the outcome of the decision:

Step 1: find the criteria in the service value cocreation environment for 3PL supplier selection. The criteria for

the best 3PL supplier selection are as follows: (1) C_1 is the mutually beneficial capacity to cooperate; (2) C_2 is the knowledge matching ability; (3) C_3 is the capacity to innovate businesses; (4) C_4 is the service quality.

Step 2: use the weights $\omega = (0.31, 0.28, 0.30, 0.11)^T$ for 3PL supplier selection criteria and IVPFUL decision matrix $R^\kappa = [\psi_{ij}^\kappa]_{m \times n} (\kappa = 1, \dots, 3)$ as given in Tables 2–4.

Step 3: aggregated given decision matrices R^κ into one decision matrix $R = [\psi_{ij}]_{3 \times 4}$, we used the information in matrix $R^\kappa (\kappa = 1, 2, 3)$ and the IVPULWG operator, where the weights of the expert are $\vartheta = (0.40, 0.30, 0.30)^T$. The collective values are given in Table 5.

Step 4: utilize the weights of index $\omega = (0.31, 0.28, 0.30, 0.11)^T$ and the IVPULHG operator with associated weights $w = (0.37, 0.31, 0.25, 0.07)^T$: $\psi_l = \text{IVPFULHG}_{\omega,w}(\psi_{l1}, \dots, \psi_{ln}) = \prod_{j=1}^4 (Y_{l\delta_j})^{\omega_j}$ to derive the total preference values ψ_l of alternative A_l , where $Y_{l\delta_j}$ is the j^{th} largest of the IVPFUL weighted argument $Y_{l\kappa} (Y_{l\kappa}) = (Y_{l\kappa})^{3\omega_\kappa} (\kappa = 1, \dots, 3)$:

$$\begin{aligned} \psi_1 &= \langle [s_{4.27}, s_{6.02}], [0.423, 0.558], [0.153, 0.254], [0.109, 0.209] \rangle, \\ \psi_2 &= \langle [s_{3.51}, s_{4.75}], [0.00, 0.376], [0.225, 0.330], [0.192, 0.412] \rangle, \\ \psi_3 &= \langle [s_{4.36}, s_{6.31}], [0.00, 0.239], [0.173, 0.299], [0.343, 0.441] \rangle, \\ \psi_4 &= \langle [s_{3.82}, s_{5.92}], [0.229, 0.423], [0.137, 0.285], [0.197, 0.303] \rangle. \end{aligned} \quad (40)$$

Step 5: determine the score value $\text{Sco}^*(\psi_l) (l = 1, \dots, 4)$ of the collective overall IVPFUL preference values $\psi_l (l = 1, \dots, 4)$:

TABLE 2: IVPFUL information decision matrix R^1 .

	C_1	C_2
A_1	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_3, s_5]; [0.6, 0.7], [0.1, 0.2], [0.0, 0.1] \rangle$
A_2	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.4], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_5]; [0.1, 0.3], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [s_5, s_8]; [0.0, 0.1], [0.1, 0.3], [0.5, 0.6] \rangle$
A_4	$\langle [s_7, s_9]; [0.4, 0.7], [0.0, 0.1], [0.1, 0.2] \rangle$	$\langle [s_2, s_4]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5] \rangle$
	C_3	C_4
A_1	$\langle [s_4, s_6]; [0.2, 0.3], [0.3, 0.4], [0.1, 0.2] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.5, 0.6], [0.2, 0.3] \rangle$
A_2	$\langle [s_3, s_5]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.4, 0.6], [0.1, 0.4] \rangle$
A_3	$\langle [s_6, s_7]; [0.2, 0.3], [0.1, 0.2], [0.3, 0.4] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.2, 0.3], [0.1, 0.2] \rangle$
A_4	$\langle [s_5, s_9]; [0.4, 0.6], [0.0, 0.1], [0.2, 0.3] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.1, 0.2], [0.4, 0.5] \rangle$

TABLE 3: IVPFUL information decision matrix R^2 .

	C_1	C_2
A_1	$\langle [s_5, s_6]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_5, s_9]; [0.4, 0.6], [0.0, 0.1], [0.2, 0.3] \rangle$
A_2	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.4], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_7]; [0.1, 0.2], [0.2, 0.3], [0.3, 0.5] \rangle$	$\langle [s_6, s_7]; [0.2, 0.3], [0.1, 0.2], [0.3, 0.4] \rangle$
A_4	$\langle [s_7, s_9]; [0.4, 0.7], [0.0, 0.1], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.1, 0.3], [0.3, 0.4], [0.2, 0.3] \rangle$
	C_3	C_4
A_1	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.4, 0.6], [0.1, 0.4] \rangle$
A_2	$\langle [s_5, s_8]; [0.0, 0.1], [0.1, 0.3], [0.5, 0.6] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.5, 0.6], [0.2, 0.3] \rangle$
A_3	$\langle [s_6, s_7]; [0.2, 0.3], [0.1, 0.2], [0.3, 0.4] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.1, 0.2], [0.4, 0.5] \rangle$
A_4	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.4], [0.1, 0.2] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.2, 0.3], [0.1, 0.2] \rangle$

TABLE 4: IVPFUL information decision matrix R^3 .

	C_1	C_2
A_1	$\langle [s_3, s_5]; [0.6, 0.7], [0.1, 0.2], [0.0, 0.1] \rangle$	$\langle [s_7, s_9]; [0.4, 0.7], [0.0, 0.1], [0.1, 0.2] \rangle$
A_2	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_8, s_9]; [0.5, 0.6], [0.1, 0.2], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_5]; [0.1, 0.3], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [s_6, s_7]; [0.2, 0.3], [0.1, 0.2], [0.3, 0.4] \rangle$
A_4	$\langle [s_3, s_6]; [0.2, 0.3], [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_5, s_9]; [0.4, 0.6], [0.0, 0.1], [0.2, 0.3] \rangle$
	C_3	C_4
A_1	$\langle [s_5, s_9]; [0.4, 0.6], [0.0, 0.1], [0.2, 0.3] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.1, 0.2], [0.4, 0.5] \rangle$
A_2	$\langle [s_4, s_5]; [0.4, 0.5], [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.2, 0.3], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_7]; [0.1, 0.2], [0.2, 0.3], [0.3, 0.5] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.5, 0.6], [0.2, 0.3] \rangle$
A_4	$\langle [s_3, s_6]; [0.1, 0.3], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.4, 0.6], [0.1, 0.4] \rangle$

TABLE 5: Collective decision matrix.

	C_1
A_1	$\langle [s_{4.27}, s_{5.26}]; [0.449, 0.553], [0.181, 0.276], [0.078, 0.180] \rangle$
A_2	$\langle [s_{2.44}, s_{3.49}]; [0.319, 0.425], [0.138, 0.366], [0.098, 0.204] \rangle$
A_3	$\langle [s_{3.99}, s_{5.52}]; [0.099, 0.268], [0.273, 0.381], [0.310, 0.367] \rangle$
A_4	$\langle [s_{5.41}, s_{6.61}]; [0.324, 0.539], [0.142, 0.243], [0.099, 0.204] \rangle$
	C_2
A_1	$\langle [s_{4.50}, s_{7.09}]; [0.472, 0.664], [0.041, 0.146], [0.098, 0.200] \rangle$
A_2	$\langle [s_{4.90}, s_{5.95}]; [0.425, 0.528], [0.180, 0.276], [0.099, 0.204] \rangle$
A_3	$\langle [s_{5.56}, s_{7.35}]; [0.00, 0.189], [0.107, 0.239], [0.386, 0.492] \rangle$
A_4	$\langle [s_{3.23}, s_{5.45}]; [0.148, 0.308], [0.182, 0.291], [0.279, 0.387] \rangle$
	C_3
A_1	$\langle [s_{4.28}, s_{6.39}]; [0.300, 0.431], [0.200, 0.301], [0.129, 0.239] \rangle$
A_2	$\langle [s_{3.80}, s_{5.72}]; [0.00, 0.197], [0.174, 0.306], [0.360, 0.643] \rangle$
A_3	$\langle [s_{3.64}, s_{6.97}]; [0.161, 0.263], [0.129, 0.239], [0.304, 0.434] \rangle$
A_4	$\langle [s_{3.25}, s_{5.71}]; [0.238, 0.429], [0.128, 0.294], [0.174, 0.273] \rangle$
	C_4
A_1	$\langle [s_{2.28}, s_{4.51}]; [0.278, 0.323], [0.181, 0.202], [0.127, 0.219] \rangle$
A_2	$\langle [s_{3.21}, s_{6.73}]; [0.114, 0.203], [0.134, 0.142], [0.390, 0.465] \rangle$
A_3	$\langle [s_{3.52}, s_{5.32}]; [0.300, 0.431], [0.200, 0.301], [0.129, 0.239] \rangle$
A_4	$\langle [s_{4.01}, s_{6.21}]; [0.00, 0.213], [0.125, 0.321], [0.327, 0.592] \rangle$

$$\begin{aligned}
 \text{Sco}^*(\psi_1) &= 3.83, \\
 \text{Sco}^*(\psi_2) &= 2.21, \\
 \text{Sco}^*(\psi_3) &= 2.64, \\
 \text{Sco}^*(\psi_4) &= 3.02.
 \end{aligned}
 \tag{41}$$

Step 6: based on the scores of IVPFUL preference values, the ranking order is as follows:

$$A_1 > A_4 > A_3 > A_2. \tag{42}$$

5.1. Comparison with the Study [83]. In addition, in order to verify the validity of the method proposed in this paper, we adopt the method proposed by Liu [83] to verify this example. Firstly, we convert the uncertain linguistic value of the interval-valued picture fuzzy uncertain linguistic variables into IVIFULNs. Then, we can use the method based on the interval-valued intuitionistic fuzzy uncertain linguistic numbers. The transformed decision matrices are given in Tables 6–8

Step 1: use the given information in the matrix R^κ ($\kappa = 1, 2, 3$) and the IVIFULWG operator. Aggregate the given decision matrices R^κ in a single decision matrix $R = [\psi_{lj}]_{4 \times 4}$, and the weights of the expert are $\theta = (0.40, 0.30, 0.30)^T$. The collective values are given in Table 9.

Step 2: use the weights index $\omega = (0.31, 0.28, 0.30, 0.11)^T$ and the IVIFULHG operator with associated weight vector $w = (0.37, 0.31, 0.25, 0.07)^T$: $\psi_l = \text{IVPULHX}_{\omega, w}(\psi_{l1}, \dots, \psi_{ln}) = \prod_{j=1}^4 (Y_{l\delta_j})^{w_j}$ to find the total preference values ψ_l of alternative A_l , where $Y_{l\delta_j}$ is the j^{th} biggest value of the IVIFUL weighted argument $Y_{l\kappa}(Y_{l\kappa}) = (\psi_{l\kappa})^{3\omega_\kappa}$, $\kappa = 1, 2, 3$:

$$\begin{aligned}
 \psi_1 &= \langle [s_{4.27}, s_{6.02}], [0.423, 0.558], [0.109, 0.209] \rangle, \\
 \psi_2 &= \langle [s_{3.49}, s_{4.73}], [0.00, 0.388], [0.181, 0.387] \rangle, \\
 \psi_3 &= \langle [s_{4.31}, s_{6.34}], [0.00, 0.238], [0.349, 0.452] \rangle, \\
 \psi_4 &= \langle [s_{3.95}, s_{5.84}], [0.240, 0.440], [0.186, 0.303] \rangle.
 \end{aligned}
 \tag{43}$$

Step 3: obtain the scores $\text{Sco}^*(\psi_l)$ ($l = 1, \dots, 4$) of the collective overall IVIFUL preference values ψ_l ($l = 1, \dots, 4$):

$$\begin{aligned}
 \text{Sco}^*(\psi_1) &= 3.42, \\
 \text{Sco}^*(\psi_2) &= 1.62, \\
 \text{Sco}^*(\psi_3) &= 1.91, \\
 \text{Sco}^*(\psi_4) &= 2.67.
 \end{aligned}
 \tag{44}$$

Step 4: according to the scores of IVIFUL preference values, the ranking order is given in Table 10.

5.2. Comparison with the Other Methods. In the coming information, the proposed MAGDM method will also discuss their similarities with established approaches. We

TABLE 6: The transformed IVIFUL decision matrix R^1 .

	C_1	C_2
A_1	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_3, s_5]; [0.6, 0.7], [0.0, 0.1] \rangle$
A_2	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_5]; [0.1, 0.3], [0.2, 0.3] \rangle$	$\langle [s_5, s_8]; [0.0, 0.1], [0.5, 0.6] \rangle$
A_4	$\langle [s_7, s_9]; [0.4, 0.7], [0.1, 0.2] \rangle$	$\langle [s_2, s_4]; [0.1, 0.2], [0.4, 0.5] \rangle$
	C_3	C_4
A_1	$\langle [s_4, s_6]; [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.2, 0.3] \rangle$
A_2	$\langle [s_3, s_5]; [0.1, 0.2], [0.4, 0.5] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.1, 0.4] \rangle$
A_3	$\langle [s_6, s_7]; [0.2, 0.3], [0.3, 0.4] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.1, 0.2] \rangle$
A_4	$\langle [s_5, s_9]; [0.4, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.4, 0.5] \rangle$

TABLE 7: The transformed IVIFUL decision matrix R^2 .

	C_1	C_2
A_1	$\langle [s_5, s_6]; [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_5, s_9]; [0.4, 0.6], [0.2, 0.3] \rangle$
A_2	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_7]; [0.1, 0.2], [0.3, 0.5] \rangle$	$\langle [s_6, s_7]; [0.2, 0.3], [0.3, 0.4] \rangle$
A_4	$\langle [s_7, s_9]; [0.4, 0.7], [0.1, 0.2] \rangle$	$\langle [s_4, s_5]; [0.1, 0.3], [0.2, 0.3] \rangle$
	C_3	C_4
A_1	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.1, 0.4] \rangle$
A_2	$\langle [s_5, s_8]; [0.0, 0.1], [0.5, 0.6] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.2, 0.3] \rangle$
A_3	$\langle [s_6, s_7]; [0.2, 0.3], [0.3, 0.4] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.4, 0.5] \rangle$
A_4	$\langle [s_2, s_3]; [0.3, 0.4], [0.1, 0.2] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.1, 0.2] \rangle$

TABLE 8: The transformed IVIFUL decision matrix R^3 .

	C_1	C_2
A_1	$\langle [s_3, s_5]; [0.6, 0.7], [0.0, 0.1] \rangle$	$\langle [s_7, s_9]; [0.4, 0.7], [0.1, 0.2] \rangle$
A_2	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_8, s_9]; [0.5, 0.6], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_5]; [0.1, 0.3], [0.2, 0.3] \rangle$	$\langle [s_6, s_7]; [0.2, 0.3], [0.3, 0.4] \rangle$
A_4	$\langle [s_3, s_6]; [0.2, 0.3], [0.1, 0.2] \rangle$	$\langle [s_5, s_9]; [0.4, 0.6], [0.2, 0.3] \rangle$
	C_3	C_4
A_1	$\langle [s_5, s_9]; [0.4, 0.6], [0.2, 0.3] \rangle$	$\langle [s_4, s_4]; [0.0, 0.1], [0.4, 0.5] \rangle$
A_2	$\langle [s_4, s_5]; [0.4, 0.5], [0.1, 0.2] \rangle$	$\langle [s_2, s_3]; [0.4, 0.6], [0.1, 0.2] \rangle$
A_3	$\langle [s_4, s_7]; [0.1, 0.2], [0.3, 0.5] \rangle$	$\langle [s_3, s_4]; [0.1, 0.2], [0.2, 0.3] \rangle$
A_4	$\langle [s_3, s_6]; [0.1, 0.3], [0.2, 0.3] \rangle$	$\langle [s_1, s_2]; [0.2, 0.3], [0.1, 0.4] \rangle$

compared our proposed advanced method with current fuzzy methods and recommended that our work be completed. Given that the IVIFS principle has an immense effect in different areas, there are still some actual problems that IVIFS have not been able to solve. Term in IVPFSs consists of the positive grade, neutral grade, and negative grade. Since, the IVPFSs are the most advanced structure, it is not possible for other established aggregation operators to solve the data contained in this problem, demonstrating the limited approach of current approaches. However, if we take on any problem with the interval-valued fuzzy information, the IVPFSs can easily solve it, converting the interval-valued data to IVPFSs, taking the values outside the IVPFSs interval to zero.

Now, we compare our developed approach to the approaches of Wan and Dong [27, 28]. We compared our proposed method to the methods which have only two terms (positive and negative). So, if we consider only the positive and negative grades, we neglect the neutral term; then, the IVPFNs reduced to IVIFNs. We take $\omega = (0.31, 0.28, 0.30, 0.11)^T$ are the criteria weight vector to facilitate the

TABLE 9: Collective decision matrix.

C_1	
A_1	$\langle [s_{4,27}, s_{5,26}]; [0.449, 0.553], [0.078, 0.180] \rangle$
A_2	$\langle [s_{2,44}, s_{3,49}]; [0.319, 0.425], [0.098, 0.204] \rangle$
A_3	$\langle [s_{3,99}, s_{5,52}]; [0.099, 0.268], [0.310, 0.367] \rangle$
A_4	$\langle [s_{5,41}, s_{6,61}]; [0.324, 0.539], [0.099, 0.204] \rangle$
C_2	
A_1	$\langle [s_{4,50}, s_{7,09}]; [0.472, 0.664], [0.098, 0.200] \rangle$
A_2	$\langle [s_{4,90}, s_{5,95}]; [0.425, 0.528], [0.099, 0.204] \rangle$
A_3	$\langle [s_{5,56}, s_{7,35}]; [0.00, 0.189], [0.386, 0.492] \rangle$
A_4	$\langle [s_{3,23}, s_{5,45}]; [0.148, 0.308], [0.279, 0.387] \rangle$
C_3	
A_1	$\langle [s_{4,28}, s_{6,39}]; [0.300, 0.431], [0.129, 0.239] \rangle$
A_2	$\langle [s_{3,80}, s_{5,72}]; [0.00, 0.197], [0.360, 0.643] \rangle$
A_3	$\langle [s_{3,64}, s_{6,97}]; [0.161, 0.263], [0.304, 0.434] \rangle$
A_4	$\langle [s_{3,25}, s_{5,71}]; [0.238, 0.429], [0.174, 0.273] \rangle$
C_4	
A_1	$\langle [s_{3,25}, s_{4,84}]; [0.00, 0.141], [0.325, 0.412] \rangle$
A_2	$\langle [s_{4,39}, s_{5,01}]; [0.241, 0.321], [0.138, 0.216] \rangle$
A_3	$\langle [s_{3,38}, s_{5,13}]; [0.273, 0.398], [0.162, 0.214] \rangle$
A_4	$\langle [s_{3,51}, s_{5,94}]; [0.124, 0.291], [0.304, 0.434] \rangle$

TABLE 10: Ranking of the existing method.

Methods	Ranking
Method by P. Liu [83]	$A_1 > A_4 > A_3 > A_2$.
Our method	$A_1 > A_4 > A_3 > A_2$.

TABLE 11: Ranking of the existing method.

Methods	Ranking
Wan and Dong [28]	$A_1 > A_3 > A_4 > A_2$.
Wan and Dong [27]	$A_1 > A_4 > A_2 > A_3$.

comparison. Using the given preferences and information, the existing methods Wan and Dong [27, 28] are applied to the data being considered, and then the final scores of the alternatives A_i ($1, \dots, 4$) are shown in Table 4. Table 4 shows that A_1 is the best alternative in any approach. Compared with these existing approaches with IVIFSs, the proposed DM method under IVPFS environment contains much more evaluation information on the alternatives by considering IVIFSs simultaneously, while the existing approaches contain IVIFS information. Therefore, we claim that our proposed PCF aggregation operators are more efficient and reliable than previous aggregation operators. The ranking order of the comparative study is given in Table 11.

6. Comparison and Conclusion

In this section, the proposed IVPFUL aggregation operators are compared with existing AOs and our work is concluded. Even IFS theory has a great influence on many fields, there are some real world problems that could not be solved by intuitionistic fuzzy setting and could not even be solved by IVIFLS. Like IVIFLS, each element of an IVIFLS is presented as a framework of an ordered pair characterized by positive and negative membership grades. The positive and negative membership

function is gripped in the form of the [83] interval. While in IVPFLS, every element consists of grade of positive, neutral, and negative. If we take the problem of Section 5, to be the most advanced structure, then the data found in the problem cannot be solved by the current fuzzy AOs, which denoted that the current AOs have the minimal method. However, if we consider any type of problem with the IVPFUL information, we can solve it easily. Therefore, IVPFUL operators are more efficient in solving unforeseeable problems. Various researchers can easily observe it from the existing approach [83], and the algorithms by using interval intuitionistic uncertain linguistic variables setting for MCGDM problems have some limitations and are unable to handle the problems in certain uncertain situations. So, their proposed approach may not produce the exact results. However, IVPFUL operators can give more precise results. We have introduced IVPFUL set, and we have also established the degree of accuracy and the score for the comparison of ICF numbers. Some IVPFUL operational laws were developed. Also we established a number of IVPFUL geometric AOs. We also studied some of its properties, such as idempotency, boundary, and monotonicity. The operator is established by considering the IVPFULNs aggregate relationship. In order to show the performance of these operators, a multicriteria group DM approach was developed using these operators with IVPFUL information. A numerical example was presented showing that the developed operators provide an alternative way to more effectively resolve the DM process. Finally, to illustrate the relevance, practicality, and usefulness of the proposed approaches, we have presented a comparison with current operators.

In the future, we extend the developed idea to many other existing approaches, such as cleaner production evaluation in gold mines using a novel distance measure method with cubic picture fuzzy numbers; fuzzy decision support modeling for Internet finance soft power evaluation based on sine trigonometric Pythagorean fuzzy information; entropy-based Pythagorean probabilistic hesitant fuzzy decision-making technique and its application for Fog-Haze factor assessment problem; picture fuzzy aggregation information based on Einstein operations and their application in decision-making; a new possibility degree measure for interval-valued q-rung orthopair fuzzy sets in decision-making; green supplier selection in steel industry with intuitionistic fuzzy taxonomy method; algorithms for probabilistic uncertain linguistic multiple attribute group decision-making based on the GRA and CRITIC method; application to location planning of electric vehicle charging stations; the maximizing deviation method for multiple attribute decision-making under q-rung orthopair fuzzy environment; uncertain database retrieval with measure-based belief function attribute values with the intuitionistic fuzzy set; the multiplicative consistency adjustment model and data envelopment analysis-driven DM process with probabilistic hesitant fuzzy preference relations; and the MABAC method for multiple attribute group decision-making under q-rung orthopair fuzzy environment.

In Appendix A, we include the abbreviation table, which includes abbreviations used in the paper.

Appendix

A. Abbreviation List

Table 12 shows the expansions for the abbreviations used in the paper that could promote readability.

B. Proof of Theorem 1

Proof. According to the basic operation of addition and multiplication of IVPFULNs, rules (1) and (2) are easily proved.

To prove rule (3), we have

$$\begin{aligned}
 \lambda(\Lambda_1 + \Lambda_2) &= \lambda \left(\begin{array}{c} [s_{\theta(\Lambda_1)+\theta(\Lambda_2)}, s_{\tau(\Lambda_1)+\tau(\Lambda_2)}], \\ [1 - (1 - a^-(\Lambda_1))(1 - a^-(\Lambda_2)), 1 - (1 - a^+(\Lambda_1)), (1 - a^+(\Lambda_2))], \\ [b^-(\Lambda_1), b^-(\Lambda_2), b^+(\Lambda_1), b^+(\Lambda_2)], [c^-(\Lambda_1), c^-(\Lambda_2), c^+(\Lambda_1), c^+(\Lambda_2)] \end{array} \right) \\
 &= \left(\begin{array}{c} [s_{\lambda(\theta(\Lambda_1)+\theta(\Lambda_2))}, s_{\lambda(\tau(\Lambda_1)+\tau(\Lambda_2))}], \\ [1 - (1 - a^-(\Lambda_1))^\lambda, (1 - a^-(\Lambda_2))^\lambda, 1 - (1 - a^+(\Lambda_1))^\lambda, (1 - a^+(\Lambda_2))^\lambda], \\ [(b^-(\Lambda_1))^\lambda, (b^-(\Lambda_2))^\lambda, (b^+(\Lambda_1))^\lambda, (b^+(\Lambda_2))^\lambda], \\ [(c^-(\Lambda_1))^\lambda, (c^-(\Lambda_2))^\lambda, (c^+(\Lambda_1))^\lambda, (c^+(\Lambda_2))^\lambda] \end{array} \right), \\
 \lambda\Lambda_1 + \lambda\Lambda_2 &= \left(\begin{array}{c} [s_{\lambda \times \theta(\Lambda_1)}, s_{\lambda \times \tau(\Lambda_2)}], [1 - (1 - a^-(\Lambda_1))^\lambda, 1 - (1 - a^+(\Lambda_1))^\lambda], \\ [b^-(\Lambda_1)^\lambda, b^+(\Lambda_1)^\lambda], [c^-(\Lambda_1)^\lambda, c^+(\Lambda_1)^\lambda] \end{array} \right) \\
 &\oplus \left(\begin{array}{c} [s_{\lambda \times \theta(\Lambda_2)}, s_{\lambda \times \tau(\Lambda_2)}], [1 - (1 - a^-(\Lambda_2))^\lambda, 1 - (1 - a^+(\Lambda_2))^\lambda], \\ [b^-(\Lambda_2)^\lambda, b^+(\Lambda_2)^\lambda], [c^-(\Lambda_2)^\lambda, c^+(\Lambda_2)^\lambda] \end{array} \right), \\
 &= \left(\begin{array}{c} [s_{\lambda \times \theta(\Lambda_1)+\lambda \times \theta(\Lambda_2)}, s_{\lambda \times (\tau(\Lambda_1)+\lambda \times \tau(\Lambda_2))}], \\ [1 - (1 - (1 - (1 - a^-(\Lambda_1))^\lambda))(1 - (1 - (1 - a^-(\Lambda_2))^\lambda))], \\ [1 - (1 - (1 - (1 - a^+(\Lambda_1))^\lambda))(1 - (1 - (1 - a^+(\Lambda_2))^\lambda))], \\ [(b^-(\Lambda_1))^\lambda, (b^-(\Lambda_2))^\lambda, (b^+(\Lambda_1))^\lambda, (b^+(\Lambda_2))^\lambda], \\ [(c^-(\Lambda_1))^\lambda, (c^-(\Lambda_2))^\lambda, (c^+(\Lambda_1))^\lambda, (c^+(\Lambda_2))^\lambda] \end{array} \right) \\
 &= \left(\begin{array}{c} [s_{\lambda(\theta(\Lambda_1)+\theta(\Lambda_2))}, s_{\lambda(\tau(\Lambda_1)+\tau(\Lambda_2))}], \\ [1 - (1 - a^-(\Lambda_1))^\lambda, (1 - a^-(\Lambda_1))^\lambda, 1 - (1 - a^+(\Lambda_1))^\lambda, (1 - a^+(\Lambda_2))^\lambda], \\ [(b^-(\Lambda_1))^\lambda, (b^-(\Lambda_2))^\lambda, (b^+(\Lambda_1))^\lambda, (b^+(\Lambda_2))^\lambda], \\ [(c^-(\Lambda_1))^\lambda, (c^-(\Lambda_2))^\lambda, (c^+(\Lambda_1))^\lambda, (c^+(\Lambda_2))^\lambda] \end{array} \right). \tag{B.1}
 \end{aligned}$$

$\therefore \lambda(\Lambda_1 + \Lambda_2) = \lambda\Lambda_1 + \lambda\Lambda_2.$

The proof of (4)-(6) is similarly to the proof of (3). \square

C. Proof of Theorem 2

Proof. Using the principle of mathematical induction to prove this theorem,

TABLE 12: Abbreviations and its expansions.

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
IFN	Intuitionistic fuzzy number
PFS	Picture fuzzy set
PFN	Picture fuzzy number
DM	Decision-making
IFOWA	Intuitionistic fuzzy ordered weighted averaging
IFOWG	Intuitionistic fuzzy ordered weighted geometric
IFHA	Intuitionistic fuzzy hybrid averaging
MADM	Multiple attribute decision-making
AOs	Aggregation operators
MCDM	Multiple criteria decision-making
MCGDM	Multiple criteria group decision-making
IVPFULSs	Interval-valued picture fuzzy uncertain linguistic sets
IVPFULNs	Interval-valued picture fuzzy uncertain linguistic numbers
IVPFULG	Interval-valued picture fuzzy uncertain linguistic geometric
IVPFULOWG	Interval-valued picture fuzzy uncertain linguistic ordered weighted geometric
IVPFULHG	Interval-valued picture fuzzy uncertain linguistic hybrid geometric
IFLS	Intuitionistic fuzzy linguistic set
PFLS	Picture fuzzy linguistic set
SCM	Supply chain management

(1) it is obvious that when $n = 1$, then it is true and

(2) when $n = 2$, then

$$\begin{aligned}
(\Lambda_1)^{\omega_1} &= \left(\begin{array}{l} [s_{\theta}(\Lambda_1)^{\omega_1}, s_{\tau}(\Lambda_1)^{\omega_1}], [(a^-(\Lambda_1))^{\omega_1}, (a^+(\Lambda_1))^{\omega_1}], \\ [1 - (1 - b^-(\Lambda_1))^{\omega_1}, 1 - (1 - b^+(\Lambda_1))^{\omega_1}], \\ [1 - (1 - c^-(\Lambda_1))^{\omega_1}, 1 - (1 - c^+(\Lambda_1))^{\omega_1}] \end{array} \right), \\
(\Lambda_2)^{\omega_2} &= \left(\begin{array}{l} [s_{\theta}(\Lambda_2)^{\omega_2}, s_{\tau}(\Lambda_2)^{\omega_2}], [(a^-(\Lambda_2))^{\omega_2}, (a^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - b^-(\Lambda_2))^{\omega_2}, 1 - (1 - b^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - c^-(\Lambda_2))^{\omega_2}, 1 - (1 - c^+(\Lambda_2))^{\omega_2}] \end{array} \right), \\
\text{IVPFULWG}_{\omega}(\Lambda_1, \Lambda_2) &= (\Lambda_1)^{\omega_1} \times (\Lambda_2)^{\omega_2} \\
&= \left(\begin{array}{l} [s_{\theta}(\Lambda_1)^{\omega_1}, s_{\tau}(\Lambda_1)^{\omega_1}], [(a^-(\Lambda_1))^{\omega_1}, (a^+(\Lambda_1))^{\omega_1}], \\ [1 - (1 - b^-(\Lambda_1))^{\omega_1}, 1 - (1 - b^+(\Lambda_1))^{\omega_1}], \\ [1 - (1 - c^-(\Lambda_1))^{\omega_1}, 1 - (1 - c^+(\Lambda_1))^{\omega_1}] \end{array} \right) \\
&\quad \times \left(\begin{array}{l} [s_{\theta}(\Lambda_2)^{\omega_2}, s_{\tau}(\Lambda_2)^{\omega_2}], [(a^-(\Lambda_2))^{\omega_2}, (a^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - b^-(\Lambda_2))^{\omega_2}, 1 - (1 - b^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - c^-(\Lambda_2))^{\omega_2}, 1 - (1 - c^+(\Lambda_2))^{\omega_2}] \end{array} \right) \\
&= \left(\begin{array}{l} [s_{\theta}(\Lambda_1)^{\omega_1 \times \theta}(\Lambda_2)^{\omega_2}, s_{\tau}(\Lambda_1)^{\omega_1 \times \tau}(\Lambda_2)^{\omega_2}], \\ [a^{\Lambda^-}(\Lambda_1)^{\omega_1} \times (a^-(\Lambda_2))^{\omega_2}, (a^+(\Lambda_1))^{\omega_1} \times (a^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - b^-(\Lambda_1))^{\omega_1} \times 1 - (1 - b^-(\Lambda_2))^{\omega_2}], \\ [1 - (1 - b^+(\Lambda_1))^{\omega_1} \times 1 - (1 - b^+(\Lambda_2))^{\omega_2}], \\ [1 - (1 - c^-(\Lambda_1))^{\omega_1} \times 1 - (1 - c^-(\Lambda_2))^{\omega_2}], \\ [1 - (1 - c^+(\Lambda_1))^{\omega_1} \times 1 - (1 - c^+(\Lambda_2))^{\omega_2}]. \end{array} \right)
\end{aligned}$$

$$= \left(\begin{array}{c} \left[S_{\prod_{j=1}^2}(\theta(\Lambda_j))^{\omega_j}, S_{\prod_{j=1}^2}(\tau(\Lambda_j))^{\omega_j} \right], \\ \left[\prod_{j=1}^2 (a^-(\Lambda_j))^{\omega_j}, \prod_{j=1}^2 (a^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^2 (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^2 (1 - b^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^2 (1 - c^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^2 (1 - c^+(\Lambda_j))^{\omega_j} \right]. \end{array} \right), \tag{C.1}$$

which is true for $n = 2$.

Now, for $n = \kappa$, we have

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_n) = \left(\begin{array}{c} \left[S_{\prod_{j=1}^\kappa}(\theta(\Lambda_j))^{\omega_j}, S_{\prod_{j=1}^\kappa}(\tau(\Lambda_j))^{\omega_j} \right], \left[\prod_{j=1}^\kappa (a^-(\Lambda_j))^{\omega_j}, \prod_{j=1}^\kappa (a^+(\Lambda_j))^{\omega_j} \right] \\ \left[1 - \prod_{j=1}^\kappa (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^\kappa (1 - b^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^\kappa (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^\kappa (1 - b^+(\Lambda_j))^{\omega_j} \right] \end{array} \right). \tag{C.2}$$

Then, when $n = \kappa + 1$,

$$\text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_\kappa, \Lambda_{\kappa+1}) = \text{IVPFULWG}_\omega(\Lambda_1, \dots, \Lambda_\kappa) \times (\Lambda_{\kappa+1})^{\omega_{\kappa+1}}$$

$$= \left(\begin{array}{c} \left[S_{\prod_{j=1}^\kappa}(\tau(\Lambda_j))^{\omega_j}, S_{\prod_{j=1}^\kappa}(\tau(\Lambda_j))^{\omega_j} \right], \left[\prod_{j=1}^\kappa (a^-(\Lambda_j))^{\omega_j}, \prod_{j=1}^\kappa (a^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^\kappa (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^\kappa (1 - b^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^\kappa (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^\kappa (1 - b^+(\Lambda_j))^{\omega_j} \right]. \end{array} \right) \otimes \left(\begin{array}{c} \left[S_{\theta(\Lambda_{\kappa+1})}^{\omega_{\kappa+1}}, S_{\tau(\Lambda_{\kappa+1})}^{\omega_{\kappa+1}} \right], \left[(a^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, (a^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right], \\ \left[1 - (1 - b^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, 1 - (1 - b^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right], \\ \left[1 - (1 - c^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, 1 - (1 - c^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right]. \end{array} \right)$$

$$\begin{aligned}
 & \left(\begin{array}{l} \left[S_{\Pi_{j=1}^{\kappa}}((\theta(\Lambda_j))^{\omega_j}) \times (\theta(\Lambda_{\kappa+1})^{\omega_{\kappa+1}}), S_{\Pi_{j=1}^{\kappa}}((\tau(\Lambda_j))^{\omega_j}) \times (\tau(\Lambda_{\kappa+1})^{\omega_{\kappa+1}}) \right], \\ \left[\prod_{j=1}^{\kappa} (a^-(\Lambda_j))^{\omega_j} \times (a^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, \prod_{j=1}^{\kappa} (a^+(\Lambda_j))^{\omega_j} \times (a^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right], \\ \left[1 - \prod_{j=1}^{\kappa} (1 - b^-(\Lambda_j))^{\omega_j} \times 1 - (1 - b^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, \right. \\ \left. 1 - \prod_{j=1}^{\kappa} (1 - b^+(\Lambda_j))^{\omega_j} \times 1 - (1 - b^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right], \\ \left[1 - \prod_{j=1}^{\kappa} (1 - c^-(\Lambda_j))^{\omega_j} \times 1 - (1 - c^-(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}}, \right. \\ \left. 1 - \prod_{j=1}^{\kappa} (1 - c^+(\Lambda_j))^{\omega_j} \times 1 - (1 - c^+(\Lambda_{\kappa+1}))^{\omega_{\kappa+1}} \right] \end{array} \right) \tag{C.3} \\
 & = \left(\begin{array}{l} \left[S_{\Pi_{j=1}^{\kappa+1}}((\theta(\Lambda_j))^{\omega_j}), S_{\Pi_{j=1}^{\kappa+1}}((\tau(\Lambda_j))^{\omega_j}) \right], \left[\prod_{j=1}^{\kappa+1} (a^-(\Lambda_j))^{\omega_j}, \prod_{j=1}^{\kappa+1} (a^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^{\kappa+1} (1 - b^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^{\kappa+1} (1 - b^+(\Lambda_j))^{\omega_j} \right], \\ \left[1 - \prod_{j=1}^{\kappa+1} (1 - c^-(\Lambda_j))^{\omega_j}, 1 - \prod_{j=1}^{\kappa+1} (1 - c^+(\Lambda_j))^{\omega_j} \right]. \end{array} \right)
 \end{aligned}$$

which is true for $n = \kappa + 1$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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