Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 8887566, 10 pages
https://doi.org/10.1155/2021/8887566

Research Article

On the Approximate Solutions of the Constant Forced (Un) Damping Helmholtz Equation for Arbitrary Initial Conditions

Alvaro H. Salas,[1] **Castillo H. Jairo E**,[2] **and M. R. Alharthi**[3]

[1]*Department of Mathematics, Universidad Nacional de Colombia, Universidad Nacional de Colombia-Nubia Campus, Department of Mathematics and Statistics FIZMAKO Research Group, Manizales, Colombia*
[2]*Grupo FIZMAKO, Universidad Distrital Francisco José de Caldas, Bogotá, Colombia*
[3]*Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to Alvaro H. Salas; ahsalass@unal.edu.co

Received 17 September 2020; Revised 7 January 2021; Accepted 20 January 2021; Published 31 January 2021

Academic Editor: Viet-Thanh Pham

This paper presents some novel solutions to the family of the Helmholtz equations (including the constant forced undamping Helmholtz equation (equation (1)) and the constant forced damping Helmholtz equation (equation (2))) which have been reported. In the beginning, equation (1) is solved analytically using two different techniques (direct and indirect solutions): in the first technique (direct solution), a new assumption is introduced to find the analytical solution of equation (1) in the form of the Weierstrass elliptic function with arbitrary initial conditions. In the second case (indirect solution), the solution of the undamping (standard) Duffing equation is devoted to determine the analytical solution to equation (1) in the form of Jacobian elliptic function with arbitrary initial conditions. Moreover, equation (2) is solved using a new ansatz and with the help of equation (1) solutions. Also, the evolution equations (equations (1) and (2)) are solved numerically via the Adomian decomposition method (ADM). Furthermore, a comparison between the approximate analytical solution and approximate numerical solutions using the fourth-order Runge–Kutta method (RK4) and ADM is reported. Furthermore, the maximum distance error for the obtained solutions is estimated. As a practical application, the Helmholtz-type equation will be derived from the fluid governing equations of quantum plasma particles with(out) taking the ionic kinematic viscosity into account for investigating the characteristics of (un)damping oscillations in a degenerate quantum plasma model.

1. Introduction

The ordinary and partial differential equations have played an important role in explaining many natural phenomena, in addition to their applications in many engineering and physical problems. Due to the great role played by these equations, many authors focused their efforts on finding some solutions to these equations [1–10]. The undamping Helmholtz equation ($\ddot{q}(t) + \sum_{i=1}^{n} k_i q^i(t) = 0$, where i is the odd number, i.e., $i = 1, 3, 5, \ldots$) and undamping Duffing equation ($\ddot{q}(t) + \sum_{i=1}^{n} k_i q^i(t) = 0$, where $i = 1, 2, 3, \ldots$) in addition to their family (including friction/damping force in addition to excitation/perturbation force) are among the most famous differential equations in dynamic, electrical, and engineering systems [11–16]. A lot of physical and engineering problems such as the human eardrum oscillations, the dynamics of the ships, the electrical circuits signal oscillations, heavy symmetric gyroscope, and microperforated panel absorber [17–21] have been investigated using different solutions of the Helmholtz-type oscillator. The Helmholtz-type oscillator is a second-order differential equation with a quadratic nonlinear term in addition to some other terms. For realistic physical situations, we cannot ignore both the friction/dissipation force and excitation/perturbation forcing. Accordingly, the general form of the Helmholtz equation reads [22–24]

$$\left\{ \ddot{q} + 2\gamma\dot{q} + \alpha q + \beta q^2 = F, q(0) = q_0 \,\&\, q'(0) = \dot{q}_0, \right. \quad (1)$$

where q denotes the displacement of the system, α is the natural frequency, β is a nonlinear system parameter, γ represents the damping factor, and F is a constant/excitation force. The first equation in system (1) is called the constant forced damping Helmholtz equation (equation (2)). If the coefficient γ of the damping term $(2\gamma\dot{q})$ is neglected, equation (2) reduces to the constant forced Helmholtz equation $(\ddot{q} + \alpha q + \beta q^2 = F)$ (equation (1)). Also, if both the coefficient γ of the damping term and the constant force F are neglected, the traditional form of Helmholtz equation is covered $(\ddot{q} + \alpha q + \beta q^2 = 0)$. The initial value problem (IVP) (1) and its family have many applications in several fields, starting from analyzing the signals that propagate in electrical circuits, plasma physics, general relativity, betatron oscillations, vibrations of shells, vibrations of the acoustically driven human eardrum, solid-state physics, etc. [26–33].

It is well known that, in the absence of both friction and the excitation forces from the IVP (1), the unforced and undamping Helmholtz equations are covered. The exact analytic solutions to the unforced and undamping Helmholtz equation have been derived in detail in the literature in terms of the Weierstrass elliptic function [34–38] and Jacobi elliptic functions [38–41]. Generally, to solve any quadratic or cubic nonlinear second-order differential equation, firstly, we should transform it to an elliptic integral and then we solve it [24]. It is known that the unforced and undamping Helmholtz equations are completely integrable, so they have exact solutions, but if the friction force (damping term) is included, then the unforced damping Helmholtz equation becomes nonintegrable and cannot support an exact solution for arbitrary values of its coefficients (γ, α, β). Thus, under certain condition, the unforced damping Helmholtz equation has been solved analytically in terms of the Jacobi elliptic functions by Johannessen [24]. Also, Almendral and Sanjuán [27] derived an exact solution to the unforced damping Helmholtz equation using the Lie theory under certain conditions for the coefficients (γ, α, β).

In many realistic physical models, both the damping and the excitation/external terms are very important to be included, and thus the problem becomes more complicated to find its analytical solutions. In this paper, we will derive some analytical solutions to equation (1) in the terms of the Weierstrass and Jacobian elliptic functions. Also, an approximate analytical solution to equation (2) for arbitrary values to the coefficients and the initial conditions will be derived in detail. Moreover, the problem under consideration will be solved numerically via using the RK4 and ADM to make a comparison between the obtained solutions and the approximate numerical solutions. Furthermore, the maximum distance error between the approximate analytical solution and the approximate numerical solutions will be estimated. Also, the dynamics of nonlinear oscillations that can be generated in the RLC electronic circuits and quantum plasma will be investigated using the solution of equation (2).

The rest of this work is organized in the following manner: in Section 2, we will introduce in detail our methodology for solving the family of the Helmholtz-type equations. Also, we will introduce some new approaches for solving equation (1) as well as the exact solution of the undamped and unforced Duffing equation, which will be devoted to finding an approximate analytical solution to equation (2). In Section 3, a comparison between the obtained solutions and the approximate numerical solution using the ADM will be investigated. In Section 4, some realistic applications will be introduced. Finally, our results will be summarized in Section 5.

2. Our Methodology for Solving the Family of the Helmholtz Equations

Before proceeding in solving the IVP (1), it is necessary to refer to two fundamental equations and their solutions: the first one is called the Duffing equation and the other is called the constant forced Helmholtz equation.

2.1. Duffing Equation and Its Solution. The analytical solution to the following IVP, which is called Duffing equation [30],

$$\begin{cases} \eta''(t) + R\eta(t) + S\eta^3(t) = 0, \\ \eta(0) = \eta_0 \,\&\, \eta'(0) = \dot{\eta}_0, \end{cases} \quad (2)$$

is given by the following formula:

$$\eta(t) = c_1 cn\left(\sqrt{R + Sc_1^2}\, t + c_2, m\right). \quad (3)$$

By inserting this relation into the IVP (2) and after several tedious but simple math operations, we finally get the values of c_1 and c_2 as follows:

$$\begin{aligned} c_1 &= \pm\sqrt{\frac{-R + \sqrt{\Delta}}{S}} t, \\ c_2 &= cn^{-1}\left(\frac{\eta_0}{c_1}, \frac{Sc_1^2}{2\left(R + Sc_1^2\right)}\right), \end{aligned} \quad (4)$$

where R, S, η_0, and $\dot{\eta}_0$ are real numbers and Δ is called the discriminant to Duffing (2):

$$\Delta = \left(R + \eta_0^2 S\right)^2 + 2\dot{\eta}_0^2 S > 0. \quad (5)$$

Solution (3) could be expressed as

$$\eta(t) = \frac{\eta_0 cn(\omega t|m) + (\dot{\eta}_0/\omega) dn(\omega t|m) sn(\omega t|m)}{1 - (1/2)\left(1 - \left(R + S\eta_0^2/\sqrt{\Delta}\right)\right) sn(\omega t|m)^2}, \quad (6)$$

where

$$\omega = \sqrt[4]{\Delta},$$
$$m = \frac{1}{2} - \frac{R}{2\sqrt{\Delta}}. \tag{7}$$

For negative discriminant ($\Delta < 0$), the solution may be written in the following form:

$$\eta(t) = \rho - \frac{2\rho}{1 + \kappa sc\left(\sqrt{\omega}t + sc^{-1}\left(\rho + \eta_0/\kappa(\rho - \eta_0)\right), m\right)}, \tag{8}$$

where

$$m = \frac{4\rho\sqrt{2S}\sqrt{\rho^2 S - R}}{2\rho\sqrt{2S}\sqrt{\rho^2 S - R} + R - 3\rho^2 S},$$
$$\omega = \frac{1}{4}\left(2\rho\sqrt{2S}\sqrt{\rho^2 S - R} + R - 3\rho^2 S\right),$$
$$\kappa = \frac{\sqrt{2\rho\sqrt{2S}\sqrt{\rho^2 S - R} - R + 3\rho^2 S}}{\sqrt{R + \rho^2 S}}, \tag{9}$$
$$\rho = \pm\sqrt[4]{\frac{2R\eta_0^2 + S\eta_0^4 + 2\dot{\eta}_0^2}{-S}}.$$

For a zero discriminant ($\Delta = 0$), the solution of Duffing equation (2) will be

$$\eta(t) = c_1 \tan h\left(c_1\sqrt{\frac{-S}{2}}t + c_2\right), \tag{10}$$

with

$$c_1 = \pm\frac{\sqrt{\sqrt{-S}\eta_0^4 - \sqrt{2}\eta_0^2\dot{\eta}_0}}{\sqrt[4]{-s}S\eta_0},$$
$$c_2 = \frac{1}{2}\log\left(\frac{\dot{\eta}_0 - \sqrt{2}\eta_0^2\sqrt{-S} - \sqrt[4]{-S}\sqrt{2\eta_0^4\sqrt{-S} - 2\sqrt{2}\eta_0^2\dot{\eta}_0}}{\dot{\eta}_0}\right), \tag{11}$$

where $\dot{\eta}_0\eta_0 \neq 0$.

When $(R + \eta_0^2 S)^2 + 2\dot{\eta}_0^2 S = \eta_0 = 0$, the solution reads

$$\eta(t) = \sqrt{\frac{-S}{2}}\dot{\eta}_0 \tan h\left(\sqrt{\frac{-S}{2}}\dot{\eta}_0 t\right). \tag{12}$$

In case $(R + \eta_0^2 S)^2 + 2\dot{\eta}_0^2 S = \dot{\eta}_0 = 0$, the solution becomes the constant function $\eta(t) = \eta_0$.

2.2. The Analytical Solution to the Constant Forced Helmholtz Equation. The solution of the constant forced Helmholtz equation

$$\begin{cases} \xi''(t) + a + b\xi(t) + c\xi^2(t) = 0, \\ \xi(0) = \xi_0 \,\&\, \xi'(0) = \dot{\xi}_0, \end{cases} \tag{13}$$

may be expressed in either one of the following forms. Note that $a = -F$.

2.2.1. First Formula. Suppose that the solution of system (13) has the following form:

$$\xi(t) = A + \frac{B}{1 + C\left((1/4)\left(d_2 - \wp'(t; g_2, g_3)/d_1 - \wp(t; g_2, g_3)\right)^2 - d_1 - \wp(t; g_2, g_3)\right)}, \tag{14}$$

as well as

$$\xi(t) = A + \frac{B}{1 + C\wp\left(t \pm \wp^{-1}(d_1; g_2, g_3); g_2, g_3\right)}, \tag{15}$$

where a represents the constant force and $\wp(t)$ gives the Weierstrass elliptic function, which satisfies the following relations:

$$\wp'(x; g_2, g_3)^2 = 4\wp^3(x; g_2, g_3) - g_1\wp(x; g_2, g_3) - g_3,$$
$$\wp''(x; g_2, g_3) = -\frac{g_2}{2} + 6\wp^2(x; g_2, g_3). \tag{16}$$

Inserting equation (14) into system (13) and after tedious straightforward calculations, we can estimate the values of B, C, g_2, g_3, d_1, and d_2 as follows:

$$B = -\frac{6(a + A(Ac + b))}{(2Ac + b)},$$
$$C = \frac{12}{2Ac + b},$$
$$g_2 = \frac{1}{12}\left(b^2 - 4ac\right),$$
$$g_3 = \frac{1}{216}(2Ac + b)\left(b^2 - 2c\left(3a + A^2 c\right) - 2Abc\right), \tag{17}$$
$$d_1 = \frac{(A - \xi_0)(2Ac + b)}{6(a + A(Ac + b))},$$
$$d_2 = -\frac{\dot{\xi}_0(2Ac + b)}{6(a + A(Ac + b))}.$$

The value of parameter A represents a root to the following quartic equation:

$$
\begin{aligned}
&4a^2b + a^2b^2 - 4a^3c + 4ab^2\xi_0 - 4b^2\xi_0^2 + 4abc\xi_0^2 \\
&-2\begin{pmatrix} 8bc\xi_0^2 - 2ab^2 - ab^3 - 4a^2c + 4a^2bc \\ -4b^2\xi_0 - 2b^3\xi_0 - 4abc\xi_0 - 2b^2c\xi_0^2 - 4ac^2\xi_0^2 \end{pmatrix}A \\
&+\begin{pmatrix} b^4 - 4b^2 + 12abc - 2ab^2c - 8a^2c^2 \\ +32bc\xi_0 + 12b^2c\xi_0 - 16c^2\xi_0^2 + 12bc^2\xi_0^2 \end{pmatrix}A^2 \\
&+2c\left(b^3 - 8b + 4ac - 4abc + 16c\xi_0 + 4bc\xi_0 + 4c^2\xi_0^2\right)A^3 \\
&-c^2\left(16 - b^2 + 4ac\right)A^4 = 0.
\end{aligned} \tag{18}
$$

2.2.2. Second Formula. As another form for the solution of system (13), let us assume that its solution is given by the following relationship:

$$\xi(t) = A + \frac{B}{1 - \eta(t)}, \tag{19}$$

where $\eta = \eta(t)$ is the solution to the following Duffing equation:

$$
\begin{cases} \eta''(t) + R\eta(t) + S\eta^3(t) = 0, \\ \eta(0) = \eta_0 \;\&\; \eta'(0) = \dot{\eta}_0. \end{cases} \tag{20}
$$

By following the same procedures in the above sections, we finally obtain

$$
\begin{aligned}
R &= \frac{3a + b(3A + B) + Ac(3A + 2B)}{B}, \\
S &= -\frac{a + A(Ac + b)}{B}, \\
\eta_0 &= \frac{B}{A - \xi_0} + 1, \\
\dot{\eta}_0 &= \frac{B\dot{\xi}_0}{(A - \xi_0)^2}.
\end{aligned} \tag{21}
$$

The value of parameter A is a solution to the following equation:

$$
\begin{aligned}
0 = {} & A^6c^3 + 3bc^2A^5 + 15ac^2A^4 \\
&- 10c^2\left(6a\xi_0 + 3b\xi_0^2 + 2c\xi_0^3 + 3\dot{\xi}_0^2\right)A^3 \\
&- 15c\left(3a^2 + 6ab\xi_0 + 3b^2\xi_0^2 + 2bc\xi_0^3 + 3b\dot{\xi}_0^2\right)A^2 \\
&- 3\begin{pmatrix} 9a^2b + 18ab^2\xi_0 - 12a^2c\xi_0 + 9b^3\xi_0^2 \\ - 6abc\xi_0^2 + 6b^2c\xi_0^3 - 4ac^2\xi_0^3 + 9b^2\dot{\xi}_0^2 - 6ac\dot{\xi}_0^2 \end{pmatrix}A \\
&+\begin{pmatrix} - 27a^3 - 54a^2b\xi_0 - 27ab^2\xi_0^2 - 72a^2c\xi_0^2 - 90abc\xi_0^3 \\ - 18b^2c\xi_0^4 - 48ac^2\xi_0^4 - 24bc^2\xi_0^5 - 8c^3\xi_0^6 - 27ab\dot{\xi}_0^2 \\ - 72ac\xi_0\dot{\xi}_0^2 - 36bc\xi_0^2\dot{\xi}_0^2 - 24c^2\xi_0^3\dot{\xi}_0^2 - 18c\dot{\xi}_0^4 \end{pmatrix}.
\end{aligned} \tag{22}
$$

Also, the value of parameter B is a solution to the following equation:

$$
\begin{aligned}
0 = {} & c^4B^6 - 9c^2\left(b^2 - 4ac\right)B^4 - \\
& 432\begin{pmatrix} -3a^2b^2 + 16a^3c - 6ab^3\xi_0 + 36a^2bc\xi_0 \\ -3b^4\xi_0^2 + 18ab^2c\xi_0^2 + 36a^2c^2\xi_0^2 - 2b^3c\xi_0^3 \\ +48abc^2\xi_0^3 + 9b^2c^2\xi_0^4 + 24ac^3\xi_0^4 + 12bc^3\xi_0^5 + \\ 4c^4\xi_0^6 - 3b^3\dot{\xi}_0^2 + 18abc\dot{\xi}_0^2 + 36ac^2\xi_0\dot{\xi}_0^2 \\ +18bc^2\xi_0^2\dot{\xi}_0^2 + 12c^3\xi_0^3\dot{\xi}_0^2 + 9c^2\dot{\xi}_0^4 \end{pmatrix}.
\end{aligned} \tag{23}
$$

2.3. The Approximate Analytical Solution for the Constant Forced and Damped Helmholtz Equation. Let us rewrite system (1) in the following traditional form:

$$\left\{ \ddot{x} + 2\varepsilon\dot{x} + \alpha x + \beta x^2 = F, x(0) = x_0 \; x'(0) = \dot{x}_0, \right. \tag{24}$$

with $\beta \neq 0$.

Also, let us assume that

$$\alpha^2 + 4F\beta \geq 0. \tag{25}$$

Now, suppose that the solution of system (24) is given by

$$x(t) = d + \exp(-\varepsilon t)y(t), \tag{26}$$

where $y = y(t)$ is a solution to the following Helmholtz equation:

$$
\begin{cases} y'' + py + \beta y^2 = 0, \\ y(0) = y_0 = x_0 - d, \\ y'(0) = \dot{y}_0 = \varepsilon x_0 + \dot{x}_0 - \mathrm{d}\varepsilon. \end{cases} \tag{27}
$$

Inserting (26) into the first equation in system (24), $\mathbb{R}(t) \equiv \ddot{x} + 2\varepsilon\dot{x} + \alpha x + \beta x^2 - F = 0$, we obtain

$$
\begin{aligned}
\mathbb{R}(t) = {} & \beta d^2 + \alpha d - F \\
& + e^{-2\varepsilon t}y(t)\left[\beta y(t) - e^{\varepsilon t}\left(-\alpha - 2\beta d + \varepsilon^2 + p + \beta y(t)\right)\right].
\end{aligned} \tag{28}
$$

Expression (28) suggests the following choices:

$$
\begin{aligned}
-\alpha - 2\beta d + \varepsilon^2 + p &= 0, \\
\beta d^2 + \alpha d - F &= 0,
\end{aligned} \tag{29}
$$

giving us the values of p and d as follows:

$$
\begin{aligned}
p &== \alpha + 2\beta d - \varepsilon^2, \\
d &= \frac{-\alpha + \sqrt{\alpha^2 + 4F\beta}}{2\beta}.
\end{aligned} \tag{30}
$$

The solution to the following IVP,

$$\begin{cases} y'' + \left(\alpha + 2\beta d - \varepsilon^2\right) y + \beta y^2 = 0, \\ y(0) = y_0 = x_0 - d, \\ y'(0) = \dot{y}_0 = \varepsilon x_0 + \dot{x}_0 - \mathrm{d}\varepsilon, \end{cases} \quad (31)$$

may be expressed in different forms as we explained in the previous section.

3. A Comparison between Our Solutions and ADM Solution

There are many numerical methods that could be used to find approximate solutions to IVP (24). Here, we make use of the ADM [42–44] to solve IVP (24) with(out) forcing term (F). According to this method, the first iteration/approximation for the unforced case (F) reads

$$x_{\mathrm{ADM0}}(t) = \left(\frac{e^{-\varepsilon t}}{\mathcal{F}}\right)\left[\left(\varepsilon x_0 + \dot{x}_0\right)\Theta_1 + x_0 \mathcal{F} \Theta_2\right], \quad (32)$$

where $\Theta_1 = \sin(t\mathcal{F})$, $\Theta_2 = \cos(t\mathcal{F})$, and $\mathcal{F} = \sqrt{\alpha - \varepsilon^2}$.

For the second iteration/approximation, we have

$$x_{\mathrm{ADM1}}(t) = \beta\left(\frac{e^{-\varepsilon t}}{\alpha\mathcal{F}}\right)^3 \frac{\left(-e^{\varepsilon t}\mathcal{F} + \varepsilon\Theta_1 + \mathcal{F}\Theta_2\right)}{\left[\alpha x_0 \mathcal{F}\Theta_2 + \left(\varepsilon x_0 \mathcal{F}^2 + \dot{x}_0 \mathcal{F}^2 + \varepsilon^3 x_0 + \varepsilon^2 \dot{x}_0\right)\Theta_1\right]^2}. \quad (33)$$

The approximate Adomian approximate solution reads

$$x(t) = x_{\mathrm{ADM0}}(t) + x_{\mathrm{ADM1}}(t) + \cdots. \quad (34)$$

For $(\varepsilon, \alpha, \beta, F) = (0.1, 4, 1, 0)$ and $x(0) = 0$ & $x'(0) = 0.2$, we can compare between the approximate/semi-analytical solution (26) (for $F = 0$) and the ADM approximate numerical solution (34) and the RK4 approximate numerical solution as shown in Figure 1. Also, the maximum distance error according to the formula $L_D = \max_{t_i \le t \le t_f} |x_{\mathrm{RK4}}(t) - x_{\mathrm{method}}(t)|$ has been estimated:

$$\begin{aligned} L_D &= \max_{0 \le t \le 40} \left| x_{\mathrm{RK4}}(t) - x_{\text{semi-analy}}(t) \right| = 0.000504825, \\ L_D &= \max_{0 \le t \le 40} \left| x_{\mathrm{RK4}}(t) - x_{\mathrm{ADM}}(t) \right| = 0.0000997768. \end{aligned} \quad (35)$$

It is clear that the semianalytical solution (26) (for $F = 0$) gives good results as compared to both the RK4 and ADM approximate numerical solutions.

Now, let us find an approximate solution for IVP (24) in the presence of the forcing term, using the ADM. Accordingly, the first approximation is given by

$$x_{\mathrm{ADM0}}(t) = \left(\frac{e^{-\varepsilon t}}{\widetilde{\mathcal{F}}}\right)\left[\left(\varepsilon\left(x_0 - d\right) + \dot{x}_0\right)\widetilde{\Theta}_1 + \left(x_0 - d\right)\widetilde{\mathcal{F}}\widetilde{\Theta}_2\right], \quad (36)$$

where $\widetilde{\Theta}_1 = \sin(t\widetilde{\mathcal{F}})$, $\widetilde{\Theta}_2 = \cos(t\widetilde{\mathcal{F}})$, $\widetilde{\mathcal{F}} = \sqrt{\Lambda - \varepsilon^2}$, $\Lambda = (\alpha + 2\beta d)$, and $d = \left(-\alpha \pm \sqrt{\alpha^2 + 4F\beta}\right)/(2\beta)$.

The second approximation to IVP (24) according to the ADM reads

$$\begin{aligned} x_{\mathrm{ADM1}}(t) = {} & \left(\frac{\beta e^{-2t\varepsilon}}{2\Lambda\widetilde{\mathcal{F}}^2}\right)\left[\left(d - x_0\right)^2\left(-2d\beta - \alpha\right) + 2\dot{x}_0\varepsilon\left(d - x_0\right) - \dot{x}_0^2\right] \\ & + \left(\frac{\beta e^{-t\varepsilon}}{\Lambda\widetilde{\mathcal{F}}\mathbb{N}}\right)\left[\varepsilon\left(d - x_0\right)^2\left(13\Lambda - 8\varepsilon^2\right) - 6\Lambda\dot{x}_0\left(d - x_0\right) + 2\dot{x}_0^2\varepsilon\right]\widetilde{\Theta}_1 \\ & + \left(\frac{\beta e^{-2t\varepsilon}}{\Lambda\widetilde{\mathcal{F}}\mathbb{N}}\right)\left[\varepsilon\left(d - x_0\right)^2\left(8\varepsilon^2 - 5\Lambda\right) + \dot{x}_0\left(d - x_0\right)\left(3\Lambda - 8\varepsilon^2\right) + 2\dot{x}_0^2\varepsilon\right]\widetilde{\Theta}_1 \\ & - \left(\frac{\beta e^{-t\varepsilon}}{\Lambda\mathbb{N}}\right)\left[\left(d - x_0\right)^2\left(3\Lambda + 8\varepsilon^2\right) + 16\dot{x}_0\varepsilon\left(x_0 - d\right) + 6\dot{x}_0^2\right]\widetilde{\Theta}_2 \\ & - \left(\frac{\beta e^{-2t\varepsilon}}{2\Lambda\mathbb{N}\widetilde{\mathcal{F}}^2}\right)\begin{pmatrix} 2\dot{x}_0\varepsilon\left(d - x_0\right)\left(7\Lambda - 8\varepsilon^2\right) + \dot{x}_0^2\left(-3\Lambda + 4\varepsilon^2\right) + \\ \left(d - x_0\right)^2\left(-18\varepsilon^2\Lambda + 3\Lambda^2 + 16\varepsilon^4\right) \end{pmatrix}\widetilde{\Theta}_2, \end{aligned} \quad (37)$$

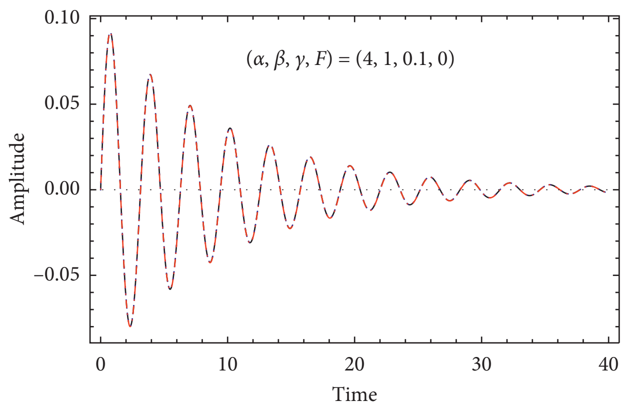

FIGURE 1: A comparison between approximate analytical solution (26) (dotted curve) for $F = 0$ and both RK4 (dashed curve) and ADM (dashed-dot curve) numerical solutions is plotted against the time.

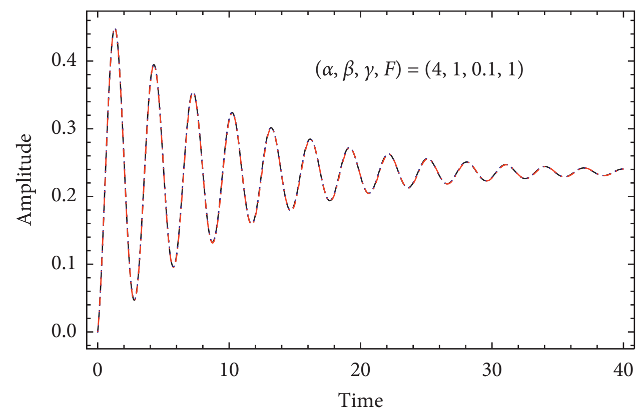

FIGURE 2: A comparison between approximate analytical solution (26) (dotted curve) for $F \neq 0$ and both RK4 (dashed curve) and ADM (dashed-dot curve) numerical solutions is plotted against the time.

where $\mathbb{N} = (8\varepsilon^2 - 9(2d\beta + \alpha))$.

The approximate Adomian approximate solution is given by

$$x(t) = x_{\mathrm{ADM0}}(t) + x_{\mathrm{ADM1}}(t) + \cdots. \tag{38}$$

For $(\varepsilon, \alpha, \beta, F) = (0.1, 2, 1, 1)$ and $x(0) = 0$ & $x'(0) = 0.2$, a comparison between the semianalytical solution (26) (for $F \neq 0$) and the ADM approximate numerical solution (38) and the RK4 approximate numerical solution has been investigated as shown in Figure 2. Furthermore, the maximum distance error has been calculated as follows:

$$\begin{aligned} L_D &= \max_{0\le t\le 40}\left|x_{\mathrm{RK4}}(t) - x_{\text{semi-analy}}(t)\right| = 0.00332902, \\ L_D &= \max_{0\le t\le 40}\left|x_{\mathrm{RK4}}(t) - x_{\mathrm{ADM}}(t)\right| = 0.00132421. \end{aligned} \tag{39}$$

Also, the semianalytical solution (26) for $F = 0$ and $F \neq 0$ gives excellent results as compared to the ADM approximate numerical solution (38).

4. Quantum Plasma Oscillations

In this section, we will reduce the fluid governing equations of a quantum plasma model to an evolution equation using the RPT [45–49]. After a suitable transformation, we will be able to convert the obtained evolution equation to a Helmholtz-type equation in order to investigate the characteristics behavior of the damping oscillations in the model under consideration. Now, let us assume that we have a collisionless and unmagnetized electron-ion quantum plasma consisting of inertialess degenerate trapped electrons which obey the Fermi–Dirac distribution and classical fluid cold positive nondegenerate ion. Thus, the basic normalized fluid equations that govern the nonlinear dynamics of various structures could be presented as [51, 52]

$$\begin{aligned} &\partial_t n + \partial_x(nu) = 0, \\ &\partial_t u + u\partial_x u + \partial_x\phi = \eta\partial_x^2 u, \\ &\partial_x^2\phi = (n_e - n), \end{aligned} \tag{40}$$

where n_e and n represent the normalized electron and ion number densities, respectively, u gives the normalized ion speed, η is the normalized kinematic viscosity of the ions, and ϕ indicates the normalized electrostatic wave potential. Therefore, we shall adopt the adiabatic trapped degeneracy for electrons, by relying on notations similar to those in [52], wherein the fundamental algebra is expressed in detail. The electron normalized number density according to Fermi–Dirac distribution reads

$$\begin{aligned} n_e &= \sqrt{(1+\phi)^3} + \frac{T^2}{\sqrt{(1+\phi)}} \\ &\approx s_0 + s_1\phi + s_2\phi^2 + s_3\phi^3, \end{aligned} \tag{41}$$

where T expresses the normalized temperature of the degenerate electron, $s_0 = (1+T^2)$, $s_1 = (3-T^2)/2$, $s_2 = 3(1+T^2)/8$, and $s_3 = -(1+5T^2)/6$. Note that expression (41) is obtained under the approximation $\phi \ll 1$ for small wave amplitude.

For investigating the nonlinear structures and oscillations in the present model, the RPM will be employed for this purpose. Accordingly, the stretching and expansions for the independent and dependent variables are, respectively, introduced as follows:

$$\begin{aligned} \xi &= \varepsilon^{(1/2)}\left(x - V_{\mathrm{ph}}t\right), \\ \tau &= \varepsilon^{(3/2)}t, \end{aligned} \tag{42}$$

$$\begin{aligned} n(x,t) &= 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \cdots, \\ u(x,t) &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \cdots, \\ \phi(x,t) &= \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + \cdots, \end{aligned} \tag{43}$$

where ε is a small ($\varepsilon \ll 1$) and real parameter and V_{ph} represents the normalized phase velocity of the unmodulated structures. It is assumed that the impact of ionic kinematic viscosity is small according to many lab experiments [53, 54]. Thus, we could set $\eta = \varepsilon^{1/2}\tilde{\eta}$, where $\tilde{\eta}$ is $O(1)$. By substituting stretching (42) and expansions (43) into system (40) and by collecting the terms of different powers of ε, we could get a system of reduced equations. Solving the system of reduced equations for the first-two orders of ε by following the same procedures in [46–49], we finally obtain the Korteweg–de Vries Burgers (KdVB) equation [50].

$$\partial_\tau \phi + A_p \phi \partial_\xi \phi + B_p \partial_\xi^3 \phi - C_p \partial_\xi^2 \phi = 0, \tag{44}$$

with

$$\begin{aligned} A_p &= B_p\left(\frac{3}{V_{\mathrm{ph}}^4} - 2s_2\right), \\ B_p &= \frac{V_{\mathrm{ph}}^3}{2}, \\ C_p &= \frac{\tilde{\eta}}{2}, \\ V_{\mathrm{ph}} &= \frac{1}{\sqrt{s_1}}, \end{aligned} \tag{45}$$

where A_p, B_p, and C_p represent the coefficients of the nonlinear, dispersion, and dissipative terms, respectively, and $\phi \equiv \phi^{(1)}$.

Using the traveling wave transformation $\phi(\xi + V_f t) = q(\zeta)$, where $\zeta = \xi + V_f t$ and V_f gives the frame velocity, into the KdVB equation (44) and integrating once over ζ, we get the constant forced and damped Helmholtz equation as follows:

$$q''(\zeta) + 2\gamma q'(\zeta) + \alpha q(\zeta) + \beta q(\zeta)^2 = \mathbb{C}, \tag{46}$$

where $\mathbb{C}$ is the integration constant, $\gamma = -C_p/(2B_p)$, $\alpha = V_f/B_p$, and $\beta = A_p/(2B_p)$. Now, we can apply the above solution of the constant forced and damped Helmholtz equation that is given in equation (26) to equation (46) for investigating the characteristics of the damped oscillations in a quantum plasma.

Note that if the ionic kinematic viscosity is neglected, i.e., $C_p = 0$, then the KdV and undamped Helmholtz equations could be covered. The KdV equation, i.e., equation (44) for $C_p = 0$, is one of the most popular soliton and cnoidal equations and has been extensively investigated. For the soliton solution, the following conditions must be fulfilled: $(q(\zeta), q'(\zeta), q''(\zeta)) \longrightarrow 0$ at $\zeta \longrightarrow \pm\infty$, so the integration constant $\mathbb{C}$ in equation (46) must vanish. Accordingly, equation (46) could be reduced to the undamped Helmholtz equation:

$$q''(\zeta) + \alpha q(\zeta) + \beta q(\zeta)^2 = 0. \tag{47}$$

It is well known that equation (47) supports many solutions such as periodic solution (see the solutions to the constant forced Helmholtz equation above) and solitons. The soliton solution to equation (47) in the form of the Weierstrass elliptic function $\wp$ could be written in the following manner:

$$\phi(x,t) = \frac{2\alpha\beta + \sqrt{\alpha^2\left(\beta^2 + 2\alpha C\right)}}{\alpha^2} - \frac{(9\beta/2\alpha)}{1 + 6\wp\left(x - \left(t\sqrt{\alpha^2\left(\beta^2 + 2C\alpha\right)}/\alpha\right); (1/12), -(1/216)\right)}, \tag{48}$$

where $(x,t) \equiv (\zeta, \tau)$.

Here, the obtained approximate analytical solution (26) to the constant forced and damped Helmholtz equation (46) will be analyzed numerically according to the quantum plasma parameters $(T, \tilde{\eta}) = (0.1, 0.055)$, i.e., $(\alpha, \beta, \gamma, F) = (1.828, 2.98, -0.05, F)$, and $(T, \tilde{\eta}) = (0.9, 0.18)$, i.e., $(\alpha, \beta, \gamma, F) = (1.146, 1.7273, -0.1, F)$. The behavior of the quantum plasma oscillations according to the approximate analytical solution (26) and the approximate numerical solution according to the RK4 method is presented in Figure 3 for different values of quantum plasma parameters. It is clear from Figure 3 that our approximate analytical solution (26) is more accurate than the RK4 numerical solution. On the contrary, the RK4 numerical solution gives poor results and with increasing time this solution becomes unstable.

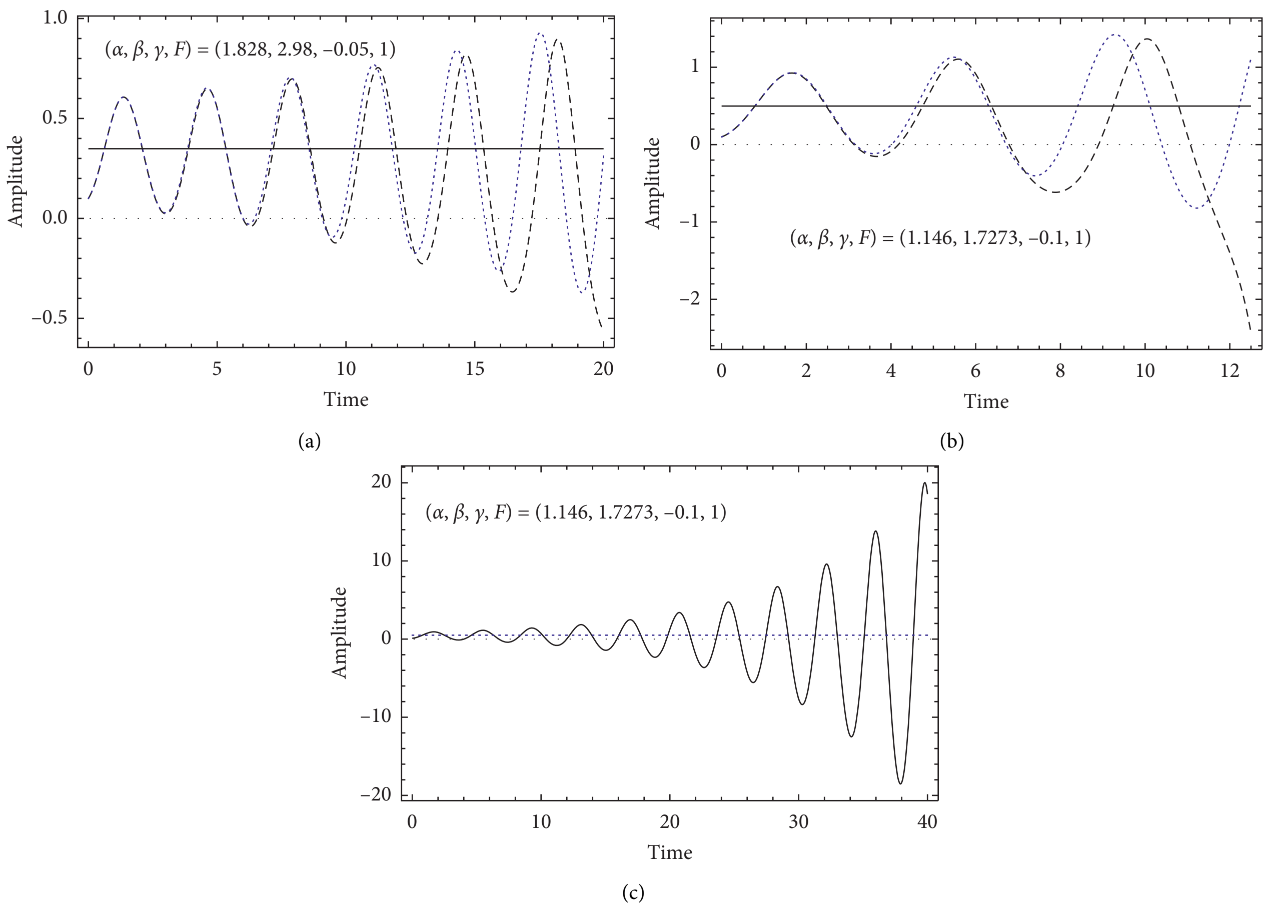

Figure 3: A comparison between the approximate analytical solution (26) (dotted curve) and the RK4 numerical solution (dashed curve) is plotted for different values of quantum plasma parameters $(T, \tilde{\eta})$, i.e., for different values of $(\alpha, \beta, \gamma, F)$. Here, $q_0 = 0.1$, and $\dot{q}_0 = 0.2$.

5. Conclusions

The Helmholtz-type equations including the constant forced undamping Helmholtz equation (equation (1)) and the constant forced damping Helmholtz equation (equation (2)) have been solved analytically and numerically. Two techniques were used to get the analytical solutions to equation (1). In the first technique, we used a new assumption to find an analytical solution to equation (1) in the form of Weierstrass elliptic function. In the second case, the solution of the standard Duffing equation has been utilized to find an analytical solution to equation (1) in the form of Jacobian elliptic function. However, the main goal of this paper is to solve equation (2), using the obtained solutions of equation (1). Moreover, both equation (1) and equation (2) have been solved numerically via the ADM. The analytical and approximate analytical solutions of equations (1) and (2) have been compared to the RK4 and ADM approximate numerical solutions. Furthermore, the maximum distance error between the RK4 approximate numerical solution and the approximate analytical solutions in addition to the approximate numerical solution using the ADM has been estimated. It was found that the obtained solutions are generally consistent with both RK4 and ADM solutions. Moreover, the obtained solutions have been applied for analyzing the oscillations that may arise in the quantum plasma. During the analysis, it was found sometimes that the approximate analytical solution is better than the RK4 numerical solution as shown in the quantum plasma model. Finally, these solutions may help us understand the oscillations that may arise in the different physical and engineering systems.

In future work, the similar approaches could be used for analyzing and solving higher-order nonlinear oscillator equations. Also, a damping Helmholtz–Duffing equation with time-dependent forced term is considered one of the most important and vital problems due to its great role in explaining many natural phenomena in different branches of science. Thus, in the next work, some new approaches will be devoted to find some solutions for these problems.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by Taif University Researchers Supporting Project (no. TURSP-2020/275), Taif University, Taif, Saudi Arabia. The authors thank Associate Professor S.A. El-Tantawy, the Chief of Research Center for Physics (RCP), Faculty of Science and Arts, Al-Baha University, Saudi Arabia, and Faculty of Science, Port Said University, Egypt, for fruitful discussions, careful reading, his contributions to applications to plasma physics, and improving their manuscript very well.

References

[1] A. Ouannas, A. A. Khennaoui, X. Wang, V.-T. Pham, S. Boulaaras, and S. Momani, "Bifurcation and chaos in the fractional form of Hénon-Lozi type map," *The European Physical Journal Special Topics*, vol. 229, no. 12-13, pp. 2261–2273, 2020.

[2] A. Ben Dhahbi, Y. Chargui, S. M. Boulaaras, and S. Ben Khalifa, "A one-sided competition mathematical model for the sterile insect technique," *Complexity*, vol. 2020, Article ID 6246808, 12 pages, 2020.

[3] A. M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing, China, 2009.

[4] S. A. El-Tantawy, "Nonlinear dynamics of soliton collisions in electronegative plasmas: the phase shifts of the planar KdV- and mkdV-soliton collisions," *Chaos, Solitons & Fractals*, vol. 93, pp. 162–168, 2016.

[5] S. A. El-Tantawy, M. H. Alshehri, F. Z. Duraihem, and L. S. El-Sherif, "Dark soliton collisions and method of lines approach for modeling freak waves in a positron beam plasma having superthermal electrons," *Results in Physics*, vol. 19, pp. 103452–103458, 2020.

[6] S. Boulaaras and M. Haiour, "L^∞-asymptotic behavior for a finite element approximation in parabolic quasi-variational inequalities related to impulse control problem," *Applied Mathematics and Computation*, vol. 217, no. 14, pp. 6443–6450, 2011.

[7] S. Boulaaras and M. Haiour, "The finite element approximation of evolutionary Hamilton-Jacobi-Bellman equations with nonlinear source terms," *Indagationes Mathematicae*, vol. 24, no. 1, pp. 161–173, 2013.

[8] S. Boulaaras, "Some new properties of asynchronous algorithms of theta scheme combined with finite elements methods for an evolutionary implicit 2-sided obstacle problem," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 18, pp. 1–9, 2017.

[9] S. Boulaaras, M. Haiour, and M. A. Bencheick Le hocine, "L^∞-error estimates of discontinuous Galerkin methods with theta time discretization scheme for an evolutionary HJB equations L^∞-error estimates of discontinuous Galerkin methods with theta time discretization scheme for an evolutionary HJB equations"" *Mathematical Methods in the Applied Sciences*, vol. 40, no. 12, p. 4310, 2017.

[10] S. Boulaaras, M. El Amine Bencheikh Le Hocine, and M. Haiour, "A new error estimate on uniform norm of a parabolic variational inequality with nonlinear source terms via the subsolution concepts," *Journal of Inequalities and Applications*, vol. 2020, 2020.

[11] K. Johannessen, "The Duffing oscillator with damping," *European Journal of Physics*, vol. 36, 2015.

[12] K. Johannessen, "The duffing oscillator with damping for a softening potential," *International Journal of Applied and Computational Mathematics*, vol. 3, no. 4, p. 3805, 2017.

[13] A. Elías-Zúñiga, "Exact solution of the quadratic mixed-parity Helmholtz-Duffing oscillator," *Applied Mathematics and Computation*, vol. 218, no. 14, pp. 7590–7594, 2012.

[14] M. Hammad, A. H. Salas, and S. A. El-Tantawy, "New method for solving strong conservative odd parity nonlinear oscillators: applications to plasma physics and rigid rotator," *AIP Advances*, vol. 10, no. 8, pp. 085001–085011, 2020.

[15] A. H. Salas and S. A. El-Tantawy, "On the approximate solutions to a damped harmonic oscillator with higher-order nonlinearities and its application to plasma physics: semi-analytical solution and moving boundary method," *The European Physical Journal-Plus*, vol. 135, pp. 833–917, 2020.

[16] Md.A. Hosen and M. S. H. Chowdhury, "Analytical approximate solutions for the helmholtz-duffing oscillator," *ARPN Journal of Engineering and Applied Sciences*, vol. 10, pp. 17363–17369, 2015.

[17] Y. Geng, "Exact solutions for the quadratic mixed-parity Helmholtz-Duffing oscillator by bifurcation theory of dynamical systems," *Chaos, Solitons & Fractals*, vol. 81, pp. 68–77, 2015.

[18] E. Metter, "Dynamic buckling," in *Handbook of Engineering Mechanics*, W. Flügge, Ed., Wiley, Hoboken, NJ, USA, 1992.

[19] M. Bikdash, B. Balachandran, and A. Nayfeh, "Melnikov analysis for a ship with a general roll-damping model," *Nonlinear Dynamics*, vol. 6, pp. 101–124, 1994.

[20] V. Ajjarapu and B. Lee, "Bifurcation theory and its application to nonlinear dynamical phenomena in an electrical power system," *IEEE Transactions on Power Systems*, vol. 7, no. 1, pp. 424–431, 1992.

[21] I. S. Kang and L. G. Leal, "Bubble dynamics in time-periodic straining flows," *Journal of Fluid Mechanics*, vol. 218, no. 1, pp. 41–69, 1990.

[22] D.-Q. Wei, X.-S. Luo, and S.-Y. Zeng, "Noise-triggered escapes in Helmholtz oscillator," *Modern Physics Letters B*, vol. 28, pp. 1450047-1450048, 2014.

[23] J. A. Almendral, J. M. Seoane, and M. A. F. Sanjúan, "Nonlinear dynamics of the helmholtz oscillator," *Journal of Sound and Vibration*, vol. 2, pp. 115–150, 2004.

[24] K. Johannessen, "The solution to the differential equation with linear damping describing a physical systems governed by a cubic energy potential," http://arxiv.org/abs/1810.10336.

[25] J. M. Seoane, S. Zambrano, S. Euzzor, R. Meucci, F. T. Arecchi, and M. A. F. Sanju, "Avoiding escapes in open dynamical systems using phase control," *Physical Review E*, vol. 75, pp. 16205–16208, 2008.

[26] N. Nayfeth and D. T. Mook, *Non-linear Oscillations*, Wiley, New York, NY, USA, 1973.

[27] J. A. Almendral and M. A. F. Sanjuan, "Integrability and symmetries for the helmholtz oscillator with friction," *Journal of Physics A: Mathematical and General*, vol. 36, no. 3, pp. 695–710, 2003.

[28] S. Morfa and J. C. Comte, "A nonlinear oscillators network devoted to image processing," *International Journal of Bifurcation and Chaos*, vol. 14, pp. 1385–1394, 2009.

[29] Y. Geng, J. Li, and L. Zhang, "Exact explicit traveling wave solutions for two nonlinear Schrödinger type equations," *Applied Mathematics and Computation*, vol. 217, no. 4, pp. 1509–1521, 2010.

[30] A. H. Salas and J. E. Castillo, "Exact solutions to cubic Duffing equation for a nonlinear electrical circuit," *Visión Electrónica: algo más que un Estado Sólido*, vol. 8, pp. 46–53, 2014.

[31] E. Gluskin, "A nonlinear resistor and nonlinear inductor using a nonlinear capacitor," *Journal of the Franklin Institute*, vol. 336, no. 7, pp. 1035–1047, 1999.

[32] A. H. Salas, “Soluciones exactas a la ecuación del oscilador de helmholtz para circuitos eléctricos con no linealidad cuadrática,” *Visión Electrónica*, vol. 9, no. 2, pp. 248–252, 2015.

[33] A. H. Salas, E. Jairo, H. Castillo, J. Darin, and P. Mosquera, “A new approach for solving the undamped Helmholtz oscillator for the given arbitrary initial conditions and its physical applications,” *Mathematical Problems in Engineering*, vol. 2020, Article ID 7876413, 7 pages, 2020.

[34] K. S. Viswanathan, “The theory of the anharmonic oscillator,” *Proceedings of the Indian Academy of Sciences A*, vol. 46, pp. 201–217, 1957.

[35] B. F. Apostol, “On anharmonic oscillators,” *Journal of Physics*, vol. 50, pp. 915–918, 2005.

[36] P. Amore and F. M. Fernández, “Exact and approximate expressions for the period of anharmonic oscillators,” *European Journal of Physics*, vol. 26, no. 4, pp. 589–601, 2005.

[37] S. Y. Vernov, “Exact solutions of nonlocal nonlinear field equations in cosmology,” *Theoretical and Mathematical Physics*, vol. 166, no. 3, pp. 392–402, 2011.

[38] A. J. Brizard, “A primer on elliptic functions with applications in classical mechanics,” *European Journal of Physics*, vol. 30, no. 4, pp. 729–750, 2009.

[39] J.-H. He, “Some asymptotic methods for strongly nonlinear equations,” *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[40] P. Walker, “The analyticity of Jacobian functions with respect to the parameter k,” *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 459, no. 2038, pp. 2569–2574, 2003.

[41] A. A. Hussein and S. Al Athel, “A note on the Helmholz oscillator,” *Chaos, Solitons & Fractals*, vol. 12, no. 10, pp. 1835–1838, 2001.

[42] G. Adomian, “A review of the decomposition method in applied mathematics,” *Journal of Mathematical Analysis and Applications*, vol. 135, no. 2, pp. 501–544, 1988.

[43] A.-M. Wazwaz, “Adomian decomposition method for a reliable treatment of the Bratu-type equations,” *Applied Mathematics and Computation*, vol. 166, no. 3, pp. 652–663, 2005.

[44] A. M. Wazwaz, “A new algorithm for calculating Adomian polynomials for nonlinear operators,” *Applied Mathematics and Computation*, vol. 111, pp. 53–69, 2000.

[45] H. Washimi and T. Taniuti, “Propagation of ion-acoustic solitary waves of small amplitude,” *Physical Review Letters*, vol. 17, no. 19, pp. 996–998, 1966.

[46] N. H. Aljahdaly and S. A. El-Tantawy, “Simulation study on nonlinear structures in nonlinear dispersive media,” *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 30, no. 5, pp. 053117–53213, 2020.

[47] B. S. Kashkari, S. A. El-Tantawy, A. H. Salas, and L. S. El-Sherif, “Homotopy perturbation method for studying dissipative nonplanar solitons in an electronegative complex plasma,” *Chaos, Solitons & Fractals*, vol. 130, pp. 109457–109510, 2020.

[48] S. A. El-Tantawy, T. Aboelenen, and S. M. E. Ismaeel, “Local discontinuous Galerkin method for modeling the nonplanar structures (solitons and shocks) in an electronegative plasma,” *Physics of Plasmas*, vol. 26, no. 2, pp. 022115–22211, 2019.

[49] S. A. El-Tantawy and A. M. Wazwaz, “Anatomy of modified Korteweg–de Vries equation for studying the modulated envelope structures in non-Maxwellian dusty plasmas: freak waves and dark soliton collisions,” *Physics of Plasmas*, vol. 25, 2018.

[50] S. A. El-Tantawy, “Effect of ion viscosity on dust ion-acoustic shock waves in a nonextensive magnetoplasma,” *Astrophysics and Space Science*, vol. 361, p. 249, 2016.

[51] S. A. El-Tantawy, S. Ali Shan, N. Akhtar, and A. T. Elgendy, “Impact of electron trapping in degenerate quantum plasma on the ion-acoustic breathers and super freak waves,” *Chaos, Solitons & Fractals*, vol. 113, pp. 356–364, 2018.

[52] H. A. Shah, M. N. S. Qureshi, and N. Tsintsadze, “Effect of trapping in degenerate quantum plasmas,” *Physics of Plasmas*, vol. 17, no. 3, pp. 032312–032316, 2010.

[53] S. A. El-Tantawy, A. H. Salas, M. Abu Hammad, S. M. E. Ismaeel, D. M. Moustafa, and E. I. El-Awady, “Impact of dust kinematic viscosity on the breathers and rogue waves in a complex plasma having kappa distributed particles,” *Waves in Random and Complex Media*, 2019.

[54] S. A. El-Tantawy, “Ion-acoustic waves in ultracold neutral plasmas: modulational instability and dissipative rogue waves,” *Physics Letters A*, vol. 381, no. 8, pp. 787–791, 2017.