Research Article

On the Controllability of Discrete-Time Leader-Follower Multiagent Systems with Two-Time-Scale and Heterogeneous Features

Mengqi Gu ^{1,2,3} and Guo-Ping Jiang ^{1,3}

¹College of Automation and College of Artificial Intelligence, Nanjing University of Posts and Telecommunications, Nanjing 210023, China

²School of Physics and Electronic Electrical Engineering, Huaiyin Normal University, Huaian 223300, China

³Jiangsu Engineering Lab for IOT Intelligent Robots (IOTRobot), Nanjing 210023, China

Correspondence should be addressed to Guo-Ping Jiang; jianggp@njupt.edu.cn

Received 19 September 2020; Revised 8 January 2021; Accepted 5 February 2021; Published 20 March 2021

Academic Editor: Wenyu Zhang

Copyright © 2021 Mengqi Gu and Guo-Ping Jiang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper investigates the controllability of discrete-time leader-follower multiagent systems (MASs) with two-time-scale and heterogeneous features, motivated by the fact that many real systems are operating in discrete-time. In this study, singularly perturbed difference systems are used to model the two-time-scale heterogeneous discrete-time MASs. To avoid the ill-posedness problem caused by the singular perturbation parameter when using the classical control theory to study the model, the singular perturbation method was first applied to decompose the system into two subsystems with slow-time-scale and fast-time-scale feature. Then, from the perspective of algebra and graph theory, several easier-to-use controllability criteria for the related MASs are proposed. Finally, the effectiveness of the main results is verified by simulation.

1. Introduction

In recent years, with the wide application of MASs in aircraft formation, multirobot cooperative control, traffic vehicle control, network resource allocation, and other fields, scholars have become particularly interested in the distributed cooperative control [1–4] of these systems. MASs are systems composed of some dynamic agents with the certain autonomous ability through information communication and interaction. The ultimate goal of studying them is reflected in the control that people can have, which makes the research on the controllability of MASs extremely important. Controllability features of MASs are related to the agents that can reach all desired final states from any initial states within a limited time through controlling a specific portion of the agents.

In the 1960s, Kalman [5] introduced the concept of controllability of linear time-invariant (LTI) dynamic systems

and pioneered the effective Kalman rank criterion for discriminating controllability of LTI dynamic systems. Then, Tanner [6] extended the concept of controllability to MASs, aiming to understand more about the issue of single-integrator continuous-time in such systems. An algebraic controllability criterion was obtained under the assumption of one leader and nearest-neighbor communication protocol and made a great contribution to the field. Rahmani and Mesbahi [7] proposed the relationship between graph symmetry and system controllability based on Tanner's results and achieved conditions that could prove that the corresponding system is uncontrollable when the topological structure graph linked to the leader agent is symmetrical. Furthermore, in [8, 9], the nontrivial equitable partition method is used to deeply explore the relationship between the controllability and network structure of MASs with multiple leaders. This is a research branch based on graph theory that should definitely be considered when regarding the issue of the controllability of

MASs. Since then, a large number of studies [10, 11] on this realm have been conducted, achieving different outcomes.

Most of the existing research studies on the controllability of MASs from the perspective of algebra consider that the agents constituting the systems are of a single type. Ni et al. [12] investigated the controllability of the first-order MASs and obtained some controllability criteria when the controllability was decoupled into two independent parts, one is about the controllability of each individual node, and the other is completely determined by the network topology. Taking the switching topology into consideration, Tian et al. [13] minutely studied the controllability of first-order MASs composed of continuous-time subsystems and discrete-time subsystems. Using the concepts of invariant subspace and controllable state set, a sufficient and necessary condition for the controllability of switched MASs was obtained. Based on the Jordan standard form of the Laplacian matrix, the effects of topology, communication strength, and the number of external inputs on the controllability of the first-order leader-based MASs are discussed in [14]. Meanwhile, a topological structure that is completely controllable regardless of the positions and number of leaders is also proposed. This structure is extremely relevant for system design in engineering practice.

As the discrete-time system is ubiquitous in life, it should not be ignored, specially considering a scenario of rapid development of information and communication technologies. Liu et al. studied the controllability of discrete-time MASs with one leader under fixed and switched topologies in [15] and concluded that when one of the agents is selected as the leader appropriately, the interconnected system is completely controllable even though each subsystem cannot be controlled. Furthermore, in [16], the concept of group controllability of multiagent systems is proposed first, and the controllability criteria of a class of first-order multiagent systems are explored only when the agents constituting the MASs are divided into different subgroups. These subgroups are divided according to different control objectives. However, the reality is that usually there are different types of individuals with different abilities in the same system. For example, heterogeneous MASs composed of unmanned air vehicles with different capabilities can often exhibit more superior performance [17]. Based on this, some recent studies have been considering the controllability of MASs and its heterogeneous characteristics. Guan et al. [18] did that and concluded that the controllability of these systems was completely dependent on the controllability of its underlying topology under the choice of specific leaders. Tian et al. [19] further studied the same features of heterogeneous MASs with switching topologies. Based on the concept of invariant subspace, the authors pointed out that if the union of all possible topologies is controllable, then so are these systems.

All the research results mentioned above consider that the agents that make up the system work on the same timescale. However, it is common that different timescales coexist in the

same system. Due to the mutual influence between the different timescale components, they cannot be analyzed separately. For example, in the field of robots, the dynamics of flexible manipulator includes two major timescales: macro rigid motion and micro flexible vibration [20]. Almost all large-scale systems have a dynamic coexistence phenomenon with large timescale differences. Prandtl [21] first proposed a singularly perturbed model to describe the two-time-scale system when studying the fluid dynamic systems. In 1968, Kokotovic and Sannuti [22] used this new model to describe system dynamics with different timescales and proposed a fast-slow combination control strategy, which established the basic control strategy of two-time-scale dynamic system. The singular perturbation method [23, 24] is a major means of study used to understand more about singularly perturbed systems. The core idea is to decompose this specific system into fast-time-scale and slow-time-scale subsystems. Specifically, it is assumed that the slow variable remains unchanged during the response period of the fast variable. When analyzing the response of the slow variable, it is considered that the fast variable has reached stability state value. Many researchers had studied issues related to the singularly perturbed system and method (e.g., feedback control of the two-time-scale system [25]). Su et al. [26] first studied the controllability of discrete-time first-order MASs with the two-time-scale feature and obtained necessary and/or sufficient controllability criteria based on the matrix theory. Furthermore, the controllability of continuous-time and discrete-time second-order MASs with the two-time-scale feature is discussed in [27, 28], respectively.

However, few studies have considered the controllability of MASs with both heterogeneous and two-time-scale features and only one study, conducted by Long et al. [29], considers such features with continuous-time. With the increasing development of cyber-physical systems, discussions in discrete-time have been ascending more and more. Inspired by this, this paper focuses on the controllability of discrete-time leader-follower MASs with heterogeneous and two-time-scale features and tries to solve this research gap that requires more attention. The essential difference between discrete-time systems and continuous-time systems will lead to different modelling methods as well as different transformation derivation methods instead of a simple generalization. The research content of this article has the potential to supplement existing results in this research field. Specifically, the significance and innovation of this research are summarized as follows:

- (1) The definition of controllability of discrete-time leader-follower MASs with heterogeneous and two-time-scale features is proposed for the first time.
- (2) A singularly perturbed difference system is used to model the discrete-time leader-follower MASs with heterogeneous and two-time-scale features, and the singular perturbation method is applied to decouple the model. This is done to avoid the ill-posedness problem when the classical control method is directly used to study the controllability of these systems.

- (3) Several easier-to-use necessary and/or sufficient conditions for the controllability of discrete-time leader-follower MASs were obtained with heterogeneous and two-time-scale features.

The rest of this paper is arranged as follows. Section 2 gives the preliminary knowledge, describes the problem to be studied, and models the discrete-time leader-follower MASs with heterogeneous and two-time-scale features. In Section 3, we first decompose the systems by the singular perturbation method to eliminate the singular perturbation parameter. Then, we define the controllability of the discrete-time leader-follower MASs with two-time-scale and heterogeneous features, and several necessary and/or sufficient conditions for controllability are stated. The effectiveness of the proposed criterion is verified by the simulation in Section 4. Conclusions are drawn in Section 5.

2. Preliminaries and Problem Formulation

2.1. Preliminaries. A graph $G(V, E)$ composed of the vertex set V and edge set E can be used to represent a concrete MAS abstractly. Each agent is represented by a vertex, and the information interaction between agents is represented as an edge. The number of vertices is recorded as $N = |V|$, and the number of edges is recorded as $M = |E|$. Each edge in the set E has a pair of vertices in the set V corresponding to it. If any vertex pair (i, j) and (j, i) correspond to the same edge, the graph is labelled as an undirected graph. If not, it is called a directed graph. The adjacency matrix $A = [a_{ij}]$ is usually used to represent the topological structure of the graph. If there is an edge from vertex j to vertex i , then $a_{ij} \neq 0$ and a neighbor of i is j . Otherwise, $a_{ij} = 0$. All neighbor vertices of the vertex i are recorded as N_i . If the graph G is undirected, then A is a symmetric matrix. The Laplacian matrix of the graph G is defined as $L = D - A$, where D is the degree matrix of G . Because the sum of each row of the Laplacian matrix L is 0, regardless of the graph being an undirected or a directed one, 0 is the eigenvalue of L , and the corresponding eigenvector is $\mathbf{1}$ (a column vector where every element is 1). The Laplacian matrix corresponding to the undirected graph is a positive semidefinite matrix, and the algebraic multiple of its eigenvalue 0 is the number of connected components of the graph.

Lemma 1 (see [30]). *If all eigenvalues $\lambda(e^A)$ of e^A satisfying $|\lambda(e^A)| < 1$, then when $\varepsilon \rightarrow 0$, there is*

$$\varepsilon \sum_{p=0}^{[1/\varepsilon]-1} (I + \varepsilon A)^{[1/\varepsilon]-1-p} = \int_0^1 e^{At} dt + O(\varepsilon), \quad (1)$$

where A is a constant matrix with appropriate dimensions.

Lemma 2 (see [30]). *If all eigenvalues $\lambda(e^A)$ of e^A satisfying $|\lambda(e^A)| < 1$, then when $\varepsilon \rightarrow 0$, there is*

$$(I + \varepsilon A)^{[1/\varepsilon]} \sim e^A + O(\varepsilon), \quad (2)$$

where A is a constant matrix with appropriate dimensions.

Next, $\mathbb{R}$, $\mathbb{C}$, and I are used to represent the real number set, the complex number set, and the identity matrix with a suitable dimension. The symbol $\otimes$ denotes the Kronecker product, and $\varepsilon (0 < \varepsilon \leq 1)$ is a singular perturbation parameter used to distinguish two timescales.

2.2. Problem Formulation. The discrete-time MAS with heterogeneous and two-time-scale features under a leader-follower framework in consideration is described below. First of all, the heterogeneity reflects that $m + m_l$ agents in the MAS are first-order integrators, and the remaining $n + n_l$ agents are second-order integrators. m_l represents the number of leaders among the first-order integrators agents, and m is the number of followers in the first-order integrators agent cluster. Similarly, n_l and n are the numbers of leaders and followers among the second-order integrator agents, respectively. In this way, the whole system is divided into four parts: first-order follower agent cluster, first-order leader agent cluster, second-order follower agent cluster, and second-order leader agent cluster, which are represented by $G(f_1)$, $G(l_1)$, $G(f_2)$, and $G(l_2)$ in Figure 1. The interaction between all agents can be expressed by the following matrix L corresponding to the entire MAS:

$$L = [l_{ij}] = \begin{bmatrix} L_{f_1 f_1} & L_{f_1 l_1} & L_{f_1 f_2} & L_{f_1 l_2} \\ L_{l_1 f_1} & L_{l_1 l_1} & L_{l_1 f_2} & L_{l_1 l_2} \\ L_{f_2 f_1} & L_{f_2 l_1} & L_{f_2 f_2} & L_{f_2 l_2} \\ L_{l_2 f_1} & L_{l_2 l_1} & L_{l_2 f_2} & L_{l_2 l_2} \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix},$$

where $l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j \in N_i} a_{ij}, & i = j \end{cases}$ represents the elements of the matrix L and $L_{p,q} (p, q \in \{f_1, f_2, l_1, l_2\})$ is the block matrix of L , which embodies the connection between $G(p)$ and $G(q)$ if $(p \neq q)$ or within $G(p)$ (if $p = q$).

Secondly, each first-order agent i operates on two different timescales simultaneously. Specifically, vectors $x_i \in \mathbb{R}^{n_x \times 1}$ and $z_i \in \mathbb{R}^{n_z \times 1}$ are used to represent the position state vectors of the first-order agent i on the slow-time-scale and the fast-time-scale. Each second-order agent o has vectors $x_o \in \mathbb{R}^{n_x \times 1}$ and $w_o \in \mathbb{R}^{n_x \times 1}$ to represent the position and velocity states of slow-time-scale while vectors $z_o \in \mathbb{R}^{n_z \times 1}$ and $d_o \in \mathbb{R}^{n_z \times 1}$ are used to represent the position

FIGURE 1: Legend for discrete-time leader-follower MAS with two-time-scale and heterogeneous features.

and velocity states of fast-time-scale. The dynamic model of all agents in this MAS is modelled by

$$\begin{cases} x_i(n+1) = x_i(n) + \varepsilon B_1 u_i(n), \\ z_i(n+1) = z_i(n) + B_2 u_i(n), \end{cases} \quad i \in g_1, \quad (4a)$$

$$\begin{cases} x_o(n+1) = x_o(n) + \varepsilon w_o(n), \\ w_o(n+1) = w_o(n) + \varepsilon B_1 u_o(n), \\ z_o(n+1) = z_o(n) + d_o(n), \\ d_o(n+1) = d_o(n) + B_2 u_o(n), \end{cases} \quad o \in g_2, \quad (4b)$$

where $g_1 \triangleq \{1, 2, \dots, m + m_l\}$ and $g_2 \triangleq \{m + m_l + 1, m + m_l + 2, \dots, m + m_l + n + n_l\}$. $u_i(t) \in \mathbb{R}^{q \times 1}$ and $u_o(t) \in \mathbb{R}^{q \times 1}$ are control inputs. Here, the input matrices of two different timescales are $B_1 \in \mathbb{R}^{n_x \times q}$ and $B_2 \in \mathbb{R}^{n_z \times q}$.

Inspired by the consensus protocol, the following communication protocol for MAS (4a) and (4b) was designed:

$$u_i = F_1 \sum_{j \in N_i} a_{ij} (x_j(n) - x_i(n)) + F_2 \sum_{j \in N_i} a_{ij} (z_j(n) - z_i(n)), \quad i \in g_1, \quad (5)$$

$$\begin{aligned} u_o = & F_1 \sum_{j \in N_o} a_{oj} (x_j(n) - x_o(n)) + F_2 \sum_{j \in N_o} a_{oj} (z_j(n) - z_o(n)) \\ & + E_1 \sum_{j \in N_o} a_{oj} (w_j(n) - w_o(n)) + E_2 \sum_{j \in N_o} a_{oj} (d_j(n) - d_o(n)), \quad o \in g_2, \end{aligned} \quad (6)$$

with $F_1 \in \mathbb{R}^{q \times n_x}$, $F_2 \in \mathbb{R}^{q \times n_z}$, $E_1 \in \mathbb{R}^{q \times n_x}$, and $E_2 \in \mathbb{R}^{q \times n_z}$ representing the position state coupling matrices and the velocity state coupling matrices, accordingly.

Let $x_{f1} = (x_1^T, x_2^T, \dots, x_m^T)^T \in \mathbb{R}^{m n_x \times 1}$, $x_{f2} = (x_{m+m_l+1}^T, x_{m+m_l+2}^T, \dots, x_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_x \times 1}$, $x_{l1} = (x_{m+1}^T, x_{m+2}^T, \dots, x_{m+m_l}^T)^T \in \mathbb{R}^{m_l n_x \times 1}$, $w_{l2} = (w_{m+m_l+1}^T, w_{m+m_l+2}^T, \dots, w_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_x \times 1}$, $z_{f1} = (z_1^T, z_2^T, \dots, z_m^T)^T \in \mathbb{R}^{m n_z \times 1}$, $x_{l2} = (x_{m+m_l+1}^T, x_{m+m_l+2}^T, \dots, x_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_x \times 1}$, z_{f2}

$= (z_{m+m_l+1}^T, z_{m+m_l+2}^T, \dots, z_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_z \times 1}$, $z_{l2} = (z_{m+m_l+1}^T, z_{m+m_l+2}^T, \dots, z_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_z \times 1}$, $w_{f2} = (w_{m+m_l+1}^T, w_{m+m_l+2}^T, \dots, w_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_x \times 1}$, $d_{f2} = (d_{m+m_l+1}^T, d_{m+m_l+2}^T, \dots, d_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_z \times 1}$, $d_{l2} = (d_{m+m_l+1}^T, d_{m+m_l+2}^T, \dots, d_{m+m_l+n}^T)^T \in \mathbb{R}^{n n_z \times 1}$, $z_{l1} = (z_{m+1}^T, z_{m+2}^T, \dots, z_{m+m_l}^T)^T \in \mathbb{R}^{m_l n_z \times 1}$. By sorting out formulas (4a), (4b)–(6), one has

$$\begin{aligned}
\begin{bmatrix} x_{f1}(n+1) \\ x_{f2}(n+1) \\ w_{f2}(n+1) \\ z_{f1}(n+1) \\ z_{f2}(n+1) \\ d_{f2}(n+1) \end{bmatrix} &= \begin{bmatrix} I + \varepsilon\theta_{11} & \varepsilon\theta_{12} & \mathbf{0} & \varepsilon\theta_{14} & \varepsilon\theta_{15} & \mathbf{0} \\ \mathbf{0} & I & \varepsilon I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \varepsilon\theta_{31} & \varepsilon\theta_{32} & I + \varepsilon\theta_{33} & \varepsilon\theta_{34} & \varepsilon\theta_{35} & \varepsilon\theta_{36} \\ \theta_{41} & \theta_{42} & \mathbf{0} & I + \theta_{44} & \theta_{45} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I & I \\ \theta_{61} & \theta_{62} & \theta_{63} & \theta_{64} & \theta_{65} & I + \theta_{66} \end{bmatrix} \times \begin{bmatrix} x_{f1}(n) \\ x_{f2}(n) \\ w_{f2}(n) \\ z_{f1}(n) \\ z_{f2}(n) \\ d_{f2}(n) \end{bmatrix} \\
&+ \begin{bmatrix} \varepsilon\vartheta_{11} & \varepsilon\vartheta_{12} & \mathbf{0} & \varepsilon\vartheta_{14} & \varepsilon\vartheta_{15} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \varepsilon\vartheta_{31} & \varepsilon\vartheta_{32} & \varepsilon\vartheta_{33} & \varepsilon\vartheta_{34} & \varepsilon\vartheta_{35} & \varepsilon\vartheta_{36} \\ \vartheta_{41} & \vartheta_{42} & \mathbf{0} & \vartheta_{44} & \vartheta_{45} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vartheta_{61} & \vartheta_{62} & \vartheta_{63} & \vartheta_{64} & \vartheta_{65} & \vartheta_{66} \end{bmatrix} \times \begin{bmatrix} x_{l1}(n) \\ x_{l2}(n) \\ w_{l2}(n) \\ z_{l1}(n) \\ z_{l2}(n) \\ d_{l2}(n) \end{bmatrix}, \quad (7)
\end{aligned}$$

where $\theta_{11} = -L_{11} \otimes B_1 F_1$, $\theta_{12} = -L_{13} \otimes B_1 F_1$, $\theta_{14} = -L_{11} \otimes B_1 F_2$, $\theta_{15} = -L_{13} \otimes B_1 F_2$, $\theta_{31} = -L_{31} \otimes B_1 F_1$, $\theta_{32} = -L_{33} \otimes B_1 F_1$, $\theta_{33} = -L_{33} \otimes B_1 E_1$, $\theta_{34} = -L_{31} \otimes B_1 F_2$, $\theta_{35} = -L_{33} \otimes B_1 F_2$, $\theta_{36} = -L_{33} \otimes B_1 E_2$, $\theta_{41} = -L_{11} \otimes B_2 F_1$, $\theta_{42} = -L_{13} \otimes B_2 F_1$, $\theta_{44} = -L_{11} \otimes B_2 F_2$, $\theta_{45} = -L_{13} \otimes B_2 F_2$, $\theta_{61} = -L_{31} \otimes B_2 F_1$, $\theta_{62} = -L_{33} \otimes B_2 F_1$, $\theta_{63} = -L_{33} \otimes B_2 E_1$, $\theta_{64} = -L_{31} \otimes B_2 F_2$, $\theta_{65} = -L_{33} \otimes B_2 F_2$, $\theta_{66} = -L_{33} \otimes B_2 E_2$, $\vartheta_{11} = -L_{12} \otimes B_1 F_1$, $\vartheta_{12} = -L_{14} \otimes B_1 F_1$, $\vartheta_{14} = -L_{12} \otimes B_1 F_2$, $\vartheta_{15} = -L_{14} \otimes B_1 F_2$, $\vartheta_{31} = -L_{32} \otimes B_1 F_1$, $\vartheta_{32} = -L_{34} \otimes B_1 F_1$, $\vartheta_{33} = -L_{34} \otimes B_1 E_1$, $\vartheta_{34} = -L_{32} \otimes B_1 F_2$, $\vartheta_{35} = -L_{34} \otimes B_1 F_2$, $\vartheta_{36} = -L_{34} \otimes B_1 E_2$, $\vartheta_{41} = -L_{12} \otimes B_2 F_1$, $\vartheta_{42} = -L_{14} \otimes B_2 F_1$, $\vartheta_{44} = -L_{12} \otimes B_2 F_2$, $\vartheta_{45} = -L_{14} \otimes B_2 F_2$, $\vartheta_{61} = -L_{32} \otimes B_2 F_1$, $\vartheta_{62} = -L_{34} \otimes B_2 F_1$, $\vartheta_{63} = -L_{34} \otimes B_2 E_1$, $\vartheta_{64} = -L_{32} \otimes B_2 F_2$, $\vartheta_{65} = -L_{34} \otimes B_2 F_2$, and $\vartheta_{66} = -L_{34} \otimes B_2 E_2$.

Then, using $r_1 = (x_{f1}^T, x_{f2}^T, w_{f2}^T)^T \in \mathbb{R}^{(m_{n_x} + 2m_{n_z}) \times 1}$, $r_2 = (z_{f1}^T, z_{f2}^T, d_{f2}^T)^T \in \mathbb{R}^{(m_{n_z} + 2m_{n_x}) \times 1}$, and $u = (x_{l1}^T, x_{l2}^T, w_{l2}^T, z_{l1}^T, z_{l2}^T, d_{l2}^T)^T \in \mathbb{R}^{((m_l + 2n_l)n_x + (m_l + 2n_l)n_z) \times 1}$ yields

$$\{r_1(n+1) = (I + \varepsilon\varphi_{11})r_1(n) + \varepsilon\varphi_{12}r_2(n) + \varepsilon\phi_1 u, \quad (8a)$$

$$\{r_2(n+1) = \varphi_{21}r_1(n) + (I + \varphi_{22})r_2(n) + \phi_2 u, \quad (8b)$$

where

$$\begin{aligned}
\varphi_{11} &= \begin{bmatrix} \theta_{11} & \theta_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ \theta_{31} & \theta_{32} & \theta_{33} \end{bmatrix}, \\
\varphi_{12} &= \begin{bmatrix} \theta_{14} & \theta_{15} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \theta_{34} & \theta_{35} & \theta_{36} \end{bmatrix}, \\
\varphi_{21} &= \begin{bmatrix} \theta_{41} & \theta_{42} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \theta_{61} & \theta_{62} & \theta_{63} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\varphi_{22} &= \begin{bmatrix} \theta_{44} & \theta_{45} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ \theta_{64} & \theta_{65} & \theta_{66} \end{bmatrix}, \\
\phi_1 &= \begin{bmatrix} \vartheta_{11} & \vartheta_{12} & \mathbf{0} & \vartheta_{14} & \vartheta_{15} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vartheta_{31} & \vartheta_{32} & \vartheta_{33} & \vartheta_{34} & \vartheta_{35} & \vartheta_{36} \end{bmatrix}, \\
\phi_2 &= \begin{bmatrix} \vartheta_{41} & \vartheta_{42} & \mathbf{0} & \vartheta_{44} & \vartheta_{45} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vartheta_{61} & \vartheta_{62} & \vartheta_{63} & \vartheta_{64} & \vartheta_{65} & \vartheta_{66} \end{bmatrix}. \quad (9)
\end{aligned}$$

3. Main Results

3.1. Decomposition of Singularly Perturbed Systems. In view of the existence of the singular perturbation parameter ε representing different timescales, systems (4a) and (4b) are labelled as a singularly perturbed system. If the classical controllability theory is used to process this system, it will cause the ill-posedness problem. Inspired by [25], systems (4a) and (4b) should first be decomposed to eliminate the singular perturbation parameter ε so that it can be discussed with the traditional controllability research method. Since ε exists in the slow-time-scale equation, n is the timescale of the fast-time-scale subsystem. If l is used to represent the timescale of the slow-time-scale subsystem, then

$$n = \left[\frac{l}{\varepsilon} \right], \quad l = 1, 2, \dots \quad (10)$$

Accordingly, the first attempt was to decouple the equation of the slow-time-scale subsystem. In this process, a reasonable assumption is that the states of the fast-time-scale subsystem have reached steady-states. Based on the matrix φ_{22} is invertible, formula (8b) can be written as

$$r_2(n) = \varphi_{21}r_1(n) + (I + \varphi_{22})r_2(n) + \phi_2u(n), \quad (11)$$

which further leads to

$$r_2(n) = -\varphi_{22}^{-1}\varphi_{21}r_1(n) - \varphi_{22}^{-1}\phi_2u(n). \quad (12)$$

Substituting (12) into (8a) yields

$$r_1(n+1) = [I + \varepsilon(\varphi_{11} - \varphi_{12}\varphi_{22}^{-1}\varphi_{21})]r_1(n) + \varepsilon(\phi_1 - \varphi_{12}\varphi_{22}^{-1}\phi_2)u(n). \quad (13)$$

Let $\varphi_s = \varphi_{11} - \varphi_{12}\varphi_{22}^{-1}\varphi_{21}$ and $\phi_s = \phi_1 - \varphi_{12}\varphi_{22}^{-1}\phi_2$. Simultaneously, the states of the agents and the input signals of the slow-time-scale subsystem are represented as $r_s(n)$ and $u_s(n)$. So,

$$r_s(n+1) = (I + \varepsilon\varphi_s)r_s(n) + \varepsilon\phi_su_s(n). \quad (14)$$

From the perspective of the timescale l , the control input signal of the slow-time-scale subsystem remains unchanged during the period $l[1/\varepsilon] \leq n < (l+1)[1/\varepsilon]$, that is $u_s(n) = \hat{u}_s(l[1/\varepsilon])$. Based on this, combined with formula (14), it gives

$$\begin{aligned} r_s\left(l\left[\frac{1}{\varepsilon}\right] + 1\right) &= (I + \varepsilon\varphi_s)r_s\left(l\left[\frac{1}{\varepsilon}\right]\right) + \varepsilon\phi_s\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right]\right), \\ r_s\left(l\left[\frac{1}{\varepsilon}\right] + 2\right) &= (I + \varepsilon\varphi_s)r_s\left(l\left[\frac{1}{\varepsilon}\right] + 1\right) + \varepsilon\phi_s\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right] + 1\right) \\ &= (I + \varepsilon\varphi_s)^2r_s\left(l\left[\frac{1}{\varepsilon}\right]\right) + \varepsilon(I + \varepsilon\varphi_s)\phi_s\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right]\right) \\ &\quad + \varepsilon\phi_s\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right]\right), \\ &\dots \\ r_s\left((l+1)\left[\frac{1}{\varepsilon}\right]\right) &= (I + \varepsilon\varphi_s)^{[1/\varepsilon]}r_s\left(l\left[\frac{1}{\varepsilon}\right]\right) \\ &\quad + \varepsilon \sum_{p=0}^{[1/\varepsilon]-1} (I + \varepsilon\varphi_s)^{[1/\varepsilon]-1-p}\phi_s\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right]\right). \end{aligned} \quad (15)$$

Using Lemmas 1 and 2 when the value $n = [1/\varepsilon]$ is large enough, and letting $\hat{\varphi}_s$ and $\hat{\phi}_s$ stand for e^{φ_s} and $\int_0^1 e^{\varphi_s t} dt \phi_s$ yields

$$r_s\left((l+1)\left[\frac{1}{\varepsilon}\right]\right) = (\hat{\varphi}_s + O(\varepsilon))r_s\left(l\left[\frac{1}{\varepsilon}\right]\right) + (\hat{\phi}_s + O(\varepsilon))\hat{u}_s\left(l\left[\frac{1}{\varepsilon}\right]\right). \quad (16)$$

The decoupled slow-time-scale subsystem becomes

$$\hat{r}_s(l+1) = \hat{\varphi}_s\hat{r}_s(l) + \hat{\phi}_s\hat{u}_s(l), \quad (17a)$$

by letting $\hat{r}_s(l) = r_s(l[1/\varepsilon])$ and ignoring the infinitesimal term.

Secondly, the decoupled fast-time-scale subsystem is obtained as follows:

$$r_f(n+1) = \varphi_f r_f(n) + \phi_f u_f(n), \quad (17b)$$

where $r_f(n) = r_2(n) + \varphi_{22}^{-1}\varphi_{21}r_1(n)$, $\varphi_f = I + \varphi_{22}$, $\phi_f = \phi_2$, and $u_f = u$.

Let us describe the decomposition process in detail. During this process, the state vector r_1 of (8a) and (8b) can be assumed to remain unchanged. By adding $\varphi_{22}^{-1}\varphi_{21}r_1(n)$ to both sides of equation (8b), it gives

$$\begin{aligned} r_2(n+1) + \varphi_{22}^{-1}\varphi_{21}r_1(n) &= r_2(n+1) + \varphi_{22}^{-1}\varphi_{21}r_1(n+1) \\ &= \varphi_{21}r_1(n) + (I + \varphi_{22})r_2(n) + \phi_2u + \varphi_{22}^{-1}\varphi_{21}r_1(n) \\ &= (\varphi_{22} + I)\varphi_{22}^{-1}\varphi_{21}r_1(n) + (I + \varphi_{22})r_2(n) + \phi_2u \\ &= (I + \varphi_{22})(r_2(n) + \varphi_{22}^{-1}\varphi_{21}r_1(n)) + \phi_2u. \end{aligned} \quad (18)$$

So far, formula (17b) can be obtained.

3.2. Controllability Analysis

Definition 1. For the discrete-time leader-follower MAS with two-time-scale and heterogeneous features as (4a) and (4b) discussed in this paper, if any nonzero states $\hat{r}_s$ and r_f of systems (17a) and (17b) meet the following conditions simultaneously, then systems (4a) and (4b) can be said to be controllable:

- (1) For the initial nonzero state $\hat{r}_s(0) = \hat{r}_s$, there exists a piecewise input $\hat{u}_s$ so that it can reach the zero state within a finite time T_1 (i.e., $\hat{r}_s(T_1) = 0$).
- (2) For the initial nonzero state $r_f(0) = r_f$, there exists a piecewise input u_f so that it can reach the zero state within a finite time T_2 (i.e., $r_f(T_2) = 0$).

In the equation, $T_2 \ll T_1$ and $T_1(T_2) \in \{1, 2, \dots\}$.

Lemma 3. The discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable if and only if the controllability matrices corresponding to two subsystems (17a) and (17b) with different timescales are all full row rank, i.e., $\text{rank}[\hat{\varphi}_s, \hat{\varphi}_s^2, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\varphi}_s] = (m+2n) \times n_x$ and $\text{rank}[\phi_f, \phi_f \varphi_f, \dots, \phi_f^{(m+2n) \times n_z - 1} \phi_f] = (m+2n) \times n_z$.

Proof. From Definition 1 and the Kalman rank criterion [5], the assertion is obvious. $\square$

Theorem 1. The discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable if and only if one of the following statements is satisfied:

- (1) Let λ_s and μ_f represent the eigenvalues of matrices $\hat{\varphi}_s$ and φ_f , respectively, then $\text{rank}[\lambda_s I - \hat{\varphi}_s, \hat{\varphi}_s] = (m+2n) \times n_x$ and $\text{rank}[\mu_f I - \varphi_f, \phi_f] = (m+2n) \times n_z$, where $s \in \{1, 2, \dots, (m+2n) \times n_x\}$ and $f \in \{1, 2, \dots, (m+2n) \times n_z\}$.
- (2) The vectors $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$ and $\zeta \in \mathbb{C}^{(m+2n)n_z \times 1}$ satisfying $\varsigma^T \hat{\varphi}_s = \sigma \varsigma^T$, $\varsigma^T \hat{\varphi}_s = \mathbf{0}$, $\zeta^T \varphi_f = \gamma \zeta^T$, and $\zeta^T \phi_f = \mathbf{0}$ must be zero vectors, where $\sigma \in \mathbb{C}$ and $\gamma \in \mathbb{C}$.

Proof of Proposition (1). Necessity: one has to prove that if MAS (4a) and (4b) are controllable, then $\text{rank}[\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] = (m+2n) \times n_x$ and $\text{rank}[\mu_f I - \varphi_f, \phi_f] = (m+2n) \times n_z$, where λ_s and μ_f represent the eigenvalues of matrices $\hat{\varphi}_s$ and φ_f , respectively. Let us look at its contrapositive proposition. By contradiction, suppose there is an eigenvalue λ_s of $\hat{\varphi}_s$ such that $\text{rank}[\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] < (m+2n) \times n_x$, which means that the rows of the matrix $[\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s]$ are linearly dependent. Consequently, there must exist a nonzero vector ρ such that

$$\rho^T [\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] = \mathbf{0}, \quad (19)$$

that is,

$$\begin{aligned} \rho^T \hat{\phi}_s &= \mathbf{0}, \\ \lambda_s \rho^T &= \rho^T \hat{\varphi}_s. \end{aligned} \quad (20)$$

Furthermore,

$$\begin{aligned} \rho^T \hat{\phi}_s &= \mathbf{0}, \\ \rho^T \hat{\varphi}_s \hat{\phi}_s &= \mathbf{0}, \dots, \rho^T \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s = \mathbf{0}. \end{aligned} \quad (21)$$

Then,

$$\rho^T [\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] = \mathbf{0}. \quad (22)$$

Since the vector ρ is nonzero, we obtain $\text{rank}[\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] < (m+2n) \times n_x$. A similar conclusion can be drawn for the matrix $[\phi_f, \varphi_f \phi_f, \dots, \varphi_f^{(m+2n) \times n_z - 1} \phi_f]$. According to Lemma 3, it can be concluded that MAS (4a) and (4b) are uncontrollable. Because the contrapositive proposition of the original proposition is true, the original proposition is also true. The necessity is proved.

Sufficiency: with λ_s and μ_f representing the eigenvalues of matrices $\hat{\varphi}_s$ and φ_f , our purpose is to prove that if $\text{rank}[\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] = (m+2n) \times n_x$ and $\text{rank}[\mu_f I - \varphi_f, \phi_f] = (m+2n) \times n_z$, then MAS (4a) and (4b) are controllable. Let us look at its contrapositive proposition. By contradiction, suppose that MAS (4a) and (4b) are uncontrollable. So, $\text{rank}[\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] < (m+2n) \times n_x$. Then, there exists a nonzero left eigenvector ρ^T corresponding to the eigenvalue λ_s of the matrix $\hat{\varphi}_s$, so that

$$\rho^T [\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] = \mathbf{0}. \quad (23)$$

Then,

$$\begin{aligned} \rho^T \hat{\phi}_s &= \mathbf{0}, \\ \rho^T \hat{\varphi}_s \hat{\phi}_s &= \lambda_s \rho^T \hat{\phi}_s = \mathbf{0}, \\ &\dots \\ \rho^T \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s &= \mathbf{0}. \end{aligned} \quad (24)$$

Motivated by (24), there holds $\rho^T [\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] = \mathbf{0}$. This means $\text{rank}[\lambda_s I - \hat{\varphi}_s, \hat{\phi}_s] < (m+2n) \times n_x$. A similar conclusion can be drawn that $\text{rank}[\mu_f I - \varphi_f, \phi_f] <$

$(m+2n) \times n_z$. Because the contrapositive proposition of the original proposition is true, the original proposition is also true. The sufficiency is proved. $\square$

Proof of Proposition (2). Necessity: our purpose is to prove that if MAS (4a) and (4b) are controllable, the vectors ς and ζ that satisfy $\varsigma^T \hat{\varphi}_s = \sigma \varsigma^T$, $\varsigma^T \hat{\phi}_s = \mathbf{0}$, $\zeta^T \varphi_f = \gamma \zeta^T$, and $\zeta^T \phi_f = \mathbf{0}$ must be zero vectors, where $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$ and $\zeta \in \mathbb{C}^{(m+2n)n_z \times 1}$. Let us look at its contrapositive proposition. Suppose that there is a nonzero vector $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$ and $\sigma \in \mathbb{C}$ satisfy $\varsigma^T \hat{\varphi}_s = \sigma \varsigma^T$ and $\varsigma^T \hat{\phi}_s = \mathbf{0}$; then,

$$\begin{aligned} \varsigma^T \hat{\phi}_s &= \mathbf{0}, \\ \varsigma^T \hat{\varphi}_s \hat{\phi}_s &= \sigma \varsigma^T \hat{\phi}_s = \mathbf{0}, \\ \varsigma^T \hat{\varphi}_s^2 \hat{\phi}_s &= \sigma \varsigma^T \hat{\varphi}_s \hat{\phi}_s = \mathbf{0}, \\ &\dots \\ \varsigma^T \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s &= \mathbf{0}, \end{aligned} \quad (25)$$

which further results in

$$\varsigma^T [\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] = \mathbf{0}. \quad (26)$$

Since the vector ς is nonzero, we obtain $\text{rank}[\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] < (m+2n) \times n_x$. A similar conclusion can be drawn for the matrix $[\phi_f, \varphi_f \phi_f, \dots, \varphi_f^{(m+2n) \times n_z - 1} \phi_f]$, and then MAS (4a) and (4b) are uncontrollable. Because the contrapositive proposition of the original proposition is true, the original proposition is also true. The necessity is proved.

Sufficiency: we want to prove that if the vectors ς and ζ that satisfy $\varsigma^T \hat{\varphi}_s = \sigma \varsigma^T$, $\varsigma^T \hat{\phi}_s = \mathbf{0}$, $\zeta^T \varphi_f = \gamma \zeta^T$, and $\zeta^T \phi_f = \mathbf{0}$ are all zero vectors, then MAS (4a) and (4b) are controllable, where $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$, $\zeta \in \mathbb{C}^{(m+2n)n_z \times 1}$, $\sigma \in \mathbb{C}$, and $\gamma \in \mathbb{C}$. Let us look at its contrapositive proposition. By contradiction, suppose that MAS (4a) and (4b) are uncontrollable. So, $\text{rank}[\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n) \times n_x - 1} \hat{\phi}_s] < (m+2n) \times n_x$. Similar to the sufficiency proof of Proposition (1), we can find that there must be a nonzero vector $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$ and $\sigma \in \mathbb{C}$ such that $\varsigma^T \hat{\varphi}_s = \sigma \varsigma^T$ and $\varsigma^T \hat{\phi}_s = \mathbf{0}$. Similar conclusions can be drawn about the matrices φ_f and ϕ_f . Because the contrapositive proposition of the original proposition is true, the original proposition is also true. The sufficiency is proved. $\square$

Theorem 2. *If the two matrices $\hat{\varphi}_s$ and φ_f do not have the same eigenvalues with Q, the discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable, where*

$$Q = \begin{bmatrix} \hat{\varphi}_s & \hat{\phi}_s & \mathbf{0} & \mathbf{0} \\ \hat{\phi}_s^T & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varphi_f & \phi_f \\ \mathbf{0} & \mathbf{0} & \phi_f^T & I \end{bmatrix}. \quad (27)$$

Proof. Let us look at its contrapositive proposition. Suppose that the slow-time-scale subsystem of MAS (4a) and (4b) is uncontrollable. So, under Proposition (2) of Theorem 1, it

gives that there exists a nonzero vector $\varsigma \in \mathbb{C}^{(m+2n)n_x \times 1}$ such that $\varsigma^T \hat{\varphi}_s = \alpha \varsigma^T$ and $\varsigma^T \hat{\phi}_s = \mathbf{0}$, where α is an eigenvalue of the matrix $\hat{\varphi}_s$.

Defining a new vector $[\varsigma^T, \mathbf{0}, \mathbf{0}, \mathbf{0}]$, then

$$\begin{aligned} [\varsigma^T \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}]Q &= [\varsigma^T \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}] \begin{bmatrix} \hat{\varphi}_s & \hat{\phi}_s & \mathbf{0} & \mathbf{0} \\ \hat{\phi}_s^T & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varphi_f & \phi_f \\ \mathbf{0} & \mathbf{0} & \phi_f^T & I \end{bmatrix} \\ &= [\varsigma^T \hat{\varphi}_s \ \varsigma^T \hat{\phi}_s \ \mathbf{0} \ \mathbf{0}] \\ &= [\alpha \varsigma^T \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}] \\ &= \alpha [\varsigma^T \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}]. \end{aligned} \quad (28)$$

Obviously, $\hat{\varphi}_s$ and Q have the same eigenvalue α . Because the contrapositive proposition of the original proposition is true, the original proposition is also true. A similar conclusion

about the matrices φ_f and Q can be obtained with a similar proof process, so it is omitted here. Theorem 2 is proved. $\square$

Theorem 3. The discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable if the eigenvalues of $\hat{\varphi}_s$ and φ_f are all different, and all rows of matrices P^{-1} and R^{-1} are not orthogonal to at least one column of matrices $\hat{\phi}_s$ and ϕ_f , respectively, where P and R are, respectively, composed of the eigenvectors of matrices $\hat{\varphi}_s$ and φ_f .

Proof. According to the knowledge of matrix theory, since the eigenvalues of $\hat{\varphi}_s$ are different, $\hat{\varphi}_s$ can be similarly diagonalized. Let $\lambda_s (\forall s \in \{1, 2, \dots, (m+2n)n_x\})$ represent the eigenvalues of $\hat{\varphi}_s$ and P as an invertible matrix composed of corresponding eigenvectors, it follows that $\hat{\varphi}_s = P\Lambda P^{-1}$ with $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{(m+2n)n_x}\}$.

Then,

$$\begin{aligned} [\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n)n_x-1} \hat{\phi}_s] &= [PP^{-1}\hat{\phi}_s, P\Lambda P^{-1}\hat{\phi}_s, \dots, P\Lambda^{(m+2n)n_x-1}P^{-1}\hat{\phi}_s] \\ &= P[P^{-1}\hat{\phi}_s, \Lambda P^{-1}\hat{\phi}_s, \dots, \Lambda^{(m+2n)n_x-1}P^{-1}\hat{\phi}_s]. \end{aligned} \quad (29)$$

Let $\Gamma = [P^{-1}\hat{\phi}_s, \Lambda P^{-1}\hat{\phi}_s, \dots, \Lambda^{(m+2n)n_x-1}P^{-1}\hat{\phi}_s]$, then motivated by that P is an invertible matrix, there holds $\text{rank}[\hat{\phi}_s, \hat{\varphi}_s \hat{\phi}_s, \dots, \hat{\varphi}_s^{(m+2n)n_x-1} \hat{\phi}_s] = \text{rank}\Gamma$. By expanding

$P^{-1}\hat{\phi}_s$ as $[\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T$ with for $\eta_i = [\eta_{1i}, \eta_{2i}, \dots, \eta_{(m+2n)n_x+i}]^T$ $\forall i \in \{1, 2, \dots, (m_l+2n_l)n_x + (m_l+2n_l)n_z\}$, one has

$$\Gamma = \begin{bmatrix} [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T, \Lambda [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T, \\ \dots, \Lambda^{(m+2n)n_x-1} [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T \end{bmatrix}. \quad (30)$$

Based on the conclusion of matrix theory that the elementary transformation of matrix does not change the rank,

$$\begin{aligned} \text{rank}\Gamma &= \text{rank} \begin{bmatrix} [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T, \Lambda [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T, \\ \dots, \Lambda^{(m+2n)n_x-1} [\eta_1, \eta_2, \dots, \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z}]^T \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} [\eta_1, \Lambda \eta_1, \dots, \Lambda^{(m+2n)n_x-1} \eta_1], [\eta_2, \Lambda \eta_2, \dots, \Lambda^{(m+2n)n_x-1} \eta_2], \dots \\ \dots, \Lambda^{(m+2n)n_x-1} \eta_{(m_l+2n_l)n_x+(m_l+2n_l)n_z} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \text{diag}(\eta_{11}, \eta_{21}, \dots, \eta_{(m+2n)n_x,1})K, \dots \\ \dots, \text{diag} \left(\eta_{1, (m_l+2n_l)n_x+(m_l+2n_l)n_z}, \eta_{2, (m_l+2n_l)n_x+(m_l+2n_l)n_z}, \dots \right) K \end{bmatrix} \end{bmatrix} \quad (31)$$

where

$$K = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{(m+2n)n_x-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{(m+2n)n_x-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{(m+2n)n_x} & \cdots & \lambda_{(m+2n)n_x}^{(m+2n)n_x-1} \end{bmatrix}. \quad (32)$$

Since the eigenvalues of the matrix $\widehat{\varphi}_s$ are different, then K is a full row rank matrix. If we let

$$P^{-1} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{(m+2n)n_x} \end{bmatrix} \in \mathbb{R}^{(m+2n)n_x \times (m+2n)n_x} \quad \text{and} \quad \widehat{\phi}_s = [\phi_1,$$

$\phi_2, \dots, \phi_{(m+2n)n_x + (m+2n)n_z}] \in \mathbb{R}^{(m+2n)n_x \times ((m+2n)n_x + (m+2n)n_z)}$, then for $\forall i \in \{1, 2, \dots, (m+2n)n_x\}$ and $\forall j \in \{1, 2, \dots, (m+2n)n_x + (m+2n)n_z\}$, $\eta_{i,j} = (p_i, \phi_j)$, that is, $\eta_{i,j}$ is the inner product of vectors p_i and ϕ_j . Meanwhile, since all rows of matrices P^{-1} are not orthogonal to at least one column of $\widehat{\phi}_s$, there must exist a diagonal matrix $\text{diag}(\eta_{1i}, \eta_{2i}, \dots, \eta_{(m+2n)n_x, i})$ whose elements are all not zero. Using this information, we can conclude the full row rank of the matrix Γ . According to the previous derivation, this is equivalent to that the row rank of the controllability matrix $[\widehat{\phi}_s, \widehat{\varphi}_s \widehat{\phi}_s, \dots, \widehat{\varphi}_s^{(m+2n)n_x-1} \widehat{\phi}_s]$ is full. Regarding φ_f and ϕ_f , similar results can be obtained.

So far, it has been proved that under the two conditions mentioned above, the discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable. $\square$

Remark 1. Compared with calculation of the rank of controllability matrices, the criteria proposed in Theorems 2 and 3 are considered to be more practical, because they are only related to the eigenvalues of some submatrices of the system matrix and the input matrix in (7). These eigenvalues are relatively simpler to calculate.

Remark 2. The conclusions of Theorems 2 and 3 give the conditions that can satisfy the relevant matrices $\widehat{\varphi}_s$, φ_f , $\widehat{\phi}_s$, and ϕ_f for the systems to be controllable. These matrices are related to $B_1, B_2, F_1, F_2, E_1, E_2$, and a_{ij} of system models (4a) and (4b). Therefore, in essence, given Theorems 2 and 3 can be used to provide specific guidance for the design of the control protocol of the system.

Theorem 4. *If the discrete-time leader-follower MAS with two-time-scale and heterogeneous features (4a) and (4b) is controllable and there is a follower agent of the fast-time-scale subsystem whose indegree is zero, it can be concluded that this follower agent must have control input from leader agents.*

Proof. Assuming that the index of the agent with zero indegree in the follower agent cluster of the fast-time-scale subsystem is i , we discuss two cases according to the type of agent i :

(1) Suppose the agent i is a first-order integrator. Then, $[0, \dots, 0, I_{n_z}, 0, \dots, 0]$ corresponds to the rows $((i-1)n_z + 1)$ th to (in_z) th of φ_f . Let us look at its contrapositive proposition. By contradiction, suppose that this follower agent does not have any input from leader agents, we obtain $((i-1)n_z + 1)$ th to (in_z) th rows of ϕ_f are all zero. Let $\xi_0 \in \mathbb{R}^{n_z \times 1}$ denote an eigenvector of I_{n_z} corresponding to eigenvalue $\lambda = 1$ and let $\xi^T = [0, \dots, 0, \xi_0^T, 0, \dots, 0] \in \mathbb{R}^{1 \times (mn_z + 2mn_z)}$, then it gives

$$\begin{aligned} \xi^T \varphi_f &= [0, \dots, 0, \xi_0^T, 0, \dots, 0] \varphi_f \\ &= [0, \dots, 0, \lambda \xi_0^T, 0, \dots, 0] \\ &= \lambda [0, \dots, 0, \xi_0^T, 0, \dots, 0] = \lambda \xi^T, \end{aligned} \quad (33)$$

$$\xi^T \phi_f = [0, \dots, 0, \xi_0^T, 0, \dots, 0] \phi_f = 0, \quad (34)$$

and as such,

$$\begin{aligned} \xi^T \varphi_f \phi_f &= \lambda \xi^T \phi_f = 0, \\ \xi^T \varphi_f^2 \phi_f &= \lambda \xi^T \varphi_f \phi_f = 0, \\ &\vdots \\ \xi^T \varphi_f^{(m+2n)n_z-1} \phi_f &= 0. \end{aligned} \quad (35)$$

$$\xi^T [\phi_f, \varphi_f \phi_f, \varphi_f^2 \phi_f, \dots, \varphi_f^{(m+2n)n_z-1} \phi_f] = 0, \quad (36)$$

and since the eigenvector ξ^T is a nonzero vector, $\text{rank}([\phi_f, \varphi_f \phi_f, \varphi_f^2 \phi_f, \dots, \varphi_f^{(m+2n)n_z-1} \phi_f]) < (m+2n)n_z$. From Lemma 3, we can see that MAS (4a) and (4b) are uncontrollable. Because the contrapositive proposition of the original proposition is true, the original proposition is true.

(2) Assume that agent i is a second-order integrator. Then, $[0, \dots, 0, I_{n_z}, 0, \dots, 0, I_{n_z}, 0, \dots, 0]$ corresponds to the rows $((i-1)n_z + 1)$ th to (in_z) th of φ_f . Let us look at its contrapositive proposition. By contradiction, suppose that this follower agent does not have any input from leader agents, we obtain $((i-1)n_z + 1)$ th to (in_z) th rows of ϕ_f are all zero. Let $\xi_0(\delta_0) \in \mathbb{R}^{n_z \times 1}$ denote two different eigenvectors of the matrix I corresponding to eigenvalue $\lambda = 1$ and $\xi^T = [0, \dots, 0, \xi_0^T, 0, \dots, 0, \delta_0^T, 0, \dots, 0] \in \mathbb{R}^{1 \times (mn_z + 2mn_z)}$, and we can discover the same result as the case (1); that is, MAS (4a) and (4b) are also uncontrollable. Because the contrapositive proposition of the original proposition is true, the original proposition is also true.

In summary, Theorem 4 is proved. $\square$

Corollary 1. *If several follower agents and their connections are regarded as a subgraph of G , then Theorem 4 can be*

extended to the case that when there is such a subgraph with 0 indegree and the subgraph has no directed path from any leader agent, then the MAS corresponding to graph G is uncontrollable.

Remark 3. Theorem 4 and Corollary 1 are more graph-focused criteria than the previous theorems.

Remark 4. Theorem 4 and Corollary 1 can be simply and visually understood as if there is one agent or multiple agents in the system without any directed path from the input signal. In this case, the system will not be completely controllable. This is consistent with the conclusion in the linear system; that is, if the system is controllable, every vertex must be reachable from one input, regardless of the dynamic form. Otherwise, the system is reducible and part of it cannot be controlled. It is noteworthy that similar properties have been demonstrated in structural controllability.

4. Simulation

The following examples are all results from MATLAB R2018a software used in a Windows 7 operating system environment.

Here, a discrete-time leader-follower MAS is considered, with two-time-scale and heterogeneous features consisting of seven agents. The interaction between agents is shown in Figure 2 and, as it can be seen, the system consists of four first-order agents and three second-order ones. In these two types of agent clusters, all agents act as followers except for the one that acts as the leader, that is, $m = 3$, $m_l = 1$, $n = 2$, and $n_l = 1$.

The interaction information between these agents can be expressed as the following matrix:

$$\begin{aligned} L_{11} &= \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}, \\ L_{12} &= \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \\ L_{13} &= \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}, \\ L_{14} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\ L_{21} &= [0 \ 0 \ 0], \\ L_{22} &= [0], \\ L_{23} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ L_{24} &= [0], \end{aligned}$$

FIGURE 2: Topology of the leader-based discrete-time MAS with heterogeneous and two-time-scale features consisting of seven agents as described in the example.

$$\begin{aligned} L_{31} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \\ L_{32} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\ L_{33} &= \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}, \\ L_{34} &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \\ L_{41} &= [0 \ 0 \ 0], \\ L_{42} &= [0], \\ L_{43} &= [0 \ 0], \\ L_{44} &= [0]. \end{aligned} \tag{37}$$

The matrices B_1 , B_2 , F_1 , F_2 , E_1 , and E_2 in system (7) are chosen as

$$\begin{aligned} B_1 &= [1 \ 2], B_2 = [3 \ 1], \\ F_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ E_1 &= \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \\ E_2 &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned} \tag{38}$$

Based on the previous derivation, it was possible to calculate the eigenvalues of matrices $\hat{\varphi}_s$, φ_f , and Q as $\{11.32, -8.1, 2.62, -0.28, -1.13, -0.97 \pm 0.31i\}$, $\{-23.71, -21.68, -18.63, -13.21, -7.43, -0.03, -0.31\}$, and $\{38.15, 18.85, 7.51, 3.08, 1.12, 1, -0.13, -0.27, -1.35, -8.3, -20.22, -26.08, -26.59, -30.66, -0.93 \pm 0.27i, -16.38 \pm 3.49i\}$, respectively. Evidently, the eigenvalues of $\hat{\varphi}_s$ and φ_f are different from those of Q . According to Theorem 2, the conclusion is that this MAS is controllable.

By setting $\varepsilon = 0.1$, results on the evolution process of the state errors of the follower agents on the slow-time-scale and

FIGURE 3: The position state errors of all follower agents on slow-time-scale. Variables x_1 , x_2 , x_3 , x_5 , and x_6 , respectively, correspond to the position states of agents 1, 2, 3, 5, and 6 in the slow-time-scale subsystem. e_p represents the deviation between the current position state and the target position state.

FIGURE 5: The position state errors of all follower agents on fast-time-scale. Variables z_1 , z_2 , z_3 , z_5 , and z_6 , respectively, correspond to the position states of agents 1, 2, 3, 5, and 6 in the fast-time-scale subsystem. The meaning of e_p is the same as in Figure 3.

FIGURE 4: The velocity state errors of second-order follower agents on slow-time-scale. Variables w_5 and w_6 , respectively, correspond to the velocity states of agents 5 and 6 in the slow-time-scale subsystem. e_v represents the deviation between the current velocity state and the target velocity state.

FIGURE 6: The velocity state errors of second-order follower agents on fast-time-scale. Variables d_5 and d_6 , respectively, correspond to the velocity states of agents 5 and 6 in the fast-time-scale subsystem. e_v has the same meaning as in Figure 4.

fast-time-scale can be obtained as shown in Figures 3–6. The differences between the current states and the desired states of each follower agent will eventually converge to 0 under the control of the leader agents, which also indicates that this MAS is controllable. It is also simple to identify that the

states of the follower agents on the fast-time-scale tend to reach the expected values more quickly than the states on the slow-time-scale.

To illustrate the controllability of this MAS more intuitively, the trajectories of five follower agents are shown in Figure 7. The first- and second-order agents are represented

FIGURE 7: The trajectories of follower agents.

by circles and asterisks, respectively, and the x -axis and y -axis coordinates of each agent correspond to its position state on slow-time-scale and fast-time-scale. The initial states of all agents were randomly selected from $(0, 1) \times (0, 1)$. The dotted line represents their trajectories, and their ultimate control goal is to form a triangle on the two-dimensional plane.

5. Conclusion

This paper studies the controllability of discrete-time MASs with two-time-scale and heterogeneous features based on the leader-follower structure. Considering the essential difference between a discrete-time system and a continuous-time system, the modelling and analysis methods applied here are also different. The content of this paper can supplement existing results regarding this issue. In this paper, the singularly perturbed difference system is first applied to model the system, so it later can be decomposed into a slow-time-scale subsystem and a fast-time-scale subsystem. This process is done using the singular perturbation method before the controllability analysis, which avoids the ill-posedness problem when the classical controllability method is used to analyze the system directly. Due to the computational burden caused by using the rank of controllability matrix to judge controllability, some more practical and operational controllability criteria were obtained based on the matrix and the graph theory. These methods only depend on the characteristics of submatrices of the system matrix and the input matrix. Finally, the validity could be verified by simulation. For future studies related to this one, a new focus will be shaped towards the controllability of multi-time-scale MASs and the robustness of controllability.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors thank Xiaoling Wang and Mingkan Long for discussions. This work was supported by the National Natural Science Foundation of China (grant nos. 61873326 and 61672298).

References

- [1] C. Deng and G.-H. Yang, "Distributed adaptive fault-tolerant control approach to cooperative output regulation for linear multi-agent systems," *Automatica*, vol. 103, pp. 62–68, 2019.
- [2] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2011.
- [3] X. Wang, X. Wang, H. Su, and J. Lam, "Coordination control for uncertain networked systems using interval observers," *IEEE Transactions on Cybernetics*, vol. 50, 2019.
- [4] Y. Hong, J. Hu, and L. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology," *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2006.
- [5] R. E. Kalman, "Mathematical description of linear dynamical systems," *Journal of the Society for Industrial and Applied Mathematics Series A Control*, vol. 1, no. 2, pp. 152–192, 1963.
- [6] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proceedings of the 2004 43rd IEEE Conference on Decision and Control (CDC)*, pp. 2467–2472, IEEE, Atlantis Paradise Island, The Bahamas, 2004.
- [7] A. Rahmani and M. Mesbahi, "On the controlled agreement problem," in *Proceedings of the 2006 American Control Conference*, p. 6, IEEE, Minneapolis, MN, USA, 2006.
- [8] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 162–186, 2009.
- [9] M. Ji and M. Egerstedt, "A graph-theoretic characterization of controllability for multi-agent systems," in *Proceedings of the 2007 American Control Conference*, pp. 4588–4593, IEEE, New York, NY, USA, 2007.
- [10] Z. Ji, H. Lin, S. Cao, Q. Qi, and H. Ma, "The complexity in complete graphic characterizations of multiagent controllability," *IEEE Transactions on Cybernetics*, vol. 51, 2020.
- [11] X. Liu, H. Lin, and B. M. Chen, "Graph-theoretic characterisations of structural controllability for multi-agent system with switching topology," *International Journal of Control*, vol. 86, no. 2, pp. 222–231, 2013.
- [12] W. Ni, X. Wang, and C. Xiong, "Consensus controllability, observability and robust design for leader-following linear multi-agent systems," *Automatica*, vol. 49, no. 7, pp. 2199–2205, 2013.
- [13] L. Tian, Y. Guan, and L. Wang, "Controllability and observability of switched multi-agent systems," *International Journal of Control*, vol. 92, no. 8, pp. 1742–1752, 2019.
- [14] S. Liu, Z. Ji, and H. Ma, "Jordan form-based algebraic conditions for controllability of multiagent systems under directed graphs," *Complexity*, vol. 2020, Article ID 7685460, 15 pages, 2020.
- [15] B. Liu, T. Chu, L. Wang, and G. Xie, "Controllability of a class of multi-agent systems with a leader," in *Proceedings of the 2006 American Control Conference*, p. 6, IEEE, Minneapolis, MN, USA, 2006.
- [16] B. Liu, Y. Han, F. Jiang, H. Su, and J. Zou, "Group controllability of discrete-time multi-agent systems," *Journal of the Franklin Institute*, vol. 353, no. 14, pp. 3524–3559, 2016.

- [17] Y. Zheng, L. Wang, and Y. Zhu, "Consensus of heterogeneous multi-agent systems," *IET Control Theory & Applications*, vol. 5, no. 16, pp. 1881–1888, 2011.
- [18] Y. Guan, Z. Ji, L. Zhang, and L. Wang, "Controllability of heterogeneous multi-agent systems under directed and weighted topology," *International Journal of Control*, vol. 89, no. 5, pp. 1009–1024, 2016.
- [19] L. Tian, Y. Guan, and L. Wang, "Controllability and observability of multi-agent systems with heterogeneous and switching topologies," *International Journal of Control*, vol. 93, no. 3, pp. 437–448, 2020.
- [20] S. K. Dwivedy and P. Eberhard, "Dynamic analysis of flexible manipulators, a literature review," *Mechanism and Machine Theory*, vol. 41, no. 7, pp. 749–777, 2006.
- [21] L. Prandtl, "On the motion of fluids of very small velocity," in *Proceedings of the The Third International Congress of Mathematician*, Heidelberg, Germany, 1904.
- [22] P. Kokotovic and P. Sannuti, "Singular perturbation method for reducing the model order in optimal control design," *IEEE Transactions on Automatic Control*, vol. 13, no. 4, pp. 377–384, 1968.
- [23] P. V. Kokotovic, R. E. O'Malley, and P. Sannuti, "Singular perturbations and order reduction in control theory—an overview," *Automatica*, vol. 12, no. 2, pp. 123–132, 1976.
- [24] V. R. Saksena, J. O'Reilly, and P. V. Kokotovic, "Singular perturbations and time-scale methods in control theory: survey 1976–1983," *Automatica*, vol. 20, no. 3, pp. 273–293, 1984.
- [25] B. Litkouhi and H. Khalil, "Multirate and composite control of two-time-scale discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 30, no. 7, pp. 645–651, 1985.
- [26] H. Su, M. Long, and Z. Zeng, "Controllability of two-time-scale discrete-time multiagent systems," *IEEE Transactions on Cybernetics*, vol. 50, no. 4, pp. 2168–2267, 2018.
- [27] M. Long, H. Su, and B. Liu, "Second-order controllability of two-time-scale multi-agent systems," *Applied Mathematics and Computation*, vol. 343, pp. 299–313, 2019.
- [28] M. Long, H. Su, and B. Liu, "Second-order controllability of two-time-scale discrete-time multi-agent systems," *IET Control Theory & Applications*, vol. 13, no. 15, pp. 2356–2364, 2019.
- [29] M. Long, H. Su, X. Wang, and B. Liu, "Controllability of heterogeneous multiagent systems with two-time-scale feature," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 29, no. 4, Article ID 43116, 2019.
- [30] P. Kokotović, H. K. Khalil, and J. O'reilly, "Singular perturbation methods in control: analysis and design," in *Proceedings of the SIAM*, San Antonio, TX, USA, 1999.