Research Article

Correlation Measures for Cubic m -Polar Fuzzy Sets with Applications

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A cubic m -polar fuzzy set (CmPFS) is a new hybrid extension of m -polar fuzzy set and cubic set. A CmPFS is a robust model to express multipolar information in terms of m fuzzy intervals representing membership grades and m fuzzy numbers representing nonmembership grades. In this article, we explore some new operational laws of CmPFSs, produce some related results, and discuss their consequences. We propose relative informational coefficients and relative noninformational coefficients for CmPFSs. These coefficients are analyzed to investigate further properties of CmPFSs. Based on these coefficients, we introduce new correlation measures and their weighted versions for CmPFSs. The value of proposed correlation measures is symmetrical and lies between -1 and 1 . Moreover, the applications of the proposed correlation in pattern recognition and medical diagnosis are developed. The feasibility and efficiency of suggested correlation measures is determined by respective illustrative examples.

1. Introduction

Modern systems and logic are the schematic study of the rules that can lead to the acceptance of a certain proposition on the basis of preassumed hypothesis. The development of the modern systems owes to the advancement of set theory and logic. To solve uncertain problems of real-life, the researchers have been established various sets and models. In 1965, Zadeh [1] extended crisp set to fuzzy set (FS) by assigning a real number in interval $[0, 1]$ which dictates the membership value of an element/object in the crisp set. Zadeh's pioneer work has a revolutionary importance during the last decades as it laid a foundation to the modern "fuzzy set theory (FST) and its applications." This theory, upon being uplifted to a certain degree, turned to fuzzy logics. Fuzzy logic and fuzzy sets have a large number of applications in computational intelligence, robotics, neural networks, data sciences, and pattern recognition. The idea of fuzzification proved to be an open end for the scientists. A lot of extensions of FST have been introduced so far. The notions of bipolar fuzzy set (BPFS) [2] and intuitionistic fuzzy set (IFS) [3] are both robust

extensions of FS. An IFS assigns a pair of membership value and a nonmembership value which lie between 0 and 1 . A BFS is a direct extension of FS which gives information which is bipolar in nature. Its positive membership value μ^+ lies in the interval $[0, 1]$ and negative membership value μ^- lies in the interval $[-1, 0]$. The positive scores depict the consent of a certain property, and the negative scores assigned reflect the consent of a counter property. However, an important point to be noted here is that which type of necessity served as a mother of this extension, that is, which limitation led to this work? The innovative idea of BFS correctly addresses this problem. It should be acknowledged that the individual viewpoint or reasoning is not limited to bipolarity. There can be more than two independent view points to a certain crisp item. For example, in an electoral system, a vote being casted may have exact degree from $[0, 1]^m$. These degrees may be assigned on the basis of personal mindset, honesty of the candidate, leadership qualities, education, and future plans. Therefore, bipolar fuzzy theory cannot answer this sort of problem and there is a need to invent a new extension. To address this limitation, m -polar fuzzy set (mPFS) theory has been

proposed by Chen et al. [4]. An m -polar fuzzy set assigns m independent values/degrees to each crisp element/object.

Jun et al. [5] introduced the concept of cubic set (CS) as an extension of intuitionistic fuzzy set (IFS). A CS assigns a membership grade which is a fuzzy interval and non-membership which is fuzzy value to each crisp element/object. Besides, both types of grades are condition free. However, CS allows each of its alternatives to follow one attribute only. Multiattributed information is denied in CS theory. This is the main reason and answer to the important question “why cubic m -polar fuzzy set (CmPFS) theory has been developed.” The CmPFS (initiated by Riaz and Hashmi [6]) is a new hybrid structure of CS and m PFS, and it manipulates not only multiattributed but also cubic information as well. Therefore, CS and m PFS are the special cases of CmPFS.

Some of the important extensions of fuzzy set are Pythagorean fuzzy set (PFS) [7], q -rung orthopair fuzzy set (q -ROFS) [8], neutrosophic set [9], single-valued neutrosophic set [10], spherical fuzzy set [11–13], and picture fuzzy sets (PicFSS) [14]. Soft set [15] is a robust independent extension of crisp sets for modeling uncertainty problems. Wang et al. [16] introduced Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight. Liu et al. [17] suggested Banzhaf–Choquet copula-based aggregation operators for managing q -ROF information. Akram et al. [18] established complex spherical fuzzy prioritized weighted aggregation operators. Sitara et al. [19] proposed novel decision-making analysis based on q -rung picture fuzzy graph structures. Garg [20, 21] developed MAGDM process based on operators for intuitionistic multiplicative set and interval-valued picture uncertain linguistic generalized Hamacher aggregation operators. Liu et al. [22] proposed group decision-making using complex q -ROF Bonferroni mean. Liu and Wang [23] established a new MAGDM method based on IF Einstein interactive operations.

Recently, Riaz et al. [24, 25] introduced some new extensions of fuzzy sets named as linear Diophantine fuzzy set (LDFS) and spherical linear Diophantine sets. Kamaci [26] introduced algebraic structure on LDFS with an interesting application to coding theory which is based on LDFS codes. Ayub et al. [27] introduced LDF-relations and their algebraic properties with decision-making application. Almagrabi et al. [28] presented a new approach to the q -LDF emergency decision support system for COVID-19.

Correlation and similarity measures are significant mathematical tools to deal with real-life problems. The development of fuzzy set theories give rise to new extension of correlation and similarity measures. The researchers have been used these measures to develop new techniques for decision-making, pattern recognition, and cluster analysis. Ganie and Singh [29] used similarity measures for MCDM in picture fuzzy environment. Mahmood and Ali [30] used correlation measures to propose the extended TOPSIS method for Pythagorean fuzzy environment. Zulqarnain et al. [31] proposed correlated TOPSIS for the mask selection.

Correlation is a statistic that assesses the extent to which two objects move in relation to each other whereas similarity measure is a function which measures the closeness of two objects. Correlation gives some important applications to statistical analysis. Statistics has effectively used probability methods for dealing with many real space issues focused on random processing of data. Similarly, some statistical problems in engineering sciences have also been addressed by the same probabilistic approaches. However, these approaches have some drawbacks, since these methods are based on random sample selection to obtain a certain degree of trust. When the data are collected on a large-scale, massive fuzziness is involved therein which makes it difficult to optimize the idealistic probability. The main question is that how to overcome such obstacles to obtain idealistic results? The answer to this question is somewhat complicated. The first thing to be noted is that in every real-world problem, no matter it is an engineering, medical, or administrative issue, uncertainty always finds its place. In statistics, correlation is one of the most important concepts we encounter with and is digested by many other subjects such as economics, finance, engineering, and management sciences. However, the problem is that this type of correlation can cope with the crisp data only and we have been managing many real-life problems with the help of this without taking into account uncertainty. Since the birth of fuzzy set, many uncertain settings have been introduced to tackle variability. We have seen above that a lot of ambiguous sets have been defined so far to handle imprecise information. Therefore, there is a need to introduce correlation measures in every environment because a real problem may be diverse in its uncertain nature. To address this issue, Ejegwa et al. [32], Gerstenkorn and Mańko [33], and Szmidt and Kacprzyk [34] proposed some correlation measures for IFSs. Garg et al. [35] presented the correlation coefficient for cubic intuitionistic fuzzy set and hence presented a decision-making algorithm. Correlation measures are used in PFSs. Lin et al. [36] and Thao [37] also discussed limitations of the previously defined correlations and put forward new directional correlations for PFSs.

The main objectives of the manuscript are listed as follows:

- (1) A cubic m -polar fuzzy set (CmPFS) is a robust fusion of cubic sets and m -polar fuzzy set to solve more complicated decision-making problems in the multipolar and cubic environment. This model is more suitable to express vague information efficiently, and the decision-makers find more flexibility to give their assessment towards multicriterion, multiattribute, multisensor, multisystems, and multirelations.
- (2) To define new operations on CmPFSs and discuss related results.
- (3) To develop relative informational coefficients and relative noninformational coefficients for CmPFSs.
- (4) To analyze these coefficients to investigate further properties of CmPFSs.

- (5) To define new correlation measures and their weighted versions for CmPFSs.
- (6) To construct new algorithms with the help of new correlation measures for solving the complicated problems of pattern recognition and medical diagnosis which cannot be tackled by existing techniques. The feasibility and efficiency of suggested correlation measures is determined by respective illustrative examples.

Figure 1 demonstrates novel hybrid extension of cubic set and m -polar fuzzy set named as cubic m -polar fuzzy set.

This paper is organized as follows. In Section 2, we first recall few basic concepts of fuzzy sets, cubic sets, and m -polar fuzzy sets. Then, we review definition and operations of CmPFSs. In Section 3, we discuss some novel features of CmPFSs depending on new operations. We also define support, core, height, level sets, and binary relations. In Section 4, we define correlation measures for CmPFSs. Their weighted versions have also been addressed. We prove that all types of correlation measures are symmetrical and lie in the interval $[-1, 1]$, the correlation of two CmPFSs is 1 only if either both sets are equal or one of the sets is a positive linear multiple of the other, and the correlation of the two sets is -1 only if one set is a positive linear multiple of the complement of the other set. In Section 5, new algorithms are proposed using proposed correlation measures under CmPF environment for solving the complicated problems of pattern recognition and medical diagnosis. The feasibility and efficiency of suggested correlation measures is determined by respective illustrative examples. The conclusion of this research studies is presented in Section 6.

2. Preliminaries

Some basic definitions and operations are discussed here in this section which can serve foundation for current research work. We shall use $\mathfrak{N}$ as a universal space.

Definition 1 (see [1]). A fuzzy set (FS) on a universe $\mathfrak{N}$ is a rule which assigns a number from the interval $[0, 1]$ to each element in $\mathfrak{N}$. This number is referred to as membership grade of the crisp element. Mathematically, an FS is described by the membership function $\mu: \mathfrak{N} \rightarrow [0, 1]$.

Definition 2 (see [4]). An m -polar fuzzy set (m PFS) on a universe $\mathfrak{N}$ is defined by

$$\mathfrak{M} = \{ \langle \kappa, (\mu_i(\kappa))_{i=1}^m \rangle : \kappa \in \mathfrak{N} \} \\ = \{ \langle \kappa, \mu_1(\kappa), \mu_2(\kappa), \dots, \mu_m(\kappa) \rangle : \kappa \in \mathfrak{N} \}, \quad (1)$$

where $\mu_i: [0, 1]^m \rightarrow [0, 1]; i = 1, 2, \dots, m$, is the i^{th} projection mapping.

An m -polar fuzzy set is a direct extension of FS that describes the multipolarity of the objects. An m PFS assigns m fuzzy grades to each object/element of universe $\mathfrak{N}$.

Definition 3 (see [5]). A cubic set (CS) is defined by a rule which assigns to each element/object of the discourse $\mathfrak{N}$ a

FIGURE 1: Cubic m -polar fuzzy set: a hybrid model.

subinterval of $[0, 1]$ called membership value which is a fuzzy subinterval and a nonmembership value from $[0, 1]$. Mathematically, we can write

$$CS(\mathfrak{N}) = \{ \langle \kappa, A_\kappa, B_\kappa \rangle : \kappa \in \mathfrak{N} \}, \quad (2)$$

where A_κ is a fuzzy subinterval and B_κ is a fuzzy value assigned to the alternative κ . There is no restriction on B_κ to be a number from A_κ . If, however, $B_\kappa \in A_\kappa$, then $CS(\mathfrak{N})$ is called internal cubic set (ICS). Otherwise, it is called external cubic set (ECS). Thus, cubic information is of two categories, that is, internal and external. So, whenever we deal with cubic sets, we must ensure its behaviour.

Definition 4 (see [6]). A cubic m -polar fuzzy set (CmPFS) is a set which deals with multipolar information in cubic environment. It assigns to each member of the crisp discourse a cubic element allowing multipolarity. If we take $\mathfrak{N}$ as universe of discourse, the CmPFS can be viewed mathematically as follows:

$$CmPFS(\mathfrak{N}) = \{ \langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m \}. \quad (3)$$

For convenience, the term $\langle [\eta_j^-, \eta_j^+], \eta_j \rangle_{j=1}^m$ is referred to as cubic m -polar fuzzy number (CmPFN).

2.1. Operations on CmPFSs. Let $A_{\mathfrak{N}}^{(1)} = \{ \langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathfrak{N} \}$ and $A_{\mathfrak{N}}^{(2)} = \{ \langle \kappa, [\zeta_i^-(\kappa), \zeta_i^+(\kappa)], \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathfrak{N} \}$ be the CmPFSs. Then, the operations for these sets under P-ordering are defined as follows:

- (1) $A_{\mathfrak{N}}^{(1)} = A_{\mathfrak{N}}^{(2)} \Leftrightarrow \eta_i^-(\kappa) = \zeta_i^-(\kappa), \eta_i^+(\kappa) = \zeta_i^+(\kappa)$ and $\eta_i(\kappa) = \zeta_i(\kappa)$. For all $i = 1, 2, \dots, m$ and $\kappa \in \mathfrak{N}$
- (2) $(A_{\mathfrak{N}}^{(1)})^c = \{ \langle \kappa, [1 - \eta_i^+(\kappa), 1 - \eta_i^-(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathfrak{N} \}$
- (3) $A_{\mathfrak{N}}^{(1)} \cup_P A_{\mathfrak{N}}^{(2)} = \{ \langle \kappa, [\eta_i^-(\kappa) \vee \zeta_i^-(\kappa), \eta_i^+(\kappa) \vee \zeta_i^+(\kappa)], \eta_i(\kappa) \vee \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathfrak{N} \}$
- (4) $A_{\mathfrak{N}}^{(1)} \cap_P A_{\mathfrak{N}}^{(2)} = \{ \langle \kappa, [\eta_i^-(\kappa) \wedge \zeta_i^-(\kappa), \eta_i^+(\kappa) \wedge \zeta_i^+(\kappa)], \eta_i(\kappa) \wedge \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathfrak{N} \}$
- (5) $A_{\mathfrak{N}}^{(1)} \subseteq_P A_{\mathfrak{N}}^{(2)} \Leftrightarrow \eta_i^-(\kappa) \leq \zeta_i^-(\kappa), \eta_i^+(\kappa) \leq \zeta_i^+(\kappa)$ and $\eta_i(\kappa) \leq \zeta_i(\kappa)$. For all $i = 1, 2, \dots, m$ and $\kappa \in \mathfrak{N}$

- (6) $A_N^{(1)} \oplus_P A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa) \zeta_i^-(\kappa), \eta_i^+(\kappa) + \zeta_i^+(\kappa) - \eta_i^+(\kappa) \zeta_i^+(\kappa)], \eta_i(\kappa) + \zeta_i(\kappa) - \eta_i(\kappa) \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (7) $A_N^{(1)} \otimes_P A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) \zeta_i^-(\kappa), \eta_i^+(\kappa) \zeta_i^+(\kappa)], \eta_i(\kappa) \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (8) $\zeta A_N^{(1)} = \{\langle \kappa, [1 - (1 - \eta_i^-(\kappa))^\zeta, 1 - (1 - \eta_i^+(\kappa))^\zeta], 1 - (1 - \eta_i(\kappa))^\zeta \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (9) $(A_N^{(1)})^\zeta = \{\langle \kappa, [(\eta_i^-(\kappa))^\zeta, (\eta_i^+(\kappa))^\zeta], (\eta_i(\kappa))^\zeta \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$

The R-ordering operations for the abovementioned CmpFSSs are given as follows:

(Note: the equality and complement operations for these sets are universal; that is, they are independent of P/R-ordering.)

- (1) $A_N^{(1)} \cup_R A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) \vee \zeta_i^-(\kappa), \eta_i^+(\kappa) \vee \zeta_i^+(\kappa)], \eta_i(\kappa) \wedge \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (2) $A_N^{(1)} \cap_R A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) \wedge \zeta_i^-(\kappa), \eta_i^+(\kappa) \wedge \zeta_i^+(\kappa)], \eta_i(\kappa) \vee \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (3) $A_N^{(1)} \subseteq_R A_N^{(2)} \Leftrightarrow \eta_i^-(\kappa) \leq \zeta_i^-(\kappa), \eta_i^+(\kappa) \leq \zeta_i^+(\kappa) \text{ and } \eta_i(\kappa) \geq \zeta_i(\kappa) \text{ for all } i = 1, 2, \dots, m \text{ and } \kappa \in \mathbb{N}.$
- (4) $A_N^{(1)} \oplus_R A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa) \zeta_i^-(\kappa), \eta_i^+(\kappa) + \zeta_i^+(\kappa) - \eta_i^+(\kappa) \zeta_i^+(\kappa)], \eta_i(\kappa) \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (5) $A_N^{(1)} \otimes_R A_N^{(2)} = \{\langle \kappa, [\eta_i^-(\kappa) \zeta_i^-(\kappa), \eta_i^+(\kappa) \zeta_i^+(\kappa)], \eta_i(\kappa) + \zeta_i(\kappa) - \eta_i(\kappa) \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (6) $\zeta A_N^{(1)} = \{\langle \kappa, [1 - (1 - \eta_i^-(\kappa))^\zeta, 1 - (1 - \eta_i^+(\kappa))^\zeta], (\eta_i(\kappa))^\zeta \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$
- (7) $(A_N^{(1)})^\zeta = \{\langle \kappa, [(\eta_i^-(\kappa))^\zeta, (\eta_i^+(\kappa))^\zeta], 1 - (1 - \eta_i(\kappa))^\zeta \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$

Note: it is interesting to note that the collection of all CmpFSSs on a universe $\mathbb{N}$ is closed under these operations.

3. Some Novel Features of CmpFSSs

In this section, we give some properties and related results for CmpFSSs.

Theorem 1. Let $A_N^{(1)}$, $A_N^{(2)}$, and $A_N^{(3)}$ be CmpFSSs. Then,

- (1) $A_N^{(1)} \cup_P A_N^{(2)} = A_N^{(2)} \cup_P A_N^{(1)}$
- (2) $A_N^{(1)} \cap_P A_N^{(2)} = A_N^{(2)} \cap_P A_N^{(1)}$
- (3) $A_N^{(1)} \subseteq_P A_N^{(2)}$ and $A_N^{(2)} \subseteq_P A_N^{(3)}$ implies $A_N^{(1)} \subseteq_P A_N^{(3)}$
- (4) $A_N^{(1)} \cup_P (A_N^{(2)} \cap_P A_N^{(3)}) = (A_N^{(1)} \cup_P A_N^{(2)}) \cap_P A_N^{(3)}$
- (5) $A_N^{(1)} \cap_P (A_N^{(2)} \cup_P A_N^{(3)}) = (A_N^{(1)} \cap_P A_N^{(2)}) \cup_P A_N^{(3)}$
- (6) $A_N^{(1)} \cup_P (A_N^{(2)} \cap_P A_N^{(3)}) = (A_N^{(1)} \cup_P A_N^{(2)}) \cap_P (A_N^{(1)} \cup_P A_N^{(3)})$
- (7) $A_N^{(1)} \cap_P (A_N^{(2)} \cup_P A_N^{(3)}) = (A_N^{(1)} \cap_P A_N^{(2)}) \cup_P (A_N^{(1)} \cap_P A_N^{(3)})$

Proof. Let $A_N^{(1)} = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$, $A_N^{(2)} = \{\langle \kappa, [\zeta_i^-(\kappa), \zeta_i^+(\kappa)], \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$, and

$A_N^{(3)} = \{\langle \kappa, [\theta_i^-(\kappa), \theta_i^+(\kappa)], \theta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$ be the three CmpFSSs on $\mathbb{N}$.

Proofs (1) and (2) are trivial.

- (3) $A_N^{(1)} \subseteq_P A_N^{(2)}$ implies $\eta_i^-(\kappa) \leq \zeta_i^-(\kappa)$, $\eta_i^+(\kappa) \leq \zeta_i^+(\kappa)$ and $\eta_i(\kappa) \leq \zeta_i(\kappa)$.

Similarly $A_N^{(2)} \subseteq_P A_N^{(3)}$ implies $\zeta_i^-(\kappa) \leq \theta_i^-(\kappa)$, $\zeta_i^+(\kappa) \leq \theta_i^+(\kappa)$ and $\zeta_i(\kappa) \leq \theta_i(\kappa)$.

Combining the two facts, we obtain $\eta_i^-(\kappa) \leq \theta_i^-(\kappa)$, $\eta_i^+(\kappa) \leq \theta_i^+(\kappa)$ and $\eta_i(\kappa) \leq \theta_i(\kappa)$ for all $i = 1, 2, \dots, m$ and $\kappa \in \mathbb{N}$. This implies that $A_N^{(1)} \subseteq_P A_N^{(3)}$.

- (4) Consider $A_N^{(1)} \cup_P (A_N^{(2)} \cap_P A_N^{(3)}) = A_N^{(1)} \cup_P (\{\langle \kappa, [\zeta_i^-(\kappa) \vee \theta_i^-(\kappa), \zeta_i^+(\kappa) \vee \theta_i^+(\kappa)], \zeta_i(\kappa) \vee \theta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}) = \{\langle \kappa, [\eta_i^-(\kappa) \vee (\zeta_i^-(\kappa) \vee \theta_i^-(\kappa)), \eta_i^+(\kappa) \vee (\zeta_i^+(\kappa) \vee \theta_i^+(\kappa))], \eta_i(\kappa) \vee (\zeta_i(\kappa) \vee \theta_i(\kappa)) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = \{\langle \kappa, [\eta_i^-(\kappa) \vee \zeta_i^-(\kappa), \eta_i^+(\kappa) \vee \zeta_i^+(\kappa)], \eta_i(\kappa) \vee \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = (A_N^{(1)} \cup_P A_N^{(2)}) \cap_P A_N^{(3)}.$

- (5) Similar to (4).

- (6) $A_N^{(1)} \cup_P (A_N^{(2)} \cap_P A_N^{(3)}) = A_N^{(1)} \cup_P \{\langle \kappa, [\zeta_i^-(\kappa) \wedge \theta_i^-(\kappa), \zeta_i^+(\kappa) \wedge \theta_i^+(\kappa)], \zeta_i(\kappa) \wedge \theta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = \{\langle \kappa, [\eta_i^-(\kappa) \vee (\zeta_i^-(\kappa) \wedge \theta_i^-(\kappa)), \eta_i^+(\kappa) \vee (\zeta_i^+(\kappa) \wedge \theta_i^+(\kappa))], \eta_i(\kappa) \vee (\zeta_i(\kappa) \wedge \theta_i(\kappa)) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = \{\langle \kappa, [(\eta_i^-(\kappa) \vee \zeta_i^-(\kappa)) \wedge (\zeta_i^-(\kappa) \wedge \theta_i^-(\kappa)), (\eta_i^+(\kappa) \vee \zeta_i^+(\kappa)) \wedge (\zeta_i^+(\kappa) \wedge \theta_i^+(\kappa))], (\eta_i(\kappa) \vee \zeta_i(\kappa)) \wedge (\zeta_i(\kappa) \wedge \theta_i(\kappa)) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = (A_N^{(1)} \cup_P A_N^{(2)}) \cap_P (A_N^{(1)} \cup_P A_N^{(3)}).$

- (7) Similar to (6). $\square$

Remark 1. In rest of the section, we will take $A_N^{(1)} = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$, $A_N^{(2)} = \{\langle \kappa, [\zeta_i^-(\kappa), \zeta_i^+(\kappa)], \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$, and $A_N^{(3)} = \{\langle \kappa, [\theta_i^-(\kappa), \theta_i^+(\kappa)], \theta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$ as CmpFSSs.

Proposition 1. Let $A_N^{(1)}$ and $A_N^{(2)}$. Then,

- (1) $(A_N^{(1)} \cup_P A_N^{(2)})^c = (A_N^{(1)})^c \cap_P (A_N^{(2)})^c$
- (2) $(A_N^{(1)} \cap_P A_N^{(2)})^c = (A_N^{(1)})^c \cup_P (A_N^{(2)})^c$

Proof. Let $A_N^{(1)} = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$ and $A_N^{(2)} = \{\langle \kappa, [\zeta_i^-(\kappa), \zeta_i^+(\kappa)], \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$.

- (1) $(A_N^{(1)} \cup_P A_N^{(2)})^c = (\{\langle \kappa, [\eta_i^-(\kappa) \vee \zeta_i^-(\kappa), \eta_i^+(\kappa) \vee \zeta_i^+(\kappa)], \eta_i(\kappa) \vee \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\})^c = \{\langle \kappa, [1 - (\eta_i^-(\kappa) \vee \zeta_i^-(\kappa)), 1 - (\eta_i^+(\kappa) \vee \zeta_i^+(\kappa))], 1 - (\eta_i(\kappa) \vee \zeta_i(\kappa)) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = \{\langle \kappa, [(1 - \eta_i^-(\kappa)) \wedge (1 - \zeta_i^-(\kappa)), (1 - \eta_i^+(\kappa)) \wedge (1 - \zeta_i^+(\kappa))], (1 - \eta_i(\kappa)) \wedge (1 - \zeta_i(\kappa)) \rangle_{i=1}^m : \kappa \in \mathbb{N}\} = (A_N^{(1)})^c \cap_P (A_N^{(2)})^c.$

- (2) Similar to (1). $\square$

Definition 5. Let $A_N = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N}\}$ be a CmpFSS with discourse $\mathbb{N}$. Then, the support of A_N is the collection of all those crisp elements from $\mathbb{N}$ which contribute to its cubic m -polar behaviour. That is,

$$\text{Supp}(A_N) = \{\kappa \in N: [\eta_i^-(\kappa), \eta_i^+(\kappa)] \neq \mathbf{0}_m \text{ or } \eta_i(\kappa) \neq 1 \text{ for at least one } i\}, \quad (4)$$

where $\mathbf{0}_m$ is the null CmPFN.

Example 1. Let $N = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be the universe, then a CmPFS A_N on N is given by

$$\begin{aligned} A_N = \{ & (\kappa_1, \langle [0.1, 0.2], [0.8, 0.9], [0, 0], 0.9, 1, 0.3 \rangle) \\ & (\kappa_2, \langle [0, 0], [0, 0], [0, 0], 1, 1, 1 \rangle) \\ & (\kappa_3, \langle [0, 0], [0, 0], [0, 0.7], 1, 1, 0.5 \rangle) \\ & (\kappa_4, \langle [1, 1], [1, 1], [1, 1], 0, 0, 0 \rangle) \}. \end{aligned} \quad (5)$$

The CmPFS A_N can be expressed in the tabular form as given by Table 1. Then, $\text{Supp}(A_N) = \{\kappa_1, \kappa_3, \kappa_4\}$.

Definition 6. The core of a CmPFS $A_N = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m: \kappa \in N\}$ is the collection of crisp elements κ from N for which $[\eta_i^-(\kappa), \eta_i^+(\kappa)] = \mathbf{1}_m$ and $\eta_i(\kappa) = \mathbf{0}_m$ for all i . Mathematically,

$$\text{Cor}(A_N) = \{\kappa \in N: [\eta_i^-(\kappa), \eta_i^+(\kappa)] = \mathbf{1}_m \text{ and } \eta_i(\kappa) = \mathbf{0}_m \text{ for all } i\}, \quad (6)$$

where $\mathbf{1}_m$ is the absolute CmPFN.

Example 2. The core of the CmPFS in Example 1 is given by

$$\text{Cor}(A_N) = \{\kappa_4\}. \quad (7)$$

Definition 7. The height of a CmPFS is the maximum membership value that a certain alternative can attain. If $A_N = \{\langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m: \kappa \in N\}$ is a CmPFS, then the height is given by

$$\underline{ht}(A_N) = \max_{\{\kappa \in N, i\}} \eta_i^+(\kappa). \quad (8)$$

If the height of a CmPFS finds its value to be 1, then it is said to be a normal CmPFS:

$$A_N \text{ is normal} \Leftrightarrow \underline{ht}(A_N) = 1. \quad (9)$$

Example 3. The CmPFS given in Example 1 can be referred as a normal one because its height is 1, that is, $\underline{ht}(A_N) = 1$.

Remark 2. If $A_N^{(1)}$ and $A_N^{(2)}$ are CmPFSs over a crisp domain N , then

$$\begin{aligned} A_N^{(1)} \subseteq_P A_N^{(2)} &\Rightarrow \underline{ht}(A_N^{(1)}) \leq \underline{ht}(A_N^{(2)}), \\ A_N^{(1)} \subseteq_R A_N^{(2)} &\Rightarrow \underline{ht}(A_N^{(1)}) \leq \underline{ht}(A_N^{(2)}). \end{aligned} \quad (10)$$

However, the converse does not hold in general. The counter example is given as follows.

Example 4. Consider the two CmPFSs, namely, $A_N^{(1)}$ and $A_N^{(2)}$ as given by

$$\begin{aligned} A_N^{(1)} = \{ & (\kappa_1, \langle [0.10, 0.12], [0.30, 0.36], [0.26, 0.40], 0.71, 0.68, 0.90 \rangle) \\ & (\kappa_2, \langle [0.40, 0.47], [0.50, 0.60], [0.47, 0.57], 0.80, 0.80, 0.76 \rangle) \\ & (\kappa_3, \langle [0.50, 0.55], [0.60, 0.66], [0.23, 0.34], 0.56, 0.69, 0.44 \rangle) \\ & (\kappa_4, \langle [0.89, 0.93], [0.10, 0.36], [0.56, 0.60], 0.70, 0.20, 0.47 \rangle) \}, \\ A_N^{(2)} = \{ & (\kappa_1, \langle [0.88, 0.10], [0.64, 0.70], [0.60, 0.74], 0.29, 0.32, 0.10 \rangle) \\ & (\kappa_2, \langle [0.53, 0.60], [0.40, 0.50], [0.43, 0.53], 0.20, 0.20, 0.24 \rangle) \\ & (\kappa_3, \langle [0.45, 0.50], [0.34, 0.40], [0.66, 0.77], 0.44, 0.31, 0.56 \rangle) \\ & (\kappa_4, \langle [0.07, 0.11], [0.64, 0.90], [0.40, 0.44], 0.30, 0.80, 0.53 \rangle) \}. \end{aligned} \quad (11)$$

Clearly $\underline{ht}(A_N^{(1)}) = 0.93 \leq 1 = \underline{ht}(A_N^{(2)})$. However, $A_N^{(1)} \not\subseteq_{P,R} A_N^{(2)}$.

Theorem 2. For a CmPFS A_N , following results are straightforward:

- (1) $A_N = A_N \cup_P A_N, A_N \cap_P A_N = A_N$
- (2) $A_N = A_N \cup_R A_N, A_N \cap_R A_N = A_N$
- (3) $A_N \oplus_P A_N = A_N \Leftrightarrow A_N = \mathbf{1}_m \text{ or } \mathbf{0}_m$
- (4) $A_N \otimes_P A_N = A_N \Leftrightarrow A_N = \mathbf{0}_m \text{ or } \mathbf{1}_m$
- (5) $A_N \oplus_R A_N = A_N \Leftrightarrow A_N = \mathbf{0}_m \text{ or } \mathbf{1}_m$
- (6) $A_N \otimes_R A_N = A_N \Leftrightarrow A_N = \mathbf{0}_m \text{ or } \mathbf{1}_m$

$$(7) \quad \zeta A_N = A_N \text{ and } (A_N)^\zeta = A_N \Leftrightarrow \zeta = 1$$

Theorem 3. Let $A_N, B_N, A_N^{(1)}$, and $A_N^{(2)}$ be the CmPFS with usual notations. Then,

- (1) If $A_N \subseteq_P A_N^{(1)}$ and $A_N \subseteq_P A_N^{(2)}$, then $A_N \subseteq_P A_N^{(1)} \cap_P A_N^{(2)}$
- (2) If $A_N^{(1)} \subseteq_P A_N$ and $A_N^{(2)} \subseteq_P A_N$, then $A_N^{(1)} \cup_P A_N^{(2)} \subseteq_P A_N$
- (3) If $A_N^{(1)} \subseteq_P A_N$ and $A_N^{(2)} \subseteq_P A_N$, then $A_N^{(1)} \cap_P A_N^{(2)} \subseteq_P A_N$

Theorem 4. Let $A_N, A_N^{(1)}$, and $A_N^{(2)}$ be the usual CmPFS. Then,

TABLE 1: CmPFS A_N .

κ_1	$\langle [0.1, 0.2], [0.8, 0.9], [0, 0], 0.9, 1, 0.3 \rangle$
κ_2	$\langle [0, 0], [0, 0], [0, 0], 1, 1, 1 \rangle$
κ_3	$\langle [0, 0], [0, 0], [0, 0.7], 1, 1, 0.5 \rangle$
κ_4	$\langle [1, 1], [1, 1], [1, 1], 0, 0, 0 \rangle$

- (1) $((A_N)^c)^c = A_N$
- (2) $A_N^{(1)} \subseteq_P A_N^{(2)} \Leftrightarrow (A_N^{(2)})^c \subseteq_P (A_N^{(1)})^c$
- (3) $A_N^{(1)} = A_N^{(2)} \Leftrightarrow (A_N^{(1)})^c = (A_N^{(2)})^c$
- (4) $(A_N^{(1)} \oplus_P A_N^{(2)})^c = (A_N^{(1)})^c \otimes_P (A_N^{(2)})^c$
- (5) $(A_N^{(1)} \oplus_P A_N^{(2)})^c = (A_N^{(1)})^c \otimes_P (A_N^{(2)})^c$
- (6) $((A_N)^c)^c = \zeta((A_N)^c)$
- (7) $(\zeta(A_N)^c)^c = (A_N)^c$
- (8) $(\zeta A_N)^c = ((A_N)^c)^c$

Proof. (1) and (3) are straightforward.

- (2) Let $A_N^{(1)} \subseteq_P A_N^{(2)}$
 $\Leftrightarrow \eta_i^-(\kappa) \leq \zeta_i^-(\kappa), \eta_i^+(\kappa) \leq \zeta_i^+(\kappa) \text{ and } \eta_i(\kappa) \leq \zeta_i(\kappa)$
 $\Leftrightarrow 1 - \zeta_i^-(\kappa) \leq 1 - \eta_i^-(\kappa), 1 - \zeta_i^+(\kappa) \leq 1 - \eta_i^+(\kappa) \text{ and } 1 - \zeta_i(\kappa) \leq 1 - \eta_i(\kappa)$
 $\Leftrightarrow (A_N^{(2)})^c \subseteq_P (A_N^{(1)})^c$.
- (4) $(A_N^{(1)} \oplus_P A_N^{(2)})^c$
 $= \{ \langle \kappa, [\eta_i^-(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa)\zeta_i^-(\kappa), \eta_i^+(\kappa) + \zeta_i^+(\kappa) - \eta_i^+(\kappa)\zeta_i^+(\kappa)],$
 $\eta_i(\kappa) + \zeta_i(\kappa) - \eta_i(\kappa)\zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - \eta_i^+(\kappa) - \zeta_i^+(\kappa) + \eta_i^+(\kappa)\zeta_i^+(\kappa), 1 - \eta_i^-(\kappa) - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa)],$
 $1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i(\kappa)\zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - \eta_i^+(\kappa), 1 - \eta_i^-(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \otimes_P \{ \langle \kappa, [1 - \zeta_i^+(\kappa), 1 - \zeta_i^-(\kappa)], 1 - \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= (A_N^{(1)})^c \otimes_P (A_N^{(2)})^c$.
- (5) $(A_N^{(1)} \oplus_P A_N^{(2)})^c$
 $= \{ \{ \langle \kappa, [\eta_i^-(\kappa)\zeta_i^-(\kappa), \eta_i^-(\kappa)\zeta_i^-(\kappa)], \eta_i(\kappa)\zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - \eta_i^+(\kappa)\zeta_i^+(\kappa), 1 - \eta_i^-(\kappa)\zeta_i^-(\kappa)], 1 - \eta_i(\kappa)\zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [1 - \eta_i^+(\kappa), 1 - \eta_i^-(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \otimes_P \{ \langle \kappa, [1 - \zeta_i^+(\kappa), 1 - \zeta_i^-(\kappa)], 1 - \zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= (A_N^{(1)})^c \otimes_P (A_N^{(2)})^c$.

- (6) $((A_N)^c)^c = \{ \langle \kappa, [(\eta_i^-(\kappa))^c, (\eta_i^+(\kappa))^c], (\eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - (\eta_i^-(\kappa))^c, 1 - (\eta_i^+(\kappa))^c], 1 - (\eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [1 - (1 - (1 - \eta_i^-(\kappa)))^c, 1 - (1 - (1 - \eta_i^+(\kappa)))^c], 1 - (1 - (1 - \eta_i(\kappa)))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [1 - \eta_i^-(\kappa), 1 - \eta_i^+(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \zeta((A_N^{(1)})^c)$.
- (7) $(\zeta(A_N)^c)^c = \{ \langle \kappa, [1 - \eta_i^-(\kappa), 1 - \eta_i^+(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - (1 - (1 - \eta_i^-(\kappa)))^c, 1 - (1 - (1 - \eta_i^+(\kappa)))^c], 1 - (1 - (1 - \eta_i(\kappa)))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [1 - (\eta_i^-(\kappa))^c, 1 - (\eta_i^+(\kappa))^c], 1 - (\eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [(\eta_i^-(\kappa))^c, (\eta_i^+(\kappa))^c], (\eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} = (A_N)^c$.
- (8) $(\zeta A_N)^c = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa))^c, 1 - (1 - \eta_i^+(\kappa))^c], 1 - (1 - \eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= \{ \langle \kappa, [1 - \eta_i^-(\kappa), 1 - \eta_i^+(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}$
 $= \{ \langle \kappa, [1 - \eta_i^-(\kappa), 1 - \eta_i^+(\kappa)], 1 - \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \}^c$
 $= ((A_N)^c)^c$. $\square$

Theorem 5. The two famous laws of crisp set theory, law of contradiction and excluded middle, do not hold in CmPF P -environment as well as R -environment. That is,

- (1) $A_N \cap_P (A_N)^c \neq \mathbf{0}_m$
- (2) $A_N \cup_P (A_N)^c \neq \mathbf{1}_m$
- (3) $A_N \cap_R (A_N)^c \neq \mathbf{1}_m$
- (4) $A_N \cup_R (A_N)^c \neq \mathbf{1}_m$

The counter example for CmPFSs is given as follows.

Example 5. Let

$$A_N = \begin{bmatrix} \kappa_1 & [0.10, 0.12], [0.30, 0.36], [0.26, 0.40], 0.71, 0.68, 0.90 \\ \kappa_2 & [0.40, 0.47], [0.50, 0.60], [0.47, 0.57], 0.80, 0.80, 0.76 \\ \kappa_3 & [0.50, 0.55], [0.60, 0.66], [0.23, 0.34], 0.56, 0.69, 0.44 \\ \kappa_4 & [0.89, 0.93], [0.10, 0.36], [0.56, 0.60], 0.70, 0.20, 0.47 \end{bmatrix} \quad (12)$$

be a CmPFS. Its complement is given by

$$(A_N)^c = \begin{bmatrix} \kappa_1 & [0.88, 0.90], [0.64, 0.70], [0.60, 0.74], 0.29, 0.32, 0.10 \\ \kappa_2 & [0.53, 0.60], [0.40, 0.50], [0.43, 0.53], 0.20, 0.20, 0.24 \\ \kappa_3 & [0.45, 0.50], [0.34, 0.40], [0.66, 0.77], 0.44, 0.31, 0.56 \\ \kappa_4 & [0.07, 0.11], [0.64, 0.90], [0.40, 0.44], 0.30, 0.80, 0.53 \end{bmatrix}. \quad (13)$$

Then,

$$\begin{aligned} A_N \cap_P (A_N)^c &= \begin{bmatrix} \kappa_1 & [0.10, 0.12], [0.30, 0.36], [0.26, 0.40], 0.29, 0.32, 0.10 \\ \kappa_2 & [0.40, 0.47], [0.40, 0.50], [0.43, 0.53], 0.20, 0.20, 0.24 \\ \kappa_3 & [0.45, 0.50], [0.34, 0.40], [0.23, 0.34], 0.44, 0.31, 0.44 \\ \kappa_4 & [0.07, 0.11], [0.10, 0.36], [0.40, 0.44], 0.30, 0.20, 0.47 \end{bmatrix} \neq \mathbf{0}_m, \\ A_N \cap_R (A_N)^c &= \begin{bmatrix} \kappa_1 & [0.10, 0.12], [0.30, 0.36], [0.26, 0.40], 0.71, 0.68, 0.90 \\ \kappa_2 & [0.40, 0.47], [0.40, 0.50], [0.43, 0.53], 0.80, 0.80, 0.76 \\ \kappa_3 & [0.45, 0.50], [0.34, 0.40], [0.23, 0.34], 0.56, 0.69, 0.56 \\ \kappa_4 & [0.07, 0.11], [0.10, 0.36], [0.40, 0.44], 0.70, 0.80, 0.53 \end{bmatrix} \neq \mathbf{0}_m. \end{aligned} \quad (14)$$

Similarly, other cases can be addressed.

Theorem 6. *Let A_N , $A_N^{(1)}$, and $A_N^{(2)}$ be the usual CmPFS. Then,*

- (1) $A_N^{(1)} \subseteq_P A_N^{(2)} \Leftrightarrow \zeta(A_N^{(1)}) \subseteq_P \zeta(A_N^{(2)})$
- (2) $A_N^{(1)} = A_N^{(2)} \Leftrightarrow \zeta(A_N^{(1)}) = \zeta(A_N^{(2)})$
- (3) $\zeta(A_N^{(1)} \oplus_P A_N^{(2)}) = (((A_N^{(1)} \oplus_P A_N^{(2)})^c)^c)^c$
- (4) $\zeta(A_N^{(1)} \otimes_P A_N^{(2)}) = (((A_N^{(1)} \otimes_P A_N^{(2)})^c)^c)^c$
- (5) $(A_N^{(1)} \otimes_P A_N^{(2)})^c = (A_N^{(1)})^c \otimes_P (A_N^{(2)})^c$
- (6) $\zeta(A_N^{(1)}) \otimes_P \zeta(A_N^{(2)}) = (((A_N^{(1)})^c)^c \otimes_P ((A_N^{(2)})^c)^c)^c$
- (7) $\zeta(A_N^{(1)} \oplus_P A_N^{(2)}) = \zeta(A_N^{(1)}) \oplus_P \zeta(A_N^{(2)})$

Proof. We discuss proof for (1), (3), (6), and (7). Rest of the parts have similar proof.

$$\begin{aligned}
(1) \quad & \begin{aligned} & c(A_N^{(1)} \otimes_{\mathbb{P}} A_N^{(2)}) \Rightarrow \eta_i(\kappa) \leq \zeta_i(\kappa), \eta_i^+(\kappa) \leq \zeta_i^+(\kappa) \text{ and } \eta_i(\kappa) \leq \zeta_i(\kappa), \text{ for all } i \in \mathbb{N} \\ & \Rightarrow (1 - \eta_i(\kappa))^c \leq 1 - \zeta_i(\kappa), (1 - \eta_i^+(\kappa))^c \leq 1 - \zeta_i^+(\kappa) \text{ and } (1 - \eta_i(\kappa))^c \leq 1 - \zeta_i(\kappa) \\ & \Rightarrow (1 - \eta_i(\kappa))^c \leq (1 - \zeta_i(\kappa))^c, (1 - \eta_i^+(\kappa))^c \leq (1 - \zeta_i^+(\kappa))^c \text{ and } (1 - \eta_i(\kappa))^c \leq (1 - \zeta_i(\kappa))^c \\ & \Rightarrow c((A_N^{(1)})^c \otimes_{\mathbb{P}} (A_N^{(2)})^c) \leq (1 - (1 - \eta_i^+(\kappa))^c) \leq (1 - (1 - \zeta_i^+(\kappa))^c) \text{ and } (1 - (1 - \eta_i(\kappa))^c) \leq (1 - (1 - \zeta_i(\kappa))^c) \\ & \Rightarrow c((A_N^{(1)})^c \otimes_{\mathbb{P}} (A_N^{(2)})^c)^c \leq (1 - (1 - \zeta_i(\kappa))^c) \end{aligned} \\
& \begin{aligned} & c(A_N^{(1)} \otimes_{\mathbb{P}} A_N^{(2)}) = c(\{ \langle \kappa, [\eta_i(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa)\zeta_i^-(\kappa), \eta_i^+(\kappa) + \zeta_i^+(\kappa) - \eta_i^-(\kappa)\zeta_i^+(\kappa), \eta_i(\kappa) + \zeta_i(\kappa) - \eta_i^-(\kappa)\zeta_i(\kappa)]_{i=1}^m \rangle : \kappa \in \mathbb{N} \}) \\ & = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa)\zeta_i^-(\kappa))^c, 1 - (1 - \eta_i^-(\kappa) + \zeta_i^+(\kappa) - \eta_i^-(\kappa)\zeta_i^+(\kappa))^c], \\ & 1 - (1 - \eta_i(\kappa) + \zeta_i(\kappa) - \eta_i^-(\kappa)\zeta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ (2) \quad & = \{ \langle \kappa, [(1 - \eta_i^-(\kappa) - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa))^c, 1 - (1 - \eta_i^-(\kappa) - \zeta_i^+(\kappa) + \eta_i^-(\kappa)\zeta_i^+(\kappa))^c], \\ & (1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i^-(\kappa)\zeta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \} \\ & = \{ \{ \langle \kappa, [1 - \eta_i^-(\kappa) - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa)]_{i=1}^m, 1 - \eta_i^-(\kappa) - \zeta_i^+(\kappa) + \eta_i^-(\kappa)\zeta_i^+(\kappa) \rangle_{i=1}^m \\ & \quad + \langle \eta_i^-(\kappa)\zeta_i^-(\kappa), 1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i^-(\kappa)\zeta_i(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \}^c \\ & = c(\{ (A_N^{(1)} \otimes_{\mathbb{P}} A_N^{(2)})^c \}^c)^c. \end{aligned} \\
(6) \quad & \begin{aligned} & c(A_N^{(1)}) \otimes_{\mathbb{P}^c} c(A_N^{(2)}) = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa))^c, 1 - (1 - \eta_i^+(\kappa))^c], \\ & (1 - (1 - \eta_i(\kappa))^c) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \otimes_{\mathbb{P}^c} \{ \langle \kappa, [1 - (1 - \eta_i^+(\kappa))^c, \\ & (1 - (1 - \eta_i^-(\kappa))^c), 1 - (1 - \zeta_i^+(\kappa))^c, 1 - (1 - \zeta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa))^c - (1 - \zeta_i^+(\kappa))^c + (1 - \eta_i^-(\kappa))^c - (1 - \zeta_i^+(\kappa))^c \\ & - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa))^c, 1 - (1 - \eta_i^+(\kappa))^c - (1 - \zeta_i^+(\kappa))^c - (1 - \eta_i^-(\kappa))^c \\ & + (1 - \eta_i^+(\kappa) - \zeta_i^+(\kappa) + \eta_i^+(\kappa)\zeta_i^+(\kappa))^c, 1 - (1 - \eta_i(\kappa))^c - (1 - \zeta_i(\kappa))^c \\ & + (1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i^-(\kappa)\zeta_i(\kappa))^c, \zeta_i^-(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = \{ \langle \kappa, [(1 - \eta_i^-(\kappa))^c + (1 - \zeta_i^+(\kappa))^c - (1 - \eta_i^-(\kappa) - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa))^c, \\ & (1 - \eta_i^+(\kappa))^c + (1 - \zeta_i^+(\kappa))^c - (1 - \eta_i^+(\kappa) - \zeta_i^+(\kappa) + \eta_i^+(\kappa)\zeta_i^+(\kappa))^c, \\ & (1 - \eta_i(\kappa))^c + (1 - \zeta_i(\kappa))^c - (1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i^-(\kappa)\zeta_i(\kappa))^c, \zeta_i^-(\kappa) \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = \{ \langle \kappa, [(1 - \eta_i^-(\kappa))^c, (1 - \eta_i^+(\kappa))^c], (1 - \eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \otimes_{\mathbb{P}^c} \{ \langle \kappa, [(1 - \eta_i^+(\kappa))^c, (1 - \eta_i^-(\kappa))^c], \\ & (1 - \eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = ((A_N^{(1)})^c)^c \otimes_{\mathbb{P}^c} ((A_N^{(2)})^c)^c \\ & = ((A_N^{(2)})^c)^c \otimes_{\mathbb{P}^c} ((A_N^{(1)})^c)^c. \end{aligned} \\
(7) \quad & \begin{aligned} & c(A_N^{(1)}) \otimes_{\mathbb{P}^c} c(A_N^{(2)}) = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa))^c, 1 - (1 - \eta_i^+(\kappa))^c], 1 - (1 - \eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \otimes_{\mathbb{P}^c} \\ & = \{ \langle \kappa, [1 - (1 - \zeta_i^-(\kappa))^c, 1 - (1 - \zeta_i^+(\kappa))^c], 1 - (1 - \eta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = \{ \langle \kappa, [1 - (1 - \eta_i^-(\kappa) - \zeta_i^-(\kappa) + \eta_i^-(\kappa)\zeta_i^-(\kappa))^c, 1 - (1 - \eta_i^-(\kappa) - \zeta_i^+(\kappa) + \eta_i^-(\kappa)\zeta_i^+(\kappa))^c], \\ & 1 - (1 - \eta_i(\kappa) - \zeta_i(\kappa) + \eta_i^-(\kappa)\zeta_i(\kappa))^c \rangle_{i=1}^m : \kappa \in \mathbb{N} \} \\ & = c(\{ \langle \kappa, [\eta_i^-(\kappa) + \zeta_i^-(\kappa) - \eta_i^-(\kappa)\zeta_i^-(\kappa), \eta_i^+(\kappa) + \zeta_i^+(\kappa) - \eta_i^-(\kappa)\zeta_i^+(\kappa), \eta_i(\kappa) + \zeta_i(\kappa) - \eta_i^-(\kappa)\zeta_i(\kappa)]_{i=1}^m \rangle : \kappa \in \mathbb{N} \}) \\ & = c(A_N^{(1)} \otimes_{\mathbb{P}} A_N^{(2)}). \end{aligned} \end{aligned}$$

Definition 8. Let $g, h \in [0, 1]$. The upper g -level set of a CmPFS,

$$A_{\aleph} = \{ \langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in \aleph \}, \quad (15)$$

is denoted by $\mathfrak{U}(A_N, g)$ and is defined as

$$\mathfrak{U}(A_N, g) = \{\kappa \in \mathbb{N}: \eta_i^-(\kappa) \geq g\}. \quad (16)$$

The lower h-level set of the set A_N is denoted by $\mathfrak{L}(A_N, h)$ and is defined as

$$\mathfrak{L}(A_N, h) = \{\kappa \in N: \eta_i(\kappa) \leq h\}. \quad (17)$$

The strong upper g-level set and the strong lower h-level set of A_N can be, respectively, defined as

$$\mathfrak{U}_s(A_N, g) = \{\kappa \in N: \eta_i^-(\kappa) > g\}, \quad (19)$$

$$\mathfrak{L}_s(A_N, h) = \{\kappa \in N: \eta_i(\kappa) < h\}.$$

Moreover, if we put the restriction $g + h \leq 1$, then we can define (g, h) -level set of A_N as

$$^{(g,h)}(A_N) = \{\kappa \in N: \eta_i^-(\kappa) \geq g, \eta_i(\kappa) \leq h\}. \quad (19)$$

Similarly, strong (g, h) -level set of A_N can be defined as

$$^{(g,h)}(A_N)_s = \{\kappa \in N: \eta_i^-(\kappa) > g, \eta_i(\kappa) < h\}. \quad (20)$$

Example 6. In Example 4, the level sets are given as follows:

$$\begin{aligned} \mathbf{U}(A_N, 0.23) &= \{\kappa_2, \kappa_3\} \cdot \mathbf{U}_s(A_N, 0.23) = \{\kappa_2\}, \\ \mathbf{Q}(A_N, 0.8) &= \{\kappa_2, \kappa_3, \kappa_4\} \cdot \mathbf{Q}_s(A_N, 0.68) = \{\kappa_3, \kappa_4\}, \\ {}^{(0.23, 0.8)}(A_N) &= \{\kappa_2, \kappa_3\} \cdot {}^{(0.23, 0.8)}(A_N)_s = \{\}. \end{aligned} \quad (21)$$

Remark 3. $^{(g,h)}(A_N) = \{\kappa \in N: \eta_i^-(\kappa) \geq g, \eta_i(\kappa) \leq h | \kappa \in N: \eta_i^-(\kappa) \geq g\} \cap \{\kappa \in N: \eta_i(\kappa) \leq h\} = \mathfrak{U}(A_N, g) \cap \mathfrak{Z}(A_N, h).$

This shows that (g, h) -level set of A_N is actually an intersection of upper g -level and lower h -level sets.

Definition 9. Two CmPFSs are given by

$$\begin{aligned} A_N^{(1)} &= \{\langle \kappa_1, [\eta_i^-(\kappa_1), \eta_i^+(\kappa_1)], \eta_i(\kappa_1) \rangle_{i=1}^m : \kappa_1 \in \mathfrak{N}_1\}, \\ A_N^{(2)} &= \{\langle \kappa_2, [\zeta_i^-(\kappa_2), \zeta_i^+(\kappa_2)], \zeta_i(\kappa_2) \rangle_{i=1}^m : \kappa_2 \in \mathfrak{N}_2\}. \end{aligned} \quad (22)$$

The four versions of Cartesian products of the two CmPFSs are given by

$$\begin{aligned}
A_N^{(1)} \times_P A_N^{(2)} &= \{ \langle (\kappa_1, \kappa_2), [\eta_i^-(\kappa_1) \zeta_i^-(\kappa_2), \eta_i^+(\kappa_1) \zeta_i^+(\kappa_2)], \eta_i(\kappa_1) \zeta_i(\kappa_2) \rangle_{i=1}^m : \kappa_2 \in \aleph_2, \kappa_1 \in \aleph_1 \}, \\
A_N^{(1)} \times_R A_N^{(2)} &= \{ \langle (\kappa_1, \kappa_2), [\eta_i^-(\kappa_1) \zeta_i^-(\kappa_2), \eta_i^+(\kappa_1) \zeta_i^+(\kappa_2)], \eta_i(\kappa_1) + \zeta_i(\kappa_2) - \eta_i(\kappa_1) \zeta_i(\kappa_2) \rangle_{i=1}^m : \kappa_2 \in \aleph_2, \kappa_1 \in \aleph_1 \}, \\
A_N^{(1)} \times_P A_N^{(2)} &= \{ \langle (\kappa_1, \kappa_2), [\eta_i^-(\kappa_1) \wedge \zeta_i^-(\kappa_2), \eta_i^+(\kappa_1) \wedge \zeta_i^+(\kappa_2)], \eta_i(\kappa_1) \wedge \zeta_i(\kappa_2) \rangle_{i=1}^m : \kappa_2 \in \aleph_2, \kappa_1 \in \aleph_1 \}, \\
A_N^{(1)} \times_R A_N^{(2)} &= \{ \langle (\kappa_1, \kappa_2), [\eta_i^-(\kappa_1) \wedge \zeta_i^-(\kappa_2), \eta_i^+(\kappa_1) \wedge \zeta_i^+(\kappa_2)], \eta_i(\kappa_1) \vee \zeta_i(\kappa_2) \rangle_{i=1}^m : \kappa_2 \in \aleph_2, \kappa_1 \in \aleph_1 \}.
\end{aligned} \tag{23}$$

Interestingly, the cubic Cartesian product of the CmPFSs is again a CmPFS.

Theorem 7. Let $A_N^{(1)}$, $A_N^{(2)}$, and A_N^3 be CmpFSSs. Then,

- (1) $A_N^{(1)} \cup_R A_N^{(2)} = A_N^{(2)} \cup_R A_N^{(1)}$
- (2) $A_N^{(1)} \cap_R A_N^{(2)} = A_N^{(2)} \cap_R A_N^{(1)}$
- (3) $A_N^{(1)} \subseteq_R A_N^{(2)}$ and $A_N^{(2)} \subseteq_R A_N^3$ implies $A_N^{(1)} \subseteq_R A_N^3$
- (4) $A_N^{(1)} \cup_R (A_N^{(2)} \cup_R A_N^3) = (A_N^{(1)} \cup_R A_N^{(2)}) \cup_R A_N^3$
- (5) $A_N^{(1)} \cap_R (A_N^{(2)} \cap_R A_N^3) = (A_N^{(1)} \cap_R A_N^{(2)}) \cap_R A_N^3$
- (6) $A_N^{(1)} \cup_R (A_N^{(2)} \cap_R A_N^3) = (A_N^{(1)} \cup_R A_N^{(2)}) \cap_R (A_N^{(1)} \cup_R A_N^3)$
- (7) $A_N^{(1)} \cap_R (A_N^{(2)} \cup_R A_N^3) = (A_N^{(1)} \cap_R A_N^{(2)}) \cup_R (A_N^{(1)} \cap_R A_N^3)$

Theorem 8. Let A_N , Φ_m , and $\mathbf{U}_m$ be the CmpFSS, and null and absolute CmpFSSs, respectively. Then,

- (1) $A_N \cup_R \Phi_m = A_N$
- (2) $A_N \cap_R \Phi_m = \Phi_m$
- (3) $A_N = A_N \cup_R A_N$
- (4) $A_N \cap_R A_N = A_N$
- (5) $A_N \cup_R \mathbf{U}_m = \mathbf{U}_m$
- (6) $A_N \cap_R \mathbf{U}_m = A_N$

Proposition 2. Let $A_N^{(1)}$ and $A_N^{(2)}$. Then,

- (1) $(A_N^{(1)} \cup_R A_N^{(2)})^c = (A_N^{(1)})^c \cap_R (A_N^{(2)})^c$
- (2) $(A_N^{(1)} \cap_R A_N^{(2)})^c = (A_N^{(1)})^c \cup_R (A_N^{(2)})^c$

Theorem 9. Let A_N , B_N , $A_N^{(1)}$, and $A_N^{(2)}$ be the CmpFSS with usual notations. Then,

- (1) If $A_N^{(1)} \subseteq_R A_N^{(2)}$ and $A_N^{(2)} \subseteq_R A_N$, then $A_N^{(1)} \subseteq_R A_N$
- (2) If $A_N \subseteq_R A_N^{(1)}$ and $A_N \subseteq_R A_N^{(2)}$, then $A_N \subseteq_R A_N^{(1)} \cap_R A_N^{(2)}$
- (3) If $A_N^{(1)} \subseteq_R A_N$ and $A_N^{(2)} \subseteq_R A_N$, then $A_N^{(1)} \cup_R A_N^{(2)} \subseteq_R A_N$
- (4) If $A_N^{(1)} \subseteq_R A_N$ and $A_N^{(2)} \subseteq_R B_N$, then $A_N^{(1)} \cup_R A_N^{(2)} \subseteq_R A_N \cup_R B_N$ and $A_N^{(1)} \cap_R A_N^{(2)} \subseteq_R A_N \cap_R B_N$

Theorem 10. Let A_N , $A_N^{(1)}$, and $A_N^{(2)}$ be the usual CmpFSS. Then,

- (1) $A_N^{(1)} \subseteq_R A_N^{(2)} \Leftrightarrow (A_N^{(2)})^c \subseteq_R (A_N^{(1)})^c$
- (2) $(A_N^{(1)} \oplus_R A_N^{(2)})^c = (A_N^{(1)})^c \otimes_R (A_N^{(2)})^c$
- (3) $(A_N^{(1)} \otimes_R A_N^{(2)})^c = (A_N^{(1)})^c \oplus_R (A_N^{(2)})^c$

Theorem 11. Let A_N , $A_N^{(1)}$, and $A_N^{(2)}$ be the usual CmpFSS. Then,

- (1) $A_N^{(1)} \subseteq_R A_N^{(2)} \Leftrightarrow \varsigma(A_N^{(1)}) \subseteq_R \varsigma(A_N^{(2)})$
- (2) $\varsigma(A_N^{(1)} \oplus_R A_N^{(2)}) = (((A_N^{(1)} \oplus_R A_N^{(2)})^c)^c)^c$
- (3) $\varsigma(A_N^{(1)} \otimes_R A_N^{(2)}) = (((A_N^{(1)} \otimes_R A_N^{(2)})^c)^c)^c$
- (4) $(A_N^{(1)} \otimes_R A_N^{(2)})^c = (A_N^{(1)})^c \otimes_R (A_N^{(2)})^c$
- (5) $\varsigma(A_N^{(1)} \otimes_R \varsigma(A_N^{(2)})) = (((A_N^{(1)})^c)^c \oplus_R ((A_N^{(2)})^c)^c)^c$

$$(6) \varsigma(A_N^{(1)} \oplus_R A_N^{(2)}) = \varsigma(A_N^{(1)}) \oplus_R \varsigma(A_N^{(2)})$$

Proof. Straightforward. $\square$

3.1. Cubic m -Polar Fuzzy Relations. Now, we define cubic m -polar fuzzy relations (CmpFRs) and give their different types.

Definition 10. Let $\mathcal{X}_c$ and $\mathcal{Y}_c$ be the two crisp universes. A CmpFR $\mathcal{X}_c \longrightarrow \mathcal{Y}_c$ is an entity given by

$$R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c} = \{ \langle (u, v), [\eta_i^-, \eta_i^+], \eta_i \rangle_{i=1}^m : u \in \mathcal{X}_c, v \in \mathcal{Y}_c \}, \quad (24)$$

where $0 \leq \eta_i^-, \eta_i^+, \eta_i \leq 1$, and $\eta_i^- \leq \eta_i^+$ for all $i = 1, 2, \dots, m$.

Definition 11. The different types of the CmpFRs are given as follows:

- (i) A CmpFR $R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}$ is said to be a reflexive if $[\eta_i^-, \eta_i^+] = \mathbf{1}$ and $\eta_i = 0$ for all i and $u \in \mathcal{X}_c, v \in \mathcal{Y}_c$
- (ii) $R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}$ is said to be CmpFR of symmetric type if $R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}(u, v) = R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}(v, u)$ for all $u \in \mathcal{X}_c, v \in \mathcal{Y}_c$
- (iii) $R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}$ is said to be CmpFR of transitive type if $R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}(u, w) \geq_p \vee_v (R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}(u, v) \wedge R_{\mathcal{X}_c \longrightarrow \mathcal{Y}_c}(v, w))$

4. Correlation Measures for CmpFSSs

In this section, we define correlation measure for two cubic m -polar fuzzy sets (CmpFSSs) which can describe the informational settings for CmpFSSs to vary or tend to vary together. If two settings tend to vary in some direction, simultaneously, they are said to be correlated. If both of them increase/decrease, simultaneously, they are positively correlated, otherwise negatively correlated. Correlation actually measures the closeness of the two variables. We establish certain properties of proposed correlation measures and also discuss novelty and effectiveness of these measures.

Definition 12. Let $A_N = \{ \langle \kappa, [\eta_i^-(\kappa), \eta_i^+(\kappa)], \eta_i(\kappa) \rangle_{i=1}^m : \kappa \in N \}$ be a CmpFSS over the discourse N . The average or mean of A_N can be defined as follows:

$$\begin{aligned} \overline{A_N} &= \langle [\overline{\eta_i^-}, \overline{\eta_i^+}], \overline{\eta_i} \rangle_{i=1}^m \\ &= \left\langle \left[\frac{1}{|N|} \sum_{\kappa} \eta_i^-(\kappa), \frac{1}{|N|} \sum_{\kappa} \eta_i^+(\kappa) \right], \frac{1}{|N|} \sum_{\kappa} \eta_i(\kappa) \right\rangle_{i=1}^m. \end{aligned} \quad (25)$$

If we take the universe $N = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and CmpFSS

$$A_N = \begin{pmatrix} \kappa_1 & [\eta_{11}^-, \eta_{11}^+], [\eta_{12}^-, \eta_{12}^+], [\eta_{13}^-, \eta_{13}^+], \eta_{14}, \eta_{15}, \eta_{16} \\ \kappa_2 & [\eta_{21}^-, \eta_{21}^+], [\eta_{22}^-, \eta_{22}^+], [\eta_{23}^-, \eta_{23}^+], \eta_{24}, \eta_{25}, \eta_{26} \\ \kappa_3 & [\eta_{31}^-, \eta_{31}^+], [\eta_{32}^-, \eta_{32}^+], [\eta_{33}^-, \eta_{33}^+], \eta_{34}, \eta_{35}, \eta_{36} \\ \kappa_4 & [\eta_{41}^-, \eta_{41}^+], [\eta_{42}^-, \eta_{42}^+], [\eta_{43}^-, \eta_{43}^+], \eta_{44}, \eta_{45}, \eta_{46} \end{pmatrix}, \quad (26)$$

then the average of A_N is given by $\overline{A_N} = [\overline{\eta_1^-}, \overline{\eta_1^+}]$, $[\overline{\eta_2^-}, \overline{\eta_2^+}]$, $[\overline{\eta_3^-}, \overline{\eta_3^+}]$, $\overline{\eta_4}$, $\overline{\eta_5}$, $\overline{\eta_6}$, where

$$\begin{aligned} \overline{\eta_1^-} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j1}^-, \\ \overline{\eta_1^+} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j1}^+, \\ \overline{\eta_2^-} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j2}^-, \\ \overline{\eta_2^+} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j2}^+, \\ \overline{\eta_3^-} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j3}^-, \\ \overline{\eta_3^+} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j3}^+, \\ \overline{\eta_4} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j4}, \\ \overline{\eta_5} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j5}, \\ \overline{\eta_6} &= \frac{1}{4} \sum_{j=1}^4 \eta_{j6}. \end{aligned} \quad (27)$$

Definition 13. Let

$$\begin{aligned} A_N &= \begin{pmatrix} \kappa_1 & [\eta_{11}^-, \eta_{11}^+], [\eta_{12}^-, \eta_{12}^+], \dots, [\eta_{1m}^-, \eta_{1m}^+], \eta_{11}, \eta_{12}, \dots, \eta_{1m} \\ \kappa_2 & [\eta_{21}^-, \eta_{21}^+], [\eta_{22}^-, \eta_{22}^+], \dots, [\eta_{2m}^-, \eta_{2m}^+], \eta_{21}, \eta_{22}, \dots, \eta_{2m} \\ \vdots & \dots \\ \kappa_n & [\eta_{n1}^-, \eta_{n1}^+], [\eta_{n2}^-, \eta_{n2}^+], \dots, [\eta_{nm}^-, \eta_{nm}^+], \eta_{n1}, \eta_{n2}, \dots, \eta_{nm} \end{pmatrix}, \\ B_N &= \begin{pmatrix} \kappa_1 & [\zeta_{11}^-, \zeta_{11}^+], [\zeta_{12}^-, \zeta_{12}^+], \dots, [\zeta_{1m}^-, \zeta_{1m}^+], \zeta_{11}, \zeta_{12}, \dots, \zeta_{1m} \\ \kappa_2 & [\zeta_{21}^-, \zeta_{21}^+], [\zeta_{22}^-, \zeta_{22}^+], \dots, [\zeta_{2m}^-, \zeta_{2m}^+], \zeta_{21}, \zeta_{22}, \dots, \zeta_{2m} \\ \vdots & \dots \\ \kappa_n & [\zeta_{n1}^-, \zeta_{n1}^+], [\zeta_{n2}^-, \zeta_{n2}^+], \dots, [\zeta_{nm}^-, \zeta_{nm}^+], \zeta_{n1}, \zeta_{n2}, \dots, \zeta_{nm} \end{pmatrix} \end{aligned} \quad (28)$$

be the two CmPFSs over $N = \{\kappa_1, \kappa_2, \kappa_3, \dots, \kappa_n\}$. The left and right relative informational coefficients (RIC) of A_N over B_N are, respectively, defined by

$$\begin{aligned} I_j^-(A_N \rightarrow B_N) &= \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-}), \\ I_j^+(A_N \rightarrow B_N) &= \sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})(\zeta_{ij}^+ - \overline{\zeta_j^+}), \quad j = 1, 2, \dots, m, \end{aligned} \quad (29)$$

and relative noninformational coefficients (RNICs) by

$$I_j(A_N \rightarrow B_N) = \sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})(\zeta_{ij} - \overline{\zeta_j}), \quad j = 1, 2, \dots, m. \quad (30)$$

Similarly, the left and right informational coefficients of A_N are, respectively, given by

$$\begin{aligned} \mathbb{C}_j^-(A_N) &= \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2, \\ \mathbb{C}_j^+(A_N) &= \sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})^2, \end{aligned} \quad (31)$$

and noninformational coefficient by

$$\mathbb{C}_j(A_N) = \sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})^2, \quad j = 1, 2, \dots, m. \quad (32)$$

Proposition 3. The RICs/RNICs and the ICs/NICs for the CmPFSs A_N and B_N follow the following properties:

- $I_j^-(A_N \rightarrow A_N) = \mathbb{C}_j^-(A_N)$, $I_j^+(A_N \rightarrow A_N) = \mathbb{C}_j^+(A_N)$, $I_j(A_N \rightarrow A_N) = \mathbb{C}_j(A_N)$
- $I_j^-(A_N \rightarrow B_N) = I_j^-(B_N \rightarrow A_N)$, $I_j^+(A_N \rightarrow B_N) = I_j^+(B_N \rightarrow A_N)$, $I_j(A_N \rightarrow B_N) = I_j(B_N \rightarrow A_N)$
- $|I_j^-(A_N \rightarrow B_N)| \leq \sqrt{\mathbb{C}_j^-(A_N) \mathbb{C}_j^-(B_N)}$, $|I_j^+(A_N \rightarrow B_N)| \leq \sqrt{\mathbb{C}_j^+(A_N) \mathbb{C}_j^+(B_N)}$, $|I_j(A_N \rightarrow B_N)| \leq \sqrt{\mathbb{C}_j(A_N) \mathbb{C}_j(B_N)}$

Proof. (a) and (b) are easy to address.

(c) The well-known Cauchy–Schwarz inequality states that $(\sum_{q=1}^k u_q v_q)^2 \leq (\sum_{q=1}^k u_q^2)(\sum_{q=1}^k v_q^2)$ for $u_q, v_q \in R$. We will use the above inequality to prove our result.

Consider

$$\begin{aligned} (I_j^-(A_N \rightarrow B_N))^2 &= \left(\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-}) \right)^2 \\ &\leq \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2 \\ &= \mathbb{C}_j^-(A_N) \mathbb{C}_j^-(B_N) \\ &\Rightarrow |I_j^-(A_N \rightarrow B_N)| \leq \sqrt{\mathbb{C}_j^-(A_N) \mathbb{C}_j^-(B_N)}, \quad \text{for all } j. \end{aligned} \quad (33)$$

The other two results are similar. $\square$

Definition 14. If A_N and B_N are the two CmPFSs with the usual notations, then their relative correlation coefficient (RCC) is given by

$$\partial(A_N, B_N) = \frac{1}{3m} \left(\sum_{j=1}^m \partial_j^-(A_N, B_N) + \sum_{j=1}^m \partial_j^+(A_N, B_N) + \sum_{j=1}^m \partial_j(A_N, B_N) \right), \quad (34)$$

where

$$\begin{aligned} \partial_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}}, \\ \partial_j^+(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})(\zeta_{ij}^+ - \overline{\zeta_j^+})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})^2 \sum_{i=1}^n (\zeta_{ij}^+ - \overline{\zeta_j^+})^2}}, \\ \partial_j(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})(\zeta_{ij} - \overline{\zeta_j})}{\sqrt{\sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})^2 \sum_{i=1}^n (\zeta_{ij} - \overline{\zeta_j})^2}}, \end{aligned} \quad (35)$$

$j = 1, 2, \dots, m$.

Theorem 12. For the CmPFSs A_N and B_N , we have

- (a) The correlation ∂ is symmetrical, that is, $\partial(A_N, B_N) = \partial(B_N, A_N)$
- (b) The correlation ∂ finds the interval $[-1, 1]$, that is, $-1 \leq \partial(A_N, B_N) \leq 1$
- (c) $\partial(A_N, B_N) = 1$ if $A_N = B_N$
- (d) $\partial(A_N, B_N) = 1$ if $A_N = (\lambda)B_N$, $\lambda > 0$
(here, $(\lambda)B_N$ means $\left\{ \langle [\lambda\zeta_{ij}^-, \lambda\zeta_{ij}^+], \lambda\zeta_{ij} \rangle_{j=1}^m \right\}$)
- (e) $\partial(A_N, B_N) = -1$ if $A_N = \lambda(B_N)^c$, $\lambda > 0$

Proof

- (a) Straightforward.
- (b) From Proposition 3, we have

$$\begin{aligned} \left| I_j^-(A_N \rightarrow B_N) \right| &\leq \sqrt{C_j^-(A_N) C_j^-(B_N)} \\ &\Rightarrow -\sqrt{C_j^-(A_N) C_j^-(B_N)} \leq I_j^-(A_N \rightarrow B_N) \leq \sqrt{C_j^-(A_N) C_j^-(B_N)} \\ &\Rightarrow -1 \leq \frac{I_j^-(A_N \rightarrow B_N)}{\sqrt{C_j^-(A_N) C_j^-(B_N)}} \leq 1. \end{aligned} \quad (36)$$

Hence, $-1 \leq \partial_j^-(A_N, B_N) \leq 1$.

Similarly, $-1 \leq \partial_j^+(A_N, B_N) \leq 1$ and $-1 \leq \partial_j(A_N, B_N) \leq 1$, for all $j = 1, 2, \dots, m$.

Consequently, $-1 \leq \partial(A_N, B_N) \leq 1$, being the mean of the quantities between -1 and 1.

- (c) If $A_N = B_N$, then $\eta_{ij}^- = \zeta_{ij}^-$, $\eta_{ij}^+ = \zeta_{ij}^+$, $\eta_{ij} = \zeta_{ij}$.
Therefore, $\partial_j^-(A_N, B_N) = \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\eta_{ij}^- - \overline{\eta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2}} = 1$.

$$\frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\eta_{ij}^- - \overline{\eta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2}} = 1.$$

Similarly, $\partial_j^+(A_N, B_N) = 1$ and $\partial_j(A_N, B_N) = 1$, and hence, $\partial(A_N, B_N) = 1$.

- (d) If $A_N = (\lambda)B_N$, $\lambda > 0$, then $\eta_{ij}^- = \lambda\zeta_{ij}^-$, $\eta_{ij}^+ = \lambda\zeta_{ij}^+$, and $\eta_{ij} = \lambda\zeta_{ij}$. Therefore,

$$\partial_j^-(A_N, B_N) = \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2 \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = \frac{\sum_{i=1}^n (\lambda\zeta_{ij}^- - \lambda\overline{\zeta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\lambda\zeta_{ij}^- - \lambda\overline{\zeta_j^-})^2 \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = 1. \quad (37)$$

Similarly, $\partial_j^+(A_N, B_N) = 1$, $\partial_j(A_N, B_N) = 1$, and hence, $\partial(A_N, B_N) = 1$.

- (e) If $A_N = \lambda(B_N)^c$, $\lambda > 0$, then $\eta_{ij}^- = \lambda(1 - \zeta_{ij}^-)$, $\eta_{ij}^+ = \lambda(1 - \zeta_{ij}^+)$, and $\eta_{ij} = \lambda(1 - \zeta_{ij})$.

Therefore, $\overline{\eta_j^-} = 1 - \overline{\zeta_j^-}$ and

$$\begin{aligned}\partial_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2} \sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = \frac{\sum_{i=1}^n (\lambda(1 - \zeta_{ij}^-) - \lambda(1 - \overline{\zeta_j^-}))(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\lambda(1 - \zeta_{ij}^-) - \lambda(1 - \overline{\zeta_j^-}))^2} \sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} \\ &= - \frac{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2} \sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = - \frac{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}{\sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2} \sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = -1.\end{aligned}\quad (38)$$

Similarly, $\partial_j^+(A_N, B_N) = -1$, $\partial_j(A_N, B_N) = -1$, and hence, $\partial(A_N, B_N) = -1$. $\square$

Definition 15. For the CmPFSs A_N and B_N , the RCC may also be defined as

$$\sigma(A_N, B_N) = \frac{1}{3m} \left(\sum_{j=1}^m \sigma_j^-(A_N, B_N) + \sum_{j=1}^m \sigma_j^+(A_N, B_N) + \sum_{j=1}^m \sigma_j(A_N, B_N) \right), \quad (39)$$

where

$$\begin{aligned}\sigma_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\max \left\{ \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2, \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2 \right\}}, \\ \sigma_j^+(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})(\zeta_{ij}^+ - \overline{\zeta_j^+})}{\max \left\{ \sum_{i=1}^n (\eta_{ij}^+ - \overline{\eta_j^+})^2, \sum_{i=1}^n (\zeta_{ij}^+ - \overline{\zeta_j^+})^2 \right\}}, \\ \sigma_j(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})(\zeta_{ij} - \overline{\zeta_j})}{\max \left\{ \sum_{i=1}^n (\eta_{ij} - \overline{\eta_j})^2, \sum_{i=1}^n (\zeta_{ij} - \overline{\zeta_j})^2 \right\}},\end{aligned}\quad (40)$$

$j = 1, 2, \dots, m$.

Theorem 13. For the CmPFSs A_N and B_N , we have

- (a) The correlation σ is symmetrical, that is, $\sigma(A_N, B_N) = \sigma(B_N, A_N)$
- (b) The correlation σ finds the interval $[-1, 1]$, that is, $-1 \leq \sigma(A_N, B_N) \leq 1$
- (c) $\sigma(A_N, B_N) = 1$ if $A_N = B_N$

Proof

(a) and (b) are straightforward.

(c) Suppose $A_N = B_N$. Then, $\eta_{ij}^- = \zeta_{ij}^-$, $\eta_{ij}^+ = \zeta_{ij}^+$, and $\eta_{ij} = \zeta_{ij}$. In this case,

$$\begin{aligned}\sigma_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\max \left\{ \sum_{i=1}^n (\eta_{ij}^- - \overline{\eta_j^-})^2, \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2 \right\}} = \frac{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})(\zeta_{ij}^- - \overline{\zeta_j^-})}{\max \left\{ \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2, \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2 \right\}} \\ &= \frac{\sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2}{\max \left\{ \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2, \sum_{i=1}^n (\zeta_{ij}^- - \overline{\zeta_j^-})^2 \right\}} = 1.\end{aligned}\quad (41)$$

Similarly, $\sigma_j^+(A_N, B_N) = 1$ and $\sigma_j(A_N, B_N) = 1$. Hence, $\sigma(A_N, B_N) = 1$. $\square$

In most real-world problems, two major aspects should be addressed, that is, uncertainty and weights. While dealing with such problems, uncertainty is a key factor which can affect our decisions and analysis. Besides, most of the

uncertain mathematical models are effected by weights. To deal with such problems, we define weighted CCs.

Definition 16. Let A_N and B_N be the two CmPFSs (usual notation followed), and the alternatives $\kappa_1, \kappa_2, \dots, \kappa_n$ are affected by the weights $\omega_1, \omega_2, \dots, \omega_n$ with the condition

$\sum_{i=1}^n \omega_i = 1$ and $0 \leq \omega_i \leq 1$. We define relative informational weighted coefficients (RIWCs) of A_N over B_N by

$$\begin{aligned} (\text{Left RIWC})^\omega I_{j(A_N \rightarrow B_N)}^- &= \sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-), \\ (\text{Right RIWC})^\omega I_{j(A_N \rightarrow B_N)}^+ &= \sum_{i=1}^n \omega_i (\eta_{ij}^+ - \bar{\eta}_j^+) (\zeta_{ij}^+ - \bar{\zeta}_j^+), \quad j = 1, 2, \dots, m, \end{aligned} \quad (42)$$

and the relative noninformational weighted coefficients (RNIWCs) by

$${}^\omega I_{j(A_N \rightarrow B_N)} = \sum_{i=1}^n \omega_i (\eta_{ij} - \bar{\eta}_j) (\zeta_{ij} - \bar{\zeta}_j), \quad j = 1, 2, \dots, m. \quad (43)$$

Similarly, the informational (left and right) and non-informational weighted coefficients of A_N are, respectively, given by

$$\begin{aligned} {}^\omega \mathbb{C}_j^-(A_N) &= \sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-)^2, \\ {}^\omega \mathbb{C}_j^+(A_N) &= \sum_{i=1}^n \omega_i (\eta_{ij}^+ - \bar{\eta}_j^+)^2, \\ {}^\omega \mathbb{C}_j(A_N) &= \sum_{i=1}^n \omega_i (\eta_{ij} - \bar{\eta}_j)^2, \quad j = 1, 2, \dots, m. \end{aligned} \quad (44)$$

Proposition 4. *The RIWCs/RNIWCs and the IWCs/NIWCs for the CmPFSs A_N and B_N follow the following properties:*

$$\begin{aligned} (a) \quad & {}^\omega I_{j(A_N \rightarrow A_N)}^- = {}^\omega \mathbb{C}_j^-(A_N), \quad {}^\omega I_{j(A_N \rightarrow A_N)}^+ = {}^\omega \mathbb{C}_j^+(A_N), \\ & {}^\omega I_{j(A_N \rightarrow A_N)} = {}^\omega \mathbb{C}_j(A_N) \\ (b) \quad & {}^\omega I_{j(A_N \rightarrow B_N)}^- = {}^\omega I_{j(B_N \rightarrow Z_N)}^-, \quad {}^\omega I_{j(A_N \rightarrow B_N)}^+ = {}^\omega I_{j(B_N \rightarrow Z_N)}^+, \\ & {}^\omega I_{j(A_N \rightarrow B_N)} = {}^\omega I_{j(B_N \rightarrow Z_N)} \\ (c) \quad & |{}^\omega I_{j(A_N \rightarrow B_N)}^-| \leq \sqrt{{}^\omega \mathbb{C}_j^-(A_N) {}^\omega \mathbb{C}_j^-(B_N)}, \quad |{}^\omega I_{j(A_N \rightarrow B_N)}^+| \leq \sqrt{{}^\omega \mathbb{C}_j^+(A_N) {}^\omega \mathbb{C}_j^+(B_N)}, \\ & |{}^\omega I_{j(A_N \rightarrow B_N)}| \leq \sqrt{{}^\omega \mathbb{C}_j(A_N) {}^\omega \mathbb{C}_j(B_N)} \end{aligned}$$

Proof

(a) and (b) are easy to address.

(c) We can prove the results in mirror of Cauchy-Schwarz inequality $(\sum_{q=1}^k u_q v_q)^2 \leq (\sum_{q=1}^k u_q^2) (\sum_{q=1}^k v_q^2)$ for $u_q, v_q \in R$.

Consider

$$\begin{aligned} \left({}^\omega I_{j(A_N \rightarrow B_N)}^- \right)^2 &= \left(\sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-) \right)^2 \\ &= \left(\sum_{i=1}^n \sqrt{\omega_i} (\eta_{ij}^- - \bar{\eta}_j^-) \sqrt{\omega_i} (\zeta_{ij}^- - \bar{\zeta}_j^-) \right)^2 \\ &\leq \sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-)^2 \sum_{i=1}^n \omega_i (\zeta_{ij}^- - \bar{\zeta}_j^-)^2 \\ &= {}^\omega \mathbb{C}_j^-(A_N) {}^\omega \mathbb{C}_j^-(B_N) \\ \Rightarrow \left| {}^\omega I_{j(A_N \rightarrow B_N)}^- \right| &\leq \sqrt{{}^\omega \mathbb{C}_j^-(A_N) {}^\omega \mathbb{C}_j^-(B_N)}, \quad \text{for all } j. \end{aligned} \quad (45)$$

The other two results are similar. $\square$

Definition 17. If A_N and B_N are the two CmPFSs, then their weighted correlation coefficient (WCC) is given by

$$\partial_\omega(A_N, B_N) = \frac{1}{3m} \left(\sum_{j=1}^m {}^\omega \partial_j^-(A_N, B_N) + \sum_{j=1}^m {}^\omega \partial_j^+(A_N, B_N) + \sum_{j=1}^m {}^\omega \partial_j(A_N, B_N) \right), \quad (46)$$

where

$$\begin{aligned}\omega \partial_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-}) (\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}}, \\ \omega \partial_j^+(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^+ - \overline{\eta_j^+}) (\zeta_{ij}^+ - \overline{\zeta_j^+})}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij}^+ - \overline{\eta_j^+})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^+ - \overline{\zeta_j^+})^2}}, \\ \omega \partial_j(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij} - \overline{\eta_j}) (\zeta_{ij} - \overline{\zeta_j})}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij} - \overline{\eta_j})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij} - \overline{\zeta_j})^2}},\end{aligned}\quad (47)$$

$j = 1, 2, \dots, m$.

Theorem 14. If A_N and B_N are CmPFSs with the preassigned weights $(\omega_1, \omega_2, \dots, \omega_n)$, then

- (a) the correlation ∂_ω is symmetrical, that is, $\partial_\omega(A_N, B_N) = \partial_\omega(B_N, A_N)$
- (b) the correlation ∂_ω finds the interval $[-1, 1]$, that is, $-1 \leq \partial_\omega(A_N, B_N) \leq 1$
- (c) $\partial_\omega(A_N, B_N) = 1$ if $A_N = (\lambda)B_N$, $\lambda > 0$ (here, $(\lambda)B_N$ means $\left\{ \langle [\lambda \zeta_{ij}^-, \lambda \zeta_{ij}^+], \lambda \zeta_{ij} \rangle_{j=1}^m \right\}$)
- (d) $\partial_\omega(A_N, B_N) = 1$ if $A_N = B_N$

Proof

(a) Obvious.

(b) By Proposition 4, we have

$$\begin{aligned}\left| \omega I_j^-(A_N \rightarrow B_N) \right| &\leq \sqrt{\omega \mathbb{C}_j^-(A_N) \omega \mathbb{C}_j^-(B_N)} \\ &\Rightarrow -\sqrt{\omega \mathbb{C}_j^-(A_N) \omega \mathbb{C}_j^-(B_N)} \leq \omega I_j^-(A_N \rightarrow B_N) \leq \sqrt{\omega \mathbb{C}_j^-(A_N) \omega \mathbb{C}_j^-(B_N)} \\ &\Rightarrow -1 \leq \omega I_j^-(A_N \rightarrow B_N) \sqrt{\omega \mathbb{C}_j^-(A_N) \omega \mathbb{C}_j^-(B_N)} \leq 1 \\ &\Rightarrow -1 \leq \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-}) (\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} \leq 1 \\ &\Rightarrow -1 \leq \omega \partial_j^-(A_N, B_N) \leq 1.\end{aligned}\quad (48)$$

(c) Suppose $A_N = (\lambda)B_N$. Then, $\eta_{ij}^- = \lambda \zeta_{ij}^-$, $\eta_{ij}^+ = \lambda \zeta_{ij}^+$, and $\eta_{ij} = \lambda \zeta_{ij}$. In this case,

$$\begin{aligned}\omega \partial_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-}) (\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \overline{\eta_j^-})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} \\ &= \frac{\sum_{i=1}^n \omega_i (\lambda \zeta_{ij}^- - \lambda \overline{\zeta_j^-}) (\zeta_{ij}^- - \overline{\zeta_j^-})}{\sqrt{\sum_{i=1}^n \omega_i (\lambda \zeta_{ij}^- - \lambda \overline{\zeta_j^-})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} \\ &= \frac{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}{\sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \overline{\zeta_j^-})^2}} = 1.\end{aligned}\quad (49)$$

Similarly, $\omega \partial_j^+(A_N, B_N) = 1$, $\omega \partial_j(A_N, B_N) = 1$, and hence, $\partial_\omega(A_N, B_N) = 1$, being the average of 1's.

(d) is similar to (c). $\square$

Definition 18. For the CmPFSs A_N and B_N , the WCC may also be defined as

$$\sigma_\omega(A_N, B_N) = \frac{1}{3m} \left(\sum_{j=1}^m \omega \sigma_j^-(A_N, B_N) + \sum_{j=1}^m \omega \sigma_j^+(A_N, B_N) + \sum_{j=1}^m \omega \sigma_j(A_N, B_N) \right), \quad (50)$$

where

$$\begin{aligned}
 {}^{\omega}\sigma_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\max \left\{ \sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-)^2, \sum_{i=1}^n \omega_i (\zeta_{ij}^- - \bar{\zeta}_j^-)^2 \right\}}, \\
 {}^{\omega}\sigma_j^+(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^+ - \bar{\eta}_j^+) (\zeta_{ij}^+ - \bar{\zeta}_j^+)}{\max \left\{ \sum_{i=1}^n \omega_i (\eta_{ij}^+ - \bar{\eta}_j^+)^2, \sum_{i=1}^n \omega_i (\zeta_{ij}^+ - \bar{\zeta}_j^+)^2 \right\}}, \\
 {}^{\omega}\sigma_j(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij} - \bar{\eta}_j) (\zeta_{ij} - \bar{\zeta}_j)}{\max \left\{ \sum_{i=1}^n \omega_i (\eta_{ij} - \bar{\eta}_j)^2, \sum_{i=1}^n \omega_i (\zeta_{ij} - \bar{\zeta}_j)^2 \right\}},
 \end{aligned} \tag{51}$$

$j = 1, 2, \dots, m$.

Proof. The proof is obvious. $\square$

Theorem 15. If A_N and B_N are CmpFSs which are dictated by the weights, then

- (a) the correlation σ_{ω} is symmetrical, that is, $\sigma_{\omega}(A_N, B_N) = \sigma_{\omega}(B_N, A_N)$
- (b) the correlation σ_{ω} finds the interval $[-1, 1]$, that is, $-1 \leq \sigma_{\omega}(A_N, B_N) \leq 1$
- (c) $\sigma_{\omega}(A_N, B_N) = 1$ if $A_N = B_N$

The two types of the weighted correlation coefficients are equally effective no matter what types of weights are assigned to the alternatives. However, the weights should follow the mentioned conditions. If there are “ n ” alternatives and are affected by the weights $((1/n), (1/n), \dots, (1/n))$, then we see that

$$\begin{aligned}
 {}^{\omega}\partial_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\sqrt{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-)^2} \sqrt{\sum_{i=1}^n \omega_i (\zeta_{ij}^- - \bar{\zeta}_j^-)^2}} \\
 &= \frac{\sum_{i=1}^n (1/n) (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\sqrt{\sum_{i=1}^n (1/n) (\eta_{ij}^- - \bar{\eta}_j^-)^2} \sqrt{\sum_{i=1}^n (1/n) (\zeta_{ij}^- - \bar{\zeta}_j^-)^2}} \\
 &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\sqrt{\sum_{i=1}^n (\eta_{ij}^- - \bar{\eta}_j^-)^2} \sqrt{\sum_{i=1}^n (\zeta_{ij}^- - \bar{\zeta}_j^-)^2}} = \partial_j^-(A_N, B_N), \\
 {}^{\omega}\sigma_j^-(A_N, B_N) &= \frac{\sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\max \left\{ \sum_{i=1}^n \omega_i (\eta_{ij}^- - \bar{\eta}_j^-)^2, \sum_{i=1}^n \omega_i (\zeta_{ij}^- - \bar{\zeta}_j^-)^2 \right\}} \\
 &= \frac{\sum_{i=1}^n (1/n) (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\max \left\{ \sum_{i=1}^n (1/n) (\eta_{ij}^- - \bar{\eta}_j^-)^2, \sum_{i=1}^n (1/n) (\zeta_{ij}^- - \bar{\zeta}_j^-)^2 \right\}} \\
 &= \frac{\sum_{i=1}^n (\eta_{ij}^- - \bar{\eta}_j^-) (\zeta_{ij}^- - \bar{\zeta}_j^-)}{\max \left\{ \sum_{i=1}^n (\eta_{ij}^- - \bar{\eta}_j^-)^2, \sum_{i=1}^n (\zeta_{ij}^- - \bar{\zeta}_j^-)^2 \right\}} = \sigma_j^-(A_N, B_N),
 \end{aligned} \tag{52}$$

and similarly others. This shows that the above defined correlations are invariant under the weights $((1/n), (1/n), \dots, (1/n))$. So, whenever we are encountered by such

weights, we can simply ignore them and proceed in an unweighted manner. That is, the weighted alternatives can be treated as unweighted ones.

Step 1: construct the pattern $\tilde{\mathfrak{P}}$ which is to be categorized.

Step 2: assemble the known patterns $\tilde{\mathfrak{P}}^{(1)}, \tilde{\mathfrak{P}}^{(2)}, \dots, \tilde{\mathfrak{P}}^{(i)}$, over a discourse $\aleph$, among which the recognition is to be performed.

Step 3 (a): find the correlation coefficients of the pattern $\tilde{\mathfrak{P}}$ from the known patterns $\tilde{\mathfrak{P}}^{(1)}, \tilde{\mathfrak{P}}^{(2)}, \dots, \tilde{\mathfrak{P}}^{(i)}$ using formula (34) or (39).

Step 3 (b): if the alternatives from the discourse $\aleph$ follow some weights, then find the respective weighted correlation coefficients using formula (46) or (50).

Step 4: if the pattern $\tilde{\mathfrak{P}}^{(i)}$ finds the greatest correlation with $\tilde{\mathfrak{P}}$, then the pattern $\tilde{\mathfrak{P}}$ belongs to $\tilde{\mathfrak{P}}^{(i)}$.

ALGORITHM 1: Steps of the proposed approach.

5. Applications of the Proposed Correlation Measures

Correlation actions are common indicators and effective approaches to address MCDM problems. In different uncertain environments, various scholars have used correlations to solve such problems efficiently and effectively. The correlations defined in this article are used here in this section to accurately recognize the patterns and medical diagnosis. Pattern recognition (PR) finds its roots in machine learning and statistics. Due to the availability of big data, the recognition problems are subjected to automatization. Bishop [38] gave a modern definition of PR as, “The field of pattern recognition is handled with the automated design of swings in data through the use of computer algorithms and with the use of these rhythms to take operations such as ordering the data into separate classes [38].” PR algorithms commonly endeavor to present a sensible answer for all attainable inputs and to achieve “most likely” matching of the inputs, using the statement of their statistical variation. They are classified according to the variety of learning procedures used to produce the output value.

5.1. Illustrative Examples. In the following example, we use our proposed correlations to categorize an unknown pattern in CmPF environment (Algorithm 1).

The flowchart for the proposed algorithm is demonstrated in Figure 2.

Example 7. In this example, we give a solution to a pattern recognition problem. Suppose a mining firm discovers a new type of carbon while working on a site. The experts analyze it completely keeping in view its structure, properties, and smell and give their opinion about its pattern. According to the expert’s point of view, suppose its pattern comes out to be a cubic 3-polar fuzzy set over the discourse $\aleph = \{\kappa_1, \kappa_2, \kappa_3\}$, given by

$$\mathfrak{P} = \begin{matrix} \kappa_1 & [0.10, 0.12], [0.30, 0.36], [0.26, 0.4], 0.71, 0.68, 0.90 \\ \kappa_2 & [0.40, 0.47], [0.50, 0.60], [0.47, 0.57], 0.80, 0.80, 0.76 \\ \kappa_3 & [0.50, 0.55], [0.60, 0.66], [0.23, 0.34], 0.56, 0.69, 0.44 \end{matrix} \quad (53)$$

The experts are aware that there are three patterns (allotropes) of carbon, that is, diamond, graphite, and fullerenes, which are already known. Now, the problem is to identify the discovered pattern among the three already known ones. For this purpose, the three known allotropes are reanalyzed and the same type of CmPF patterns is assigned to them:

$$\begin{aligned} A_{\aleph}(\text{diamond})\tilde{\mathfrak{P}}^{(1)} &= \begin{matrix} \kappa_1 & [0.20, 0.40], [0.29, 0.48], [0.32, 0.55], 0.57, 0.66, 0.42 \\ \kappa_2 & [0.32, 0.58], [0.25, 0.39], [0.48, 0.56], 0.60, 0.50, 0.40 \\ \kappa_3 & [0.49, 0.62], [0.31, 0.43], [0.36, 0.48], 0.20, 0.35, 0.50 \end{matrix} \\ (\text{graphite})\tilde{\mathfrak{P}}^{(2)} &= \begin{matrix} \kappa_1 & [0.13, 0.69], [0.33, 0.42], [0.46, 0.50], 0.39, 0.47, 0.68 \\ \kappa_2 & [0.17, 0.48], [0.25, 0.36], [0.17, 0.35], 0.42, 0.39, 0.72 \\ \kappa_3 & [0.42, 0.53], [0.30, 0.42], [0.22, 0.37], 0.48, 0.56, 0.75 \end{matrix} \\ (\text{fullerenes})\tilde{\mathfrak{P}}^{(3)} &= \begin{matrix} \kappa_1 & [0.39, 0.48], [0.40, 0.60], [0.32, 0.55], 0.16, 0.53, 0.65 \\ \kappa_2 & [0.31, 0.67], [0.39, 0.51], [0.23, 0.49], 0.50, 0.66, 0.77 \\ \kappa_3 & [0.44, 0.55], [0.42, 0.69], [0.28, 0.48], 0.39, 0.49, 0.70 \end{matrix} \end{aligned} \quad (54)$$

FIGURE 2: Flowchart for the proposed algorithm.

We find the correlations of $\mathfrak{P}$ from $\tilde{\mathfrak{P}}^{(1)}, \tilde{\mathfrak{P}}^{(2)}, \tilde{\mathfrak{P}}^{(3)}$ (using formulae (34) or (39), respectively) as follows:

$$\begin{aligned}
 \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(1)}) &= 0.33, \\
 \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(2)}) &= -0.49, \\
 \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(3)}) &= 0.16, \\
 \sigma(\mathfrak{P}, \tilde{\mathfrak{P}}^{(1)}) &= 0.24, \\
 \sigma(\mathfrak{P}, \tilde{\mathfrak{P}}^{(2)}) &= -0.20, \\
 \sigma(\mathfrak{P}, \tilde{\mathfrak{P}}^{(3)}) &= 0.10.
 \end{aligned} \tag{55}$$

This shows that $\mathfrak{P}$ finds the greatest correlations with $\tilde{\mathfrak{P}}^{(1)}$, so the pattern $\mathfrak{P}$ belongs to $\tilde{\mathfrak{P}}^{(1)}$.

However, if the alternatives $\kappa_1, \kappa_2, \kappa_3$ follow the weights $\omega = (0.29, 0.33, 0.38)$, then we find the respective weighted correlations (using formulae (46) and (50), respectively) as follows:

$$\begin{aligned}
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(1)}) &= 0.35, \\
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(2)}) &= -0.47, \\
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(3)}) &= 0.18, \\
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(1)}) &= 0.24, \\
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(2)}) &= -0.19, \\
 \omega \partial(\mathfrak{P}, \tilde{\mathfrak{P}}^{(3)}) &= 0.11.
 \end{aligned} \tag{56}$$

The same result is obtained using the weighted correlations. Hence, our proposed correlations are consistent with each other.

Example 8. Suppose a patient P is to be diagnosed in a medical center. A medical team examines him completely, and the following symptoms are observed in the patient:

$$S = \{(\kappa_1, \text{headache}), (\kappa_2, \text{cough}), (\kappa_3, \text{chest pain}), (\kappa_4, \text{shortness of breath})\}. \tag{57}$$

TABLE 2: Rating values of the patient in the CmPFS environment.

P	Patient
κ_1	[0.20, 0.40], [0.18, 0.39], [0.15, 0.27], 0.69, 0.72, 0.88
κ_2	[0.42, 0.60], [0.30, 0.50], [0.19, 0.33], 0.40, 0.60, 0.80
κ_3	[0.55, 0.62], [0.42, 0.55], [0.18, 0.25], 0.12, 0.27, 0.39
κ_4	[0.49, 0.69], [0.52, 0.64], [0.42, 0.49], 0.43, 0.55, 0.67

TABLE 3: Rating values of each diagnosis in the CmPFS environment.

Y_1	Corona
κ_1	[0.32, 0.44], [0.22, 0.27], [0.09, 0.27], 0.42, 0.53, 0.64
κ_2	[0.14, 0.26], [0.30, 0.39], [0.42, 0.51], 0.48, 0.59, 0.68
κ_3	[0.16, 0.43], [0.33, 0.40], [0.32, 0.47], 0.20, 0.30, 0.40
κ_4	[0.21, 0.33], [0.49, 0.56], [0.40, 0.50], 0.17, 0.39, 0.44
Y_2	Diarrhea
κ_1	[0.16, 0.24], [0.23, 0.36], [0.27, 0.39], 0.39, 0.47, 0.55
κ_2	[0.04, 0.17], [0.14, 0.24], [0.37, 0.47], 0.40, 0.45, 0.60
κ_3	[0.24, 0.36], [0.39, 0.47], [0.45, 0.57], 0.37, 0.49, 0.54
κ_4	[0.32, 0.48], [0.42, 0.54], [0.40, 0.51], 0.25, 0.27, 0.33
Y_3	Asthma
κ_1	[0.31, 0.43], [0.40, 0.55], [0.60, 0.71], 0.41, 0.39, 0.55
κ_2	[0.24, 0.36], [0.26, 0.40], [0.42, 0.49], 0.47, 0.56, 0.63
κ_3	[0.13, 0.28], [0.09, 0.13], [0.37, 0.47], 0.44, 0.53, 0.72
κ_4	[0.15, 0.33], [0.29, 0.37], [0.36, 0.56], 0.80, 0.90, 0.70
Y_4	Sore throat
κ_1	[0.41, 0.54], [0.29, 0.47], [0.68, 0.73], 0.39, 0.48, 0.57
κ_2	[0.38, 0.59], [0.39, 0.47], [0.40, 0.51], 0.41, 0.47, 0.56
κ_3	[0.62, 0.80], [0.55, 0.65], [0.32, 0.46], 0.62, 0.55, 0.39
κ_4	[0.40, 0.70], [0.38, 0.42], [0.62, 0.70], 0.60, 0.72, 0.54

The set of diagnoses which the patient can observe is given by Y_1 (corona), Y_2 (diarrhea), Y_3 (asthma), Y_4 (sore throat). Suppose the expert's point of views about the patient, taking these symptoms into account, comes out to be a CmPFS (Table 2).

Therefore, each diagnosis must also be viewed as CmPFS, given by Table 3.

Our aim is to find that to which disease the patient belongs, that is, to examine that from which disease the patient is suffering from. For this purpose, we find the following correlation coefficients (using formulae (34) or (39), respectively):

$$\begin{aligned}
\partial(P, Y_1) &= 0.42, \\
\partial(P, Y_2) &= 0.33, \\
\partial(P, Y_3) &= -0.50, \\
\partial(P, Y_4) &= 0.28, \\
\sigma(P, Y_1) &= 0.35, \\
\sigma(P, Y_2) &= 0.27, \\
\sigma(P, Y_3) &= -0.31, \\
\sigma(P, Y_4) &= 0.21.
\end{aligned} \tag{58}$$

However, if the symptoms bear some credibility weights $\omega = (0.17, 0.18, 0.25, 0.40)$, then we find the respective weighted correlations (using formulae (46) and (50), respectively) as follows:

$$\begin{aligned}
\omega \partial(P, Y_1) &= 0.42, \\
\omega \partial(P, Y_2) &= 0.35, \\
\omega \partial(P, Y_3) &= -0.43, \\
\omega \partial(P, Y_4) &= 0.29, \\
\omega \partial(P, Y_1) &= 0.37, \\
\omega \partial(P, Y_2) &= 0.28, \\
\omega \partial(P, Y_3) &= -0.24, \\
\omega \partial(P, Y_4) &= 0.22.
\end{aligned} \tag{59}$$

We have used all of the four proposed correlations, and we found Y_1 as optimal solution. Hence, the patient P is suffering from corona.

6. Conclusion

The paper aims to present the novel correlation coefficient between the pairs of the cubic m -polar fuzzy sets. The considered CmPFS is a more integrated set to handle the uncertainties in the data than the existing m -polar fuzzy set and the cubic set. The advantage of the CmPFS over these existing sets is that it represents the information about the element by using m fuzzy intervals as well as m fuzzy numbers. Keeping these advantages, we investigate some operational laws about the CmPFS and hence investigate their properties. Furthermore, we elaborate on the information energy and hence define the correlation coefficient between the pairs of CmPFS. The major advantage of the proposed measure is that it considered the range of correlation measures as $[-1, 1]$ instead of $[0, 1]$ only. Therefore, the proposed measure has a stronger measure of the negative relationship between the variables also. Furthermore, the applicability of the proposed measures has been demonstrated through a case study of pattern recognition as well as medical diagnosis problems. The results of the proposed method are compared with several existing approaches which reveal the effectiveness of the presented work. In future studies, we shall study applications of the proposed methodology under the different fuzzy environment and solve some more practical problems [39–45].

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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